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Max-convolution semigroups and extreme values in limit theorems for the free multiplicative convolution

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We investigate relations between additive convolution semigroups and max-convolution semigroups through the law of large numbers for the free multiplicative convolution. Based on these relations, we give a formula related with the Belinschi–Nica semigroup and the max-Belinschi–Nica semigroup. Finally, we give several limit theorems for classical, free and Boolean extreme values.

Keywords: Belinschi–Nica semigroup; Bercovici–Pata bijection; extreme values; free multiplicative convolution; max-convolution

1. Introduction

Denote by $\mathcal{P}$ and $\mathcal{P}_+$ the set of all probability measures on $\mathbb{R}$ and $[0, \infty)$, respectively. If $\mu \in \mathcal{P}_+$, then the distribution function of μ takes zero on the negative real line. In this paper, we denote by $\mu([0, \cdot])$ the distribution function of $\mu \in \mathcal{P}_+$ and we understand $\mu([0, x]) = 0$ if $x < 0$.

In classical probability theory, many mathematicians studied limit laws of addition and multiplication of large numbers of independent random variables. In free probability theory, we obtain limit laws of addition of large numbers of freely independent real random variables (selfadjoint operators): for any $\mu \in \mathcal{P}$ with the moment α, we have $D_{1/n}(\mu^{\boxplus n}) \xrightarrow{w} \delta_\alpha$ as $n \to \infty$ (see Lindsay and Pata [24]), where $\boxplus$ is called the *free additive convolution* and $\mu^{\boxplus n} := \overbrace{\mu \boxplus \cdots \boxplus \mu}^{n \text{ times}}$ (for details, see Voiculescu [37], Maassen [25], Bercovici and Voiculescu [11]) and D_c is the *dilation* that is, $D_c(\mu)(B) := \mu(c^{-1}B)$ for all $c > 0$ and Borel sets B in $\mathbb{R}$. Similarly, Tucci studied limit laws of multiplication of large numbers of freely independent bounded positive random variables (see Tucci [34]). After that, Haagerup and Möller extended Tucci's limit theorem to (unbounded) positive random variables: for any $\mu \in \mathcal{P}_+$, there exists a unique $\nu \in \mathcal{P}_+$ such that $(\mu^{\boxtimes n})^{1/n} \xrightarrow{w} \nu$ as $n \to \infty$ (see Haagerup and Möller [20]), where $\boxtimes$ is called the *free multiplicative convolution* and $\mu^{\boxtimes n} := \overbrace{\mu \boxtimes \cdots \boxtimes \mu}^{n \text{ times}}$ (for details, see Voiculescu [38], Bercovici and Voiculescu [11]) and $\mu^{1/n}$ is the distribution of $X^{1/n}$ if $X \sim \mu$. We denote by $\Phi(\mu)$ the weak limit law of $(\mu^{\boxtimes n})^{1/n}$ as $n \to \infty$.

Ben Arous and Voiculescu [8] introduced the *free max-convolution* $\boxed{\vee}$. This operation means the distribution of the maximum of freely independent real random variables with respect to the spectral order (which was introduced by Olson [29]). One of the most important classes of distributions in free max-probability is that of the *free extreme value distributions*. This distribution has many similarities to the classical extreme value distribution which are the limit laws of the maximum of large numbers of identically distributed random variables (see, e.g., Resnick [30]). Benaych-Georges and Cabanal-Duvillard [9] constructed random matrix models which realize free extreme value distributions. Furthermore, Grela and Nowak [15] obtained relations between free extreme values, order statistics (e.g., Peak-Over-Threshold method and generalized extreme values) and random matrices.

Vargas and Voiculescu [36] introduced the *Boolean max-convolution* $\cup\!\!\!\!\vee$. This convolution is the distribution of the maximum of Boolean independent random variables with respect to spectral order. The maximum works for Boolean independent nonnegative random variables. In the same way as in the free max-case, Voiculescu and Vargas identified the extreme value distributions with respect to the Boolean max-convolution as the *Dagum distribution*.

In Section 3.1, we give a formula linking the free additive convolution $\boxplus$ and the free max-convolution $\boxed{\vee}$ through the above operator $\Phi : \mathcal{P}_+ \to \mathcal{P}_+$.

Theorem 1.1. *Consider $\mu \in \mathcal{P}_+$. Then we have*

$$\Phi\big(D_{1/t}\big(\mu^{\boxplus t}\big)\big) = \Phi(\mu)^{\boxed{\vee} t}, \quad t \geq 1,$$

where the measure $\mu^{\boxplus t}$ is defined in Section 2.1 and the measure $\mu^{\boxed{\vee} t}$ is defined by (2.4) for $t \geq 1$. In Section 3.2, we prove that the operator Φ connects the Boolean additive convolution $\uplus$ (see Speicher and Woroudi [33]) and the Boolean max-convolution $\cup\!\!\!\!\vee$ as follows.

Theorem 1.2. *Consider $\mu \in \mathcal{P}_+$. Then we have*

$$\Phi\big(D_{1/t}\big(\mu^{\uplus t}\big)\big) = \Phi(\mu)^{\cup\!\!\!\!\vee t}, \quad t > 0,$$

where the measure $\mu^{\uplus t}$ is defined in Section 2.1 and the measure $\mu^{\cup\!\!\!\!\vee t}$ is defined by (2.5) for $t > 0$.

Next, we define two operators B_t and $B_t^{\vee}$ as follows:

$$B_t(\mu) := \big(\mu^{\boxplus(1+t)}\big)^{\uplus \frac{1}{1+t}}, \quad t \geq 0, \mu \in \mathcal{P};$$

$$B_t^{\vee}(\mu) := \big(\mu^{\boxed{\vee}(1+t)}\big)^{\cup\!\!\!\!\vee \frac{1}{1+t}}, \quad t \geq 0, \mu \in \mathcal{P}_+.$$

It is known that $B_t \circ B_s = B_{t+s}$ and $B_t^{\vee} \circ B_s^{\vee} = B_{t+s}^{\vee}$ for all $t, s \geq 0$, so that the families $\{B_t\}_{t\geq 0}$ and $\{B_t^{\vee}\}_{t\geq 0}$ are semigroups with respect to the composition of operators. The semigroups $\{B_t\}_{t\geq 0}$ and $\{B_t^{\vee}\}_{t\geq 0}$ are called the *Belinschi–Nica semigroup* (see Belinschi and Nica [7]) and the *max-Belinschi–Nica semigroup* (see Ueda [35]), respectively. Considering these semigroups is important to understand relations between free and Boolean type limit theorems or free-max and Boolean-max type limit theorems. In Section 3.3, by using the operator Φ, we show that the Belinschi–Nica semigroup $\{B_t\}_{t\geq 0}$ and the max-Belinschi–Nica semigroup $\{B_t^{\vee}\}_{t\geq 0}$ are closely intertwined with each other.

Theorem 1.3. *Consider $\mu \in \mathcal{P}_+$. Then we have*

$$\Phi \circ B_t(\mu) = B_t^{\vee} \circ \Phi(\mu), \quad t \geq 0.$$

In Section 3.4, we construct an operator which connects the classical additive convolution $*$ and the classical max-convolution $\vee$. A probability measure μ is said to be (classically) *infinitely divisible* if for each $n \in \mathbb{N}$ there is $\mu_n \in \mathcal{P}$ such that $\mu = \mu_n^{*n}$. Let ID_+ be the set of all (classically) *infinitely divisible distributions* on $[0, \infty)$ (for details of infinitely divisible distributions, see, e.g., Sato [32]). An operator $\Psi : \mathrm{ID}_+ \to \mathcal{P}_+$ is defined by

$$\Psi := \mathcal{X}^{\vee} \circ \Phi \circ \mathcal{X}^{-1},$$

where the operator $\mathcal{X}$ is defined by

$$\mathcal{X} := \Lambda^{-1} \circ B_1,$$

and Λ is the *Bercovici–Pata bijection* (see Bercovici and Pata [10]). Note that the operator $\mathcal{X}$ is bijective from $\mathcal{P}_+$ to ID_+ and it is called the *Boolean-classical Bercovici–Pata bijection* (see Bercovici and Pata [10], Belinschi and Nica [7]). Moreover, for any $\mu \in \mathcal{P}_+$, the measure $\mathcal{X}^\vee(\mu) \in \mathcal{P}_+$ is characterized by

$$\mathcal{X}^\vee(\mu)\big([0,\cdot]\big) := \exp\left[1 - \frac{1}{\mu([0,\cdot])}\right].$$

Moreover, it is understood that $\mathcal{X}^\vee(\mu)([0,x]) = 0$ if $\mu([0,x]) = 0$ for some $x \in \mathbb{R}$. The operator $\mathcal{X}^\vee$ is called the *Boolean-classical max-Bercovici–Pata bijection* which was firstly introduced by Vargas and Voiculescu [36]. Then we obtain the following formula.

Theorem 1.4. *Consider $\mu \in \mathrm{ID}_+$. Then we have*

$$\Psi\big(D_{1/t}\big(\mu^{*t}\big)\big) = \Psi(\mu)^{\vee t}, \quad t > 0.$$

In Section 4, we compute some probability measures in the classes $\Phi(\mathcal{P}_+)$ and $\Psi(\mathcal{P}_+)$. As one of the most important computations, we show that the operator Φ connects the free/Boolean stable laws and the free/Boolean extreme values, respectively. As one more, we show that the operator Ψ maps the classical stable laws to the classical extreme values.

In Section 5, we give a few of limit theorems for the free and Boolean extreme values by using limit theorems for the free multiplicative convolution. Through the discussion in Section 5, we establish that the Marchenko–Pastur law is closely related with the free and Boolean extreme values.

2. Preliminaries

A pair $(\mathcal{A}, \varphi)$ is called the *noncommutative probability space* if $\mathcal{A}$ is a unital $*$-algebra over $\mathbb{C}$ and $\varphi : \mathcal{A} \to \mathbb{C}$ is a linear functional on $\mathcal{A}$ such that $\varphi(1_{\mathcal{A}}) = 1$ and $\varphi(X^*X) \geq 0$ for all $X \in \mathcal{A}$, where $1_{\mathcal{A}}$ is the unit element in $\mathcal{A}$. An element $X \in \mathcal{A}$ is called a *noncommutative random variable*. In particular, an element $X \in \mathcal{A}$ is called a *real (noncommutative) random variable* if X is selfadjoint, that is, $X = X^*$. For a real (noncommutative) random variable X and $\mu \in \mathcal{P}$, we denote by $X \sim \mu$ if X is distributed as μ (see Nica and Speicher [28]). We also denote by $\mu = \mu_X$ when $X \sim \mu$ (in Section 2.2).

A family $\{\mathcal{A}_i\}_{i \in I}$ of unital $*$-subalgberas of $\mathcal{A}$, is called *freely independent* if

$$\varphi(X_1 X_2 \cdots X_k) = 0,$$

whenever we have

- k is a positive integer;
- $X_j \in \mathcal{A}_{i_j}$ $(i_j \in I)$ and $\varphi(X_j) = 0$ for all $j = 1, \dots, k$;
- $i_1 \neq i_2, i_2 \neq i_3, \cdots, i_{k-1} \neq i_k$.

A family $\{X_i\}_{i \in I}$ of noncommutative random variables in $\mathcal{A}$, is called freely independent if a family of unital *-subalgebras generated by X_i $(i \in I)$, is freely independent (see Voiculescu, Dykema and Nica [39], Nica and Speicher [28] for details).

A family $\{\mathcal{A}_i\}_{i\in I}$ of (usually not unital) *-subalgebras of $\mathcal{A}$ is called *Boolean independent* if

$$\varphi(X_1X_2\cdots X_k)=\varphi(X_1)\varphi(X_2)\cdots\varphi(X_k),$$

for any $k\geq 1$ whenever $X_j\in\mathcal{A}_{i_j}$ with $i_j\in I$ and $i_1\neq i_2,\ i_2\neq i_3,\ \cdots,\ i_{k-1}\neq i_k$. Furthermore, a family $\{X_i\}_{i\in I}$ of noncommutative random variables in $\mathcal{A}$, is called Boolean independent if a family of *-subalgebras generated by the element $X_i\in\mathcal{A}$ $(i\in I)$, is Boolean independent (see Speicher and Woroudi [33] for details).

2.1. Atoms of the free and Boolean additive convolutions

Let $X\sim\mu$ and $Y\sim\nu$ be freely independent noncommutative real random variables. We define $\mu\boxplus\nu$ as the distribution of $X+Y$ and the operation $\boxplus$ is called the *free additive convolution* which was introduced by Voiculescu [37] (see also Maassen [25], Bercovici and Voiculescu [11]). For $n\in\mathbb{N}$ and $\mu\in\mathcal{P}$, the n-fold free convolution $\overbrace{\mu\boxplus\cdots\boxplus\mu}^{n\text{ times}}$ is denoted by $\mu^{\boxplus n}$. It is known that for any $\mu\in\mathcal{P}$, the discrete semigroup $\{\mu^{\boxplus n}\}_{n\in\mathbb{N}}$ can be embedded in a continuous family $\{\mu_t\}_{t\geq 1}$ of probability measures on $\mathbb{R}$ such that $\mu_1=\mu$ and $\mu_t\boxplus\mu_s=\mu_{t+s}$ for $t,s\geq 1$. The existence of μ_t for large values of t was shown in Bercovici and Voiculescu [12] in case μ has compact support. After that, Nica and Speicher [27] proved the existence of μ_t for $t\geq 1$ in case μ has compact support. Finally, Belinschi and Bercovici [6] proved the existence of μ_t for $t\geq 1$ in case μ is a general probability measure on $\mathbb{R}$. More precisely, for any $\mu\in\mathcal{P}$ and $t\geq 1$, there exists a probability measure μ_t satisfying $R_{\mu_t}(z)=tR_\mu(z)$ for z in the common domain of the two functions, where the function R_μ is called the R-transform (or free cumulant transform) of μ. We denote by $\mu^{\boxplus t}$ the above probability measure μ_t for $\mu\in\mathcal{P}$ and $t\geq 1$.

By Belinschi and Bercovici [6], we get a location of an atom of $\mu^{\boxplus t}$.

Lemma 2.1. *Consider $\mu\in\mathcal{P}$, $t>1$ and $\alpha\in\mathbb{R}$. Then $\mu^{\boxplus t}$ has an atom α if and only if $\mu(\{\alpha/t\})>1-t^{-1}$. In this case, we have*

$$\mu^{\boxplus t}(\{\alpha\})=t\mu\left(\left\{\frac{\alpha}{t}\right\}\right)-(t-1).$$

For $\mu\in\mathcal{P}$, we define the following functions:

$$G_\mu(z):=\int_{\mathbb{R}}\frac{1}{z-x}\,d\mu(x),\qquad F_\mu(z):=\frac{1}{G_\mu(z)},\quad z\in\mathbb{C}^+.$$

The function G_μ is called the *Cauchy transform of* μ and it is analytic on the upper complex plane $\mathbb{C}^+$ taking values in the lower complex plane $\mathbb{C}^-$.

For a function $f:\mathbb{C}^+\to\mathbb{C}\cup\{\infty\}$, and $x\in\mathbb{R}$, we say that the *nontangential limit of f at x* exists if the limit $\lim_{z\to x,z\in\Gamma_\lambda(x)}f(z)$ exists for all $\lambda>0$, where $\Gamma_\lambda(x):=\{z\in\mathbb{C}^+:|\operatorname{Re}z-x|<\lambda\operatorname{Im}z\}$. We denote by $\sphericalangle\text{-}\lim_{z\to x}f(z)$ the nontangential limit of f at x.

In particular, for $\alpha\in\mathbb{R}$, we denote by $F_\mu(\alpha)$ the nontangential limit of F_μ at α if the limit exists, that is, $F_\mu(\alpha):=\sphericalangle\text{-}\lim_{z\to\alpha}F_\mu(z)$. The following lemma provides a useful criterion for locating an atom α of μ. The result is not new, but since we do not know a reference, we give a proof.

Lemma 2.2. *Consider $\mu \in \mathcal{P}$ and $\alpha \in \mathbb{R}$. Then μ has an atom α if and only if $F_\mu(\alpha) = 0$ and the Julia–Carathéodory derivative $F'_\mu(\alpha)$ exists and is finite where*

$$F'_\mu(\alpha) := \sphericalangle\text{-}\lim_{z\to\alpha} \frac{F_\mu(z) - F_\mu(\alpha)}{z-\alpha}.$$

In this case, we have $\mu(\{\alpha\}) = F'_\mu(\alpha)^{-1}$.

Proof. If μ has an atom α, then the nontangential limit of the Cauchy transform G_μ at α is infinite. Therefore, $F_\mu(\alpha) = 0$. Furthermore, we obtain

$$\sphericalangle\text{-}\lim_{z\to\alpha} \frac{F_\mu(z)}{z-\alpha} = \sphericalangle\text{-}\lim_{z\to\alpha} \frac{1}{(z-\alpha)G_\mu(z)} = \frac{1}{\mu(\{\alpha\})},$$

where it follows from Lemma 7.1 in Bercovici and Voiculescu [13] that the last equation holds. Therefore, $F'_\mu(\alpha)$ is finite and $\mu(\{\alpha\}) = F'_\mu(\alpha)^{-1}$. The converse implication follows from Lemma 7.1 in Bercovici and Voiculescu [13]. □

Let $X \sim \mu$ and $Y \sim \nu$ be Boolean independent noncommutative real random variables. We define $\mu \uplus \nu$ as the distribution of $X + Y$ and the operation $\uplus$ is called the *Boolean additive convolution* which was introduced by Speicher and Woroudi [33]. This convolution is characterized by the *self-energy function* which is defined by

$$E_\mu(z) := z - F_\mu(z), \quad \mu \in \mathcal{P}, z \in \mathbb{C}^+,$$

that is, $E_{\mu\uplus\nu} = E_\mu + E_\nu$ for all $\mu, \nu \in \mathcal{P}$. Speicher and Woroudi [33] showed that for all $\mu \in \mathcal{P}$ and $t > 0$, there exists a unique $\mu_t \in \mathcal{P}$ such that $E_{\mu_t} = tE_\mu$. Write $\mu^{\uplus t} := \mu_t$ for all $\mu \in \mathcal{P}$ and $t \geq 0$. A family $\{\mu^{\uplus t}\}_{t\geq 0}$ is a semigroup such that $\mu_0 = \delta_0$ and $\mu^{\uplus t} \uplus \mu^{\uplus s} = \mu^{\uplus(t+s)}$ for all $t, s > 0$. Therefore, we have

$$F_{\mu^{\uplus t}}(z) = (1-t)z + tF_\mu(z), \quad z \in \mathbb{C}^+. \tag{2.1}$$

According to the above discussion, we obtain the following implication.

Corollary 2.1. *Consider $\mu \in \mathcal{P} \setminus \{\delta_0\}$. Then $\mu(\{0\}) \neq 0$ if and only if $\mu^{\uplus t}(\{0\}) \neq 0$ for some (any) $t > 0$. In this case, we have*

$$\mu^{\uplus t}\big(\{0\}\big) = \frac{\mu(\{0\})}{t - (t-1)\mu(\{0\})}, \quad t > 0. \tag{2.2}$$

Proof. Consider firstly $\mu(\{0\}) \neq 0$. By Lemma 2.2, we have $F_\mu(0) = 0$ and $F'_\mu(0) < \infty$. By the equation (2.1), we have

$$\sphericalangle\text{-}\lim_{z\to 0} F_{\mu^{\uplus t}}(z) = \sphericalangle\text{-}\lim_{z\to 0}\big((1-t)z + tF_\mu(z)\big) = tF_\mu(0) = 0.$$

Therefore, $F_{\mu^{\uplus t}}(0)$ exists and it takes zero. By using the equation (2.1) again, we have

$$\sphericalangle\text{-}\lim_{z\to 0} \frac{F_{\mu^{\uplus t}}(z)}{z} = \sphericalangle\text{-}\lim_{z\to 0} \frac{(1-t)z + tF_\mu(z)}{z}$$

$$
\begin{aligned}
&= \sphericalangle\text{-}\lim_{z\to 0}\left((1-t)+t\frac{F_\mu(z)}{z}\right)\\
&= (1-t)+tF'_\mu(0)<\infty.
\end{aligned}
$$

Hence, $F'_{\mu^{\uplus t}}(0)<\infty$. Consequently, we have $\mu^{\uplus t}(\{0\})\neq 0$ by Lemma 2.2.

Next, we assume that $\mu^{\uplus t}(\{0\})\neq 0$ for any (some) $t>0$. Then $F_{\mu^{\uplus t}}(0)=0$ and $F'_{\mu^{\uplus t}}(0)<\infty$ by Lemma 2.2. By the equation (2.1), we have

$$
\sphericalangle\text{-}\lim_{z\to 0} F_\mu(z) = \sphericalangle\text{-}\lim_{z\to 0}\frac{F_{\mu^{\uplus t}}(z)-(1-t)z}{t}=0.
$$

Thus $F_\mu(0)$ exists and it takes zero. Furthermore, we obtain

$$
\sphericalangle\text{-}\lim_{z\to 0}\frac{F_\mu(z)}{z} = \sphericalangle\text{-}\lim_{z\to 0}\frac{F_{\mu^{\uplus t}}(z)-(1-t)z}{tz} = \frac{1}{t}F'_{\mu^{\uplus t}}(0)-\frac{1-t}{t}<\infty.
$$

Hence, $F'_\mu(0)<\infty$. Consequently, we have $\mu(\{0\})\neq 0$ by Lemma 2.2.

Finally, if $\mu(\{0\})\neq 0$, (and therefore $\mu^{\uplus t}(\{0\})\neq 0$), then we obtain

$$
\begin{aligned}
\mu^{\uplus t}\big(\{0\}\big) &= \frac{1}{F'_{\mu^{\uplus t}}(0)} = \sphericalangle\text{-}\lim_{z\to 0}\frac{z}{F_{\mu^{\uplus t}}(z)}\\
&= \sphericalangle\text{-}\lim_{z\to 0}\frac{z}{(1-t)z+tF_\mu(z)}\\
&= \sphericalangle\text{-}\lim_{z\to 0}\frac{\frac{z}{F_\mu(z)}}{(1-t)\frac{z}{F_\mu(z)}+t} = \frac{\mu(\{0\})}{t-(t-1)\mu(\{0\})}.
\end{aligned}
$$

□

2.2. Limit theorem for the free multiplicative convolution

For probability measures $\mu\in\mathcal{P}_+$ and $\nu\in\mathcal{P}$, we write $\mu\boxtimes\nu\in\mathcal{P}$ as the distribution of $\sqrt{X}Y\sqrt{X}$, where $X\geq 0$ and Y are freely independent random variables distributed as μ and ν, respectively. The operation $\boxtimes$ is called the *free multiplicative convolution*. This was first introduced by Voiculescu [38] as the distribution of multiplication of bounded random variables. Finally, it was extended to unbounded random variables (see Bercovici and Voiculescu [11]).

Consider $\mu\in\mathcal{P}_+\setminus\{\delta_0\}$. We define

$$
\Psi_\mu(z) := \int_0^\infty \frac{tz}{1-tz}\,d\mu(t),\quad z\in\mathbb{C}\setminus[0,\infty).
$$

Then its inverse function (namely Ψ_μ^{-1}) exists in a neighborhood of $(\mu(\{0\})-1,0)$. We define the *S-transform of* μ by setting

$$
S_\mu(z)=\frac{z+1}{z}\Psi_\mu^{-1}(z),\quad z\in\big(\mu\big(\{0\}\big)-1,0\big).
$$

Bercovici and Voiculescu [11] proved that for any $\mu,\nu\in\mathcal{P}_+\setminus\{\delta_0\}$, we have $S_{\mu\boxtimes\nu}=S_\mu S_\nu$ on the common interval where all three S-transforms are defined. From Belinschi [5], we know that an atom of $\mu \boxtimes \nu$ at 0 satisfies

$$(\mu \boxtimes \nu)\big(\{0\}\big) = \max\big\{\mu\big(\{0\}\big), \nu\big(\{0\}\big)\big\}.$$

Therefore, we get $\mu^{\boxtimes n}(\{0\}) = \mu(\{0\})$ for all $n \in \mathbb{N}$ by induction. Moreover, the following lemma provides the S-transforms of free/Boolean additive convolution powers of probability measures.

Lemma 2.3 (see Belinschi and Nica [7]). *For $\mu \in \mathcal{P}_+ \setminus \{\delta_0\}$, we have*

$$S_{\mu^{\boxplus t}}(z) = \frac{1}{t} S_\mu\left(\frac{z}{t}\right), \quad t \geq 1,$$
$$S_{\mu^{\uplus t}}(z) = \frac{1}{t} S_\mu\left(\frac{z}{t - z + tz}\right), \quad t > 0.$$

We give a functional property of the S-transform.

Lemma 2.4 (see Haagerup and Larsen [19], Haagerup and Möller [20]). *Consider $\mu \in \mathcal{P}_+$ not being a Dirac measure. Then S_μ is strictly decreasing on $(\mu(\{0\}) - 1, 0)$. Moreover, we have $S_\mu((\mu(\{0\}) - 1, 0)) = (b_\mu^{-1}, a_\mu^{-1})$, where $0 \leq a_\mu < b_\mu \leq \infty$ are defined by*

$$a_\mu := \left(\int_0^\infty x^{-1}\, d\mu(x)\right)^{-1}, \qquad b_\mu := \int_0^\infty x\, d\mu(x). \tag{2.3}$$

Note that if $\mu(\{0\}) > 0$, then we understand $a_\mu^{-1} = \infty$.

Tucci [34] and Haagerup and Möller [20] give the following limit theorem for the free multiplicative convolution.

Proposition 2.1. *Consider $\mu \in \mathcal{P}_+$. The sequence of probability measures $(\mu^{\boxtimes n})^{1/n}$, converges weakly to some probability measure (denoted by $\Phi(\mu)$) on $[0, \infty)$. If μ is a Dirac measure on $[0, \infty)$, then $\Phi(\mu) = \mu$. If μ is not a Dirac measure on $[0, \infty)$, then $\Phi(\mu)$ is uniquely determined by the identities*

$$\Phi(\mu)\big(\{0\}\big) = \mu\big(\{0\}\big), \qquad \Phi(\mu)\left(\left[0, \frac{1}{S_\mu(x-1)}\right]\right) = x,$$

for any $x \in (\mu(\{0\}), 1)$. The support of $\Phi(\mu)$ is the closure of (a_μ, b_μ).

Remark 2.1. Let $(\mathcal{M}, \tau)$ be a finite von Neumann algebra $\mathcal{M}$ with a normal faithful tracial state τ on $\mathcal{M}$. It was proved that the distribution $(\mu_{(T^*)^n T^n})^{1/n}$ converges weakly to the distribution $\mu_{|T|^2}$ as $n \to \infty$ for all bounded operators $T \in \mathcal{M}$ (see Haagerup and Schultz [22]). After that, Haagerup and Möller [20] extended this result to unbounded R-diagonal elements T affiliated with $\mathcal{M}$ such that $\tau(\log^+ |T|) < \infty$, by using Proposition 2.1 and

$$\mu_{(T^*)^n T^n} = \mu_{T^*T}^{\boxtimes n},$$

for all $n \in \mathbb{N}$ and unbounded R-diagonal elements T affiliated with $\mathcal{M}$ with $\tau(\log^+ |T|) < \infty$ (see Haagerup and Schultz [21]).

We prove that the operator Φ commutes with the dilation.

Lemma 2.5. *For all $c > 0$, we have $D_c \circ \Phi = \Phi \circ D_c$ on $\mathcal{P}_+$.*

Proof. For all $\mu \in \mathcal{P}_+$ and for all $n \in \mathbb{N}$, we have

$$D_c\big((\mu^{\boxtimes n})^{1/n}\big) = \big(D_{c^n}(\mu^{\boxtimes n})\big)^{1/n} = \big(D_c(\mu)^{\boxtimes n}\big)^{1/n}.$$

As $n \to \infty$, we obtain $D_c \circ \Phi(\mu) = \Phi \circ D_c(\mu)$. □

We give a relation between Φ and the S-transform as follows.

Lemma 2.6. *For all $\mu \in \mathcal{P}_+$ not being a Dirac measure and for all $x \in (a_\mu, b_\mu)$, we have*

$$\Phi(\mu)\big([0,x]\big) = S_\mu^{-1}\left(\frac{1}{x}\right) + 1,$$

and therefore $(a_\mu, b_\mu) = \{x : \Phi(\mu)([0,x]) \in (\mu(\{0\}), 1)\}$.

Proof. For all $x \in (a_\mu, b_\mu)$ we have $S_\mu^{-1}(1/x) \in (\mu(\{0\}) - 1, 0)$ by Lemma 2.4. Since

$$x = \frac{1}{S_\mu(S_\mu^{-1}(1/x) + 1 - 1)},$$

we have

$$\Phi(\mu)\big([0,x]\big) = \Phi(\mu)\left(\left[0, \frac{1}{S_\mu(S_\mu^{-1}(1/x) + 1 - 1)}\right]\right) = S_\mu^{-1}\left(\frac{1}{x}\right) + 1,$$

for all $x \in (a_\mu, b_\mu)$. □

Remark 2.2. By Proposition 2.1 and Lemma 2.6, the support of $\Phi(\mu)$ is the closure of $\{x : \Phi(\mu)([0,x]) \in (\mu(\{0\}), 1)\}$.

2.3. Max-convolutions

2.3.1. *Classical max-convolution*

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a classical probability space. For independent ($\mathcal{F}$-measurable) real random variables X and Y, we have

$$\mathbb{P}(X \vee Y \le x) = \mathbb{P}(X \le x, Y \le x) = \mathbb{P}(X \le x)\mathbb{P}(Y \le x), \quad x \in \mathbb{R},$$

where $X \vee Y := \max\{X, Y\}$. According to the above calculation, we define the *classical max-convolution* $\mu \vee \nu$ of $\mu, \nu \in \mathcal{P}$ as

$$\mu \vee \nu\big((-\infty, \cdot]\big) := \mu\big((-\infty, \cdot]\big)\nu\big((-\infty, \cdot]\big).$$

For $n \in \mathbb{N}$ and $\mu \in \mathcal{P}$, we define $\mu^{\vee n} := \overbrace{\mu \vee \cdots \vee \mu}^{n \text{ times}}$. More generally, for $t > 0$, we define

$$\mu^{\vee t}\big((-\infty, \cdot]\big) := \mu\big((-\infty, \cdot]\big)^t.$$

A non-degenerate distribution function F is said to be *max-stable* if for any $n \in \mathbb{N}$, there exist $a_n > 0$ and $b_n \in \mathbb{R}$ such that

$$F^{\vee n}(a_n \cdot + b_n) \xrightarrow{w} F(\cdot), \quad n \to \infty,$$

where $\xrightarrow{w}$ means convergence at every point of continuity of F. The max-stable distributions are characterized as follows.

Proposition 2.2 (see Fisher and Tippett [16], Fréchet [17], Gnedenko [18]). *A non-degenerate distribution function F is max-stable, if and only if there exist $a > 0$ and $b \in \mathbb{R}$ such that $F(ax + b)$ is one of the following distributions*:

$$C_I(x) := \exp\bigl(-\exp(-x)\bigr), \quad \textit{for } x \in \mathbb{R} \ (\textit{Gumbel distribution});$$

$$C_{II,\alpha}(x) := \begin{cases} \exp\bigl(-x^{-\alpha}\bigr), & x > 0, \\ 0, & x \le 0, \end{cases} \quad \textit{for } \alpha > 0 \ (\textit{Fr échet distribution});$$

$$C_{III,\alpha}(x) := \begin{cases} 1, & x > 0, \\ \exp\bigl(-(-x)^{\alpha}\bigr), & x \le 0, \end{cases} \quad \textit{for } \alpha > 0 \ (\textit{Weibull distribution}).$$

The above distributions are called *extreme value distributions*. In mathematical statistics, extreme value distributions are often used to analyze statistical data of rare phenomenon.

2.3.2. *Free max-convolution*

In free probability theory, the max-convolution was introduced by Ben Arous and Voiculescu [8]. Let $(\mathcal{M}, \tau)$ be a tracial W^*-probability space, that is, $\mathcal{M}$ is a von Neumann algebra and τ is a normal faithful tracial state on $\mathcal{M}$. We may assume that $\mathcal{M}$ acts on a Hilbert space $\mathcal{H}$. Denote by $\mathrm{Proj}(\mathcal{M})$ the set of all projections in $\mathcal{M}$ and denote by $\mathcal{M}_{sa}$ the set of all selfadjoint operators in $\mathcal{M}$. For $P, Q \in \mathrm{Proj}(\mathcal{M})$, we define $P \vee Q$ as the selfadjoint operator onto $(P \vee Q)\mathcal{H} := \mathrm{cl}(P\mathcal{H} + Q\mathcal{H})$. Then $P \vee Q \in \mathrm{Proj}(\mathcal{M})$ and it is the maximum of P and Q with respect to the usual operator order. However, the maximum of two selfadjoint operators in $\mathcal{M} = \mathcal{B}(\mathcal{H})$ with respect to the operator order does not necessarily exist (see Kadison [23]). Instead of the operator order, Olson [29] introduced the spectral order to define the maximum of (bounded) selfadjoint operators on $\mathcal{H}$. Moreover, Ando [1] investigated the spectral order on the matrix algebras in more detail. After that, Ben Arous and Voiculescu [8] extended the spectral order to general von Neumann algebras as follows. For $X, Y \in \mathcal{M}_{sa}$, we define $X \prec Y$ to mean that

$$E_X\bigl((x,\infty)\bigr) \le E_Y\bigl((x,\infty)\bigr), \quad x \in \mathbb{R},$$

where E_X is the spectral projection of X and $\le$ is the usual operator order. The order $\prec$ is called the *spectral order*. For any $X, Y \in \mathcal{M}_{sa}$, we define $X \vee Y \in \mathcal{M}_{sa}$ by

$$E_{X\vee Y}\bigl((x,\infty)\bigr) := E_X\bigl((x,\infty)\bigr) \vee E_Y\bigl((x,\infty)\bigr), \quad x \in \mathbb{R}.$$

The operator $X \vee Y$ is well-defined since the right-hand side in the above identity is projection-valued, decreasing and right-continuous in the strong operator topology as a function of x. Moreover, $X \vee Y$ is the maximum of X and Y with respect to the spectral order. Finally, Ben Arous and Voiculescu extended the spectral order to the set of all (unbounded) selfadjoint operators affiliated with $\mathcal{M}$. A (unbounded) selfadjoint operator X on $\mathcal{H}$ is said to be *affiliated with* $\mathcal{M}$ if $f(X) \in \mathcal{M}$ for all bounded Borel functions f on $\mathbb{R}$, where $f(X)$ is defined in terms of the functional calculus of X. Note that X is a bounded selfadjoint operator affiliated with $\mathcal{M}$ if and only if $X \in \mathcal{M}$. For any Borel sets B in $\mathbb{R}$, if $f = \mathbf{1}_B$, then $f(X) = E_X(B)$. For a selfadjoint operator X affiliated with $\mathcal{M}$, we define a (spectral) distribution function by

$$x \mapsto \tau\big(E_X\big((-\infty, x]\big)\big), \quad x \in \mathbb{R}.$$

If $X \sim \mu$, then we have $\tau(E_X((-\infty, x])) = \mu((-\infty, x])$ for all $x \in \mathbb{R}$.

Proposition 2.3 (see Ben Arous and Voiculescu [8]). *Let $(\mathcal{M}, \tau)$ be a tracial W^*-probability space and X, Y be freely independent real random variables (selfadjoint operators) affiliated with $\mathcal{M}$. Then we have*

$$\tau\big(E_{X\vee Y}\big((-\infty, x]\big)\big) = \max\big\{\tau\big(E_X\big((-\infty, x]\big)\big) + \tau\big(E_Y\big((-\infty, x]\big)\big) - 1, 0\big\}, \quad x \in \mathbb{R}.$$

For any distribution functions F, G on $\mathbb{R}$, we define

$$F \boxed{\vee} G := \max\{0, F + G - 1\}.$$

The operation $\boxed{\vee}$ is called the *free max-convolution*. We write $\mu \boxed{\vee} \nu$ for the distribution of the maximum of freely independent real random variables $X \sim \mu$ and $Y \sim \nu$, that is,

$$\mu \boxed{\vee} \nu\big((-\infty, \cdot]\big) := \mu\big((-\infty, \cdot]\big) \boxed{\vee} \nu\big((-\infty, \cdot]\big).$$

For $n \in \mathbb{N}$ and $\mu \in \mathcal{P}$, we define $\mu^{\boxed{\vee} n} := \overbrace{\mu \boxed{\vee} \cdots \boxed{\vee} \mu}^{n \text{ times}}$. More generally, for $t \geq 1$, we define

$$\mu^{\boxed{\vee} t}\big((-\infty, \cdot]\big) := \max\big\{t\mu\big((-\infty, \cdot]\big) - (t-1), 0\big\}. \tag{2.4}$$

For $\mu \in \mathcal{P}_+$, we get an atom of $\mu^{\boxed{\vee} t}$ at 0 as follows.

Corollary 2.2. *Consider $t > 1$ and $\mu \in \mathcal{P}_+$. Then $\mu^{\boxed{\vee} t}$ has an atom at 0 if and only if $\mu(\{0\}) > 1 - t^{-1}$. In this case, we have*

$$\mu^{\boxed{\vee} t}\big(\{0\}\big) = t\mu\big(\{0\}\big) - (t-1).$$

Proof. By definition (2.4), we have

$$\mu^{\boxed{\vee} t}\big(\{0\}\big) = \max\big\{t\mu\big(\{0\}\big) - (t-1), 0\big\}.$$

Therefore $\mu^{\boxed{\vee} t}$ has an atom at 0 if and only if $t\mu(\{0\}) - (t-1) > 0$, that is, $\mu(\{0\}) > 1 - t^{-1}$. □

A non-degenerate distribution function F is said to be *freely max-stable* if for any $n \in \mathbb{N}$, there exist $a_n > 0$ and $b_n \in \mathbb{R}$ such that

$$F^{\boxed{\vee} n}(a_n \cdot + b_n) \xrightarrow{w} F(\cdot), \quad n \to \infty.$$

The freely max-stable distributions are characterized as follows.

Proposition 2.4 (see Ben Arous and Voiculescu [8]). *A non-degenerate distribution function F is freely max-stable, if and only if there exist $a > 0$ and $b \in \mathbb{R}$ such that $F(ax+b)$ is one of the following distributions*:

$$F_I(x) := \max\{0, 1-e^{-x}\}, \quad \text{for } x \in \mathbb{R} \text{ (Exponential distribution)},$$

$$F_{II,\alpha}(x) := \begin{cases} 1-x^{-\alpha}, & x \geq 1, \\ 0, & x < 1, \end{cases} \quad \text{for } \alpha > 0 \text{ (Pareto distribution)};$$

$$F_{III,\alpha}(x) := \begin{cases} 1, & x > 0, \\ 1-(-x)^{\alpha}, & -1 \leq x \leq 0, \\ 0, & x < -1, \end{cases} \quad \text{for } \alpha > 0 \text{ (Beta law)}.$$

The above distributions are called free extreme value distributions.

Define a function $\Lambda^{\vee}$ on $[0, 1]$ by setting $\Lambda^{\vee}(0) := 0$ and $\Lambda^{\vee}(x) := \max\{0, 1+\log x\}$ for $x \in (0, 1]$. If F is a distribution function on $\mathbb{R}$, so is $\Lambda^{\vee}(F)$. Note that $\Lambda^{\vee}$ maps the classical extreme values to the corresponding type of free extreme values.

For a probability measure μ on $\mathbb{R}$, we define the probability measure $\Lambda^{\vee}(\mu)$ such that

$$\Lambda^{\vee}(\mu)\big((-\infty, \cdot]\big) := \Lambda^{\vee}\big(\mu\big((-\infty, \cdot]\big)\big).$$

Note the slight abuse of notation resulting from the dual use of the symbol $\Lambda^{\vee}$. It is known that the operator $\Lambda^{\vee}$ is a homomorphism from $(\mathcal{P}, \vee)$ to $(\mathcal{P}, \boxed{\vee})$, that is,

$$\Lambda^{\vee}(\mu \vee \nu) = \Lambda^{\vee}(\mu) \boxed{\vee} \Lambda^{\vee}(\nu),$$

for all probability measures μ and ν on $\mathbb{R}$ (see Ben Arous and Voiculescu [8]).

The function $\Lambda^{\vee}$ is surjective on $[0, 1]$, but it is not injective on $[0, 1]$. The proof of these facts is very simple. First, we show that it is surjective. Let $\Pi^{\vee}$ be the function on $[0, 1]$ defined by

$$\Pi^{\vee}(x) := \exp\big(-(1-x)\big), \quad x \in [0, 1].$$

For all $y \in [0, 1]$, we have

$$\Lambda^{\vee}\big(\Pi^{\vee}(y)\big) = \max\big\{0, 1+\log\big(\exp\big(-(1-y)\big)\big)\big\} = y.$$

Next, we show that it is not injective. If we take real numbers x and y such that $x \neq y$ and $0 \leq x, y \leq e^{-1}$, then $\Lambda^{\vee}(x) = 0 = \Lambda^{\vee}(y)$. Therefore, it is not injective.

For $\mu \in \mathcal{P}_+$, we define

$$\Pi^{\vee}(\mu)\big([0, \cdot]\big) := \Pi^{\vee}\big(\mu\big([0, \cdot]\big)\big) = \exp\big(-\big(1-\mu\big([0, \cdot]\big)\big)\big).$$

It is understood that $\Pi^{\vee}(\mu)$ has an atom at 0 with $\Pi^{\vee}(\mu)(\{0\}) = e^{-1}$, and $\Pi^{\vee}(\mu)([0, x]) = 0$ for $x < 0$. Note the slight abuse of notation resulting from the dual use of the symbol $\Pi^{\vee}$. The measure $\Pi^{\vee}(\mu)$ is called the *max-compound Poisson law associated with* μ.

2.3.3. *Boolean max-convolution*

Let $\mathcal{H}$ be a Hilbert space and $\xi \in \mathcal{H}$ a unit vector. Define the vector state φ_ξ by setting

$$\varphi_\xi(T) := \langle T\xi, \xi\rangle,$$

for all operators T on $\mathcal{H}$, where $\langle\cdot,\cdot\rangle$ is the inner product on $\mathcal{H}$. Then the pair $(\mathcal{B}(\mathcal{H}), \varphi_\xi)$ is a W^*-probability space. Vargas and Voiculescu [36] found Boolean independent positive operators X and Y on the W^*-probability space $(\mathcal{B}(\mathcal{H}), \varphi_\xi)$ such that the maximum $X \vee Y$ (with respect to spectral order) is defined.

In this section, we consider the (spectral) distribution function of a selfadjoint operator T on $\mathcal{H}$ given by

$$x \mapsto \varphi_\xi\big(E_T\big((-\infty, x]\big)\big), \quad x \in \mathbb{R}.$$

If T is a positive operator, then $\varphi_\xi(E_T((-\infty, x])) = 0$ for $x < 0$. In this case, we write $\varphi_\xi(E_T([0,\cdot]))$ as the distribution function of T, and it is understood that $\varphi_\xi(E_T([0,x])) = 0$ for $x < 0$.

Proposition 2.5 (see Vargas and Voiculescu [36]). *Let $X \geq 0$, $Y \geq 0$ be Boolean independent random variables on $(\mathcal{H}, \xi)$. Then we have*

$$\begin{aligned}\varphi_\xi\big(E_{X\vee Y}\big([0,\cdot]\big)\big) &= \frac{\varphi_\xi(E_X([0,\cdot]))\varphi_\xi(E_Y([0,\cdot]))}{\varphi_\xi(E_X([0,\cdot])) + \varphi_\xi(E_Y([0,\cdot])) - \varphi_\xi(E_X([0,\cdot]))\varphi_\xi(E_Y([0,\cdot]))} \\ &=: \varphi_\xi\big(E_X\big([0,\cdot]\big)\big) \cup\!\!\!\!\vee\, \varphi_\xi\big(E_Y\big([0,\cdot]\big)\big).\end{aligned}$$

Here it is understood that $\varphi_\xi(E_X([0,x])) \cup\!\!\!\!\vee\, \varphi_\xi(E_Y([0,x])) = 0$ when $x \in \mathbb{R}$ satisfies $\varphi_\xi(E_X([0,x])) = 0$ or $\varphi_\xi(E_Y([0,x])) = 0$. We write $\mu \cup\!\!\!\!\vee\, \nu$ as the distribution of the maximum of Boolean independent positive random variables $X \sim \mu \in \mathcal{P}_+$ and $Y \sim \nu \in \mathcal{P}_+$, that is,

$$\mu \cup\!\!\!\!\vee\, \nu\big([0,\cdot]\big) := \mu\big([0,\cdot]\big) \cup\!\!\!\!\vee\, \nu\big([0,\cdot]\big).$$

The operation $\cup\!\!\!\!\vee$ is called the *Boolean max-convolution*. For $n \in \mathbb{N}$ and $\mu \in \mathcal{P}_+$, we define $\mu^{\cup\!\!\!\!\vee n} := \overbrace{\mu \cup\!\!\!\!\vee \cdots \cup\!\!\!\!\vee \mu}^{n \text{ times}}$. More generally, for $t > 0$, we define

$$\mu^{\cup\!\!\!\!\vee t}\big([0,\cdot]\big) := \frac{\mu([0,\cdot])}{t - (t-1)\mu([0,\cdot])}. \tag{2.5}$$

By the definition (2.5), we get the following corollary.

Corollary 2.3. *Consider $t > 0$ and $\mu \in \mathcal{P}_+$. Then μ has an atom at 0, if and only if $\mu^{\cup\!\!\!\!\vee t}$ also has an atom at 0. In this case, we have*

$$\mu^{\cup\!\!\!\!\vee t}\big(\{0\}\big) = \frac{\mu(\{0\})}{t - (t-1)\mu(\{0\})}.$$

A non-degenerate distribution function F on $[0,\infty)$ is said to be *Boolean max-stable* if for any $n \in \mathbb{N}$, there exists $a_n > 0$ such that

$$F^{\cup\!\!\!\!\vee n}(a_n \cdot) \xrightarrow{w} F(\cdot), \quad n \to \infty.$$

Proposition 2.6 (see Vargas and Voiculescu [36]). *A non-degenerate distribution function F on $[0,\infty)$ is Boolean max-stable, if and only if there exist $a>0$ and $b\in\mathbb{R}$ such that $F(ax+b)$ is the following distribution*:

$$B_{II,\alpha}(x) := \begin{cases} \left(1+x^{-\alpha}\right)^{-1}, & x>0,\\ 0, & x\le 0,\end{cases}$$

for some $\alpha>0$. This is called the Dagum distribution or the (type II) Boolean extreme value distribution.

Problem 2.1. *Can we find Boolean independent (not necessarily positive) selfadjoint operators X and Y on some Hilbert space such that the maximum $X\vee Y$ (with respect to spectral order) is defined? Moreover, we should find other types of Boolean extreme values.*

Define a function $\mathcal{X}^{\vee}$ on $[0,1]$ by setting $\mathcal{X}^{\vee}(0):=0$ and $\mathcal{X}^{\vee}(x):=\exp(1-x^{-1})$ for $x\in(0,1]$. If F is a distribution function on $\mathbb{R}$, so is $\mathcal{X}^{\vee}(F)$. Note that $\mathcal{X}^{\vee}$ maps the Boolean extreme values to the classical extreme values.

For a probability measure μ on $[0,\infty)$, we define the probability measure $\mathcal{X}^{\vee}(\mu)\in\mathcal{P}_+$ such that

$$\mathcal{X}^{\vee}(\mu)\big([0,\cdot]\big) := \mathcal{X}^{\vee}\big(\mu\big([0,\cdot]\big)\big).$$

Note the slight abuse of notation resulting from the dual use of the symbol $\mathcal{X}^{\vee}$. The operator $\mathcal{X}^{\vee}$ is a homomorphism from $(\mathcal{P}_+, \cup\!\!\!\!\vee)$ to $(\mathcal{P}_+,\vee)$, that is,

$$\mathcal{X}^{\vee}(\mu \cup\!\!\!\!\vee \nu) = \mathcal{X}^{\vee}(\mu)\vee\mathcal{X}^{\vee}(\nu),$$

for all probability measures μ and ν on $[0,\infty)$ (see Vargas and Voiculescu [36]). Moreover, it is clear that the function $\mathcal{X}^{\vee}$ is a bijection and its inverse function is given by

$$\left(\mathcal{X}^{\vee}\right)^{-1}(x) = \frac{1}{1-\log x}, \quad x\in(0,1],$$
$$\left(\mathcal{X}^{\vee}\right)^{-1}(0) = 0.$$

The operator $\mathcal{X}^{\vee}$ is called the *Boolean-classical max-Bercovici–Pata bijection* (see Vargas and Voiculescu [36], Ueda [35]).

2.3.4. *Max-Belinschi–Nica semigroup*

We firstly recall the *Belinschi–Nica semigroup* $\{B_t\}_{t\ge 0}$ introduced by Belinschi and Nica [7]:

$$B_t(\mu) := \left(\mu^{\boxplus(1+t)}\right)^{\uplus\frac{1}{1+t}}, \quad \mu\in\mathcal{P}_+, t\ge 0.$$

Belinschi and Nica [7] proved that $B_t\circ B_s = B_{t+s}$ for all $t,s\ge 0$ and $B_1(\mu\uplus\nu) = B_1(\mu)\boxplus B_1(\nu)$ for all $\mu,\nu\in\mathcal{P}_+$. In addition, B_t is a homomorphism with respect to $\boxtimes$, that is, $B_t(\mu\boxtimes\nu) = B_t(\mu)\boxtimes B_t(\nu)$ for all $t\ge 0$ and $\mu,\nu\in\mathcal{P}_+$. Furthermore, B_1 connects the Boolean type limit theorems to free type ones. Bercovici and Pata [10] showed that for a sequence $\{\mu_n\}_n$ in $\mathcal{P}$ and $\{k_n\}_n$ in $\mathbb{N}$ with $k_1<k_2<\cdots$ and $k_n\to\infty$ as $n\to\infty$, there exists $\mu\in\mathcal{P}$ such that $\mu_n^{\uplus k_n}\xrightarrow{w}\mu$ if and only if there exists a unique $\nu\in\mathcal{P}$ (which is freely infinitely divisible with respect to $\boxplus$, for short, FID) such that $\mu_n^{\boxplus k_n}\xrightarrow{w}\nu$ as

$n \to \infty$. Finally, Belinschi and Nica [7] proved that $\nu = B_1(\mu)$ in the above setting. Note that B_1 is a bijection from $\mathcal{P}$ to the set of all FID distributions on $\mathbb{R}$. For example, B_1 maps the Boolean stable laws to the corresponding freely stable laws.

Next, we consider the one parameter family $\{B_t^{\vee}\}_{t\geq 0}$ of operators on $\mathcal{P}_+$ defined by

$$B_t^{\vee}(\mu) := \left(\mu^{\boxvee(1+t)}\right)^{\cup\!\!\!\vee \frac{1}{1+t}}, \quad \mu \in \mathcal{P}_+, t \geq 0,$$

which was introduced by Ueda [35]. It is known that $B_t^{\vee} \circ B_s^{\vee} = B_{t+s}^{\vee}$ for all $t, s \geq 0$ (see Ueda [35]). The family $\{B_t^{\vee}\}_{t\geq 0}$ is called the *max-Belinschi–Nica semigroup*. This semigroup is very similar to the original Belinschi–Nica semigroup. For example, $B_1^{\vee}$ is a homomorphism from $(\mathcal{P}_+, \cup\!\!\!\vee)$ to $(\mathcal{P}_+, \boxvee)$, that is,

$$B_1^{\vee}(\mu \cup\!\!\!\vee \nu) = B_1^{\vee}(\mu) \boxvee B_1^{\vee}(\nu),$$

for all $\mu, \nu \in \mathcal{P}_+$ (see Ueda [35]). Moreover, for a sequence $\{\mu_n\}_n$ in $\mathcal{P}_+$ and $\{k_n\}_n$ in $\mathbb{N}$ with $k_1 < k_2 < \cdots$ and $k_n \to \infty$ as $n \to \infty$, if there exists $\mu \in \mathcal{P}_+$ such that $\mu_n^{\cup\!\!\!\vee k_n} \xrightarrow{w} \mu$, then $\mu_n^{\boxvee k_n} \xrightarrow{w} B_1^{\vee}(\mu)$ as $n \to \infty$. In particular, we have $B_1^{\vee}(B_{\mathrm{II},\alpha}) = F_{\mathrm{II},\alpha}$ for all $\alpha > 0$ (see Ueda [35]). In addition, we have the following relation.

Lemma 2.7 (see Ueda [35]). *We have* $B_1^{\vee} = \Lambda^{\vee} \circ \mathcal{X}^{\vee}$.

3. Proof of main theorems

3.1. Proof of Theorem 1.1

In this section, we first show that the operator Φ provides the relation between the free additive convolution and the free max-convolution stated in Theorem 1.1.

Proof of Theorem 1.1. We may assume that $t > 1$. If μ is a Dirac measure, then $\Phi(D_{1/t}(\mu^{\boxplus t})) = \mu = \Phi(\mu)^{\boxvee t}$. Therefore, we may assume that μ is not a Dirac measure. Note that the closure of the interval (α_t, ω) is the support of $\Phi(\mu)^{\boxvee t}$, where

$$\alpha_t := \inf\left\{x : \Phi(\mu)\big([0,x]\big) > 1 - \frac{1}{t}\right\}, \qquad \omega := \sup\left\{x : \Phi(\mu)\big([0,x]\big) < 1\right\},$$

for each $t > 1$. Next, we define

$$A_t := \left\{y : \Phi\big(D_{1/t}\big(\mu^{\boxplus t}\big)\big)\big([0,y]\big) \in \big(\mu^{\boxplus t}(\{0\}), 1\big)\right\} = \left(\frac{a_{\mu^{\boxplus t}}}{t}, \frac{b_{\mu^{\boxplus t}}}{t}\right), \quad t > 1,$$

where $a_{\mu^{\boxplus t}}$ and $b_{\mu^{\boxplus t}}$ were defined by (2.3), and the last equation holds by Lemma 2.6. By Remark 2.2, the closure of A_t is the support of $\Phi(D_{1/t}(\mu^{\boxplus t}))$ for each $t > 1$.

We show that $A_t = (\alpha_t, \omega)$ to state that the support of $\Phi(D_{1/t}(\mu^{\boxplus t}))$ coincides with the support of $\Phi(\mu)^{\boxvee t}$. Moreover, we prove that

$$\Phi\big(D_{1/t}\big(\mu^{\boxplus t}\big)\big)\big([0,x]\big) = t\Phi(\mu)\big([0,x]\big) - (t-1),$$

for all $x \in A_t = (\alpha_t, \omega)$ and each $t > 1$. For an arbitrary fixed $t > 1$, we divide the statement into two cases.

Case I. $0 \le \mu(\{0\}) \le 1 - t^{-1}$: By Lemma 2.1, we have $\mu^{\boxplus t}(\{0\}) = 0$. First, we show that $(\alpha_t, \omega) \subseteq A_t$. For all $x \in (\alpha_t, \omega)$, we get

$$t\Phi(\mu)\big([0, x]\big) - (t-1) \in (0, 1).$$

Since $(\alpha_t, \omega) \subseteq (a_\mu, b_\mu)$, we have

$$x = \frac{1}{S_\mu(\Phi(\mu)([0, x]) - 1)},$$

for $x \in (\alpha_t, \omega)$ by Lemma 2.6. Therefore Lemma 2.3 and Proposition 2.1 imply that

$$\begin{aligned}
\Phi\big(\mu^{\boxplus t}\big)\big([0, tx]\big) &= \Phi\big(\mu^{\boxplus t}\big)\left(\left[0, \frac{t}{S_\mu(\Phi(\mu)([0, x]) - 1)}\right]\right) \\
&= \Phi\big(\mu^{\boxplus t}\big)\left(\left[0, \frac{1}{S_{\mu^{\boxplus t}}(t(\Phi(\mu)([0, x]) - 1))}\right]\right) \\
&= \Phi\big(\mu^{\boxplus t}\big)\left(\left[0, \frac{1}{S_{\mu^{\boxplus t}}(\{t\Phi(\mu)([0, x]) - (t-1)\} - 1)}\right]\right) \\
&= t\Phi(\mu)\big([0, x]\big) - (t-1).
\end{aligned}$$

By Lemma 2.5, we have

$$\begin{aligned}
\Phi\big(D_{1/t}\big(\mu^{\boxplus t}\big)\big)\big([0, x]\big) &= D_{1/t} \circ \Phi\big(\mu^{\boxplus t}\big)\big([0, x]\big) \\
&= \Phi\big(\mu^{\boxplus t}\big)\big([0, tx]\big) \\
&= t\Phi(\mu)\big([0, x]\big) - (t-1) \in (0, 1) = \big(\mu^{\boxplus t}\big(\{0\}\big), 1\big),
\end{aligned}$$

and therefore we have $x \in A_t$. Hence, we have $(\alpha_t, \omega) \subseteq A_t$.

Next, we show that for all $x \in (a_{\mu^{\boxplus t}}, b_{\mu^{\boxplus t}})$, we have $x/t \in (\alpha_t, \omega)$. If we prove this, then it follows that $A_t \subseteq (\alpha_t, \omega)$. For any $x \in (a_{\mu^{\boxplus t}}, b_{\mu^{\boxplus t}})$, we have

$$\Phi\big(\mu^{\boxplus t}\big)\big([0, x]\big) = S^{-1}_{\mu^{\boxplus t}}\left(\frac{1}{x}\right) + 1,$$

by Lemma 2.6. Since $S_{\mu^{\boxplus t}}(S^{-1}_{\mu^{\boxplus t}}(1/x)) = 1/x$, we have

$$S_\mu\left(\frac{1}{t}S^{-1}_{\mu^{\boxplus t}}\left(\frac{1}{x}\right)\right) = \frac{t}{x},$$

by Lemma 2.3. Therefore, we obtain

$$S^{-1}_{\mu^{\boxplus t}}\left(\frac{1}{x}\right) = tS^{-1}_\mu\left(\frac{t}{x}\right),$$

by Lemma 2.4. Thus, we get $\Phi(\mu^{\boxplus t})([0,x]) = tS_\mu^{-1}(t/x)+1$, and therefore

$$x = \frac{t}{S_\mu(\frac{1}{t}\Phi(\mu^{\boxplus t})([0,x]) - \frac{1}{t} + 1 - 1)}.$$

Since

$$\frac{1}{t}\Phi\big(\mu^{\boxplus t}\big)\big([0,x]\big) - \frac{1}{t} + 1 \in \left(1 - \frac{1}{t}, 1\right) \subseteq \big(\mu\big(\{0\}\big), 1\big),$$

we have

$$\begin{aligned}\Phi(\mu)\big([0,x/t]\big) &= \Phi(\mu)\left(\left[0, \frac{1}{S_\mu(\frac{1}{t}\Phi(\mu^{\boxplus t})([0,x]) - \frac{1}{t} + 1 - 1)}\right]\right) \\ &= \frac{1}{t}\Phi\big(\mu^{\boxplus t}\big)\big([0,x]\big) - \frac{1}{t} + 1 \in \left(1 - \frac{1}{t}, 1\right),\end{aligned}$$

by Proposition 2.1. This implies that $x/t \in (\alpha_t, \omega)$. Therefore, we have

$$A_t = \left(\frac{a_{\mu^{\boxplus t}}}{t}, \frac{b_{\mu^{\boxplus t}}}{t}\right) \subseteq (\alpha_t, \omega).$$

Finally, we get the following properties:

$$\Phi\big(D_{1/t}\big(\mu^{\boxplus t}\big)\big)\big([0,x]\big) = t\Phi(\mu)\big([0,x]\big) - (t-1),$$

for all $x \in A_t = (\alpha_t, \omega)$. Moreover, the support of $\Phi(D_{1/t}(\mu^{\boxplus t}))$ is equal to the support of $\Phi(\mu)^{\boxtimes t}$ since $A_t = (\alpha_t, \omega)$. Therefore, $\Phi(D_{1/t}(\mu^{\boxplus t})) = \Phi(\mu)^{\boxtimes t}$.

Case II. $\mu(\{0\}) > 1 - t^{-1}$: By Lemma 2.1, we have

$$\mu^{\boxplus t}\big(\{0\}\big) = t\mu\big(\{0\}\big) - (t-1) > 0,$$

and therefore $a_{\mu^{\boxplus t}} = 0$. We also have $\alpha_t = 0$ since the condition $\mu(\{0\}) > 1 - t^{-1}$ means that $\Phi(\mu)([0,x]) > 1 - t^{-1}$ for all $x \geq 0$.

Firstly we show that $(\alpha_t, \omega) = (0, \omega) \subseteq A_t$. For any $x \in (0, \omega)$, we get

$$t\Phi(\mu)\big([0,x]\big) - (t-1) \in \big(t\mu\big(\{0\}\big) - (t-1), 1\big) = \big(\mu^{\boxplus t}\big(\{0\}\big), 1\big).$$

Since $(0, \omega) \subseteq (0, b_\mu) = (a_\mu, b_\mu)$, for $x \in (0, \omega)$, we get

$$\Phi\big(D_{1/t}\big(\mu^{\boxplus t}\big)\big)\big([0,x]\big) = t\Phi(\mu)\big([0,x]\big) - (t-1) \in \big(\mu^{\boxplus t}\big(\{0\}\big), 1\big),$$

in the same way as in Case I. Hence, we have $x \in A_t$. Therefore, $(0, \omega) \subseteq A_t$.

Next for $x \in (a_{\mu^{\boxplus t}}, b_{\mu^{\boxplus t}}) = (0, b_{\mu^{\boxplus t}})$, we show that $x/t \in (0, \omega)$. If we prove this, then it follows that $A_t \subseteq (0, \omega)$. By Proposition 2.1 and Lemma 2.6, we have $\Phi(\mu^{\boxplus t})([0,x]) \in (\mu^{\boxplus t}(\{0\}), 1)$ for all $x \in (0, b_{\mu^{\boxplus t}})$. Therefore, we obtain

$$\Phi(\mu)\left(\left[0, \frac{x}{t}\right]\right) = \frac{1}{t}\Phi\big(\mu^{\boxplus t}\big)\big([0,x]\big) - \frac{1}{t} + 1 \in \left(\frac{1}{t}\mu^{\boxplus t}\big(\{0\}\big) - \frac{1}{t} + 1, 1\right)$$

$$= \left(\frac{1}{t}\big(t\mu(\{0\}) - (t-1)\big) - \frac{1}{t} + 1, 1 \right)$$
$$= \big(\mu(\{0\}), 1\big),$$

where the first equation holds in the same way as in Case I. Consequently, we have $x/t \in (0, \omega)$ since $\mu(\{0\}) > 1 - t^{-1}$. Hence,

$$A_t = \left(0, \frac{b_{\mu^{\boxplus t}}}{t}\right) \subseteq (0, \omega).$$

Finally, we get the following properties:

$$\Phi\big(D_{1/t}(\mu^{\boxplus t})\big)\big([0, x]\big) = t\Phi(\mu)\big([0, x]\big) - (t-1),$$

for all $x \in A_t = (0, \omega)$. Moreover, the support of $\Phi(D_{1/t}(\mu^{\boxplus t}))$ is equal to the support of $\Phi(\mu)^{\boxtimes t}$ since $A_t = (0, \omega)$. Therefore, $\Phi(D_{1/t}(\mu^{\boxplus t})) = \Phi(\mu)^{\boxtimes t}$. □

3.2. Proof of Theorem 1.2

In this section, we show Theorem 1.2 as follows.

Proof of Theorem 1.2. If μ is a Dirac measure, then $\Phi(D_{1/t}(\mu^{\uplus t})) = \mu = \Phi(\mu)^{\cup\!\!\!\!\vee t}$. Therefore, we may assume that μ is not a Dirac measure. By Corollary 2.1, the equation (2.2) and Corollary 2.3, if $\mu(\{0\}) = 0$ then $\mu^{\uplus t}(\{0\}) = \mu^{\cup\!\!\!\!\vee t}(\{0\}) = 0$ and if $\mu(\{0\}) \neq 0$, then

$$\mu^{\uplus t}\big(\{0\}\big) = \mu^{\cup\!\!\!\!\vee t}\big(\{0\}\big) = \frac{\mu(\{0\})}{t - (t-1)\mu(\{0\})}.$$

Define $M_t := \{x : \Phi(\mu)^{\cup\!\!\!\!\vee t}([0, x]) \in (\mu^{\cup\!\!\!\!\vee t}(\{0\}), 1)\}$. For all $y \in M_t$, we have

$$\Phi(\mu)^{\cup\!\!\!\!\vee t}\big([0, y]\big) \in \big(\mu^{\cup\!\!\!\!\vee t}\big(\{0\}\big), 1\big) = \left(\frac{\mu(\{0\})}{t - (t-1)\mu(\{0\})}, 1 \right).$$

Then

$$\Phi(\mu)\big([0, y]\big) = \frac{t\Phi(\mu)^{\cup\!\!\!\!\vee t}([0, y])}{1 + (t-1)\Phi(\mu)^{\cup\!\!\!\!\vee t}([0, y])} \in \left(\frac{\frac{t\mu(\{0\})}{t-(t-1)\mu(\{0\})}}{1 + (t-1)\frac{\mu(\{0\})}{t-(t-1)\mu(\{0\})}}, 1 \right)$$
$$= \big(\mu\big(\{0\}\big), 1\big).$$

Hence $y \in \{x : \Phi(\mu)([0, x]) \in (\mu(\{0\}), 1)\} = (a_\mu, b_\mu)$, and therefore $M_t \subseteq (a_\mu, b_\mu)$. It is clear that $(a_\mu, b_\mu) \subseteq M_t$. Consequently, we have

$$M_t = \big\{x : \Phi(\mu)\big([0, x]\big) \in \big(\mu\big(\{0\}\big), 1\big)\big\} = (a_\mu, b_\mu).$$

The set M_t does not depend on t, and therefore we denote it by M.

It follows from Corollary 2.3 that

$$\Phi(\mu)^{\uplus t}\big([0,x]\big) = \frac{\Phi(\mu)([0,x])}{t-(t-1)\Phi(\mu)([0,x])} \in \left(\frac{\mu(\{0\})}{t-(t-1)\mu(\{0\})}, 1\right)$$
$$= \big(\mu^{\uplus t}\big(\{0\}\big), 1\big), \quad x > 0.$$

Moreover, the distribution function $x \mapsto \Phi(\mu)^{\uplus t}([0,x])$ is strictly increasing by Lemma 2.4 and Lemma 2.6. Therefore the support of $\Phi(\mu)^{\uplus t}$ coincides with the closure of the set $\{x : \Phi(\mu)^{\uplus t}([0,x]) \in (\mu^{\uplus t}(\{0\}), 1)\} = M$. Note that it follows from the above fact and Remark 2.2 that the support of $\Phi(\mu)^{\uplus t}$ also coincides with the support of $\Phi(\mu)$.

Next, we define $A_t := \{x : \Phi(D_{1/t}(\mu^{\boxplus t}))([0,x]) \in (\mu^{\boxplus t}(\{0\}), 1)\}$. Then we have

$$A_t = \left(\frac{a_{\mu^{\boxplus t}}}{t}, \frac{b_{\mu^{\boxplus t}}}{t}\right).$$

By Remark 2.2, the closure of A_t is the support of $\Phi(D_{1/t}(\mu^{\boxplus t}))$ for each $t > 0$. To conclude the proof, we show that $A_t = M$ and

$$\Phi\big(D_{1/t}\big(\mu^{\boxplus t}\big)\big)\big([0,x]\big) = \frac{\Phi(\mu)([0,x])}{t-(t-1)\Phi(\mu)([0,x])},$$

for all $x \in A_t = M$ and each $t > 0$. Let an arbitrary fixed $t > 0$ be given.

For all $x \in M$, we have

$$\frac{\Phi(\mu)([0,x])}{t-(t-1)\Phi(\mu)([0,x])} \in \big(\mu^{\uplus t}\big(\{0\}\big), 1\big) = \big(\mu^{\boxplus t}\big(\{0\}\big), 1\big).$$

Since $M = (a_\mu, b_\mu)$, by Lemma 2.6, we have

$$x = \frac{1}{S_\mu(\Phi(\mu)([0,x]) - 1)}.$$

Applying Lemma 2.3 and Proposition 2.1, we have

$$\begin{aligned}
\Phi\big(\mu^{\boxplus t}\big)\big([0,tx]\big) &= \Phi\big(\mu^{\boxplus t}\big)\left(\left[0, \frac{t}{S_\mu(\Phi(\mu)([0,x]) - 1)}\right]\right) \\
&= \Phi\big(\mu^{\boxplus t}\big)\left(\left[0, \frac{1}{S_{\mu^{\boxplus t}}(\frac{\Phi(\mu)([0,x])}{t-(t-1)\Phi(\mu)([0,x])} - 1)}\right]\right) \\
&= \frac{\Phi(\mu)([0,x])}{t-(t-1)\Phi(\mu)([0,x])} \\
&= \Phi(\mu)^{\uplus t}\big([0,x]\big).
\end{aligned}$$

By Lemma 2.5, we have

$$\begin{aligned}
\Phi\big(D_{1/t}\big(\mu^{\boxplus t}\big)\big)\big([0,x]\big) &= D_{1/t} \circ \Phi\big(\mu^{\boxplus t}\big)\big([0,x]\big) \\
&= \Phi\big(\mu^{\boxplus t}\big)\big([0,tx]\big) \\
&= \Phi(\mu)^{\uplus t}\big([0,x]\big).
\end{aligned}$$

Next we show that $A_t = M$. By Lemma 2.4, we have

$$\left(\frac{1}{b_{\mu^{\uplus t}}}, \frac{1}{a_{\mu^{\uplus t}}}\right) = S_{\mu^{\uplus t}}\big((\mu^{\uplus t}(\{0\}) - 1, 0)\big)$$

$$= S_{\mu^{\uplus t}}\left(\left(\frac{t(\mu(\{0\}) - 1)}{t - (t-1)\mu(\{0\})}, 0\right)\right) = \frac{1}{t}\left(\frac{1}{b_\mu}, \frac{1}{a_\mu}\right),$$

where the last equation holds by Lemma 2.3. Therefore, $M = (a_\mu, b_\mu) = A_t$. Thus, the support of $\Phi(D_{1/t}(\mu^{\uplus t}))$ is equal to the support of $\Phi(\mu)^{\cup\!\!\!\!\vee t}$. □

Remark 3.1. Consider $\mu \in \mathcal{P}_+ \setminus \{\delta_0\}$. Since $A_t = (a_\mu, b_\mu)$ in the proof of Theorem 1.2, we get

$$t\int_0^\infty x^{-1}\, d\mu^{\uplus t}(x) = \int_0^\infty x^{-1}\, d\mu(x), \quad t > 0.$$

3.3. Proof of Theorem 1.3

By using Theorems 1.1 and 1.2, we will see that the Belinschi–Nica semigroup is closely intertwined with the max-Belinschi–Nica semigroup via the operator Φ.

Proof of Theorem 1.3. If μ is a Dirac measure, then $\Phi(B_t(\mu)) = \mu = B_t^\vee(\Phi(\mu))$. Therefore we may assume that μ is not a Dirac measure. By Theorems 1.1, 1.2 and Lemma 2.5, we have

$$\begin{aligned}\Phi\big(B_t(\mu)\big) &= D_{\frac{1}{1+t}}\Big(\Phi\big(\mu^{\boxplus(1+t)}\big)^{\cup\!\!\!\!\vee \frac{1}{1+t}}\Big)\\ &= D_{\frac{1}{1+t}}\Big(\big(D_{1+t}\big(\Phi(\mu)^{\boxed{\vee}(1+t)}\big)\big)^{\cup\!\!\!\!\vee \frac{1}{1+t}}\Big)\\ &= D_{\frac{1}{1+t}} \circ D_{1+t}\Big(\big(\Phi(\mu)^{\boxed{\vee}(1+t)}\big)^{\cup\!\!\!\!\vee \frac{1}{1+t}}\Big) = B_t^\vee\big(\Phi(\mu)\big).\end{aligned}$$ □

3.4. Proof of Theorem 1.4

In this section, we prove Theorem 1.4. Before a proof of this theorem, we introduce the following important maps. Let FID_+ be the set of all freely infinitely divisible distributions (with respect to $\boxplus$) on $[0, \infty)$. Recall that $B_t(\mu) := (\mu^{\boxplus(1+t)})^{\uplus\frac{1}{1+t}} \in \mathrm{FID}_+$ for any $\mu \in \mathcal{P}_+$ and $t \geq 1$ (see Belinschi and Nica [7]). Hence, we can define the operator $\mathcal{X} : \mathcal{P}_+ \to \mathrm{ID}_+$ by

$$\mathcal{X} := \Lambda^{-1} \circ B_1,$$

where Λ is called the *Bercovici–Pata bijection* and $\Lambda^{-1}(\mathrm{FID}_+) = \mathrm{ID}_+$ (for details, see Bercovici and Pata [10], Barndorff-Nielsen and Thorbjørnsen [4]).

By Belinschi and Nica [7], for any $\nu \in \mathrm{FID}_+$ and $t \geq 0$, the measure $\nu^{\uplus(1+t)}$ is also in FID_+ and we obtain $B_t^{-1}(\nu) = (\nu^{\uplus(1+t)})^{\boxplus\frac{1}{1+t}}$. Hence $\mathcal{X}^{-1} = B_1^{-1} \circ \Lambda$, and therefore $\mathcal{X}$ is a bijection from $\mathcal{P}_+$ to ID_+. It is called the *Boolean-classical Bercovici–Pata bijection* (see Bercovici and Pata [10], Belinschi and Nica [7]). Note that $\mathcal{X}^{-1} \circ D_c = D_c \circ \mathcal{X}^{-1}$ for $c > 0$. Furthermore, we have $\mathcal{X}^{-1}(\mu^{*t}) = \mathcal{X}^{-1}(\mu)^{\uplus t}$ for all $\mu \in \mathrm{ID}_+$ and $t > 0$ since $\Lambda(\mu^{*t}) = \Lambda(\mu)^{\boxplus t}$ for all $\mu \in \mathrm{ID}_+$ and $t > 0$ (see, e.g., Bercovici and Voiculescu [11], Barndorff-Nielsen and Thorbjørnsen [4]) and $B_1^{-1}(\nu^{\boxplus t}) = B_1^{-1}(\nu)^{\uplus t}$ for all $\nu \in \mathrm{FID}_+$

and $t > 0$ (see Belinschi and Nica [7]). These facts are true even if we replace $\mathcal{P}_+$, ID_+ and FID_+ by $\mathcal{P}$, ID (the set of all classically infinitely divisible distributions on $\mathbb{R}$) and FID, respectively.

Finally, we define the operator $\Psi : \mathrm{ID}_+ \to \mathcal{P}_+$ by setting

$$\Psi := \mathcal{X}^\vee \circ \Phi \circ \mathcal{X}^{-1},$$

where $\mathcal{X}^\vee$ was defined in Section 2.3.3.

Proof of Theorem 1.4. Consider $\mu \in \mathcal{P}_+$ and $t > 0$. By Theorem 1.2, we have

$$\begin{aligned}
\Psi\big(D_{1/t}(\mu^{*t})\big)([0,\cdot]) &= \mathcal{X}^\vee \circ \Phi \circ \mathcal{X}^{-1}\big(D_{1/t}(\mu^{*t})\big)([0,\cdot]) \\
&= \mathcal{X}^\vee \circ \Phi\big(D_{1/t}(\mathcal{X}^{-1}(\mu)^{\uplus t})\big)([0,\cdot]) \\
&= \mathcal{X}^\vee\big(\Phi(\mathcal{X}^{-1}(\mu))^{\cup\!\!\!\vee t}\big)([0,\cdot]) \\
&= \exp\left[1 - \frac{1}{\Phi(\mathcal{X}^{-1}(\mu))^{\cup\!\!\!\vee t}([0,\cdot])}\right] \\
&= \exp\left[t\left(1 - \frac{1}{\Phi(\mathcal{X}^{-1}(\mu))([0,\cdot])}\right)\right].
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\Psi(\mu)^{\vee t}([0,\cdot]) &= \big(\mathcal{X}^\vee \circ \Phi \circ \mathcal{X}^{-1}(\mu)([0,\cdot])\big)^t \\
&= \exp\left[t\left(1 - \frac{1}{\Phi(\mathcal{X}^{-1}(\mu))([0,\cdot])}\right)\right].
\end{aligned}$$

Therefore, we have $\Psi(D_{1/t}(\mu^{*t})) = \Psi(\mu)^{\vee t}$. □

In addition, we conclude that the operator Φ is intertwined with the operator Ψ as follows.

Proposition 3.1. *We have* $\Lambda^\vee \circ \Psi = \Phi \circ \Lambda$.

Proof. By Lemma 2.7 and Theorem 1.3, we have

$$\begin{aligned}
\Lambda^\vee \circ \Psi &= \big(\Lambda^\vee \circ \mathcal{X}^\vee\big) \circ \Phi \circ \mathcal{X}^{-1} \\
&= \big(B_1^\vee \circ \Phi \circ B_1^{-1}\big) \circ \Lambda \\
&= \Phi \circ \Lambda.
\end{aligned}$$

Therefore, we conclude this proposition. □

According to the discussions in Section 3, we have the commutative diagram.

$$\begin{array}{ccccc}
\text{(Classical)} & \text{(Free)} & \text{(Boolean)} & \\
\mathrm{ID}_+ \xrightarrow{\ \Lambda\ } & \mathrm{FID}_+ & \xleftarrow{\ B_1\ } \mathrm{BID}_+ & \text{(Additive)} \\
\Psi \big\downarrow & \big\downarrow \Phi & \big\downarrow \Phi & \\
\mathcal{P}_+ \xrightarrow[\Lambda^\vee]{} & \mathcal{P}_+ & \xleftarrow[B_1^\vee]{} \mathcal{P}_+ & \text{(Max)}
\end{array}$$

The class BID_+ is the set of all Boolean infinitely divisible distributions on $[0,\infty)$. It is well known that $\mathrm{BID}_+ = \mathcal{P}_+$ (see Speicher and Woroudi [33]).

4. Examples

In this section, we give several examples of probability measures in the classes $\Phi(\mathcal{P}_+)$ and $\Psi(\mathcal{P}_+)$

4.1. Stable laws and extreme values

In this section, we give relations between (strictly) stable laws and extreme value distributions. Let $\mathcal{A}$ be the set of *admissible parameters*:

$$\mathcal{A} := \{(\alpha,\rho) : \alpha \in (0,1], \rho \in [0,1]\} \cup \{(\alpha,\rho) : \alpha \in (1,2], \rho \in [1-\alpha^{-1}, \alpha^{-1}]\}.$$

Consider $(\alpha,\rho) \in \mathcal{A}$. Denote by $c_{\alpha,\rho}$ the *classical strictly stable law* (see e.g. Sato [32]), $f_{\alpha,\rho}$ the *free strictly stable law* (see Bercovici and Voiculescu [11], Bercovici and Pata [10]) and $b_{\alpha,\rho}$ the *Boolean strictly stable law* (see Speicher and Woroudi [33]). In particular, we define $c_\alpha^+ := c_{\alpha,1} \in \mathrm{ID}_+$, $f_\alpha^+ := f_{\alpha,1} \in \mathrm{FID}_+$ and $b_\alpha^+ := b_{\alpha,1} \in \mathcal{P}_+$. Note that $c_1^+ = f_1^+ = b_1^+ = \delta_1$. For all $\alpha \in (0,1)$, the strictly stable laws c_α^+, f_α^+ and b_α^+ are not Dirac measures. Thus, we may assume that $\alpha \in (0,1)$ in this section.

From Arizmendi and Hasebe [2], we know a relation between the Boolean stable laws and the Boolean extreme value distributions via the operator Φ.

Example 4.1. Consider $\alpha \in (0,1)$. Then $\Phi(b_\alpha^+)([0,\cdot]) = B_{\mathrm{II},\frac{\alpha}{1-\alpha}}(\cdot)$.

Next, we give a relation between the free stable laws and the free extreme value distributions via the operator Φ.

Example 4.2. Consider $\alpha \in (0,1)$. Then $\Phi(f_\alpha^+)([0,\cdot]) = F_{\mathrm{II},\frac{\alpha}{1-\alpha}}(\cdot)$.

Proof. By Belinschi and Nica [7] and Bercovici and Pata [10], we have $B_1(b_\alpha^+) = f_\alpha^+$. Moreover, by Ueda [35], we have $B_1^\vee(B_{\mathrm{II},\frac{\alpha}{1-\alpha}}) = F_{\mathrm{II},\frac{\alpha}{1-\alpha}}$. By Theorem 1.3 and Example 4.1, we have

$$\begin{aligned}\Phi(f_\alpha^+)([0,\cdot]) &= \Phi(B_1(b_\alpha^+))([0,\cdot]) \\ &= B_1^\vee(\Phi(b_\alpha^+))([0,\cdot])) \\ &= B_1^\vee(B_{\mathrm{II},\frac{\alpha}{1-\alpha}}(\cdot)) = F_{\mathrm{II},\frac{\alpha}{1-\alpha}}(\cdot).\end{aligned}$$

Therefore, $\Phi(f_\alpha^+)([0,\cdot]) = F_{\mathrm{II},\frac{\alpha}{1-\alpha}}(\cdot)$. □

Finally, we obtain a relation between the classical stable laws and the classical extreme value distributions via the operator Ψ.

Example 4.3. Consider $\alpha \in (0,1)$. Then $\Psi(c_\alpha^+)([0,\cdot]) = C_{\mathrm{II},\frac{\alpha}{1-\alpha}}(\cdot)$.

Proof. By Belinschi and Nica [7] and Bercovici and Pata [10], we have $\mathcal{X}^{-1}(c_\alpha^+) = b_\alpha^+$. Moreover, by Vargas and Voiculescu [36], we have $\mathcal{X}^\vee(B_{\mathrm{II},\frac{\alpha}{1-\alpha}}) = C_{\mathrm{II},\frac{\alpha}{1-\alpha}}$. By Example 4.1, we have

$$\begin{aligned}\Psi(c_\alpha^+)([0,\cdot]) &= \mathcal{X}^\vee \circ \Phi(b_\alpha^+)([0,\cdot])\\ &= \mathcal{X}^\vee\big(B_{\mathrm{II},\frac{\alpha}{1-\alpha}}(\cdot)\big) = C_{\mathrm{II},\frac{\alpha}{1-\alpha}}(\cdot).\end{aligned}$$

Therefore, $\Psi(c_\alpha^+)([0,\cdot]) = C_{\mathrm{II},\frac{\alpha}{1-\alpha}}(\cdot)$. □

4.2. The Marchenko–Pastur law and the uniform distribution

In this section, we show that Φ connects the Marchenko–Pastur law to the uniform distribution on $(0,1)$. Denote by π the *Marchenko–Pastur law* (or *free Poisson law*), that is,

$$\pi(dx) := \frac{1}{2\pi}\sqrt{\frac{4-x}{x}}\mathbf{1}_{(0,4)}(x)\,dx.$$

This distribution appears as the limit of eigenvalue distributions of Wishart matrices as the size of the random matrices goes to infinity.

It is known that the S-transform of the Marchenko–Pastur law is given by

$$S_\pi(z) = \frac{1}{z+1}, \qquad z \in (-1,0).$$

Then we have

$$S_\pi^{-1}\left(\frac{1}{x}\right) = x - 1, \qquad x \in (0,1).$$

By Lemma 2.6, we obtain the measure $\Phi(\pi)$ as follows.

Example 4.4. We have

$$\Phi(\pi)\big([0,x]\big) = x, \qquad x \in (0,1).$$

Thus $\Phi(\pi) = U(0,1)$, where $U(0,1)$ is the uniform distribution on $(0,1)$.

4.3. Poisson law and max-compound Poisson law

We find a relation between the Poisson law and the max-compound Poisson law with the uniform distribution on $(0,1)$ via the map Ψ. Denote by $\mathrm{Po}(\lambda)$ the *Poisson law with parameter* $\lambda > 0$, that is,

$$\mathrm{Po}(\lambda) = \sum_{k=0}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!}\delta_k.$$

Note that $\Lambda(\mathrm{Po}(1)) = \pi$ (see Bercovici and Pata [10]) and $\mathcal{X}^{-1}(\mathrm{Po}(1)) = \frac{1}{2}\delta_0 + \frac{1}{2}\delta_2$ (see Speicher and Woroudi [33]). Hence, we obtain the following relation.

Example 4.5. We have

$$\Psi\big(\mathrm{Po}(1)\big)=\Pi^{\vee}\big(U(0,1)\big)=\frac{1}{e}\delta_0+e^{-1+x}\mathbf{1}_{(0,1)}(x)\,dx.$$

Proof. We simply calculate the left-hand side as follows. For all $x\in(0,1)$,

$$\begin{aligned}\Psi\big(\mathrm{Po}(1)\big)\big([0,x]\big)&=\big(\mathcal{X}^{\vee}\circ\Phi\big)\big(\mathcal{X}^{-1}\big(\mathrm{Po}(1)\big)\big)\big([0,x]\big)\\&=\mathcal{X}^{\vee}\left(\Phi\left(\frac{1}{2}\delta_0+\frac{1}{2}\delta_2\right)\right)\big([0,x]\big)\\&=\exp\left[1-\frac{1}{\Phi(\frac{1}{2}\delta_0+\frac{1}{2}\delta_2)([0,x])}\right].\end{aligned}$$

Put $\sigma:=\frac{1}{2}\delta_0+\frac{1}{2}\delta_2$. Then we get $\Psi_\sigma(z)=\frac{z}{1-2z}$, and hence $\Psi_\sigma^{-1}(z)=\frac{z}{1+2z}$ for all $z\in(-\frac{1}{2},0)$. Hence, we have

$$S_\sigma(z)=\frac{1+z}{1+2z},\quad z\in\left(-\frac{1}{2},0\right).$$

Thus, we get

$$\Phi(\sigma)\left(\left[0,\frac{2x-1}{x}\right]\right)=x,\quad x\in\left(\frac{1}{2},1\right),$$

and therefore

$$\Phi(\sigma)\big([0,x]\big)=\frac{1}{2-x},\quad x\in(0,1).\tag{4.1}$$

Hence, we obtain

$$\Psi\big(\mathrm{Po}(1)\big)\big([0,x]\big)=\exp(-1+x),\quad x\in(0,1).$$

Therefore, its density function is given by

$$\frac{d\Psi(\mathrm{Po}(1))}{dx}(x)=e^{-1+x}\mathbf{1}_{(0,1)}(x).$$

Moreover, $\Psi(\mathrm{Po}(1))(\{0\})=\mathcal{X}^{\vee}(\sigma)(\{0\})=e^{-1}$.

Note that the probability measure on the right-hand side of Example 4.5 is the max-compound Poisson law associated with the uniform distribution on $(0,1)$. □

By Theorem 1.4, for $\lambda>0$, we get

$$\begin{aligned}\Psi\big(\mathrm{Po}(\lambda)\big)\big([0,x]\big)&=\Psi\big(\mathrm{Po}(1)^{*\lambda}\big)\big([0,x]\big)\\&=\Psi\big(\mathrm{Po}(1)\big)^{\vee\lambda}\big(\big[0,\lambda^{-1}x\big]\big)\\&=\Psi\big(\mathrm{Po}(1)\big)\big(\big[0,\lambda^{-1}x\big]\big)^{\lambda}\\&=\exp\big(\lambda\big(-1+\lambda^{-1}x\big)\big)=\exp(-\lambda+x),\end{aligned}$$

for all $x \in (0, \lambda)$. Thus, for all $\lambda > 0$, we have

$$\Psi\big(\mathrm{Po}(\lambda)\big) = e^{-\lambda}\delta_0 + e^{-\lambda+x}\mathbf{1}_{(0,\lambda)}(x)\,dx.$$

4.4. Free regular distributions

By Example 4.4 and (4.1), we get

$$\Phi(\pi)\big([0,x]\big) = \max\left\{0, 2 - \frac{1}{\Phi(\frac{1}{2}\delta_0 + \frac{1}{2}\delta_2)([0,x])}\right\}, \quad x \geq 0,$$

where the right-hand side is interpreted as zero for $x < 0$. More generally, we get a similar formula in the case of free regular distributions. A probability measure μ on $[0, \infty)$ is said to be *free regular* if μ is freely infinitely divisible (with respect to $\boxplus$) and $\mu^{\boxplus t} \in \mathcal{P}_+$ for all $t > 0$. For example, the Marchenko–Pastur law π and the positive free stable law f_α^+ are free regular. Recently, it was proved that the Fuss–Catalan distribution $\mu(p, r)$ is free regular for $1 \leq r \leq \min\{p-1, p/2\}$ or $p = r = 1$ (see Młotkowski et al. [26]). The free regular distributions were introduced by Sakuma [31]. It is known that these distributions are a marginal laws of free subordinators (see Arizmendi et al. [3]). The class of free regular distributions does not coincide with FID_+ (see also Sakuma [31]). By Theorem 1.3, we get the following formula.

Proposition 4.1. *For any free regular distribution μ, there exists a unique $\sigma \in \mathcal{P}_+$ such that*

$$\Phi(\mu)\big([0,x]\big) = \max\left\{0, 2 - \frac{1}{\Phi(\sigma)([0,x])}\right\}, \quad x \geq 0.$$

If $\Phi(\sigma)([0,x]) = 0$, then it is understood that the right-hand side is zero.

Proof. If μ is a Dirac measure, then we can take $\sigma = \mu$ to satisfy the above equation. Therefore, we may assume that μ is not a Dirac measure. By Arizmendi et al. [3], there exists a unique $\sigma \in \mathcal{P}_+$, not being a Dirac measure, such that $\mu = B_1(\sigma)$.

Moreover, we have

$$B_1^\vee(\nu)\big([0,\cdot]\big) = \Lambda^\vee \circ \mathcal{X}^\vee(\nu)\big([0,\cdot]\big) = \max\left\{0, 2 - \frac{1}{\nu([0,\cdot])}\right\},$$

for probability measures ν on $[0, \infty)$ by Lemma 2.7 and the definitions of $\Lambda^\vee$ and $\mathcal{X}^\vee$. It is understood that $B_1^\vee(\nu)([0,x]) = 0$ when $\nu([0,x]) = 0$ for some $x \geq 0$.

Consequently, we have

$$\begin{aligned}\max\left\{0, 2 - \frac{1}{\Phi(\sigma)([0,\cdot])}\right\} &= B_1^\vee\big(\Phi(\sigma)\big)\big([0,\cdot]\big) \\ &= \Phi\big(B_1(\sigma)\big)\big([0,\cdot]\big) = \Phi(\mu)\big([0,\cdot]\big),\end{aligned}$$

where the second equation holds by Theorem 1.3. Therefore, we obtain the above equation. $\square$

4.5. Infinitely divisible distributions with regular Lévy–Khintchine representations

In the proof of Example 4.5, we saw that

$$\Psi\big(\mathrm{Po}(1)\big)\big([0,x]\big)=\exp\left[1-\frac{1}{\Phi(\frac{1}{2}\delta_0+\frac{1}{2}\delta_2)([0,x])}\right],\quad x\geq 0,$$

where the right-hand side is interpreted as zero for $x<0$. More generally, we get the above formula in the case of infinitely divisible distributions with regular Lévy–Khintchine representations, which are the marginal distributions of subordinators (see, e.g., Bertoin [14], Sato [32]). A probability measure $\mu\in\mathrm{ID}_+$ is said to have a *regular Lévy–Khintchine representation* (in this paper, for short, we say that μ is *regular*) if its characteristic function has the form

$$\int_{\mathbb{R}} e^{izx}\mu(dx)=\exp\left(i\eta z+\int_0^\infty\big(e^{izx}-1\big)\nu(dx)\right),\quad z\in\mathbb{R},$$

where $\eta\geq 0$ and ν is the Lévy measure satisfying $\int_0^\infty(1\wedge x)\nu(dx)<\infty$ and $\nu((-\infty,0])=0$. It is known that $\mu\in\mathrm{ID}_+$ is regular if and only if $\mu^{*t}\in\mathcal{P}_+$ for all $t>0$. For example, the Poisson laws $\mathrm{Po}(\lambda)$, the positive stable laws c_α^+ and Gamma distributions are regular. It is known that if μ is regular, then $\Lambda(\mu)$ is free regular (see Arizmendi et al. [3]). Therefore, we get the following formula.

Proposition 4.2. *For a regular distribution μ, there exists a unique $\sigma\in\mathcal{P}_+$ such that*

$$\Psi(\mu)\big([0,x]\big)=\exp\left[1-\frac{1}{\Phi(\sigma)([0,x])}\right],\quad x\geq 0.$$

If $\Phi(\sigma)([0,x])=0$, then it is understood that the right-hand side is zero.

Proof. If μ is a Dirac measure, then we take $\sigma=\mu$. We may assume that μ is not a Dirac measure. For a regular distribution $\mu\in\mathcal{P}_+$, the measure $\Lambda(\mu)$ is free regular. By Arizmendi et al. [3], there exists a unique $\sigma\in\mathcal{P}_+$ such that $\Lambda(\mu)=B_1(\sigma)$. Then

$$\begin{aligned}\Psi(\mu)\big([0,\cdot]\big)&=\mathcal{X}^\vee\circ\Phi\circ B_1^{-1}\big(\Lambda(\mu)\big)\big([0,\cdot]\big)\\&=\mathcal{X}^\vee\circ\Phi\circ B_1^{-1}\big(B_1(\sigma)\big)\big([0,\cdot]\big)\\&=\mathcal{X}^\vee\circ\Phi(\sigma)\big([0,\cdot]\big)\\&=\exp\left[1-\frac{1}{\Phi(\sigma)([0,\cdot])}\right].\end{aligned}$$

Thus, the above formula follows. □

5. Limit theorems for extreme values

5.1. Extreme values and Marchenko–Pastur laws

Let $(\mathcal{M},\tau)$ be a tracial W^*-probability space. Suppose that $\{U_n\}_n$ is a sequence of freely independent identically distributed (bounded and positive) random variables in $(\mathcal{M},\tau)$ and $U_1\sim U(0,1)$. Define

$$\tilde{U}_n:=U_1\vee\cdots\vee U_n,\quad n\in\mathbb{N}.$$

By (2.4), we have

$$\tau\big(E_{\tilde{U}_n}([0,x])\big) = \max\big\{n\tau\big(E_{U_1}([0,x])\big) - (n-1), 0\big\}$$
$$= \begin{cases} 0, & 0 \le x < 1-1/n, \\ nx-(n-1) & 1-1/n \le x < 1, \\ 1, & x \ge 1. \end{cases}$$

Proposition 5.1. *Let F be a distribution function on $\mathbb{R}$. Then*

$$\tau\big(E_{\tilde{U}_n}\big(\big[0, F^{1/n}\big]\big)\big) \xrightarrow{w} \Lambda^{\vee}(F), \quad n \to \infty.$$

In particular, we have

$$\tau\big(E_{\tilde{U}_n}\big(\big[0, C_I^{1/n}\big]\big)\big) \xrightarrow{w} F_I,$$
$$\tau\big(E_{\tilde{U}_n}\big(\big[0, C_{II,\alpha}^{1/n}\big]\big)\big) \xrightarrow{w} F_{II,\alpha},$$
$$\tau\big(E_{\tilde{U}_n}\big(\big[0, C_{III,\alpha}^{1/n}\big]\big)\big) \xrightarrow{w} F_{III,\alpha},$$

as $n \to \infty$ and $\alpha > 0$.

Proof. Denote by $C(F)$ the set of all points of continuity of F. For all $x \in C(F)$, we have

$$\tau\big(E_{\tilde{U}_n}\big(\big[0, F(x)^{1/n}\big]\big)\big) = \begin{cases} 0, & 0 \le F(x) < (1-1/n)^n, \\ nF(x)^{1/n} - (n-1), & (1-1/n)^n \le F(x) < 1, \\ 1, & F(x) \ge 1, \end{cases}$$
$$\xrightarrow{n\to\infty} \begin{cases} 0, & 0 \le F(x) < e^{-1}, \\ 1 + \log F(x), & e^{-1} \le F(x) < 1, \\ 1, & F(x) \ge 1, \end{cases}$$
$$= \Lambda^{\vee}\big(F(x)\big).$$

The last assertion is clear since $\Lambda^{\vee}$ maps the classical extreme values to the corresponding type free extreme values. □

Remark 5.1. There is no need to use that $x \in C(F)$ in the proof of Proposition 5.1 if F is the extreme value distribution since it is continuous on $\mathbb{R}$.

Recall that $\Phi(\pi)$ is the uniform distribution on $(0, 1)$. Hence, we get the following convergence.

Corollary 5.1. *For any distribution function F on $\mathbb{R}$,*

$$\Phi\big(\pi^{\boxplus n}\big)\big(\big[0, nF^{1/n}\big]\big) \xrightarrow{w} \Lambda^{\vee}(F), \quad n \to \infty.$$

Proof. By Theorem 1.1, we have

$$\Phi\big(\pi^{\boxplus n}\big)\big(\big[0, nF^{1/n}\big]\big) = \Phi\big(D_{1/n}\big(\pi^{\boxplus n}\big)\big)\big(\big[0, F^{1/n}\big]\big) = \Phi(\pi)^{\boxed{\vee} n}\big(\big[0, F^{1/n}\big]\big).$$

By the above discussion and by Proposition 5.1, we obtain the claimed convergence. □

Next, we suppose that $\{V_n\}_n$ is a sequence of Boolean independent identically distributed (bounded and positive) random variables in $(\mathcal{M}, \tau)$ and $V_1 \sim U(0, 1)$. Define

$$\tilde{V}_n := V_1 \vee \cdots \vee V_n, \quad n \in \mathbb{N}.$$

By (2.5), we have

$$\begin{aligned}\tau\big(E_{\tilde{V}_n}([0, x])\big) &= \frac{\tau(E_{V_1}([0, x]))}{n - (n-1)\tau(E_{V_1}([0, x]))} \\ &= \begin{cases} 1, & x \geq 1, \\ \dfrac{x}{n - (n-1)x}, & 0 \leq x < 1. \end{cases}\end{aligned}$$

Proposition 5.2. *Let F be a distribution function on $\mathbb{R}$. Then*

$$\tau\big(E_{\tilde{V}_n}\big([0, F^{1/n}]\big)\big) \xrightarrow{w} \big(\mathcal{X}^{\vee}\big)^{-1}(F), \quad n \to \infty,$$

where it is understood that $((\mathcal{X}^{\vee})^{-1}(F))(x) = 0$ if $F(x) = 0$ (see Section 2.3.3). In particular, we have $\tau(E_{\tilde{V}_n}([0, C_{II,\alpha}^{1/n}])) \xrightarrow{w} B_{II,\alpha}$ as $n \to \infty$ and $\alpha > 0$.

Proof. For any distribution function F on $\mathbb{R}$,

$$\begin{aligned}\tau\big(E_{\tilde{V}_n}\big([0, F^{1/n}]\big)\big) &= \frac{F^{1/n}}{n - (n-1)F^{1/n}} \\ &= \frac{1}{1 - n(1 - F^{-1/n})} \\ &\xrightarrow{w} \frac{1}{1 - \log F} = \big(\mathcal{X}^{\vee}\big)^{-1}(F).\end{aligned}$$

The last assertion is clear since $(\mathcal{X}^{\vee})^{-1}(C_{\mathrm{II},\alpha}) = B_{\mathrm{II},\alpha}$. □

By the same argument as in the proof of Corollary 5.1, we get the following convergence.

Corollary 5.2. *For any distribution function F on $\mathbb{R}$,*

$$\Phi\big(\pi^{\uplus n}\big)\big([0, nF^{1/n}]\big) \xrightarrow{w} \big(\mathcal{X}^{\vee}\big)^{-1}(F), \quad n \to \infty,$$

where it is understood that $((\mathcal{X}^{\vee})^{-1}(F))(x) = 0$ if $F(x) = 0$ (see Section 2.3.3).

Proof. By Theorem 1.2, we have

$$\Phi\big(\pi^{\uplus n}\big)\big([0, nF^{1/n}]\big) = \Phi\big(D_{1/n}\big(\pi^{\uplus n}\big)\big)\big([0, F^{1/n}]\big) = \Phi(\pi)^{\cup\!\!\!\!\vee n}\big([0, F^{1/n}]\big).$$

By the above discussion and by Proposition 5.2, we obtain the claimed convergence. □

5.2. Extreme values and free multiplicative convolution

In this section, we obtain a relation between the free/Boolean extreme values and the free multiplicative convolution.

Proposition 5.3. *For $n \in \mathbb{N}$ and $\mu \in \mathcal{P}_+$, we have*

$$\Phi(\mu^{\boxtimes n})([0, x^n]) = \Phi(\mu)([0, x]), \quad x \in (a_\mu, b_\mu), \tag{5.1}$$

where a_μ and b_μ are defined in (2.3).

Proof. If μ is a Dirac measure δ_a for some $a \geq 0$, then $\mu^{\boxtimes n} = \delta_{a^n}$. Therefore, we get the equation (5.1). Therefore, we may assume that μ is not a Dirac measure. Note that $\mu^{\boxtimes n}(\{0\}) = \mu(\{0\})$ for all $n \in \mathbb{N}$. By Proposition 2.1 and a property of the S-transform, we have

$$z = \Phi(\mu^{\boxtimes n})\left(\left[0, \frac{1}{S_{\mu^{\boxtimes n}}(z-1)}\right]\right) = \Phi(\mu^{\boxtimes n})\left(\left[0, \frac{1}{S_\mu(z-1)^n}\right]\right), \tag{5.2}$$

for all $z \in (\mu(\{0\}), 1)$. Take $z = S_\mu^{-1}(1/x) + 1 \in (\mu(\{0\}), 1)$ in (5.2), where $x \in (a_\mu, b_\mu)$. Then

$$\Phi(\mu^{\boxtimes n})\left(\left[0, \frac{1}{S_\mu(S_\mu^{-1}(1/x)+1-1)^n}\right]\right) = S_\mu^{-1}\left(\frac{1}{x}\right) + 1, \quad x \in (a_\mu, b_\mu). \tag{5.3}$$

Since $S_\mu^{-1}(1/x) + 1 = \Phi(\mu)([0, x])$ by Lemma 2.6, the equation (5.3) holds if and only if (5.1) does. □

Take $\mu = f_\alpha^+$ or $\mu = b_\alpha^+$ in (5.1). By Examples 4.1, 4.2 and Proposition 5.3, we get the following formulas.

Corollary 5.3. *For $\alpha \in (0, 1)$ and $n \in \mathbb{N}$, we have*

$$\Phi({f_\alpha^+}^{\boxtimes n})([0, x^n]) = F_{II, \frac{\alpha}{1-\alpha}}(x), \quad x \geq 0,$$

and

$$\Phi({b_\alpha^+}^{\boxtimes n})([0, x^n]) = B_{II, \frac{\alpha}{1-\alpha}}(x), \quad x \geq 0.$$

Remark 5.2. It follows from Arizmendi and Hasebe [2] that

$$\pi^{\frac{1-\alpha}{\alpha}} \boxtimes f_\alpha^+ = b_\alpha^+, \quad \alpha \in (0, 1).$$

Therefore, we have

$$\Phi(\pi \boxtimes f_{1/2}^+)([0, \cdot]) = \Phi(b_{1/2}^+)([0, \cdot]) = B_{\mathrm{II},1}(\cdot),$$

where the last equation holds when we take $n = 1$ and $\alpha = 1/2$ in Corollary 5.3.

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