Bernoulli **27**(1), 2021, 442–468
https://doi.org/10.3150/20-BEJ1245

On the error bound in the normal approximation for Jack measures

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In this paper, we obtain uniform and non-uniform bounds on the Kolmogorov distance in the normal approximation for Jack deformations of the character ratio, by using Stein's method and zero-bias couplings. Our uniform bound comes very close to that conjectured by Fulman (*J. Combin. Theory Ser. A* **108** (2004) 275–296). As a by-product of the proof of the non-uniform bound, we obtain a Rosenthal-type inequality for zero-bias couplings.

Keywords: Jack deformation; Jack measure; Kolmogorov distance; non-uniform bound; rate of convergence; Stein's method; uniform bound; zero-bias coupling

1. Introduction and main results

Let G be a finite group, and G^* the set of all the irreducible representations of G. Then

$$\sum_{\pi\in G^*} \dim(\pi)^2 = |G|,$$

where $\dim(\pi)$ denotes the dimension of the irreducible representation π (Sagan [28], Proposition 1.10.1). The Plancherel measure is a probability measure on G^* defined by

$$\mathbb{P}\big(\{\pi\}\big) = \frac{\dim(\pi)^2}{|G|}.$$

Let n be a positive integer. An important special case is the finite symmetric group $\mathcal{S}_n$. For this group, the irreducible representations are parametrized by partitions λ of n, and the dimension of the representation associated to λ is known to be equal to the number of standard λ-tableaux (Sagan [28], Theorem 2.6.5). We also denote the number of standard λ-tableaux by $\dim(\lambda)$, and write a partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_m)$ of n simply $\lambda \vdash n$. The hooklength of a box s in the partition λ is defined as $h(s) = a(s) + l(s) + 1$. Here $a(s)$ denotes the number of boxes in the same row of s and to the right of s (the "arm" of s) and $l(s)$ denotes the number of boxes in the same column of s and below s (the "leg" of s). The Plancherel measure in this case is

$$\mathbb{P}\big(\{\lambda\}\big) = \frac{\dim(\lambda)^2}{n!}.$$

By the hook formula (see, e.g., Sagan [28]) which states that

$$\dim(\lambda) = \frac{n!}{\prod_{s\in\lambda} h(s)},$$

where the product is over boxes in the partition and $h(s)$ is the hooklength of a box s, we also have

$$\mathbb{P}\big(\{\lambda\}\big) = \frac{n!}{\prod_{s\in\lambda} h^2(s)}. \tag{1.1}$$

A random partition λ chosen by the Plancherel measure has interesting connections to the Gaussian unitary ensemble (GUE) of random matrix theory. We recall that the joint probability density of the eigenvalues $x_1 \geq x_2 \geq \cdots \geq x_n$ of the Gaussian orthogonal ensemble (GOE), Gaussian unitary ensemble (GUE), and Gaussian symplectic ensemble (GSE) is given by

$$\frac{1}{Z_\beta}\exp\left(-\frac{x_1^2+\cdots+x_n^2}{2}\right)\prod_{1\leq i<j\leq n}(x_i-x_j)^\beta \tag{1.2}$$

with $\beta = 1, 2, 4$, respectively. Here Z_β is a normalization constant. Let π be a permutation chosen from the uniform measure of the symmetric group $\mathcal{S}_n$ and $l(\pi)$ the length of the longest increasing subsequence in π. Baik, Deift and Johansson [1] proved that $(l(\pi) - 2\sqrt{n})/n^{1/6}$ converges to the Tracy–Widom distribution as $n \to \infty$. It follows from the Robinson–Schensted–Knuth correspondence (see Sagan [28]) that the first row of a random partition distributed according to the Plancherel measure has the same distribution as the longest increasing subsequence of a random permutation distributed according to the uniform measure. So the result of Baik, Deift and Johansson [1] says that a suitably normalized length of the first row of a random partition distributed according to the Plancherel measure converges to the Tracy–Widom distribution. Borodin, Okounkov and Olshanski [2], Johansson [21] proved that the joint distribution of suitably normalized lengths of the rows of a random partition distributed according to the Plancherel measure converges to the joint distribution of the eigenvalues $x_1 \geq x_2 \geq \cdots \geq x_n$ of a $n \times n$ GUE matrix.

Jack$_\alpha$ measure is an extension of the Plancherel measure. For $\alpha > 0$, the Jack$_\alpha$ measure is a probability measure on the set of all partitions of a positive integer n, which chooses a partition λ with probability

$$\mathbb{P}_\alpha\big(\{\lambda\}\big) = \frac{\alpha^n n!}{\prod_{s\in\lambda}(\alpha a(s) + l(s) + 1)(\alpha a(s) + l(s) + \alpha)},$$

where the product is over all boxes in the partition. For example, the partition

$$\lambda = \begin{array}{lll} \square & \square & \square \\ \square & \square & \\ \square & & \end{array}$$

of 6 has Jack$_\alpha$ measure

$$\frac{720\alpha^3}{(3\alpha+2)(2\alpha+3)(\alpha+2)^2(2\alpha+1)^2}.$$

We notice that the Jack measure with parameter $\alpha = 1$ agrees the Plancherel measure of the symmetric group since it coincides with (1.1). It is mentioned in Matsumoto [26] that for any positive real number $\beta > 0$, the Jack$_\alpha$ measure with $\alpha = 2/\beta$ is the counterpart of the Gaussian β-ensemble ($G\beta E$) with the probability density function proportional to (1.2).

Let λ be a partition of n chosen from the Plancherel measure of the symmetric group $\mathcal{S}_n$, and $\chi^\lambda(12)$ the character of the irreducible representation associated to λ evaluated on the transposition (12). Characters of the irreducible representations of a symmetric group are of interest in the literature because they play central roles in representation theory and other fields of mathematics such as random walks (Diaconis and Shahshahani [8]) and the moduli space of curves (Eskin and Okounkov [11]). The quantity $\chi^\lambda(12)/\dim(\lambda)$, which is a normalization of $\chi^\lambda(12)$, is called a character ratio. As λ is distributed according to the Plancherel measure, $\chi^\lambda(12)$ is a random variable.

In Kerov [23], it is stated that

$$\frac{\sqrt{\binom{n}{2}}\chi^\lambda(12)}{\dim(\lambda)} \tag{1.3}$$

is asymptotically normal with mean 0 and variance 1 as $n\to\infty$. A proof of Kerov's central limit theorem can be found in Hora [18], which uses the method of moments and combinatorics. More recently, a proof in Śniady [30] uses the genus expansion of random matrix theory, and another in Hora and Obata [19] uses quantum probability.

By a formula due to Frobenius [12] (see also Fulman [14]), we have

$$\frac{\chi^\lambda(12)}{\dim(\lambda)} = \frac{1}{\binom{n}{2}}\sum_i\left(\binom{\lambda_i}{2}-\binom{\lambda_i'}{2}\right). \tag{1.4}$$

Now, for $\alpha>0$, the random variable we will study in this paper is

$$W_{n,\alpha} = W_{n,\alpha}(\lambda) = \frac{\sum_i\left(\alpha\binom{\lambda_i}{2}-\binom{\lambda_i'}{2}\right)}{\sqrt{\alpha\binom{n}{2}}}, \tag{1.5}$$

where λ is chosen from the Jack$_\alpha$ measure on partitions of a positive integer n, λ_i is the length of the i-th row of λ and λ_i' is the length of the i-th column of λ. By (1.4), $W_{n,\alpha}$ coincides with (1.3) when $\alpha=1$. Therefore, the value $W_{n,\alpha}$ is regarded as a Jack deformation of the character ratio. Moreover, as remarked by Fulman [13], when $\alpha=2$, $W_{n,2}$ is the value of a spherical function corresponding to the Gelfand pair (S_{2n},H_{2n}), where H_{2n} is the hyperoctahedral group of size $2^n n!$.

Normally approximation for $W_{n,\alpha}$ has been studied by Fulman [13,14], Shao and Su [29], and Fulman and Goldstein [15] by using Stein's method (see, e.g., Stein [31]). In Fulman [13], the author proved that for any fixed $\alpha\ge 1$,

$$\sup_{x\in\mathbb{R}}\left|\mathbb{P}_\alpha(W_{n,\alpha}\le x)-\Phi(x)\right| \le \frac{C_\alpha}{n^{1/4}}, \tag{1.6}$$

where C_α is a constant depending only on α, $\Phi(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^x\exp(-t^2/2)\,\mathrm{d}t$ is the distribution function of the standard normal distribution.

The bound $C_\alpha n^{-1/4}$ was later improved in Fulman [14] to $C_\alpha n^{-1/2}$ using an inductive approach to Stein's method. We note that in all these results, $\alpha>0$ is fixed, but we do not know how C_α depends on α. An explicit constant is obtained by Shao and Su [29] only when $\alpha=1$. More precisely, when $\alpha=1$, Shao and Su [29] obtained the rate $761n^{-1/2}$ by using Stein's method for exchangeable pairs. More recently, Dołęga and Féray [9] proved the Berry–Esseen bound for the multivariate case with rate $C_\alpha n^{-1/4}$, and Dołęga and Śniady [10] proved a general multivariate central limit theorem for the case where $\alpha = \alpha(n)$ varying with n, satisfying

$$\frac{-\sqrt{\alpha} + 1/\sqrt{\alpha}}{\sqrt{n}} = g_1 + \frac{g_2}{\sqrt{n}} + o\left(\frac{1}{\sqrt{n}}\right),$$

where g_1 and g_2 are constants.

Fulman [13] conjectured that for general $\alpha \geq 1$, the correct bound is a universal constant multiplied by $\max\{\frac{1}{\sqrt{n}}, \frac{\sqrt{\alpha}}{n}\}$. While this bound was conjectured for the Kolmogorov distance in (1.6), using Stein's method and zero-bias couplings, Fulman and Goldstein [15] proved that it is indeed the correct bound for the Wasserstein distance for $W_{n,\alpha}$. By the result in Fulman and Goldstein [15], the central limit theorem for $W_{n,\alpha}$ holds for $\alpha = \alpha(n)$ varying with n as long as $\sqrt{\alpha}/n \to 0$. As observed by Fulman [13], this is necessary for $W_{n,\alpha}$ to be asymptotically normal. The bound conjectured by Fulman [13] for the Kolmogorov distance remains unsolved as bounds on the Kolmogorov distance are usually harder to obtain than bounds on the Wasserstein distance. This paper is an attempt to prove the conjecture of Fulman [13] for the Kolmogorov distance. We use Stein's method and zero-bias couplings to obtain both uniform and non-uniform error bounds on the Kolmogorov distance for $W_{n,\alpha}$. We have obtained a uniform error bound which comes very close to that conjectured by Fulman [13]. Besides, we have obtained a very small constant. As a by-product of the proof of the non-uniform bound, we obtain a Rosenthal-type inequality for zero-bias couplings.

Throughout this paper, Z denotes the standard normal random variable and $\Phi(x) = \frac{1}{\sqrt{2\pi}} \times \int_{-\infty}^{x} \exp(-t^2/2)\,\mathrm{d}t$ its distribution function. For a positive number x, $\log x$ denotes the natural logarithm of x, and $\lfloor x \rfloor$ denotes the greatest integer number that is less than or equal to x. For a set S, the indicator function of S is denoted by $\mathbf{1}(S)$ and the cardinality of S denoted by $|S|$. For $p \geq 1$ and a random variable X, $(\mathbb{E}|X|^p)^{1/p}$ is denoted by $\|X\|_p$. The symbol C_p denotes a generic positive constant bounded by B^p for some constant B which can be different for each appearance. We denote Jack$_\alpha$ measure by $\mathbb{P}_\alpha$.

Theorem 1.1. *Let $n \geq 3$ be an integer. Let $\alpha > 0$ and $W_{n,\alpha}$ be as in* (1.5). *Then*

$$\sup_{x \in \mathbb{R}} \left|\mathbb{P}_\alpha(W_{n,\alpha} \leq x) - \Phi(x)\right| \leq 8.2 \max\left\{\frac{1}{\sqrt{n}}, \frac{\max\{\sqrt{\alpha}, 1/\sqrt{\alpha}\} \log n}{n}\right\}.$$

Remark 1.2. If $\frac{\log^2 n}{n} \leq \alpha \leq \frac{n}{\log^2 n}$, then the bound in Theorem 1.1 is $\frac{8.2}{\sqrt{n}}$. For $\alpha \geq 1$, the bound in Theorem 1.1 is $8.2 \max\{\frac{1}{\sqrt{n}}, \frac{\sqrt{\alpha}\log n}{n}\}$, which is very close to that conjectured by Fulman [13].

We prove Theorem 1.1 by using Stein's method for zero bias couplings. Non-uniform bounds on the Kolmogorov distance in the normal approximation for independent random variables using Stein's method were first investigated by Chen and Shao [5]. Stein's method has also been used to study non-uniform bounds on the Kolmogorov distance (Chen and Shao [6]) and concentration inequalities (Chatterjee and Dey [3]) for dependent random variables. The method developed in this paper also allows us to obtain a non-uniform bound on the Kolmogorov distance, which we state in the following theorem.

Theorem 1.3. *Let $n \geq 3$ be an integer. Let $p \geq 2$, $1/n^2 < \alpha < n^2$ and $W_{n,\alpha}$ be as in* (1.5). *Then for all $x \in \mathbb{R}$, we have*

$$\left|\mathbb{P}_\alpha(W_{n,\alpha} \leq x) - \Phi(x)\right| \leq \frac{C_p}{1 + |x|^p} \left(\frac{p^2}{\log p}\right)^p \max\left\{\frac{1}{\sqrt{n}}, \frac{\max\{\sqrt{\alpha}, 1/\sqrt{\alpha}\} \log n}{n}\right\}.$$

Remark 1.4. If $\alpha \geq n^2$, then, as shown by Fulman [13], the random variables $W_{n,\alpha}$ are no longer asymptotically normal as $n, \alpha \to \infty$. However, we can still derive an upper bound for the tail probability $\mathbb{P}_\alpha(|W_{n,\alpha}| \geq x)$. Indeed, for $\alpha \geq n^2$, $n \geq 3$, and $p \geq 2$, it will be shown in the Appendix that

$$\mathbb{E}|W_{n,\alpha}|^p \leq C_p \left(\frac{p^2}{\log p}\right)^p \left(\frac{\sqrt{\alpha}}{n}\right)^{p-2}, \tag{1.7}$$

so that, by Markov's inequality,

$$\mathbb{P}_\alpha\big(|W_{n,\alpha}| \geq x\big) \leq \frac{\mathbb{E}(1 + |W_{n,\alpha}|^p)}{1 + x^p} \leq \frac{C_p}{1 + x^p}\left(\frac{p^2}{\log p}\right)^p \left(\frac{\sqrt{\alpha}}{n}\right)^{p-2}$$

for all $x \geq 0$.

2. A Rosenthal-type inequality for zero-bias couplings

It was shown in Goldstein and Reinert [17] that for any mean zero random variable W with positive finite variance σ^2, there exists a random variable W^* which satisfies

$$\mathbb{E} W f(W) = \sigma^2 \mathbb{E} f'\big(W^*\big) \tag{2.1}$$

for all absolutely continuous f with $\mathbb{E}|W f(W)| < \infty$. The random variable W^* and its distribution are called W-zero biased. Goldstein and Reinert [17] (see also Proposition 2.1 of Chen, Goldstein and Shao [4]) showed that the distribution of W^* is absolutely continuous with the density $g(x) = \mathbb{E}[W\mathbf{1}(W > x)]/\sigma^2$.

In this section, we prove a Rosenthal-type inequality for zero-bias couplings, which we state as a proposition below. We will show later that this proposition can be applied to obtain the Rosenthal inequality for sums of independent random variables. The use of a Rosenthal-type inequality is crucial for obtaining a non-uniform bound on the Kolmogorov distance.

Proposition 2.1. *Let W be a random variable with mean zero and variance $\sigma^2 > 0$ and let W^* be W-zero biased. Assume that W and W^* are defined on the same probability space. Let $T = W^* - W$. Then for every $p \geq 2$,*

$$\mathbb{E}|W|^p \leq \kappa_p\big(\sigma^p + \sigma^2 \mathbb{E}|T|^{p-2}\big), \tag{2.2}$$

where

$$\kappa_p = \frac{(\log 8)^3}{196}\left(\frac{7p}{4\log p}\right)^p.$$

Proof. Let

$$f(x) = \begin{cases} x^{p-1} & \text{if } x \geq 0, \\ -(-x)^{p-1} & \text{if } x < 0. \end{cases} \tag{2.3}$$

Then $f'(x) = (p-1)|x|^{p-2}$ and $x f(x) = |x|^p$.

If $2 \le p \le 4$, then

$$\begin{aligned}\mathbb{E}|W|^p &= \mathbb{E}Wf(W) = \sigma^2\mathbb{E}f'(W+T)\\ &= \sigma^2(p-1)\mathbb{E}|W+T|^{p-2}\\ &\le \sigma^2(p-1)\max\{1,2^{p-3}\}\big(\mathbb{E}|W|^{p-2}+\mathbb{E}|T|^{p-2}\big)\\ &\le \sigma^2(p-1)\max\{1,2^{p-3}\}\big(\sigma^{p-2}+\mathbb{E}|T|^{p-2}\big)\\ &= (p-1)\max\{1,2^{p-3}\}\big(\sigma^p+\sigma^2\mathbb{E}|T|^{p-2}\big). \end{aligned}\tag{2.4}$$

Elementary calculus shows that

$$(p-1)\max\{1,2^{p-3}\} \le \frac{(\log 8)^3}{196}\left(\frac{7p}{4\log p}\right)^p$$

for all $2 \le p \le 4$. Therefore, from (2.4), we see that (2.2) holds for all $2 \le p \le 4$.

If $p > 4$, by Jensen's inequality, we have for all $0 < \theta < 1$,

$$\begin{aligned}\mathbb{E}|W|^p &= \mathbb{E}Wf(W) = \sigma^2\mathbb{E}f'(W+T)\\ &= \sigma^2(p-1)\mathbb{E}|W+T|^{p-2}\\ &\le \sigma^2(p-1)\left(\theta\mathbb{E}\left(\frac{|W|}{\theta}\right)^{p-2}+(1-\theta)\mathbb{E}\left(\frac{|T|}{1-\theta}\right)^{p-2}\right)\\ &= \sigma^2(p-1)\left(\frac{\mathbb{E}|W|^{p-2}}{\theta^{p-3}}+\frac{\mathbb{E}|T|^{p-2}}{(1-\theta)^{p-3}}\right). \end{aligned}\tag{2.5}$$

By using the following inequality

$$x^\alpha y^{1-\alpha} \le x+y \quad \text{for all } 0<\alpha<1,\ x\ge 0,\ y\ge 0,\tag{2.6}$$

we have

$$\begin{aligned}\mathbb{E}\big(\sigma^2|T|^{p-4}\big) &= \mathbb{E}\Big(\big(\sigma^{p-2}\big)^{\frac{2}{p-2}}\big(|T|^{p-2}\big)^{\frac{p-4}{p-2}}\Big)\\ &\le \mathbb{E}\big(\sigma^{p-2}+|T|^{p-2}\big)\\ &= \sigma^{p-2}+\mathbb{E}|T|^{p-2}. \end{aligned}\tag{2.7}$$

For the case where $4 < p \le 6$, (2.4) and (2.7) yield

$$\begin{aligned}\mathbb{E}|W|^{p-2} &\le (p-3)\max\{1,2^{p-5}\}\big(\sigma^{p-2}+\sigma^2\mathbb{E}|T|^{p-4}\big)\\ &\le (p-3)\max\{1,2^{p-5}\}\big(2\sigma^{p-2}+\mathbb{E}|T|^{p-2}\big). \end{aligned}\tag{2.8}$$

By letting $\theta := \theta_1 = 1/2$, we have from (2.5) that

$$\mathbb{E}|W|^p \le \sigma^2(p-1)2^{p-3}\big(\mathbb{E}|W|^{p-2}+\mathbb{E}|T|^{p-2}\big).\tag{2.9}$$

Combining (2.9) and (2.8), we obtain

$$\mathbb{E}|W|^p \le (p-1)(p-3)2^{p-2}\max\{1,2^{p-5}\}\big(\sigma^p+\sigma^2\mathbb{E}|T|^{p-2}\big).\tag{2.10}$$

Numerical calculations show that

$$(p-1)(p-3)2^{p-2}\max\{1,2^{p-5}\}\le\frac{(\log 8)^3}{196}\left(\frac{7p}{4\log p}\right)^p$$

for all $4<p\le 6$. Therefore, from (2.10), we see that (2.2) holds in this case.

For the case where $6<p\le 8$, (2.7) and (2.10) yield

$$\mathbb{E}|W|^{p-2}\le (p-3)(p-5)2^{p-4}\max\{1,2^{p-7}\}\left(2\sigma^{p-2}+\mathbb{E}|T|^{p-2}\right). \tag{2.11}$$

By letting $\theta:=\theta_2=2/3$, we have from (2.5) that

$$\mathbb{E}|W|^p\le\sigma^2(p-1)\left(\frac{3}{2}\right)^{p-3}\left(\mathbb{E}|W|^{p-2}+2^{p-3}\mathbb{E}|T|^{p-2}\right). \tag{2.12}$$

Combining (2.11) and (2.12), we obtain

$$\mathbb{E}|W|^p\le (p-1)(p-3)(p-5)3^{p-3}\max\{1,2^{p-7}\}\left(\sigma^p+\sigma^2\mathbb{E}|T|^{p-2}\right). \tag{2.13}$$

Numerical calculations also show that

$$(p-1)(p-3)(p-5)3^{p-3}\max\{1,2^{p-7}\}\le\frac{(\log 8)^3}{196}\left(\frac{7p}{4\log p}\right)^p$$

for $6<p\le 8$. Therefore, from (2.13), we see that (2.2) holds in this case.

For the case where $p>8$, we prove the result by induction. Assume that (2.2) holds with p replaced by $p-2$. By (2.7), we have

$$\begin{aligned}\mathbb{E}|W|^{p-2}&\le\kappa_{p-2}\left(\sigma^{p-2}+\sigma^2\mathbb{E}|T|^{p-4}\right)\\&\le\kappa_{p-2}\left(2\sigma^{p-2}+\mathbb{E}|T|^{p-2}\right).\end{aligned} \tag{2.14}$$

Combining (2.5) and (2.14), we obtain

$$\mathbb{E}|W|^p\le (p-1)\left(\frac{2\kappa_{p-2}}{\theta^{p-3}}\sigma^p+\left(\frac{\kappa_{p-2}}{\theta^{p-3}}+\frac{1}{(1-\theta)^{p-3}}\right)\sigma^2\mathbb{E}|T|^{p-2}\right). \tag{2.15}$$

The proof is completed if we can choose $0<\theta<1$ such that

$$\frac{2(p-1)\kappa_{p-2}}{\theta^{p-3}}\le\kappa_p\quad\text{and}\quad\frac{p-1}{(1-\theta)^{p-3}}\le\frac{\kappa_p}{2}. \tag{2.16}$$

By Lemma A.1 in the Appendix, we have

$$\kappa_p\ge 8\left(\frac{p-1}{\log(p-1)}\right)^2\kappa_{p-2}. \tag{2.17}$$

Let

$$\theta=\theta(p):=\left(\frac{\log^2(p-1)}{4(p-1)}\right)^{1/(p-3)}.$$

Then $0<\theta<1$ and the first half of (2.16) holds by (2.17). By Lemma A.2 (in the Appendix), the second half of (2.16) holds.

The proof of the proposition is completed. □

We now present a simple proof of the Rosenthal inequality (Rosenthal [27]) for sums of mean zero independent random variables by using Proposition 2.1. If $\{X_i, 1 \le i \le n\}$ are independent symmetric random variables, Johnson, Schechtman and Zinn [22] proved that

$$\left\| \sum_{i=1}^n X_i \right\|_p \le \frac{Kp}{\log p} \max\left\{ \left\| \sum_{i=1}^n X_i \right\|_2, \left(\sum_{i=1}^n \|X_i\|_p^p \right)^{1/p} \right\} \quad \text{for all } p \ge 2, \tag{2.18}$$

where K is a universal constant satisfying $\frac{1}{e\sqrt{2}} \le K \le 7.35$. Johnson, Schechtman and Zinn [22] also proved that the rate $p/\log p$ is optimal. Latała [25] showed that (2.18) holds with K approximately equal to $2e$ (see Theorem 2 and Corollary 3 ibidem). In Ibragimov and Sharakhmetov [20], the authors proved that the constant K in (2.18) is approximated by $1/e$ when p is large enough (see the corollary in page 295 ibidem). However, we are not aware of any result in the literature (even with assuming the symmetry of the random variables) which proved (2.18) holds with $K \le 3.5$ for all $p \ge 2$ as given in the following proposition.

Proposition 2.2. *Let $p \ge 2$ and $\{X_i, 1 \le i \le n\}$ be a collection of n independent mean zero random variables with $\mathbb{E}|X_i|^p < \infty$, $1 \le i \le n$. Then*

$$\left\| \sum_{i=1}^n X_i \right\|_p \le \frac{3.5p}{\log p} \max\left\{ \left\| \sum_{i=1}^n X_i \right\|_2, \left(\sum_{i=1}^n \|X_i\|_p^p \right)^{1/p} \right\}. \tag{2.19}$$

Proof. Let $W = \sum_{i=1}^n X_i$ and $\sigma^2 = \mathrm{Var}(W)$. Denote $\mathrm{Var}(X_i)$ by σ_i^2, $1 \le i \le n$. Let X_i^* have the X_i-zero biased distribution with $\{X_i^*, 1 \le i \le n\}$ mutually independent and X_i^* independent of $\{X_j, j \ne i\}$. Let I be a random index, independent of $\{X_i, X_i^*, 1 \le i \le n\}$, with the distribution

$$\mathbb{P}(I = i) = \frac{\sigma_i^2}{\sigma^2}.$$

The argument proving part (v) of Lemma 2.1 in Goldstein and Reinert [17] shows that removing X_I and replacing it by X_I^* gives a random variable W^* with the W-zero biased distribution, that is,

$$W^* = W - X_I + X_I^*$$

has the W-zero biased distribution.

Let κ_p be as in Proposition 2.1. By Proposition 2.1, we have

$$\begin{aligned}
\mathbb{E}|W|^p &\le \kappa_p\left(\sigma^p + \sigma^2 \mathbb{E}\left|W^* - W\right|^{p-2}\right) \\
&= \kappa_p\left(\sigma^p + \sigma^2 \mathbb{E}\left|X_I - X_I^*\right|^{p-2}\right) \\
&= \kappa_p\left(\sigma^p + \sigma^2 \sum_{i=1}^n \mathbb{E}\left|X_i - X_i^*\right|^{p-2} \frac{\sigma_i^2}{\sigma^2}\right) \\
&\le \kappa_p\left(\sigma^p + \max\{1, 2^{p-3}\} \sum_{i=1}^n \sigma_i^2 \left(\mathbb{E}|X_i|^{p-2} + \mathbb{E}\left|X_i^*\right|^{p-2}\right)\right) \\
&\le \kappa_p\left(\sigma^p + 2^{p-2} \sum_{i=1}^n \sigma_i^2 \left(\mathbb{E}|X_i|^{p-2} + \mathbb{E}\left|X_i^*\right|^{p-2}\right)\right).
\end{aligned} \tag{2.20}$$

By Hölder's inequality, we have

$$\sigma_i^2 \mathbb{E}|X_i|^{p-2} \le \big(\mathbb{E}|X_i|^p\big)^{2/p} \big(\mathbb{E}|X_i|^p\big)^{(p-2)/p} = \mathbb{E}|X_i|^p \tag{2.21}$$

for all $1 \le i \le n$. With the function f as defined in (2.3), it follows from (2.1) that

$$(p-1)\sigma_i^2 \mathbb{E}\big|X_i^*\big|^{p-2} = \mathbb{E}|X_i|^p. \tag{2.22}$$

Combining (2.20)–(2.22), we have

$$\mathbb{E}|W|^p \le \kappa_p \left(\sigma^p + 2^{p-1} \sum_{i=1}^n \mathbb{E}|X_i|^p \right) \le 2^p \kappa_p \max\left\{ \sigma^p, \sum_{i=1}^n \mathbb{E}|X_i|^p \right\},$$

which proves (2.19). □

3. Uniform and non-uniform Kolmogorov bounds for zero-bias couplings

Optimal bounds on the Kolmogorov distance for zero-bias couplings have already been obtained by Goldstein [16] provided the difference between the original random variable and its zero bias transform is properly bounded. In this section, we improved the mentioned result by Goldstein [16] in two directions: firstly, a truncation argument is used to go beyond the boundedness, and secondly, non-uniform bounds with polynomial decay are provided. The following theorem gives the Kolmogorov bound in normal approximation for W^*.

Theorem 3.1. *Let W be such that $\mathbb{E}W = 0$ and $\operatorname{Var}(W) = 1$, and let W^* be W-zero biased and be defined on the same probability space as W. Let $T = W^* - W$.*

(i) *We have*

$$\sup_{x\in\mathbb{R}} \big|\mathbb{P}\big(W^* \le x\big) - \Phi(x)\big| \le \left(1 + \frac{\sqrt{2\pi}}{4}\right)\sqrt{\mathbb{E}T^2}. \tag{3.1}$$

(ii) *Let $p \ge 2$. Then for all $x \in \mathbb{R}$,*

$$\big|\mathbb{P}\big(W^* \le x\big) - \Phi(x)\big| \le \frac{C_p}{1+|x|^p}\left(\frac{p}{\log p}\right)^p \left(\sqrt{\mathbb{E}T^2} + \sqrt{\mathbb{E}|T|^{2p+2}}\right). \tag{3.2}$$

Proof. For $x \in \mathbb{R}$, let f_x be the unique bounded solution of the Stein equation

$$f'(w) - wf(w) = 1(w \le x) - \Phi(x), \tag{3.3}$$

and let

$$g_x(w) = \big(wf_x(w)\big)'. \tag{3.4}$$

We have $0 < f_x(w) \le \sqrt{2\pi}/4$ and $|f_x'(w)| \le 1$ for all $w \in \mathbb{R}$ (see Stein [31]). Therefore

$$\big|g_x(w)\big| = \big|f_x(w) + wf_x'(w)\big| \le 1 + |w| \quad \text{for all } w \in \mathbb{R}, \tag{3.5}$$

$$\mathbb{E}\big|Tf_x(W+T)\big| \le \frac{\sqrt{2\pi}}{4}\mathbb{E}|T| \le \frac{\sqrt{2\pi}}{4}\sqrt{\mathbb{E}T^2}, \tag{3.6}$$

and

$$\mathbb{E}\big|(W\big(f_x(W+T)-f_x(W)\big)\big| \le \mathbb{E}|WT| \le \sqrt{\mathbb{E}W^2\mathbb{E}T^2} = \sqrt{\mathbb{E}T^2}. \tag{3.7}$$

Since

$$\begin{aligned}\big|\mathbb{P}\big(W^* \le x\big) - \Phi(x)\big| &= \big|\mathbb{E}f_x'\big(W^*\big) - \mathbb{E}W^* f_x\big(W^*\big)\big| \\ &= \big|\mathbb{E}Wf_x(W) - \mathbb{E}(W+T)f_x(W+T)\big| \\ &\le \mathbb{E}\big|W\big(f_x(W+T)-f_x(W)\big)\big| + \mathbb{E}\big|Tf_x(W+T)\big|,\end{aligned} \tag{3.8}$$

the conclusion (3.1) follows by combining (3.6), (3.7), and (3.8). Theorem 3.1(i) is proved.

To prove Theorem 3.1(ii), it suffices to consider the case where $x \ge 0$ since we can simply apply the result to $-W^*$ when $x < 0$ (see (2.59) in Chen, Goldstein and Shao [4]). In view of the uniform bound (3.1), it suffices to consider the case where $x \ge 2$. By applying Markov's inequality and Proposition 2.1, we have

$$\begin{aligned}\big|P\big(W^* \le x\big) - \Phi(x)\big| &\le \max\big\{P\big(W^* > x\big), 1-\Phi(x)\big\} \\ &\le \max\left\{\frac{\mathbb{E}|W^*|^{p+1}}{x^{p+1}}, 1-\Phi(x)\right\} \\ &= \max\left\{\frac{\mathbb{E}|W|^{p+3}}{(p+2)x^{p+1}}, 1-\Phi(x)\right\} \\ &\le \max\left\{\frac{\kappa_{p+3}\big(1+\mathbb{E}|T|^{p+1}\big)}{(p+2)x^{p+1}}, 1-\Phi(x)\right\}.\end{aligned} \tag{3.9}$$

By using the fact that $\sqrt{2\pi}(1-\Phi(x)) \le e^{-x^2/2}/x$ for all $x > 0$, we have

$$\max_{x>0} x^{p+1}\big(1-\Phi(x)\big) \le \frac{1}{\sqrt{2\pi}} \max_{x>0} x^p e^{-x^2/2} = \frac{1}{\sqrt{2\pi}}\left(\frac{\sqrt{p}}{\sqrt{e}}\right)^p. \tag{3.10}$$

Combining (3.9) and (3.10), we obtain

$$\big|P\big(W^* \le x\big) - \Phi(x)\big| \le \frac{C_p}{1+x^p}\left(\frac{p}{\log p}\right)^p \Big(1+\sqrt{\mathbb{E}|T|^{2p+2}}\Big). \tag{3.11}$$

If $\mathbb{E}|T|^{2p+2} \ge 1$, then $1+\sqrt{\mathbb{E}|T|^{2p+2}} \le 2\sqrt{\mathbb{E}|T|^{2p+2}}$. Therefore, (3.2) holds by (3.11). It remains to consider the case where $\mathbb{E}|T|^{2p+2} < 1$. In this case, by applying Proposition 2.1 and Jensen's inequality, we have

$$\begin{aligned}\mathbb{E}|W|^{2p} &\le \kappa_{2p}\big(1+\mathbb{E}|T|^{2p-2}\big) \\ &\le 2\kappa_{2p} \le C_p\left(\frac{p}{\log p}\right)^{2p},\end{aligned} \tag{3.12}$$

and

$$\begin{aligned}\mathbb{E}|W|^{2p+2} &\le \kappa_{2p+2}\big(1+\mathbb{E}|T|^{2p}\big) \\ &\le 2\kappa_{2p+2} \le C_p\left(\frac{p}{\log p}\right)^{2p}.\end{aligned} \tag{3.13}$$

Since

$$\mathbb{P}\big(W^* \le x\big) - \Phi(x) = \mathbb{E}\big\{W f_x(W) - W^* f_x\big(W^*\big)\big\}$$
$$= -\mathbb{E}\int_0^T g_x(W+t)\,\mathrm{d}t,$$

we have

$$\big|\mathbb{P}\big(W^* \le x\big) - \Phi(x)\big| \le R_1 + R_2, \tag{3.14}$$

where

$$R_1 = \left|\mathbb{E}\int_0^T g_x(W+t)\left(\mathbf{1}(W+t\le 0) + \mathbf{1}\left(0 < W+t \le \frac{x}{2}\right)\right)\mathrm{d}t\right| \tag{3.15}$$

and

$$R_2 = \left|\mathbb{E}\int_0^T g_x(W+t)\mathbf{1}\left(W+t > \frac{x}{2}\right)\mathrm{d}t\right|. \tag{3.16}$$

From the definition of f_x and g_x, we have (see Chen and Shao [5])

$$g_x(w) = \begin{cases} \big(\sqrt{2\pi}\big(1+w^2\big)e^{w^2/2}\big(1-\Phi(w)\big) - w\big)\Phi(x) & \text{if } w \ge x, \\ \big(\sqrt{2\pi}\big(1+w^2\big)e^{w^2/2}\Phi(w) + w\big)\big(1-\Phi(x)\big) & \text{if } w < x. \end{cases} \tag{3.17}$$

Chen and Shao [5] proved that $g_x \ge 0$, $g_x(w) \le 2(1-\Phi(x))$ for $w \le 0$, and g_x is increasing for $0 \le w < x$. From (3.17) and the fact that $\sqrt{2\pi}(1-\Phi(x)) \le e^{-x^2/2}/x$ for all $x > 0$, we have

$$g_x(x/2) = \left(\sqrt{2\pi}\left(1+\frac{x^2}{4}\right)e^{x^2/8}\Phi(x/2) + \frac{x}{2}\right)\big(1-\Phi(x)\big)$$
$$\le \left(\frac{1}{x} + \frac{x}{4}\right)e^{-3x^2/8} + \frac{1}{2\sqrt{2\pi}}e^{-x^2/2}. \tag{3.18}$$

For all $r \ge 1$, a straightforward calculation shows that

$$\max_{x>0} x^r e^{-x^2/2} < \max_{x>0} x^r e^{-3x^2/8} = \left(\frac{2\sqrt{r}}{\sqrt{3e}}\right)^r.$$

Therefore, from (3.15) and (3.18) and a similar argument as the one used in (3.10), we have

$$R_1 \le \mathbb{E}\int_0^{|T|}\big(2\big(1-\Phi(x)\big) + g_x(x/2)\big)\,\mathrm{d}t$$
$$\le \frac{C_p}{1+x^p}\left(\frac{p}{\log p}\right)^p \mathbb{E}|T| \le \frac{C_p}{1+x^p}\left(\frac{p}{\log p}\right)^p \sqrt{\mathbb{E}T^2}. \tag{3.19}$$

To bound R_2, we estimate

$$\mathbf{1}\left(W+t > \frac{x}{2}\right) \le \frac{C_p}{1+x^p}\big(|W|^p + |T|^p\big) \quad \text{for all } 0 \le t \le |T|. \tag{3.20}$$

Combining (3.5) and (3.20), we have

$$R_2 \le \frac{C_p}{1+x^p}\mathbb{E}\int_0^{|T|}\big(1+|W|+|T|\big)\big(|W|^p+|T|^p\big)\,\mathrm{d}t$$
$$= \frac{C_p}{1+x^p}\mathbb{E}\big(1+|W|+|T|\big)\big(|W|^p|T|+|T|^{p+1}\big). \tag{3.21}$$

We bound each term in (3.21) as follows. Firstly, we have

$$\mathbb{E}|T|^{p+1} \le \sqrt{\mathbb{E}T^{2p+2}} \quad \text{and} \quad \mathbb{E}|W||T|^{p+1} \le \sqrt{\mathbb{E}W^2\mathbb{E}T^{2p+2}} = \sqrt{\mathbb{E}T^{2p+2}}. \tag{3.22}$$

Secondly, by using the Cauchy–Schwarz inequality, (3.12) and (3.13), and by noting that $\mathbb{E}|T|^{2p+2} < 1$, we have

$$\mathbb{E}|W|^p|T| \le \sqrt{\mathbb{E}|W|^{2p}\mathbb{E}T^2} \le C_p\left(\frac{p}{\log p}\right)^p\sqrt{\mathbb{E}T^2}, \tag{3.23}$$

$$\mathbb{E}|W|^{p+1}|T| \le \sqrt{\mathbb{E}|W|^{2p+2}\mathbb{E}T^2} \le C_p\left(\frac{p}{\log p}\right)^p\sqrt{\mathbb{E}T^2}, \tag{3.24}$$

and

$$\mathbb{E}|T|^{p+2} \le \sqrt{\mathbb{E}T^2\mathbb{E}T^{2p+2}} \le \sqrt{\mathbb{E}T^2}. \tag{3.25}$$

Finally,

$$\mathbb{E}|W|^p|T|^2 = \mathbb{E}\Big(\big(|W|^{p+1}|T|\big)^{p/(p+1)}\big(|T|^{p+2}\big)^{1/(p+1)}\Big)$$
$$\le \mathbb{E}\big(|W|^{p+1}|T|+|T|^{p+2}\big)$$
$$\le C_p\left(\frac{p}{\log p}\right)^p\sqrt{\mathbb{E}T^2}, \tag{3.26}$$

where we have used (2.6) in the first inequality, and (3.24) and (3.25) in the second inequality. From (3.21)–(3.26), we have

$$R_2 \le \frac{C_p}{1+x^p}\left(\frac{p}{\log p}\right)^p\Big(\sqrt{\mathbb{E}T^2}+\sqrt{\mathbb{E}|T|^{2p+2}}\Big). \tag{3.27}$$

Combining (3.14), (3.19) and (3.27), we obtain (3.2). □

Theorem 3.1 is a normal approximation for W^*. When $T = W^* - W$ has fast decaying tails, by using Theorem 3.1, we can obtain useful bounds in normal approximation for W. This gives us the following theorem.

Theorem 3.2. *Let W be such that $\mathbb{E}W = 0$ and $\mathrm{Var}(W) = 1$, and let W^* be W-zero biased and defined on the same probability space as W. Let $T = W^* - W$ and $\varepsilon > 0$ be arbitrary.*

(i) *We have*

$$\sup_{x\in\mathbb{R}}\big|\mathbb{P}(W \le x) - \Phi(x)\big| \le \left(1+\frac{\sqrt{2\pi}}{4}\right)\sqrt{\mathbb{E}T^2} + \frac{\varepsilon}{\sqrt{2\pi}} + \mathbb{P}\big(|T| > \varepsilon\big). \tag{3.28}$$

(ii) *Let $p \geq 2$. Then for all $x \in \mathbb{R}$,*

$$\begin{aligned}&\left|\mathbb{P}(W \leq x) - \Phi(x)\right| \\ &\quad \leq \frac{C_p}{1+|x|^p}\left(\frac{p}{\log p}\right)^p \left(\sqrt{\mathbb{E}T^2} + \sqrt{\mathbb{E}|T|^{2p+2}} + \varepsilon + \sqrt{\mathbb{P}\big(|T| > \varepsilon\big)}\right). \end{aligned} \tag{3.29}$$

Remark 3.3. If $|T| \leq \varepsilon$ almost surely, then (3.28) reduces to

$$\sup_{x\in\mathbb{R}}\left|\mathbb{P}(W \leq x) - \Phi(x)\right| \leq \left(1 + \frac{\sqrt{2\pi}}{4} + \frac{1}{\sqrt{2\pi}}\right)\varepsilon. \tag{3.30}$$

In Theorem 1.1 in Goldstein [16], the author considered the following distance between W and the standard normal random variable Z

$$\mathrm{d}(W, Z) = \sup_{h\in\mathcal{H}}\left|\mathbb{E}h(W) - \mathbb{E}h(Z)\right|,$$

where $\mathcal{H}$ is a class of measurable functions on the real line which contains the collection of indicators of all half lines. When $\mathcal{H}$ coincides with the collection of indicators of all half lines, the author proved that (see the first half of (10) in Goldstein [16])

$$\sup_{x\in\mathbb{R}}\left|\mathbb{P}(W \leq x) - \Phi(x)\right| \leq (127 + 12\varepsilon)\varepsilon. \tag{3.31}$$

Proof of Theorem 3.2. Let $\varepsilon > 0$ be arbitrary. Then by (3.1), we have

$$\begin{aligned}\mathbb{P}(W \leq x) - \Phi(x) &= \mathbb{P}\big(W^* \leq x + W^* - W\big) - \Phi(x) \\ &\leq \mathbb{P}\big(W^* \leq x + \varepsilon\big) - \Phi(x+\varepsilon) + \Phi(x+\varepsilon) - \Phi(x) \\ &\quad + \mathbb{P}\big(W^* - W > \varepsilon\big) \\ &\leq \left(1 + \frac{\sqrt{2\pi}}{4}\right)\sqrt{\mathbb{E}T^2} + \frac{\varepsilon}{\sqrt{2\pi}} + \mathbb{P}\big(\left|W^* - W\right| > \varepsilon\big), \end{aligned} \tag{3.32}$$

and

$$\begin{aligned}\mathbb{P}(W \leq x) - \Phi(x) &\geq \mathbb{P}\big(W^* \leq x - \varepsilon\big) - \Phi(x-\varepsilon) + \Phi(x-\varepsilon) - \Phi(x) \\ &\quad - \mathbb{P}\big(W^* - W < -\varepsilon\big) \\ &\geq -\left(1 + \frac{\sqrt{2\pi}}{4}\right)\sqrt{\mathbb{E}T^2} - \frac{\varepsilon}{\sqrt{2\pi}} - \mathbb{P}\big(\left|W^* - W\right| > \varepsilon\big). \end{aligned} \tag{3.33}$$

Combining (3.32) and (3.33), we obtain (3.28).

To prove (3.29), it suffices to consider $x \geq 2$, as in the proof of (3.2). Similar to the proof of (3.11), we have

$$\left|P(W \leq x) - \Phi(x)\right| \leq \frac{C_p}{1+x^p}\left(\frac{p}{\log p}\right)^p \left(1 + \sqrt{\mathbb{E}|T|^{2p+2}}\right). \tag{3.34}$$

Therefore, if either $\mathbb{E}|T|^{2p+2} \geq 1$ or $\varepsilon \geq 1$, then (3.29) holds. It remains to consider the case where $\mathbb{E}|T|^{2p+2} < 1$ and $\varepsilon < 1$. In this case, similar to (3.12), we have

$$\mathbb{E}\big|W^*\big|^{2p} = \frac{\mathbb{E}|W|^{2p+2}}{(2p+1)\mathbb{E}W^2} \leq \frac{2\kappa_{2p+2}}{2p+1} \leq C_p\left(\frac{p}{\log p}\right)^{2p}. \tag{3.35}$$

Since

$$\begin{aligned}\mathbb{P}\big(W^* > x+\varepsilon\big) &= \mathbb{P}\big(W^* > x+\varepsilon, T > \varepsilon\big) + \mathbb{P}\big(W^* > x+\varepsilon, T \leq \varepsilon\big)\\ &\leq \mathbb{P}\big(W^* > x, T > \varepsilon\big) + \mathbb{P}(W > x),\end{aligned}$$

we have

$$\begin{aligned}\mathbb{P}(W \leq x) - \Phi(x) &= 1 - \mathbb{P}(W > x) - \Phi(x)\\ &\leq 1 - \mathbb{P}\big(W^* > x+\varepsilon\big) - \Phi(x) + \mathbb{P}\big(W^* > x, T > \varepsilon\big). \end{aligned}\tag{3.36}$$

Combining (3.2), (3.10), (3.35) and (3.36), we have

$$\begin{aligned}\mathbb{P}(W \leq x) - \Phi(x) &\leq \mathbb{P}\big(W^* \leq x+\varepsilon\big) - \Phi(x+\varepsilon)\\ &\quad + \Phi(x+\varepsilon) - \Phi(x) + \mathbb{P}\big(W^* > x, T > \varepsilon\big)\\ &\leq \frac{C_p}{1+x^p}\left(\frac{p}{\log p}\right)^p\left(\sqrt{\mathbb{E}T^2} + \sqrt{\mathbb{E}|T|^{2p+2}}\right)\\ &\quad + \frac{\varepsilon e^{-x^2/2}}{\sqrt{2\pi}} + \sqrt{\mathbb{P}\big(|T| > \varepsilon\big)}\sqrt{\mathbb{P}\big(\big|W^*\big| > x\big)}\\ &\leq \frac{C_p}{1+x^p}\left(\frac{p}{\log p}\right)^p\left(\sqrt{\mathbb{E}T^2} + \sqrt{\mathbb{E}|T|^{2p+2}} + \varepsilon\right)\\ &\quad + \frac{\sqrt{\mathbb{P}(|T| > \varepsilon)}\sqrt{\mathbb{E}|W^*|^{2p}}}{x^p}\\ &\leq \frac{C_p}{1+x^p}\left(\frac{p}{\log p}\right)^p\left(\sqrt{\mathbb{E}T^2} + \sqrt{\mathbb{E}|T|^{2p+2}} + \varepsilon + \sqrt{\mathbb{P}\big(|T| > \varepsilon\big)}\right). \end{aligned}\tag{3.37}$$

Similarly, by noting that $x - \varepsilon > x - 1 \geq 1$, we can show that

$$\begin{aligned}&\mathbb{P}(W \leq x) - \Phi(x)\\ &\quad \geq -\frac{C_p}{1+x^p}\left(\frac{p}{\log p}\right)^p\left(\sqrt{\mathbb{E}T^2} + \sqrt{\mathbb{E}|T|^{2p+2}} + \varepsilon + \sqrt{\mathbb{P}\big(|T| > \varepsilon\big)}\right). \end{aligned}\tag{3.38}$$

Combining (3.37) and (3.38), we obtain (3.29). □

4. Proofs of the main results

The rate in the following proposition is better than that of Theorem 1.1 in the case where $\alpha \geq n^{1+\delta}$ for some $\delta > 0$ fixed. We would like to note here that when $1 \leq \alpha \leq n/\log^2 n$ or $\alpha \geq n^{1+\delta}$ for some $\delta > 0$ fixed, the convergence rate obtained in Proposition 4.1 is exactly the rate in Fulman's conjecture. Chen, Goldstein and Röllin [7] also obtained the bound $O(\sqrt{\alpha}/n)$ for the case $\alpha \geq n^{1+\delta}$ by applying induction with Stein's method.

Proposition 4.1. *Let $n \geq 3$ be an integer. Let $\alpha \geq 1$ and $W_{n,\alpha}$ be as in (1.5). Then*

$$\sup_{x\in\mathbb{R}}\left|\mathbb{P}_\alpha(W_{n,\alpha} \leq x) - \Phi(x)\right| \leq 8.2 \max\left\{\frac{1}{\sqrt{n}}, \frac{\sqrt{\alpha}\log n}{n}\right\}. \tag{4.1}$$

If, in addition, $\alpha \geq n^{1+\delta}$ for some $\delta := \delta(\alpha, n) > 0$, then

$$\sup_{x\in\mathbb{R}}\left|\mathbb{P}_\alpha(W_{n,\alpha} \leq x) - \Phi(x)\right| \leq \left(4.7 + \frac{3.1}{\delta}\right)\frac{\sqrt{\alpha}}{n}. \tag{4.2}$$

Remark 4.2. If $\alpha > n$, then we can write $\alpha = n^{1+\delta}$, where

$$\delta = \frac{\log\alpha - \log n}{\log n} > 0.$$

Applying (4.2), we have

$$\sup_{x\in\mathbb{R}}\left|\mathbb{P}_\alpha(W_{n,\alpha} \leq x) - \Phi(x)\right| \leq \frac{4.7\log\alpha - 1.6\log n}{\log\alpha - \log n}\frac{\sqrt{\alpha}}{n}. \tag{4.3}$$

We make some notes as follows.

(i) If $\alpha \sim Kn$ for some $K > 1$ fixed, then the rate obtained in (4.3) is $O(\frac{\sqrt{\alpha}\log n}{n})$ which is the same as the rate obtained in (4.1).

(ii) If $\alpha \sim n(\log n)^K$ for some $K > 0$ fixed, then the rate obtained in (4.3) is $O(\frac{\sqrt{\alpha}\log n}{n\log(\log n)})$ which is better than the rate obtained in (4.1).

(iii) If $\alpha \geq n^{1+\delta}$ for some $\delta > 0$ fixed, then the convergence rate obtained in (4.3) is $O(\frac{\sqrt{\alpha}}{n})$ which is exactly the rate in Fulman's conjecture.

We will prove Proposition 4.1 by applying Theorem 3.2. In Kerov [24], the author proved that there is a growth process giving a sequence of partitions $(\lambda(1), \dots, \lambda(n))$ with $\lambda(j)$ distributed according to the Jack$_\alpha$ measure on partitions of size j. We refer to Fulman [13] for details. Given Kerov's process, let $X_{1,\alpha} = 0$, $X_{j,\alpha} = c_\alpha(a)$ where a is the box added to $\lambda(j-1)$ to obtain $\lambda(j)$ and the "α-content" $c_\alpha(a)$ of a box a is defined to be α(column number of $a - 1$) $-$ (row number of $a - 1$), $j \geq 2$. Then one can write (see Fulman [14], Fulman and Goldstein [15])

$$W_{n,\alpha} = \frac{\sum_{j=1}^n X_{j,\alpha}}{\sqrt{\alpha\binom{n}{2}}}. \tag{4.4}$$

Therefore, constructing ν from the Jack$_\alpha$ measure on partitions of $n-1$ and then taking one step in Kerov's growth process yields λ with the Jack$_\alpha$ measure on partitions of n, we have

$$W_{n,\alpha} = V_{n,\alpha} + \eta_{n,\alpha}, \tag{4.5}$$

where

$$V_{n,\alpha} = \frac{\sum_{x\in\nu} c_\alpha(x)}{\sqrt{\alpha\binom{n}{2}}} = \sqrt{\frac{n-2}{n}}W_{n-1,\alpha}, \qquad \eta_{n,\alpha} = \frac{X_{n,\alpha}}{\sqrt{\alpha\binom{n}{2}}} = \frac{c_\alpha(\lambda/\nu)}{\sqrt{\alpha\binom{n}{2}}}, \tag{4.6}$$

and $c_\alpha(\lambda/\nu)$ denotes the α-content of the box added to ν to obtain λ. Fulman [14] proved that

$$EW_{n,\alpha} = 0, \qquad EW_{n,\alpha}^2 = 1, \tag{4.7}$$

$$E\eta_{n,\alpha} = 0, \qquad E\eta_{n,\alpha}^2 = \frac{2}{n}, \tag{4.8}$$

and

$$E\eta_{n,\alpha}^4 = \frac{2}{n^2}\left(\frac{4n-6}{n-1} + \frac{2(\alpha-1)^2}{\alpha(n-1)}\right). \tag{4.9}$$

From Theorems 3.1 and 4.1 in Fulman and Goldstein [15], there exists a random variable $\eta_{n,\alpha}^*$ defined on the same probability space with $\eta_{n,\alpha}$, and satisfying that $\eta_{n,\alpha}^*$ has $\eta_{n,\alpha}$-zero biased distribution and that

$$W_{n,\alpha}^* = V_{n,\alpha} + \eta_{n,\alpha}^* \tag{4.10}$$

has $W_{n,\alpha}$-zero biased distribution. Hereafter, we denote

$$T_{n,\alpha} = \eta_{n,\alpha} - \eta_{n,\alpha}^*.$$

The following lemma gives a bound for $\mathbb{E}(\eta_{n,\alpha}^*)^2$.

Lemma 4.3. *For $\alpha \geq 1$, we have*

$$\begin{aligned}\mathbb{E}(\eta_{n,\alpha}^*)^2 &= \frac{1}{3n}\left(\frac{4n-6}{n-1} + \frac{2(\alpha-1)^2}{\alpha(n-1)}\right)\\ &\leq \frac{1}{3n}\left(4 + \frac{2\alpha}{n-1}\right).\end{aligned} \tag{4.11}$$

Proof. Applying (2.1) with $f(x) = x^3$, we have

$$\mathbb{E}(\eta_{n,\alpha}^*)^2 = \frac{\mathbb{E}(\eta_{n,\alpha})^4}{3\mathbb{E}\eta_{n,\alpha}^2}. \tag{4.12}$$

Combining (4.8), (4.9) and (4.12), we obtain (4.11). □

For a partition λ of a positive integer n, we recall that the length of row i of λ and the length of column i of λ are denoted by λ_i and λ_i', respectively.

From a computation in the proof of Lemma 6.6 in Fulman [13] and Stirling's formula, we have the following lemma.

Lemma 4.4. *Let $\alpha > 0$. Then for $1 \leq l \leq n$, we have*

$$\mathbb{P}_\alpha(\lambda_1 = l) \leq \frac{\alpha}{2\pi}\left(\frac{ne^2}{\alpha l^2}\right)^l. \tag{4.13}$$

Proof. It is proved by Fulman [13] that

$$\mathbb{P}_\alpha(\lambda_1 = l) \leq \left(\frac{n}{\alpha}\right)^l \frac{\alpha l}{(l!)^2}. \tag{4.14}$$

By Stirling's formula, we have for all $l \geq 1$,

$$l! \geq \sqrt{2\pi l}\left(\frac{l}{e}\right)^{l}. \tag{4.15}$$

Combining (4.14) and (4.15), we have (4.13). □

In order to apply Theorem 3.2, we need to bound $\mathbb{P}(|T_{n,\alpha}| > \varepsilon)$ for suitably chosen ε. The following lemma shows that $|T_{n,\alpha}|$ has a very light tail.

Lemma 4.5. *For all $\alpha \geq 1$ and $q > 1$, we have*

$$\mathbb{P}_{\alpha}\left(|T_{n,\alpha}| > \frac{2e\sqrt{2q}}{\sqrt{n-1}}\right) \leq \frac{\alpha}{\pi(q-1)q^{e\sqrt{qn/\alpha}}} + \frac{\alpha^2 q\big(e\sqrt{qn/\alpha}(q-1)+q+1\big)}{\pi(n-1)(q-1)^3 q^{e\sqrt{qn/\alpha}}}.$$

Proof. First, we take an arbitrary $\alpha > 0$. It follows from (4.13) that

$$\mathbb{P}_{\alpha}(\lambda_1 = k+1) \leq \frac{\alpha}{2\pi q^{k+1}} \tag{4.16}$$

for all $k \geq e\sqrt{qn/\alpha}$. Therefore,

$$\begin{aligned}
\mathbb{P}_{\alpha}(\lambda_1 - 1 > e\sqrt{qn/\alpha}) &= \mathbb{P}_{\alpha}\big(\lambda_1 - 1 \geq \lfloor e\sqrt{qn/\alpha}\rfloor + 1\big) \\
&= \sum_{k \geq \lfloor e\sqrt{qn/\alpha}\rfloor + 1} \mathbb{P}_{\alpha}(\lambda_1 = k+1) \\
&\leq \frac{\alpha}{2\pi} \sum_{k \geq \lfloor e\sqrt{qn/\alpha}\rfloor + 1} \frac{1}{q^{k+1}} \\
&= \frac{q\alpha}{2\pi(q-1)q^{\lfloor e\sqrt{qn/\alpha}\rfloor + 2}} \\
&\leq \frac{\alpha}{2\pi(q-1)q^{e\sqrt{qn/\alpha}}}.
\end{aligned} \tag{4.17}$$

We note that from the definition of Jack measure, $\mathbb{P}_{\alpha}(\lambda) = \mathbb{P}_{1/\alpha}(\lambda^t)$, where λ^t is the transpose partition of λ. Applying (4.17) with α replaced by $1/\alpha$, we have

$$\mathbb{P}_{\alpha}\big(\lambda_1' - 1 > e\sqrt{q\alpha n}\big) \leq \frac{1}{2\pi\alpha(q-1)q^{e\sqrt{q\alpha n}}}. \tag{4.18}$$

Since $|X_{n,\alpha}| \leq \max\{\alpha(\lambda_1 - 1), \lambda_1' - 1\}$, it follows from (4.17) and (4.18) that

$$\begin{aligned}
\mathbb{P}_{\alpha}\left(|\eta_{n,\alpha}| > \frac{e\sqrt{2q}}{\sqrt{n-1}}\right) &= \mathbb{P}_{\alpha}\left(\frac{\sqrt{2}|X_{n,\alpha}|}{\sqrt{\alpha n(n-1)}} > \frac{e\sqrt{2q}}{\sqrt{n-1}}\right) \\
&\leq \mathbb{P}_{\alpha}\big(\max\{\alpha(\lambda_1 - 1), \lambda_1' - 1\} > e\sqrt{q\alpha n}\big) \\
&\leq \mathbb{P}_{\alpha}(\lambda_1 - 1 > e\sqrt{qn/\alpha}) + \mathbb{P}_{\alpha}\big(\lambda_1' - 1 > e\sqrt{q\alpha n}\big) \\
&\leq \frac{\alpha}{2\pi(q-1)q^{e\sqrt{qn/\alpha}}} + \frac{1}{2\pi\alpha(q-1)q^{e\sqrt{q\alpha n}}}.
\end{aligned} \tag{4.19}$$

For $\alpha \geq 1$, it reduces to

$$\mathbb{P}_\alpha\left(|\eta_{n,\alpha}| > \frac{e\sqrt{2q}}{\sqrt{n-1}}\right) \leq \frac{\alpha}{\pi(q-1)q^{e\sqrt{qn/\alpha}}}. \tag{4.20}$$

Recall that if X is a random variable with $\mathbb{E}X = 0$, $\mathbb{E}X^2 = \sigma^2$ and if X^* has X-zero-biased distribution, then for $x > 0$, applying (2.1) with $f_x(w) = (w-x)\mathbf{1}(w > x)$, we have

$$\mathbb{P}\big(X^* > x\big) = \mathbb{E}\big[X(X-x)\mathbf{1}(X > x)\big]/\sigma^2. \tag{4.21}$$

By using (4.21) and (4.16), and noting that

$$\eta_{n,\alpha} \leq \frac{\sqrt{2\alpha}(\lambda_1 - 1)}{\sqrt{n(n-1)}},$$

we have

$$\begin{aligned}
&\mathbb{P}_\alpha\left(\eta^*_{n,\alpha} > \frac{e\sqrt{2q}}{\sqrt{n-1}}\right) \\
&= \frac{n}{2}\mathbb{E}\left(\eta_{n,\alpha}\left(\eta_{n,\alpha} - \frac{e\sqrt{2q}}{\sqrt{n-1}}\right)\mathbf{1}\left(\eta_{n,\alpha} > \frac{e\sqrt{2q}}{\sqrt{n-1}}\right)\right) \\
&\leq \frac{\alpha}{n-1}\mathbb{E}\big((\lambda_1 - 1)(\lambda_1 - 1 - e\sqrt{qn/\alpha})\mathbf{1}(\lambda_1 - 1 > e\sqrt{qn/\alpha})\big) \\
&\leq \frac{\alpha}{n-1}\sum_{k=1}^{\infty} k\big(k + \lfloor e\sqrt{qn/\alpha}\rfloor\big)\mathbb{P}_\alpha\big(\lambda_1 = k + \lfloor e\sqrt{qn/\alpha}\rfloor + 1\big) \\
&\leq \frac{\alpha^2}{2\pi(n-1)}\sum_{k=1}^{\infty}\frac{k\big(k + \lfloor e\sqrt{qn/\alpha}\rfloor\big)}{q^{k+\lfloor e\sqrt{qn/\alpha}\rfloor+1}} \\
&= \frac{\alpha^2\big(\lfloor e\sqrt{qn/\alpha}\rfloor(q-1) + q + 1\big)}{2\pi(n-1)(q-1)^3 q^{\lfloor e\sqrt{qn/\alpha}\rfloor}} \\
&\leq \frac{\alpha^2 q\big(e\sqrt{qn/\alpha}(q-1) + q + 1\big)}{2\pi(n-1)(q-1)^3 q^{e\sqrt{qn/\alpha}}}.
\end{aligned} \tag{4.22}$$

Applying (4.13) again, we have

$$\mathbb{P}_\alpha\big(\lambda_1' = k+1\big) = \mathbb{P}_{1/\alpha}(\lambda_1 = k+1) \leq \frac{1}{2\pi\alpha q^{k+1}} \tag{4.23}$$

for all $k \geq e\sqrt{q\alpha n}$. By using (4.23) and noting that

$$\eta_{n,\alpha} \geq -\frac{\sqrt{2}(\lambda_1' - 1)}{\sqrt{\alpha n(n-1)}},$$

we have

$$\begin{aligned}
&\mathbb{P}_\alpha\left(-\eta^*_{n,\alpha} > \frac{e\sqrt{2q}}{\sqrt{n-1}}\right)\\
&= \frac{n}{2}\mathbb{E}\left[-\eta_{n,\alpha}\left(-\eta_{n,\alpha} - \frac{e\sqrt{2q}}{n-1}\right)\mathbf{1}\left(-\eta_{n,\alpha} > \frac{e\sqrt{2q}}{\sqrt{n-1}}\right)\right]\\
&\le \frac{1}{\alpha(n-1)}\mathbb{E}\left(\left(\lambda_1'-1\right)\left(\lambda_1'-1-e\sqrt{q\alpha n}\right)\mathbf{1}\left(\lambda_1'-1>e\sqrt{q\alpha n}\right)\right)\\
&\le \frac{1}{\alpha(n-1)}\sum_{k=1}^{\infty}k\left(k+\lfloor e\sqrt{q\alpha n}\rfloor\right)\mathbb{P}_\alpha\left(\lambda_1' = k+\lfloor e\sqrt{q\alpha n}\rfloor+1\right)\\
&\le \frac{1}{2\pi\alpha^2(n-1)}\sum_{k=1}^{\infty}\frac{k(k+\lfloor e\sqrt{q\alpha n}\rfloor)}{q^{k+\lfloor e\sqrt{q\alpha n}\rfloor+1}}\\
&= \frac{\lfloor e\sqrt{q\alpha n}\rfloor(q-1)+q+1}{2\pi\alpha^2(n-1)(q-1)^3 q^{\lfloor e\sqrt{q\alpha n}\rfloor}}\\
&\le \frac{q(e\sqrt{q\alpha n}(q-1)+q+1)}{2\pi\alpha^2(n-1)(q-1)^3 q^{e\sqrt{q\alpha n}}}.
\end{aligned} \tag{4.24}$$

For $\alpha \ge 1$, (4.22) and (4.24) reduce to

$$\mathbb{P}_\alpha\left(\left|\eta^*_{n,\alpha}\right| > \frac{e\sqrt{2q}}{\sqrt{n-1}}\right) \le \frac{\alpha^2 q(e\sqrt{qn/\alpha}(q-1)+q+1)}{\pi(n-1)(q-1)^3 q^{e\sqrt{qn/\alpha}}}. \tag{4.25}$$

The conclusion of the lemma follows from (4.20) and (4.25). □

Proof of Proposition 4.1. It suffices to consider $x \ge 0$ since we can simply apply the result to $-W_{n,\alpha}$ when $x < 0$. For a random variable W with $\mathbb{E}W = 0$ and $\mathrm{Var}(W) = 1$, Chen and Shao [5] proved that

$$\sup_{x\ge 0}\left|\mathbb{P}(W\le x) - \Phi(x)\right| \le \sup_{x\ge 0}\left|\frac{1}{1+x^2} - \left(1-\Phi(x)\right)\right| \le 0.55. \tag{4.26}$$

Firstly, we prove (4.1). From (4.26), it follows that it suffices to prove the proposition for $n \ge 200$. Let

$$K \ge e^{1/4}, \qquad q = K^2\max\left\{1, \frac{\alpha\log^2 n}{n}\right\}, \qquad \varepsilon = \frac{2e\sqrt{2q}}{\sqrt{n-1}}. \tag{4.27}$$

By applying Lemma 4.5 and noting that $qn/\alpha \ge K^2\log^2 n$, we have

$$\begin{aligned}
\mathbb{P}_\alpha\left(|T_{n,\alpha}| > \frac{2e\sqrt{2q}}{\sqrt{n-1}}\right) &\le \frac{qn}{\pi(q-1)q^{eK\log n}K^2\log^2 n}\\
&\quad + \frac{n^2q^3(e(q-1)K\log n + q+1)}{\pi(n-1)(q-1)^3 q^{eK\log n}K^4\log^4 n} := f(q).
\end{aligned} \tag{4.28}$$

Since $eK\log n > 1$, $f(q)$ is decreasing on $(1,\infty)$. Therefore,

$$f(q) \le f\big(K^2\big) = \frac{n}{\pi(K^2-1)K^{2eK\log n}\log^2 n} + \frac{n^2K^2(e(K^2-1)K\log n + K^2+1)}{\pi(n-1)(K^2-1)^3K^{2eK\log n}\log^4 n}. \tag{4.29}$$

By choosing $K = e^{1/4}$ (the general case will be used later) and noting that $n > 200$, we have

$$\begin{aligned}\mathbb{P}_\alpha\big(|T_{n,\alpha}| > \varepsilon\big) &\le f\big(e^{1/2}\big)\\ &= \frac{n}{\pi(\sqrt{e}-1)n^{0.5e^{5/4}}\log^2 n} + \frac{n^2\sqrt{e}\big(e^{5/4}(\sqrt{e}-1)\log n + \sqrt{e}+1\big)}{\pi(n-1)(\sqrt{e}-1)^3 n^{0.5e^{5/4}}\log^4 n}\\ &\le \frac{0.05}{\sqrt{n}},\end{aligned} \tag{4.30}$$

and

$$\begin{aligned}\frac{\varepsilon}{\sqrt{2\pi}} &= \frac{2e^{5/4}\sqrt{n}}{\sqrt{\pi(n-1)}}\max\left\{\frac{1}{\sqrt{n}}, \frac{\sqrt{\alpha}\log n}{n}\right\}\\ &\le 3.95\max\left\{\frac{1}{\sqrt{n}}, \frac{\sqrt{\alpha}\log n}{n}\right\}.\end{aligned} \tag{4.31}$$

By (4.8) and (4.11), we have

$$\begin{aligned}\sqrt{\mathbb{E}T_{n,\alpha}^2} &= \sqrt{\mathbb{E}\big(\eta_{n,\alpha}^* - \eta_{n,\alpha}\big)^2}\\ &\le \sqrt{\mathbb{E}\big(\eta_{n,\alpha}^*\big)^2} + \sqrt{\mathbb{E}(\eta_{n,\alpha})^2}\\ &\le \left(\frac{4}{3n} + \frac{2\alpha}{3n(n-1)}\right)^{1/2} + \left(\frac{2}{n}\right)^{1/2}\\ &\le \left(\left(\frac{4}{3} + \frac{2n}{3(n-1)\log^2 n}\right)^{1/2} + \sqrt{2}\right)\max\left\{\frac{1}{\sqrt{n}}, \frac{\sqrt{\alpha}\log n}{n}\right\}.\end{aligned} \tag{4.32}$$

Since $n > 200$, it follows from (4.32) that

$$\left(1 + \frac{\sqrt{2\pi}}{4}\right)\sqrt{\mathbb{E}T_{n,\alpha}^2} \le 4.2\max\left\{\frac{1}{\sqrt{n}}, \frac{\sqrt{\alpha}\log n}{n}\right\}. \tag{4.33}$$

Applying Theorem 3.2(i), (4.1) follows from (4.30), (4.31) and (4.33).

Now we prove (4.2). If either $\delta \ge 1$ or $0 < \delta < 1$ and $n \le 200$, then (4.2) holds by (4.26). Therefore, we may assume that $0 < \delta < 1$ and $n > 200$. Let

$$0 < L \le 1, \qquad q = \frac{\alpha}{(L\delta)^2 n}, \qquad \varepsilon' = \frac{2e\sqrt{2q}}{\sqrt{n-1}}. \tag{4.34}$$

Since $n > 200$ and $0 < \delta < 1$, elementary calculus shows that

$$q \geq \frac{n^\delta}{\delta^2} > 51.$$

By applying Lemma 4.5, we have

$$\mathbb{P}_\alpha\big(|T_{n,\alpha}| > \varepsilon'\big) \leq \frac{\alpha}{\pi(q-1)q^{e/(L\delta)}} + \frac{(L\delta)^2 q^2 n\alpha\big(e(q-1)/(L\delta)+q+1\big)}{\pi(n-1)(q-1)^3 q^{e/(L\delta)}}. \tag{4.35}$$

By choosing $L = 1$ (the general case will be used later) and noting $n > 200$, $q > 51$, we have from (4.35) that

$$\begin{aligned}
&\mathbb{P}_\alpha\big(|T_{n,\alpha}| > \varepsilon'\big) \\
&\quad\leq \frac{\sqrt{\alpha}}{\pi n q^{e/\delta-3/(2\delta)-1/2}}\left(\frac{1}{q-1} + \frac{ne}{n-1}\frac{q^2}{(q-1)^2} + \frac{n}{n-1}\frac{q^2(q+1)}{(q-1)^3}\right) \\
&\quad\leq \frac{0.08\sqrt{\alpha}}{n},
\end{aligned} \tag{4.36}$$

and

$$\frac{\varepsilon'}{\sqrt{2\pi}} = \frac{2\sqrt{n}e}{\delta\sqrt{\pi(n-1)}}\frac{\sqrt{\alpha}}{n} \leq \frac{3.1}{\delta}\frac{\sqrt{\alpha}}{n}. \tag{4.37}$$

Using the second inequality in (4.32) and noting again that $\alpha > n > 200$, we also have

$$\begin{aligned}
\sqrt{\mathbb{E}T_{n,\alpha}^2} &\leq \left(\frac{4}{3n} + \frac{2\alpha}{3n(n-1)}\right)^{1/2} + \left(\frac{2}{n}\right)^{1/2} \\
&\leq \left(\sqrt{\frac{4}{3} + \frac{2\times 201}{3\times 200}} + \sqrt{2}\right)\frac{\sqrt{\alpha}}{n}.
\end{aligned} \tag{4.38}$$

It follows from (4.38) that

$$\left(1 + \frac{\sqrt{2\pi}}{4}\right)\sqrt{\mathbb{E}T_{n,\alpha}^2} \leq 4.62\frac{\sqrt{\alpha}}{n}. \tag{4.39}$$

Applying Theorem 3.2(i) with ε' playing the role of ε, (4.2) follows from (4.36), (4.37) and (4.39). □

The following proposition establishes non-uniform bounds on the Kolmogorov distance for Jack measures.

Proposition 4.6. *Let $n \geq 3$ be an integer. Let $p \geq 2$, $1 \leq \alpha < n^2$ and $W_{n,\alpha}$ be as in (1.5). Then for all $x \in \mathbb{R}$, we have*

$$\big|\mathbb{P}_\alpha(W_{n,\alpha} \leq x) - \Phi(x)\big| \leq \frac{C_p}{1+|x|^p}\left(\frac{p^2}{\log p}\right)^p \max\left\{\frac{1}{\sqrt{n}}, \frac{\sqrt{\alpha}\log n}{n}\right\}. \tag{4.40}$$

If, in addition, there exists $\delta > 0$ such that $n^{1+\delta} \leq \alpha < n^2$, then for all $x \in \mathbb{R}$, we have

$$\big|\mathbb{P}_\alpha(W_{n,\alpha} \leq x) - \Phi(x)\big| \leq \left(1 + \frac{1}{\delta^{p+1}}\right)\frac{C_p}{1+|x|^p}\left(\frac{p^2}{\log p}\right)^p\frac{\sqrt{\alpha}}{n}. \tag{4.41}$$

Proof. We observe that if $\alpha > n^{5/4}$, then (4.41) implies (4.40). Therefore, once we have proved (4.41), we only need to prove (4.40) for the case where $1 \le \alpha < n^{5/4}$. For $n \ge 3$ and $1 \le \alpha < n^{5/4}$, we have

$$\max\left\{\frac{1}{\sqrt{n}}, \frac{\sqrt{\alpha}\log n}{n}\right\} < 1. \tag{4.42}$$

Let $K = p + 2$ and let q, ε be as in (4.27). Then for $n \ge 3$,

$$\varepsilon \le 10(p+2)\max\left\{\frac{1}{\sqrt{n}}, \frac{\sqrt{\alpha}\log n}{n}\right\}. \tag{4.43}$$

From (4.28) and (4.29), we have

$$\begin{aligned}\mathbb{P}_\alpha\big(|T_{n,\alpha}| > \varepsilon\big) &\le \frac{n}{\pi\big((p+2)^2-1\big)n^{2e(p+2)}\log^2 n} \\ &\quad + \frac{n^2(p+2)^2\big(e(p+2)((p+2)^2-1)\log n + (p+2)^2+1\big)}{\pi(n-1)\big((p+2)^2-1\big)^3 n^{2e(p+2)}\log^4 n} \\ &\le \frac{C_p}{n^{2e(p+2)-1}}.\end{aligned} \tag{4.44}$$

To apply Theorem 3.2(ii), we also need to bound $\mathbb{E}|T_{n,\alpha}|^{2p+2}$. Since $|X_{n,\alpha}| \le \alpha(n-1)$, we have $|\eta_{n,\alpha}| \le \sqrt{2\alpha}$ and therefore $|\eta^*_{n,\alpha}| \le \sqrt{2\alpha}$ (see (2.58) in Chen, Goldstein and Shao [4]). Combining (4.42)–(4.44), we have

$$\begin{aligned}\mathbb{E}\big(|T_{n,\alpha}|^{2p+2}\big) &\le \varepsilon^{2p+2} + (8\alpha)^{p+1}\mathbb{P}\big(|T_{n,\alpha}| > \varepsilon\big) \\ &\le \varepsilon^{2p+2} + \frac{C_p\alpha^{p+1}}{n^{2e(p+2)-1}} \\ &\le C_p p^{2p}\left(\max\left\{\frac{1}{\sqrt{n}}, \frac{\sqrt{\alpha}\log n}{n}\right\}\right)^{2p+2} \\ &\le C_p p^{2p}\left(\max\left\{\frac{1}{\sqrt{n}}, \frac{\sqrt{\alpha}\log n}{n}\right\}\right)^{2}.\end{aligned} \tag{4.45}$$

Applying Theorem 3.2(ii), (4.40) follows from (4.32) and (4.43)–(4.45).

To prove (4.41), we will need the following lemma.

Lemma 4.7. *Let $n \ge 3$. If there exists $0 < \delta \le 1$ such that $\alpha \ge n^{1+\delta}$, then for all $p \ge 0$, we have*

$$\mathbb{E}\big(|T_{n,\alpha}|^p\big) \le \frac{C_p p^p}{\delta^p}\left(\frac{\sqrt{\alpha}}{n}\right)^p. \tag{4.46}$$

Proof. Let $L = 1/(p+2)$ and let q, ε' be as in (4.34). Then

$$q \ge (p+2)^2\frac{n^\delta}{\delta^2} > 8 \quad \text{and} \quad \varepsilon' \le \frac{10(p+2)\sqrt{\alpha}}{\delta n}. \tag{4.47}$$

Noting that $L\delta \le 1$ and $q = \alpha/(L^2\delta^2 n) \ge n^\delta$, we have from (4.35) and the first half of (4.47) that

$$\begin{aligned}\mathbb{P}_\alpha\big(|T_{n,\alpha}| > \varepsilon'\big) &\le \frac{\alpha}{\pi(q-1)n^{e(p+2)}} + \frac{eq^2(q-1)+q^2(q+1)}{\pi(q-1)^3(n-1)} \frac{\alpha}{n^{e(p+2)-1}} \\ &\le \frac{\alpha}{n^{2p+4}}. \end{aligned} \tag{4.48}$$

Similar to (4.45), (4.48) combined with the second half of (4.47) yields

$$\begin{aligned}\mathbb{E}\big(|T_{n,\alpha}|^p\big) &\le \big(\varepsilon'\big)^p + \big(2\sqrt{2\alpha}\big)^p \mathbb{P}\big(|T_{n,\alpha}| > \varepsilon'\big) \\ &\le \big(\varepsilon'\big)^p + \frac{C_p \alpha^{p/2+1}}{n^{2p+4}} \\ &\le \frac{C_p p^p}{\delta^p}\left(\frac{\sqrt{\alpha}}{n}\right)^p. \end{aligned} \tag{4.49}$$

The proof of Lemma 4.7 is completed. □

Now, we will prove (4.41). Let L, q, ε' be as in the proof of Lemma 4.7. From the latter, we have

$$\mathbb{E}\big(|T_{n,\alpha}|^{2p+2}\big) \le \frac{C_p p^{2p}}{\delta^{2p+2}}\left(\frac{\sqrt{\alpha}}{n}\right)^{2p+2} \le \frac{C_p p^{2p}}{\delta^{2p+2}}\left(\frac{\sqrt{\alpha}}{n}\right)^2. \tag{4.50}$$

Applying Theorem 3.2(ii) with ε' playing the role of ε, (4.41) follows from the second inequality in (4.32), (4.48), (4.50) and the second half of (4.47). □

Proofs of Theorem 1.1 and Theorem 1.3. When $\alpha \ge 1$, Theorem 1.1 is a direct consequence of Proposition 4.1. We also see that (4.1) holds if we replace $W_{n,\alpha}$ by $-W_{n,\alpha}$. To obtain Theorem 1.1 for $0 < \alpha < 1$, we note that from the definition of Jack measure, $\mathbb{P}_\alpha(\lambda) = \mathbb{P}_{1/\alpha}(\lambda^t)$, where λ^t is the transpose partition of λ. It also follows from (4.4) and the definition of α-content that $W_{n,\alpha}(\lambda) = -W_{n,1/\alpha}(\lambda^t)$. Therefore,

$$\begin{aligned}\mathbb{P}_\alpha(W_{n,\alpha} = x) &= \mathbb{P}_\alpha\big\{\lambda : W_{n,\alpha}(\lambda) = x\big\} \\ &= \mathbb{P}_{1/\alpha}\big\{\lambda^t : W_{n,1/\alpha}\big(\lambda^t\big) = -x\big\} \\ &= \mathbb{P}_{1/\alpha}(W_{n,1/\alpha} = -x). \end{aligned}$$

From this, we conclude that $\mathbb{P}_\alpha(W_{n,\alpha} \le x) = \mathbb{P}_{1/\alpha}(W_{n,1/\alpha} \ge -x)$. Therefore,

$$\begin{aligned}\sup_{x\in\mathbb{R}}\big|\mathbb{P}_\alpha(W_{n,\alpha} \le x) - \Phi(x)\big| &= \sup_{x\in\mathbb{R}}\big|\mathbb{P}_{1/\alpha}(W_{n,1/\alpha} \ge -x) - \Phi(x)\big| \\ &= \sup_{x\in\mathbb{R}}\big|\mathbb{P}_{1/\alpha}(-W_{n,1/\alpha} \le x) - \Phi(x)\big| \\ &\le 8.2 \max\left\{\frac{1}{\sqrt{n}}, \frac{\log n}{\sqrt{\alpha n}}\right\}. \end{aligned}$$

Therefore, Theorem 1.1 also holds when $0 < \alpha < 1$. This completes the proof of Theorem 1.1.

When $1 \le \alpha < n^2$, Theorem 1.3 is a direct consequence of Proposition 4.6. When $1/n^2 < \alpha < 1$, one can use a similar argument as that in the proof of Theorem 1.1, and this completes the proof of Theorem 1.3. □

Appendix

In this section, we will prove (1.7) and two lemmas which are used in Section 2.

Proof of (1.7). For $p \geq 2$, applying Proposition 2.1 and Lemma 4.7 with noting that $\alpha \geq n^2$ (so that $\delta = 1$), we have

$$\begin{aligned}
\mathbb{E}|W_{n,\alpha}|^p &\leq \kappa_p \bigl(1 + \mathbb{E}|T_{n,\alpha}|^{p-2}\bigr) \\
&\leq \kappa_p \left(1 + C_p p^p \left(\frac{\sqrt{\alpha}}{n}\right)^{p-2}\right) \\
&\leq C_p \left(\frac{p^2}{\log p}\right)^p \left(\frac{\sqrt{\alpha}}{n}\right)^{p-2}
\end{aligned} \tag{A.1}$$

establishing (1.7). □

Lemma A.1. *Let $p > 8$ and let κ_p be as in Proposition 2.1, then*

$$\frac{\kappa_p}{\kappa_{p-2}} \geq 8\left(\frac{p-1}{\log(p-1)}\right)^2. \tag{A.2}$$

Proof. Letting

$$h(p) = \log(\kappa_p) = p\bigl(\log p - \log(\log p) + \log(7/4)\bigr) + \log\bigl((\log 8)^3/196\bigr),$$

we have

$$\begin{aligned}
h(p) - h(p-2) &= \int_{p-2}^{p} h'(t)\,\mathrm{d}t \\
&= 2 + 2\log(7/4) + \int_{p-2}^{p} \log t\,\mathrm{d}t - \int_{p-2}^{p} \log(\log t)\,\mathrm{d}t - \int_{p-2}^{p} \frac{\mathrm{d}t}{\log t}.
\end{aligned} \tag{A.3}$$

Since the function $t \mapsto \log(\log t)$ is concave,

$$\int_{p-2}^{p} \log(\log t)\,\mathrm{d}t \leq 2\log\bigl(\log(p-1)\bigr). \tag{A.4}$$

Next, since $p > 8$, we have

$$\int_{p-2}^{p} \frac{\mathrm{d}t}{\log t} \leq \int_{6}^{8} \frac{\mathrm{d}t}{\log t}, \tag{A.5}$$

and

$$\begin{aligned}
\int_{p-2}^{p} \log t\,\mathrm{d}t &= \int_{-1}^{1} \log(p-1+s)\,\mathrm{d}s \\
&= 2\log(p-1) + \int_{-1}^{1} \log\left(1 + \frac{s}{p-1}\right)\mathrm{d}s
\end{aligned}$$

$$= 2\log(p-1) + \int_0^1 \log\left(1 - \frac{s^2}{(p-1)^2}\right) \mathrm{d}s$$
$$\geq 2\log(p-1) + \int_0^1 \log\left(1 - \frac{s^2}{49}\right) \mathrm{d}s. \tag{A.6}$$

Combining (A.3)–(A.6), numerical calculation gives

$$h(p) - h(p-2) \geq 2 + 2\log(7/4) + 2\log(p-1) - 2\log\bigl(\log(p-1)\bigr)$$
$$- \int_6^8 \frac{\mathrm{d}t}{\log t} + \int_0^1 \log\left(1 - \frac{s^2}{49}\right) \mathrm{d}s$$
$$> 2\log(p-1) - 2\log\bigl(\log(p-1)\bigr) + \log 8$$

for all $p > 8$. This implies (A.2). □

Lemma A.2. *Let $p > 8$ and let κ_p be as in Proposition 2.1. Then*

$$\frac{2(p-1)}{(1-\theta)^{p-3}} \leq \kappa_p, \tag{A.7}$$

where

$$\theta = \theta(p) := \left(\frac{\log^2(p-1)}{4(p-1)}\right)^{1/(p-3)}.$$

Proof. Firstly, we will prove that

$$\frac{1}{1-\theta} \leq \frac{7p}{4\log p} \tag{A.8}$$

which is equivalent to

$$\frac{1}{p-3} \log\left(\frac{\log^2(p-1)}{4(p-1)}\right) \leq \log\left(1 - \frac{4\log p}{7p}\right). \tag{A.9}$$

Since $\frac{4\log p}{7p}$ is decreasing when $p > 8$, we have

$$\frac{4\log p}{7p} \leq 0.1486. \tag{A.10}$$

On the other hand, it is easy to prove that

$$\log(1-x) \geq \frac{-13x}{12} \quad \text{for all } 0 \leq x \leq 0.1486. \tag{A.11}$$

From (A.10) and (A.11), we have

$$\log\left(1 - \frac{4\log p}{7p}\right) \geq -\frac{13\log p}{21p}. \tag{A.12}$$

Therefore, to prove (A.9), it suffices to prove that

$$\frac{1}{p-3} \log\left(\frac{\log^2(p-1)}{4(p-1)}\right) \leq -\frac{13\log p}{21p} \tag{A.13}$$

which is equivalent to

$$R_1(p) + R_2(p) \geq 0, \tag{A.14}$$

where

$$R_1(p) = \frac{13\log(p-1)}{21(p-3)} - \frac{13\log p}{21p},$$

and

$$R_2(p) = \frac{(8/21)\log(p-1) - 2\log(\log(p-1)) + \log(4)}{p-3}.$$

Elementary calculus shows that $R_1(p) \geq 0$ and $R_2(p) \geq 0$ for all $p > 8$. Therefore (A.13) holds, completing the proof of (A.8).

Now, we will prove (A.7). Since $p > 8$, we have from (A.8) that

$$\frac{2(p-1)}{(1-\theta)^{p-3}} \leq 2(p-1)\left(\frac{4\log p}{7p}\right)^3 \left(\frac{7p}{4\log p}\right)^p \leq \frac{(\log 8)^3}{196}\left(\frac{7p}{4\log p}\right)^p = \kappa_p.$$

The proof of the lemma is completed. □

Acknowledgements

The first author and the third author were partially supported by Grant R-146-000-182-112 and Grant R-146-000-230-114 from the National University of Singapore. A substantial part of this paper was written when the third author was at the Institute for Mathematical Sciences (IMS) and Department of Mathematics, National University of Singapore. He would like to thank the IMS staff and the Department of Mathematics for their hospitality.

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Received February 2019 and revised June 2020