

Bump detection in the presence of dependency: Does it ease or does it load?

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We provide the asymptotic minimax detection boundary for a bump, i.e. an abrupt change, in the mean function of a stationary Gaussian process. This will be characterized in terms of the asymptotic behavior of the bump length and height as well as the dependency structure of the process. A major finding is that the asymptotic minimax detection boundary is generically determined by the value of its spectral density at zero. Finally, our asymptotic analysis is complemented by non-asymptotic results for AR(p) processes and confirmed to serve as a good proxy for finite sample scenarios in a simulation study. Our proofs are based on laws of large numbers for non-independent and non-identically distributed arrays of random variables and the asymptotically sharp analysis of the precision matrix of the process.

Keywords: ARMA processes; change point detection; minimax testing; time series; Toeplitz matrices; weak laws of large numbers

1. Introduction

1.1. Model and problem statement

In this paper we consider observations of a triangular array of Gaussian vectors, $Y = \mu_n + \xi_n$, $n \in \mathbb{N}$ with the coordinates

$$Y_{i,n} = \mu_{i,n} + \xi_{i,n}, \quad \xi_n = (\xi_{1,n}, \dots, \xi_{n,n})^T \sim \mathcal{N}_n(0, \Sigma_n), \quad (1.1)$$

with a known positive definite covariance matrix $\Sigma_n \in \mathbb{R}^{n \times n}$, but an unknown mean vector $\mu_n = (\mu_{1,n}, \dots, \mu_{n,n})^T \in \mathbb{R}^n$. We will furthermore assume that the noise ξ_n in (1.1) consists of n consecutive samples of a stationary process $(Z_t)_{t \in \mathbb{Z}}$.

For a proper asymptotic treatment, we will assume that μ_n is obtained from equidistantly sampling a function $m_n : [0, 1] \rightarrow \mathbb{R}$ at sampling points $\frac{i}{n}, i = 1, \dots, n$, i.e. $\mu_n = (m_n(\frac{1}{n}), \dots, m_n(\frac{n}{n}))^T$. Our goal is to analyze how difficult it is to detect abrupt changes of the function m_n based on the observations $Y = (Y_{1,n}, \dots, Y_{n,n})^T$ coming from (1.1). Therefore, we focus on functions m_n of the form

$$m_n(x) = \begin{cases} \delta_n & \text{if } x \in I_n, \\ 0 & \text{else,} \end{cases} \tag{1.2}$$

i.e. m_n has a bump located at the interval $I_n \subset [0, 1]$ of height $\delta_n \in \mathbb{R}$, see also Figure 1 for an illustration. We assume throughout the paper that the matrix Σ_n in (1.1) as well as the length of the bump $\lambda_n \in (0, 1)$ are known, but that its amplitude δ_n and the exact position of the bump itself are unknown.

To formalize the detection problem, let us introduce some notation. For an interval $I \subset [0, 1]$ we use $\mathbf{1}_I \in \mathbb{R}^n$ as abbreviation for the vector with entries

$$\mathbf{1}_I(i) = \begin{cases} 1 & \text{if } \frac{i}{n} \in I, \\ 0 & \text{else,} \end{cases} \quad 1 \leq i \leq n.$$

Consequently, $\mu_n = \delta_n \mathbf{1}_{I_n}$ whenever m_n is of the form (1.2). Furthermore let

$$\mathcal{I} := \{[a, b) \mid 0 \leq a < b \leq 1\}$$

be the set of all right-open intervals in $[0, 1]$, and for a given length $\lambda \in (0, 1)$ we introduce by

$$\mathcal{I}(\lambda) := \{[a, b) \mid 0 \leq a < b \leq 1, b - a = \lambda\}$$

the set of all right-open intervals in $[0, 1]$ of length λ .

Now the problem to detect a bump of length λ_n in the signal μ_n from (1.1) can be understood as the hypothesis testing problem

$$\begin{aligned} H_0^n : Y &\sim \mathcal{N}_n(0, \Sigma_n) \\ \text{against} & \\ H_1^n : \exists I \in \mathcal{I}(\lambda_n), \exists \delta \in \mathbb{R} : |\delta| \geq \Delta_n \quad \text{such that} \quad Y &\sim \mathcal{N}_n(\delta \mathbf{1}_I, \Sigma_n) \end{aligned} \tag{1.3}$$

with a minimal amplitude value $\Delta_n > 0$ to ensure distinguishability of H_0^n and H_1^n . Note that I and δ in (1.3) are allowed to depend on n (as the length λ_n and the minimal amplitude value Δ_n do), but we suppress this dependency in the following. Similarly we write H_0 instead of H_0^n as Σ_n is assumed to be known. Note that we will consider the situation $\lambda_n \rightarrow 0$ as $n \rightarrow \infty$ below, corresponding to a vanishing bump, which avoids trivial cases such as $\mathbb{E}_{H_1^n}[Y_i] = \delta > 0$ for all $1 \leq i \leq n$ in (1.3).

The aim of this paper is to provide insight on how the dependency structure in (1.1) encoded in terms of Σ_n influences the detection of such a bump. More precisely, we would like to derive

asymptotic conditions¹ on the minimal detectable bump amplitude Δ_n depending on Σ_n , λ_n and n . To the best of our knowledge, there is no systematic understanding of this problem from the minimax point of view. We will therefore provide (asymptotic) lower and upper bounds for the amplitude of asymptotically detectable signals in the following sense (cf. [23,24]). Let $\alpha, \beta \in (0, 1)$ be arbitrary error levels.

upper detection bound: Whenever the bump amplitude Δ_n satisfies $\Delta_n = c\varphi_n$, $c \geq c^*$ with a constant $c^* > 0$ and a rate φ_n depending on n , λ_n and Σ_n , then *there is a sequence of tests* for (1.3) with (asymptotic) type I error $\leq \alpha$ and (asymptotic) type II error $\leq \beta$.

lower detection bound: Whenever the bump amplitude Δ_n satisfies $\Delta_n = c\tilde{\varphi}_n$, $c \leq c_*$ with a constant $c_* > 0$ and a rate $\tilde{\varphi}_n$ depending on n , λ_n and Σ_n , then *no sequence of tests* for (1.3) can have type (asymptotic) I error $\leq \alpha$ and at the same time (asymptotic) type II error $\leq \beta$.

Precise definitions of the (asymptotic) type I and type II errors and comments on the validity of these particular notions of the detection bounds will be given in Section 2.1. Note that the minimax separation rate φ_n might depend on the prescribed significance levels α and β , and that the definitions become trivial if $\beta \geq 1 - \alpha$, as then any standard Bernoulli experiment with success probability α defines a corresponding test. However, in our case neither the constants c_* and c^* nor the rate depend on the error levels α and β . That is why in the following we will always choose $\alpha = \beta \in (0, \frac{1}{2})$ and argue in Section 2.1 that this is sufficient.

If $\tilde{\varphi}_n = \varphi_n$ in the above upper and lower bound, then we speak of the (asymptotic) *minimax separation rate* $\Delta_n \sim \varphi_n$. If furthermore $c^* = c_*$, then $\Delta_n \asymp c_*\varphi_n = c^*\tilde{\varphi}_n$ is called the (asymptotic) *minimax detection boundary* over all possible amplitudes $\Delta_n > 0$ and positions $I \in \mathcal{I}(\lambda_n)$. We will provide explicit expressions for this under weak assumptions on the covariance matrix Σ_n .

We will provide lower and upper bounds in terms of sums over diagonal blocks within Σ_n (cf. Section 2.3 and Lemmas 5.1 and 5.2), and for the case of noise generated by subsequent samples of a stationary time series we will show that these lower and upper bounds coincide.

In case of i.i.d. observations, this is $\Sigma_n = \sigma^2 \text{id}_n$ in (1.1), the minimax detection boundary is well-known and given by (see [8,11,15])

$$\Delta_n \asymp \sigma \sqrt{\frac{-2 \log \lambda_n}{n\lambda_n}}. \tag{1.4}$$

Here, and in the following, we require

$$\lambda_n \rightarrow 0 \quad \text{and} \quad n\lambda_n \rightarrow \infty \quad \text{as} \quad n \rightarrow \infty. \tag{1.5}$$

Signals for which the left-hand side in (1.4) is asymptotically larger than the right-hand side can be detected consistently (in the sense of an upper detection bound as described above), whereas they can not be detected consistently once the left-hand side in (1.4) is asymptotically smaller than the right-hand side (in the sense of a lower detection bound as described above). Although

¹Let $(a_n)_{n \in \mathbb{N}}$ and $(b_n)_{n \in \mathbb{N}}$ two sequences of positive numbers. In the following we write $a_n \sim b_n$ if $0 < \liminf_{n \rightarrow \infty} a_n/b_n \leq \limsup_{n \rightarrow \infty} a_n/b_n < \infty$, and $a_n \asymp b_n$ if $\lim_{n \rightarrow \infty} a_n/b_n = 1$.

(1.4) is known for a long time when the errors are i.i.d., to the best of our knowledge, the influence of the error dependency structure on the detection boundary (1.4) is an issue that is much less investigated systematically, although many methods to estimate such abrupt changes in the signal corrupted by serially dependent errors have been suggested (see Section 1.3). In this sense, this paper contributes a benchmark to such methods. Let us illustrate the effect of the dependency on (1.4) with ξ_n in (1.1) arising from an AR(1) process with unit variance and auto-correlation coefficient ρ , this is $\xi_n = (1 - \rho^2)^{1/2}(Z_1, \dots, Z_n)^T$ where $Z_t - \rho Z_{t-1} = \zeta_t$ with i.i.d. standard Gaussian noise $\zeta_t, t \in \mathbb{Z}$. In Figure 1 we illustrate three different situations encoded in terms of ρ , namely positively correlated noise ($\rho = 0.7$), independent noise ($\rho = 0$), and negatively correlated noise ($\rho = -0.7$). It seems intuitively clear that the value of ρ influences the difficulty of detecting a bump substantially, and especially positively correlated noise hinders efficient detection dramatically. Compare e.g. the first plot in Figure 1, where noise and bump appear hardly to distinguish. Furthermore, due to the positive correlation, there appear several regions which suggest a bump in signal, which is not there. In contrast, the middle and bottom plot allow for simpler identification of the bump region. Our main result makes these intuitive findings precise.

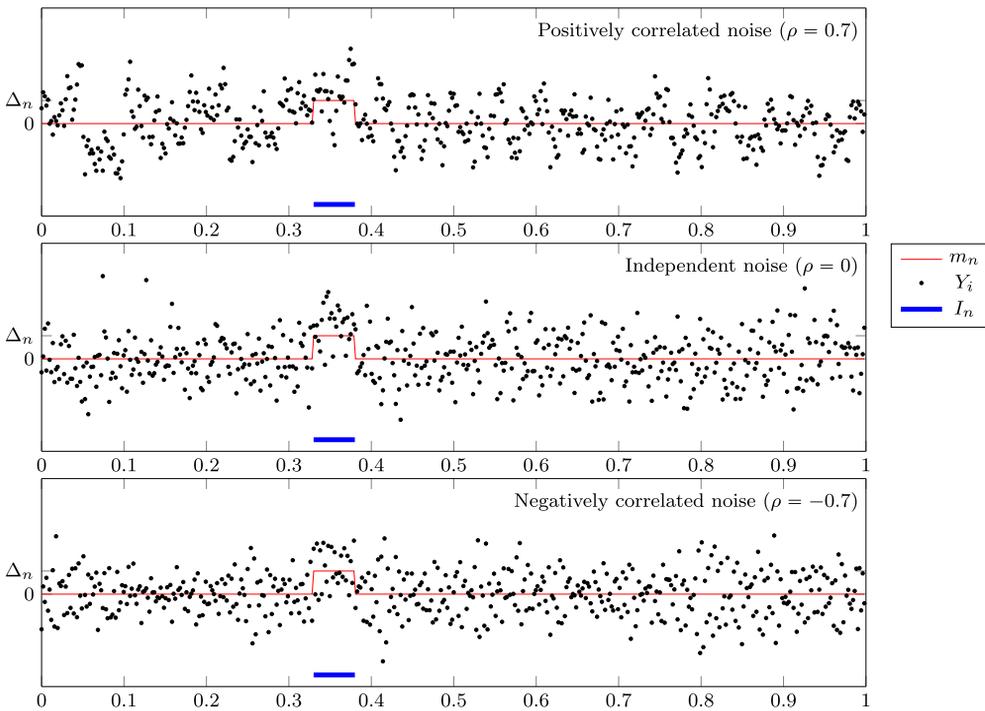


Figure 1. Model (1.1) in case of AR(1) noise for different values of ρ : Data together with the function m_n , where the model parameters are set to be $n = 512$ and $\Delta_n = 1, \sigma = 1$.

1.2. Results

To describe our results concerning the detection boundary for serially dependent data we require some more terminology. Let the autocovariance function γ_Z of the stationary process $(Z_t)_{t \in \mathbb{Z}}$ be given by $\gamma_Z(h) = \text{Cov}[Z_t, Z_{t+h}]$ for $h \in \mathbb{Z}$. Assume that γ_Z is square summable, then the process Z has the spectral density $f_Z \in \mathbf{L}_2[-1/2, 1/2)$ defined by

$$f_Z(v) = \sum_{h=-\infty}^{\infty} \gamma_Z(h)e^{-2\pi ihv}, \quad v \in [-1/2, 1/2).$$

In fact, f_Z can also be considered as a function on the unit sphere, i.e. one naturally has $\lim_{v \rightarrow 1/2} f_Z(v) = f_Z(-1/2)$. We will also assume that the autocovariance function is symmetric, which is equivalent to f_Z being real-valued. In the following, we will omit the subscript Z in the notation of the spectral density of Z when it does not create ambiguities.

With this notation introduced, we will show under mild conditions that the detection boundary for the hypothesis testing problem (1.3) is given by

$$\Delta_n \asymp \sqrt{\frac{-2f(0) \log \lambda_n}{n\lambda_n}}.$$

It is immediately clear, that in case of independent observations where $\Sigma_n = \sigma^2 \text{id}_n$, one has $f(0) = \sigma^2$, which reproduces (1.4). In the general case, note that

$$f(0) = \sum_{h \in \mathbb{Z}} \gamma(h),$$

i.e. the detection boundary solely depends on the value of the spectral density at zero which is known as *long-run variance*.²

In case of the AR(1)-based noise $\xi_n := (1 - \rho^2)^{1/2}(Z_1, \dots, Z_n)^T$ with unit variance as shown in Figure 1, the auto-covariance of the underlying AR(1) process Z_t is given by $\gamma_Z(h) = \gamma_Z(0)\rho^{|h|}$, where $\gamma_Z(0) = (1 - \rho^2)^{-1}$. Thus the spectral density at zero of the noise process $\xi = ((1 - \rho^2)^{1/2}Z_i)_{i \in \mathbb{N}}$ is

$$f_\xi(0) = (1 - \rho^2) \sum_{h=-\infty}^{\infty} \gamma_Z(h) = \frac{1 + \rho}{1 - \rho},$$

and hence the detection boundary is given by

$$\Delta_n \asymp \sqrt{\frac{1 + \rho}{1 - \rho}} \sqrt{\frac{-2 \log \lambda_n}{n\lambda_n}}. \tag{1.6}$$

²The long-run variance of a process $(Z_t)_{t \in \mathbb{Z}}$ with spectral density f is defined as $\lim_{n \rightarrow \infty} n^{-1} \text{Var}[S_n]$, where $S_n = \sum_{i=1}^n Z_i$. It holds that $\lim_{n \rightarrow \infty} n^{-1} \text{Var}[S_n] = f(0)$ (see [22] and Section 6.1.3 for details).

As an immediate consequence, this shows that bump detection is easier under a negative correlation ρ than in case of positive correlations. For the three values employed in Figure 1 we compute for the factor $\sqrt{\frac{1+\rho}{1-\rho}}$ in (1.6) the values 2.38 when $\rho = 0.7$ and 0.42 when $\rho = -0.7$. This means that the amplitude of detectable signals for $\rho = 0.7$ and $\rho = -0.7$ differs approximately by a factor of 5.6. Also, given the bump length λ_n , the detection of a bump of the same size Δ_n for $\rho = 0.7$ requires approximately a 6 times larger sample size than for $\rho = 0$, and even a 31 times larger sample size than for $\rho = -0.7$. This is in good agreement with the intuitive findings from Figure 1 and confirmed in finite sample situations in Section 4. In the simulations we also investigate the influence of several bumps instead of one, and find that independent of ρ , multiple bumps always help detection, as to be expected.

Remarkably, as in the case of i.i.d. noise with variance σ^2 , where we have $f(0) = \sigma^2$, certain dependent error processes might also satisfy $f(0) = \sigma^2$, and hence obey the same difficulty to detect a bump as for the independent case. As an example, consider the stationary and causal AR(2) process given by $Z_t = \frac{1}{2}Z_{t-1} - \frac{1}{2}Z_{t-2} + \zeta_t$, where $\zeta_t \sim \mathcal{N}(0, 1)$ for $t \in \mathbb{Z}$. In this case $f_Z(0) = \frac{1}{2} - \frac{1}{2} + 1 = 1$, even though the process Z_t is clearly not independent (see Section 3 for a comprehensive treatment of ARMA processes).

Proof strategy. To prove a lower detection bound, we will employ techniques dating back to Ingster [23] and Dümbgen and Spokoiny [11] developed for independent observations. To generalize this approach to our dependent case, we will use a recent weak law of large numbers due to Wang and Hu [40] for triangular arrays of random variables that are non-independent within each row and non-identically distributed between rows (see Section 6.1.1 for the precise statement and also [18,30,33,37] for related results).

For the upper detection bound, we will provide an explicit test based on the supremum of the moving average process $(\mathbf{1}_I^T Y)_{I \in \mathcal{I}(\lambda_n)}$. A valid critical value will be given based on a chaining technique. Note that this cannot be obtained by a continuous upper bound of the stochastic process (as e.g. provided in Theorem 6.1 in [11]) due to the fact that the dependency structure is allowed to change with n and hence there is no continuous analog of $(\mathbf{1}_I^T Y)_{I \in \mathcal{I}(\lambda_n)}$.

1.3. Related work

Bump detection for dependent data appears to be relevant to a variety of applications where piece-wise constant signals (i.e. several bumps) are observed under dependent noise. Exemplary, we mention molecular dynamics (MD) simulations, where collective motion characteristics of protein atoms are studied over time (see e.g. [28] and the references therein). For certain proteins it has been shown that the noise process can be well modeled by a stationary ARMA(p, q) process with small p and q , see [36]. Another application is the analysis of ion channel recordings, where one aims to identify opening and closing states of physiologically relevant channels (see [29] and the references therein). Here, the dependency structure is induced by a known band-pass filter, ensuring that Σ_n in (1.1) can be precomputed explicitly (which corresponds to our setting of known Σ_n), and allowing for a good approximation by stationary and m -dependent noise with small m , see [32].

In fact, bump detection as discussed here is closely related to estimation of a signal which consists of piece-wise constant segments, often denoted as change point *estimation*. We refer to the classical works of Ibragimov and Has'minskii [21], Csörgő and Horváth [9], Brodsky and Darkhovsky [5], Carlstein, Müller and Siegmund [6], and Siegmund [35] for a survey of the existing results as well as to the review article by Aue and Horváth [1]. Indeed, if the bumps have been properly identified by a detection method, posterior estimation of the signal is relatively easy, see [15] for such a combined approach in case of i.i.d. errors, and [10] in case of dependent data. We also mention [7], who presented a robust approach for AR(1) errors.

Model (1.1) can be seen as prototypical for the more complex situation when several bumps are to be detected. We do not intend to provide novel methodology for this situation in this paper, rather Theorem 2.1 provides a benchmark for *detecting* such a bump which then can be used to benchmark the detection power of any method designed for this task. Minimax detection has a long history, see e.g. the seminal series of papers by Ingster [23] or the monograph by Tsybakov [39]. More recently, Goldenshluger, Juditsky and Nemirovski [17] provided a general approach based on convex optimization. In case of independent observations, the problem of detecting a bump has been considered in [2,3,8,12,15,25], and our strategy of proof for the lower bound is adopted from [11]. We also mention [14] for a model with a simultaneous bump in the variance, and [31] for heterogeneous noise, however still restricted to independent observations.

The literature on minimax detection for dependent noise is much less developed, and most similar in spirit to our work are the papers by Hall and Jin [19] and Keshavarz, Scott and Nguyen [27]. In the former, the minimax detection boundary for an unstructured version of the model (1.1) in a Bayesian setting is derived, that is $\mathbb{P}[m_n(\frac{i}{n}) = \Delta_n] = \rho_n$ and $\mathbb{P}[m_n(\frac{i}{n}) = 0] = 1 - \rho_n$ with a probability ρ_n tending to 0. In contrast to [19], in the present setting we can borrow strength from neighboring observations in a bump. Still, we can exploit a result in [19] about the decay behavior of inverses of covariance matrices (see Section 6.1.2) to validate Assumption 2. Keshavarz, Scott and Nguyen [27] deal with the classical change-point in mean problem, i.e. with the problem to detect whether $m_n(i/n) \equiv 0$ for all $1 \leq i \leq n$, or if there exists $\tau \in [1, n]$ such that $m_n(i/n) = -\frac{1}{2}\Delta_n \mathbf{1}\{i \leq \tau\} + \frac{1}{2}\Delta_n \mathbf{1}\{i > \tau\}$ for $1 \leq i \leq n$. The authors derive upper and lower bounds for detection from dependent data as in (1.1), similar in spirit to our Theorem 2.1. Their bounds, however, do not coincide with ours, i.e. they do not derive the precise minimax detection boundary, as they are mostly interested in the rate of estimation. However, as we see from Theorem 2.1, the $\sqrt{-\log \lambda_n}$ rate does not change, it is the constant $f(0)$ which matters. We will employ several of their computations concerning covariance structures of time series (while correcting a couple of technical inaccuracies).

We finally comment on the assumption of knowing Σ_n and the length λ_n . If λ_n is unknown, estimation of the function m_n can be performed in the independent noise case by SMUCE [15] via a multiscale approach. SMUCE is known to achieve the asymptotic detection boundary (1.4) in case of i.i.d. Gaussian errors. For the dependent case with a (partially) unknown covariance matrix Σ_n , further methods for estimation of m_n such as H-SMUCE [31], J-SMURF [20] or JULES [32] have been developed. They all rely on a local estimation of the covariance structure in combination with a multiscale approach. None of these methods achieves the detection boundary derived in this paper, and hence it remains unclear if not knowing Σ_n and / or λ_n would affect it. Developing a test which achieves a corresponding upper bound by multiscale methods is beyond the scope of this paper and is postponed to future work.

1.4. Organization of the paper

The remaining part of this paper is organized as follows: In Section 2 we give a precise statement of our assumptions and formulate our main theorem. Also non-asymptotic results are discussed here. The implications for ARMA models are then given in Section 3, where the previously mentioned non-asymptotic results are specified for $AR(p)$ noise. In Section 4 we present some simulations which support that our asymptotic theory is already useful for small samples. All proofs are deferred to Section 5 and Section 6 in the Supplementary Material [13].

2. Main results

2.1. Notation and assumptions

To treat the testing problem (1.3), we will consider tests $\Phi_n : \mathbb{R}^n \rightarrow \{0, 1\}$, $n \in \mathbb{N}$, where $\Phi_n(Y) = 0$ means that the null hypothesis H_0 is accepted, and $\Phi_n(Y) = 1$ means that the null hypothesis is rejected, i.e. the presence of a bump is concluded.

Denote by \mathbb{P}_0 the measure $\mathcal{N}_n(0, \Sigma_n)$ of Y under the null hypothesis and by $\mathbb{P}_{I,\delta}$ the measure $\mathcal{N}_n(\delta \mathbf{1}_I, \Sigma_n)$ of Y given that there is a bump of height δ within the interval I . With this we will denote the corresponding expectations accordingly by \mathbb{E}_0 and $\mathbb{E}_{I,\delta}$. We define the type I error of Φ_n by

$$\bar{\alpha}(\Phi_n, \Sigma_n) := \mathbb{E}_0[\Phi_n(Y)] = \mathbb{P}_0[\Phi_n(Y) = 1].$$

Furthermore, we say that a sequence $(\Phi_n)_{n \in \mathbb{N}}$ of such tests has asymptotic level $\alpha \in [0, 1]$ if $\limsup_{n \rightarrow \infty} \bar{\alpha}(\Phi_n, \Sigma_n) \leq \alpha$. The type II error depending on the parameters Σ_n, Δ_n and λ_n is defined as

$$\bar{\beta}(\Phi_n, \Sigma_n, \Delta_n, \lambda_n) := \sup_{I \in \mathcal{I}(\lambda_n)} \sup_{|\delta| \geq \Delta_n} \mathbb{P}_{I,\delta}[\Phi_n(Y) = 0].$$

For a sequence $(\Phi_n)_{n \in \mathbb{N}}$ of such tests we define its asymptotic type II error to be $\limsup_{n \rightarrow \infty} \bar{\beta}(\Phi_n, \Sigma_n, \Delta_n, \lambda_n)$. The asymptotic power of such a family is then given by $1 - \limsup_{n \rightarrow \infty} \bar{\beta}(\Phi_n, \Sigma_n, \Delta_n, \lambda_n)$. For the sake of brevity, we might suppress the dependency on the parameters in the following and write only $\bar{\alpha}(\Phi_n)$ and $\bar{\beta}(\Phi_n)$, respectively.

With this notation, we can now precisely recall the requirements for lower and upper bounds on detectability as discussed in the introduction:

upper detection bound: For any $\alpha \in (0, \frac{1}{2})$, there exist $c^* > 0$ and a sequence of tests $\Phi_{n,\alpha}^*$, $n \in \mathbb{N}$ of asymptotic level α such that $\forall c > c^*$,

$$\limsup_{n \rightarrow \infty} \bar{\beta}(\Phi_n, \Sigma_n, c\varphi_n, \lambda_n) \leq \alpha.$$

Note that this notion of the upper detection bound is in accordance with the usual minimax testing paradigm (cf. Ingster and Suslina [24]), as it implies that

$$\lim_{n \rightarrow \infty} \inf_{\Phi \in \Psi_n} [\bar{\alpha}(\Phi, \Sigma_n) + \bar{\beta}(\Phi, \Sigma_n, c\varphi_n, \lambda_n)] = 0,$$

as $n \rightarrow \infty$, since α was arbitrary. Here, Ψ_n is the collection of all tests for the testing problem (1.3) given n observations.

lower detection bound: For any $\alpha \in (0, \frac{1}{2})$, there exists $c_* > 0$ such that $\forall c < c_*$, and for any sequence of tests $\Phi_n, n \in \mathbb{N}$ of asymptotic level α ,

$$\liminf_{n \rightarrow \infty} \bar{\beta}(\Phi_n, \Sigma_n, c\tilde{\varphi}_n, \lambda_n) \geq 1 - \alpha.$$

This implies that

$$\lim_{n \rightarrow \infty} \inf_{\Phi \in \Psi_n} [\bar{\alpha}(\Phi, \Sigma_n) + \bar{\beta}(\Phi, \Sigma_n, c\varphi_n, \lambda_n)] = 1.$$

The choice of $1 - \alpha$ as the lower bound of the limit of the type II errors in the lower detection bound is justified by the fact that the minimax testing risk is bounded from below as follows (see [24], p. 55, Theorem 2.1):

$$\inf_{\Phi \in \Psi} [\bar{\alpha}(\Phi, \Sigma_n) + \bar{\beta}(\Phi, \Sigma_n, c\varphi_n, \lambda_n)] \geq 1 - \frac{1}{2} \|[P_0], [P_1]\|_1,$$

where $\|[P_0], [P_1]\|_1$ is the L_1 -distance between the convex hulls of measures corresponding to the null and the alternative hypotheses and Ψ is the set of all possible tests. It implies that the type II error of the α -level test will be always greater or equal $1 - \alpha$ for non-distinguishable null and alternative hypotheses.

To derive lower and upper bounds in this sense, we will now pose some assumptions on the possible lengths λ_n of intervals and the covariance structure Σ_n :

Assumption 1. We assume that

- (i) $\frac{n\lambda_n}{\log n} \rightarrow \infty$ as $n \rightarrow \infty$,
- (ii) $\lambda_n = o(\frac{1}{\log n})$ as $n \rightarrow \infty$.

The first part of Assumption 1 assures that the number of observations within any interval of length λ_n is at least of logarithmic order as $n \rightarrow \infty$. The second condition of Assumption 1, however, gives a bound for the maximal length of the considered intervals, which ensures less than $n/\log n$ observations in the bump interval. Roughly speaking both conditions are required to have enough complementary observations (outside respectively inside the bump) to guarantee asymptotic detection. Note that, in particular, Assumption 1(ii) means that $\lambda_n \rightarrow 0$ as $n \rightarrow \infty$, i.e. Assumption 1 especially implies (1.5). We emphasize that conditions as in (ii) restricting λ_n from being too large are common. Assumption 1 plays a crucial role in the proof of the upper bound, whereas the lower bound can be established under milder conditions (1.5).

However, note that when we consider a slightly modified version of the testing problem (1.3) where the bump may not occur in any interval of length λ_n , but only within a candidate set $I_k := [(k - 1)\lambda_n, k\lambda_n), 1 \leq k \leq \lfloor 1/\lambda_n \rfloor$ of non-overlapping intervals, then Assumption 1 can be replaced by (1.5) and the detection boundary will remain the same (cf. Section 2.3).

Instead of posing assumptions on Σ_n directly, we will again employ the spectral density f of the underlying stationary process Z as mentioned in the introduction. To do so, we require some

more terminology. For a function $g \in \mathbf{L}_2[-1/2, 1/2)$, we denote by $\mathcal{T}(g)$ the Toeplitz matrix generated by g , i.e. the matrix with entries $(\mathcal{T}(g))_{i,j \in \mathbb{N}} = g_{j-i}$, where

$$g_k = \int_{-1/2}^{1/2} g(u)e^{-2\pi iku} du, \quad k \in \mathbb{Z},$$

is the k th Fourier coefficient of g . Note that this allows us to encode the covariance matrix Σ_n completely in terms of f . More precisely, the covariance matrix Σ_n of the noise ξ_n in (1.1) has entries $\Sigma_n(i, j) = \gamma(|i - j|) = f_{|i-j|}$, and we see that $\Sigma_n =: \mathcal{T}_n(f)$ is the n th truncated Toeplitz matrix generated by f , i.e. the upper left $n \times n$ submatrix, of $\mathcal{T}(f)$. Consequently, we will also pose the corresponding assumptions in terms of the function f , which allows us to derive results for any sequence $(\Sigma_n)_{n \geq 1}$ of covariance matrices which are generated by such an f (and not only for specific dependent processes):

Assumption 2. Let $(\Sigma_n)_{n \geq 1}$ be a sequence of covariance matrices such that $\Sigma_n = \mathcal{T}_n(f)$ as introduced above with a function $f : [-1/2, 1/2) \rightarrow \mathbb{R}$, that is continuous and satisfies $\lim_{v \rightarrow 1/2} f(v) = f(-1/2)$ and $\text{ess\,inf}_{v \in [-1/2, 1/2)} f(v) > 0$. Further, suppose that the Fourier coefficients $f_h, h \in \mathbb{Z}$ of f decay sufficiently fast, i.e. there are constants $C > 0$ and $\kappa > 0$, such that

$$|f_h| \leq C(1 + |h|)^{-(1+\kappa)}, \quad h \in \mathbb{Z}.$$

Assumption 2 ensures that the dependency between $\mathbf{1}_I^T Y$ and $\mathbf{1}_{I'}^T Y$ for two candidate intervals $I, I' \in \mathcal{I}(\lambda_n)$, will be asymptotically small as soon as they are disjoint. It excludes trivial cases such as total dependence described by $\Sigma_n(i, j) = 1$ for all $i, j \in \{1, \dots, n\}$, but also permits spectral densities f with only slowly decaying Fourier coefficients such as discontinuous functions.

Note that also sequences of covariance matrices of the form $(\Sigma_n)_{i,j} = g(\frac{|i-j|}{n}), 1 \leq i, j \leq n, n \in \mathbb{N}$, where g is some kernel function, are prohibited due to this assumption. Covariance matrices of this kind would have the undesired effect to make the dependency between $\mathbf{1}_I^T Y$ and $\mathbf{1}_{I'}^T Y$ even for disjoint candidate intervals $I, I' \in \mathcal{I}(\lambda_n)$ stronger as the length λ_n vanishes.

2.2. Asymptotic detection boundary

Our main theorem will be the following.

Theorem 2.1. *If Assumptions 1 and 2 hold for the bump regression model (1.1), then the asymptotic minimax detection boundary for the testing problem (1.3) is given by*

$$\Delta_n \asymp \sqrt{\frac{-2f(0) \log \lambda_n}{n\lambda_n}},$$

as $n \rightarrow \infty$.

For the details of the proof we refer to Section 5. The upper bound will be achieved by a specific test Φ_n^a , which scans over all intervals of length λ_n , given by

$$\Phi_n^a(Y) = \begin{cases} 1 & \text{if } \sup_{I \in \mathcal{I}(\lambda_n)} \frac{|\mathbf{1}_I^T Y|}{\sqrt{\mathbf{1}_I^T \Sigma_n \mathbf{1}_I}} > c_{\alpha,n}, \\ 0 & \text{else,} \end{cases} \tag{2.1}$$

where the threshold $c_{\alpha,n}$ will be determined in the proof of Lemma 5.1 in Section 5. Note that this test is not a likelihood ratio type test (as the LRT relies on $\mathbf{1}_I^T \Sigma_n^{-1} Y$ instead of $\mathbf{1}_I^T Y$).

For the proof of the lower bound we employ a strategy from [11], and use a very specific law of large numbers for arrays of non-independent and non-identically distributed random variables.

2.3. Non-asymptotic results

Note that Theorem 2.1 yields only an asymptotic result. In this section we give non-asymptotic results in the case of a seemingly simpler testing problem with possible bumps that belong to a set of non-overlapping intervals. This is formalized by considering the set \mathcal{I}^0 of non-overlapping candidate intervals given by

$$\mathcal{I}^0 := \{I_k \mid 1 \leq k \leq \lfloor \lambda_n^{-1} \rfloor\}, \quad I_k := [(k-1)\lambda_n, k\lambda_n], \quad 1 \leq k \leq \lfloor \lambda_n^{-1} \rfloor. \tag{2.2}$$

The goal is still to detect the presence of the bump (but with position being only in \mathcal{I}^0) and to derive non-asymptotic results on the detection boundary for the testing problem

$$H_0 : Y \sim \mathcal{N}_n(0, \Sigma_n) \tag{2.3}$$

against

$$H_1^n : \exists 1 \leq k \leq \lfloor \lambda_n^{-1} \rfloor, \exists \delta \in \mathbb{R} : |\delta| \geq \Delta_n \quad \text{such that} \quad Y \sim \mathcal{N}_n(\delta \mathbf{1}_{I_k}, \Sigma_n).$$

Note that this testing problem might seem much simpler than (1.3) at a first glance, but we will see, however, that the (asymptotic) detection boundary is in fact the same. Concerning the lower bound, this can be seen readily from the proof of Theorem 2.1, cf. Lemma 5.2.

To detect a bump, we will here employ the maximum likelihood ratio test

$$\Phi_n^d(Y) = \mathbf{1}\{T_n^0(Y) > c_{\alpha,n}\}$$

based on the statistic

$$T_n^0(Y) = \sup_{I \in \mathcal{I}^0} \frac{|\mathbf{1}_I^T \Sigma_n^{-1} Y|}{\sqrt{\mathbf{1}_I^T \Sigma_n^{-1} \mathbf{1}_I}} = \sup_{1 \leq k \leq \lfloor \lambda_n^{-1} \rfloor} \frac{|\mathbf{1}_{I_k}^T \Sigma_n^{-1} Y|}{\sqrt{\tilde{\sigma}_k}}, \tag{2.4}$$

where we denote

$$\tilde{\sigma}_k = \mathbf{1}_{I_k}^T \Sigma_n^{-1} \mathbf{1}_{I_k}, \quad k = 1, \dots, \lfloor \lambda_n^{-1} \rfloor. \tag{2.5}$$

The quantities $\tilde{\sigma}_k$ are in fact the variances of $\mathbf{1}_I^T \Sigma_n^{-1} Y$ corresponding to the sum of $\lfloor n\lambda_n \rfloor$ random variables with covariance structure given by the I_k -block of Σ_n^{-1} . The type I and II errors of the test Φ_n^d are defined as

$$\tilde{\alpha}(\Phi_n^d) := \mathbb{P}_0[\Phi_n^d(Y) = 1] \quad \text{and} \quad \tilde{\beta}(\Phi_n^d) := \sup_{I \in \mathcal{I}^0} \sup_{|\delta| \geq \Delta_n} \mathbb{P}_{I, \delta}[\Phi_n^d(Y) = 0].$$

Then the following result establishes basic properties of the test Φ_n^d .

Theorem 2.2. *Consider the testing problem (2.3) and let $\alpha \in (0, 1)$ be any fixed significance level. For the maximum likelihood ratio test Φ_n^d set*

$$c_{\alpha, n} := \sqrt{2 \log \frac{2}{\alpha \lambda_n}}. \tag{2.6}$$

Then it holds $\tilde{\alpha}(\Phi_n^d) \leq \alpha$ for all $n \in \mathbb{N}$ and

$$\tilde{\beta}(\Phi_n^d) \leq \mathbb{P}\left[|Z| > \Delta_n \inf_{1 \leq k \leq \lfloor \lambda_n^{-1} \rfloor} \sqrt{\tilde{\sigma}_k} - c_{\alpha, n}\right]$$

with a standard Gaussian random variable $Z \sim \mathcal{N}(0, 1)$ and $\tilde{\sigma}_k$ as in (2.5).

The proof is obtained by straightforward computations, see Section 6.2.2 of the Supplementary Material [13] for details. Theorem 2.2 yields explicit non-asymptotic bounds for the test Φ_n^d , but those do also yield an asymptotic upper bound for the detection boundary:

Corollary 2.3. *Let $(\varepsilon_n)_{n \in \mathbb{N}}$ be a positive sequence satisfying*

$$\varepsilon_n \sqrt{-\log \lambda_n} \geq \sqrt{\log \frac{2}{\alpha}} + \sqrt{\log \frac{1}{\alpha}}, \tag{2.7}$$

and suppose that the bump altitude Δ_n in the testing problem (2.3) obeys

$$\Delta_n \inf_{1 \leq k \leq \lfloor \lambda_n^{-1} \rfloor} \sqrt{\tilde{\sigma}_k} \geq \sqrt{2}(1 + \varepsilon_n) \sqrt{-\log \lambda_n}. \tag{2.8}$$

Then the asymptotic type II error of Φ_n^d with $c_{\alpha, n}$ as in (2.6) satisfies

$$\limsup_{n \rightarrow \infty} \tilde{\beta}(\Phi_n^d) \leq \alpha.$$

This shows that the upper bound to be obtained by Φ_n^d depends only on the asymptotic behavior of $\inf_{1 \leq k \leq \lfloor \lambda_n^{-1} \rfloor} \sqrt{\tilde{\sigma}_k}$ with $\tilde{\sigma}_k$ as in (2.5). Inspecting the proof of Lemma 5.2, we find that we can derive an according lower bound depending only on the asymptotic behavior of $\sup_{1 \leq k \leq \lfloor \frac{1}{\lambda_n} \rfloor} \sqrt{\tilde{\sigma}_k}$. In case of AR(p) noise we will see in Section 2.3 that these quantities can be computed explicitly and will asymptotically equal in agreement with Theorem 2.1.

3. ARMA processes and finite sample results

3.1. Application to ARMA processes

Suppose that the noise vector $\xi_n = (Z_1, \dots, Z_n)^T$ in (1.1) is sampled from n consecutive realizations of a stationary ARMA(p, q) time series Z_t , with $p \geq 0, q \geq 0$ defined as

$$\phi(B)Z_t = \theta(B)\zeta_t, \quad \zeta_t \stackrel{\text{i.i.d.}}{\sim} \mathcal{N}(0, 1), t \in \mathbb{Z}. \tag{3.1}$$

Here B is the so-called backshift operator, defined by $BX_t = X_{t-1}$, and $\phi(z)$ and $\theta(z), z \in \mathbb{C}$, are polynomials of degrees p and q , respectively, given by

$$\phi(z) = 1 + \sum_{i=1}^p \phi_i z^i, \quad \theta(z) = 1 + \sum_{i=1}^q \theta_i z^i. \tag{3.2}$$

We further suppose that ϕ and θ have no common roots, and that all roots of both ϕ and θ lie outside of the unit circle $\{z \in \mathbb{C} : |z| \leq 1\}$ (see [4] for more details).

Denote by γ the auto-covariance function of Z , i.e. $\gamma(h) = \mathbb{E}[Z_t Z_{t+h}]$ for $h \in \mathbb{Z}$ (as clearly $\mathbb{E}[Z_t] = 0$ for all $t \in \mathbb{Z}$). It is well-known (see for example [4], Theorem 4.4.2), that in the case of an ARMA(p, q) time series, its spectral density is given by

$$f(v) = \frac{|\theta(e^{-2\pi i v})|^2}{|\phi(e^{-2\pi i v})|^2}, \quad v \in [-1/2, 1/2). \tag{3.3}$$

Note that the spectral density f is continuous at 0 as well as the function $1/f$, since the process is reversible and causal under the posed assumptions on ϕ and θ . Thus, applying Theorem 2.1 to this setting immediately yields the following:

Theorem 3.1. *Assume that we are given observations from (1.1), where the noise ξ_n is given by n consecutive samples of an ARMA(p, q) time series as in (3.1) with the polynomials ϕ and θ in (3.2) having no common roots and no roots within the unit circle. Furthermore, assume that Assumption 1 holds. Then the asymptotic detection boundary of the hypothesis testing problem (1.3) is given by*

$$\Delta_n \asymp \sqrt{\frac{-2f(0) \log \lambda_n}{n\lambda_n}} = \left| \frac{1 + \sum_{i=1}^q \theta_i}{1 + \sum_{i=1}^p \phi_i} \right| \sqrt{\frac{-2 \log \lambda_n}{n\lambda_n}}, \tag{3.4}$$

as $n \rightarrow \infty$.

We find that the presence of dependency either eases or loads the bump detection, depending on $f(0) = |\theta(1)/\phi(1)|^2$ (which is 1 in the independent noise case). If $f(0) < 1$, then the detection becomes simpler (and smaller bumps are still consistently detectable), but if $f(0) > 1$ detection becomes more difficult. For AR(1) noise, this issue was already discussed in the introduction.

3.2. Non-asymptotic results for AR(p)

In this Section we will derive non-asymptotic results for the specific case of AR(p) noise. Let us therefore specify (3.1) to a stationary AR(p) process Z_t ,

$$\sum_{i=0}^p \phi_i Z_{t-i} = \zeta_t, \quad t \in \mathbb{Z} \quad (3.5)$$

with independent standard Gaussian innovations ζ_t . In the notation of (3.1), we have $\phi(z) = \sum_{i=0}^p \phi_i z^i$ and $\theta(z) \equiv 1$. Again, we work under the standard assumptions that the characteristic polynomial $\phi(z)$ has no zeros inside the unit circle $\{z \in \mathbb{C} : |z| \leq 1\}$. Note that in this case $f(0) = |\sum_{i=0}^p \phi_i|^{-2}$.

We have seen in the discussion succeeding Theorem 2.2 that the upper and lower bounds depend on the quantities $\tilde{\sigma}_k = \mathbf{1}_{I_k}^T \Sigma_n^{-1} \mathbf{1}_{I_k}$ and correspondingly, their minimal and maximal values. Theorem 3.1 gives the detection boundary condition for ARMA noise with an asymptotic risk constant. Since $\tilde{\sigma}_k$ is just the sum over the block of Σ_n^{-1} , using the exact inverse of Σ_n (see the appendix for the exact formula of Σ_n^{-1} obtained by [34]), we can calculate the minimax risk constants exactly.

Lemma 3.2. *Let Σ_n be the auto-covariance matrix induced by an AR(p) process Z_t and $\tilde{\sigma}_k = \mathbf{1}_{I_k}^T \Sigma_n^{-1} \mathbf{1}_{I_k}$, $k = 1, \dots, \lfloor \lambda_n^{-1} \rfloor$. Assume that $1 \leq \lfloor n\lambda_n \rfloor \leq n - 2p$ and $n > 3p$.*

1. *If $\lfloor n\lambda_n \rfloor \leq p$, then*

$$\inf_{1 \leq k \leq \lfloor \lambda_n^{-1} \rfloor} \tilde{\sigma}_k = \sum_{i=1}^{\lfloor n\lambda_n \rfloor} \left(\sum_{t=0}^{i-1} \phi_t \right)^2, \quad (3.6)$$

$$\sup_{1 \leq k \leq \lfloor \lambda_n^{-1} \rfloor} \tilde{\sigma}_k = \inf_{1 \leq k \leq \lfloor \lambda_n^{-1} \rfloor} \tilde{\sigma}_k + \sum_{i=0}^{p-\lfloor n\lambda_n \rfloor} \left(\sum_{t=1}^{\lfloor n\lambda_n \rfloor} \phi_{t+i} \right)^2 + \sum_{i=p-\lfloor n\lambda_n \rfloor}^p \left(\sum_{t=0}^{p-i} \phi_{t+i} \right)^2. \quad (3.7)$$

2. *If $p < \lfloor n\lambda_n \rfloor \leq n - 2p$, then*

$$\inf_{1 \leq k \leq \lfloor \lambda_n^{-1} \rfloor} \tilde{\sigma}_k = (\lfloor n\lambda_n \rfloor - p) \left(\sum_{t=0}^p \phi_t \right)^2 + \sum_{i=1}^p \left(\sum_{t=0}^{i-1} \phi_t \right)^2, \quad (3.8)$$

$$\sup_{1 \leq k \leq \lfloor \lambda_n^{-1} \rfloor} \tilde{\sigma}_k = \inf_{1 \leq k \leq \lfloor \lambda_n^{-1} \rfloor} \tilde{\sigma}_k + \sum_{i=1}^p \left(\sum_{t=0}^{p-i} \phi_{t+i} \right)^2. \quad (3.9)$$

We can now use the results of Theorem 2.2 and get the exact detection boundaries for two different regimes, when $\lfloor n\lambda_n \rfloor \leq p$ and $p < \lfloor n\lambda_n \rfloor \leq n - 2p$. Note that condition (5.4) is automatically satisfied since the inverse covariance matrix Σ_n^{-1} is $2p + 1$ -diagonal.

Corollary 3.3. *Assume that possible locations k of the bump $I_k \in \mathcal{I}^0$ are separated from the endpoints of the interval: $p < k < n - p - \lfloor n\lambda_n \rfloor$. Then the upper and lower bound constants match in both cases and are given by formulas (3.6) and (3.8) for the case of $\lfloor n\lambda_n \rfloor \leq p$ and $p < \lfloor n\lambda_n \rfloor \leq n - 2p$, respectively. This follows immediately from the discussion in Section 6.3.1, in particular equations (6.4) and (6.5).*

Remark 3.4. It seems reasonable, that, in case of bumps of length smaller than p , we would need to analyze the type I error with some finer technique than just the union bound.

On the other hand, we observe that if $n\lambda_n \rightarrow \infty$ and $\lambda_n \rightarrow 0$ as $n \rightarrow \infty$, then

$$\sup_{1 \leq k \leq \lfloor \lambda_n^{-1} \rfloor} \tilde{\sigma}_k \asymp n\lambda_n \left(\sum_{t=0}^p \phi_t \right)^2 \asymp \inf_{1 \leq k \leq \lfloor \lambda_n^{-1} \rfloor} \tilde{\sigma}_k,$$

in accordance with Theorem 3.1.

4. Simulations

In this Section we will perform numerical studies to examine the finite sample accuracy of the asymptotic upper bounds for the detection boundary. We focus on the situation that the noise ξ_n in (1.1) is generated by an AR(1) process, given by $\phi(z) = 1 - \rho z$ and $\theta(z) \equiv 1$ (in the notation of (3.1)), where $|\rho| < 1$. More precisely, the AR(1) process is given by the equation $Z_t - \rho Z_{t-1} = \zeta_t$, $t \in \mathbb{Z}$ where $\zeta_t \sim \mathcal{N}(0, 1)$ are i.i.d.. Note the slight difference to the setting considered in the introduction and Figure 1, as here the noise does not have standardized margins.

From Theorem 3.1 we obtain the detection boundary

$$\Delta_n \asymp \frac{\sqrt{2}}{1 - \rho} \sqrt{\frac{-\log \lambda_n}{n\lambda_n}}. \tag{4.1}$$

In the following we fix the value of the detection rate $\sqrt{-\log \lambda_n / (n\lambda_n)}$ in (4.1) to be roughly 1/6 and consider three different situations, namely small sample size ($\lambda_n = 0.1$, $n = 829$), medium sample size ($\lambda_n = 0.05$, $n = 2157$) and large sample size ($\lambda_n = 0.025$, $n = 5312$). Thus, the remaining free parameters are ρ and Δ_n , and the detection boundary (4.1) connects them by the asymptotic relation

$$\Delta_n \asymp \frac{\sqrt{2}}{1 - \rho} \cdot \frac{1}{6} \approx \frac{0.236}{1 - \rho}. \tag{4.2}$$

Let us now specify the investigated tests. For the (general) test from Section 2.2, the critical value $c_{\alpha,n}$ is only given implicitly, cf. (5.2). To simplify, in view of $c_{\alpha,n} = \sqrt{2 \log \frac{2}{\alpha \lambda_n}} (1 + o(1))$,

we will therefore use

$$\Phi_n^a(Y) := \begin{cases} 1 & \text{if } \sup_{I \in \mathcal{I}(\lambda_n)} \frac{|\mathbf{1}_I^T Y|}{\sqrt{\mathbf{1}_I^T \Sigma_n \mathbf{1}_I}} > \sqrt{2 \log \frac{2}{\alpha \lambda_n}}, \\ 0 & \text{else,} \end{cases} \tag{4.3}$$

as an asymptotic version. Further we would like to investigate the maximum likelihood ratio test relying only on non-overlapping intervals $I_k = [(k - 1)\lfloor n\lambda_n \rfloor + 1, k\lfloor n\lambda_n \rfloor)$ from Section 2.3 given by

$$\Phi_n^d(Y) := \begin{cases} 1 & \text{if } \sup_{1 \leq k \leq \lfloor \frac{1}{\lambda_n} \rfloor} \frac{|\mathbf{1}_{I_k}^T \Sigma_n^{-1} Y|}{\sqrt{\mathbf{1}_{I_k}^T \Sigma_n^{-1} \mathbf{1}_{I_k}}} > \sqrt{2 \log \frac{2}{\alpha \lambda_n}}, \\ 0 & \text{else.} \end{cases} \tag{4.4}$$

Note that the latter requires to scan only over $\lfloor 1/\lambda_n \rfloor$ intervals, whereas the former requires to scan over $\lfloor n(1 - \lambda_n) \rfloor$ intervals. Consequently, the maximum likelihood ratio test from Section 2.3 can be computed faster by a factor of

$$\frac{n(1 - \lambda_n)}{1/\lambda_n} = n\lambda_n(1 - \lambda_n)$$

independent of Σ_n . For the three situations mentioned above this yields values of ≈ 74 in the small sample regime, ≈ 102 in the medium sample regime, and ≈ 129 in the large sample regime. However, our results from Theorem 2.1 and the discussion succeeding Theorem 2.2 imply, that the testing problems (1.3) and (2.3) are of the same difficulty in the sense that they both have the same separation rate.

In the following we examine the type I and type II errors $\bar{\alpha}(\Phi_*)$ and $\bar{\beta}(\Phi_*)$ with $* \in \{a, d\}$ by 2000 simulation runs for $\alpha = 0.05$ with different choices of ρ , n , λ_n and Δ_n . The position $I \in \mathcal{I}(\lambda_n)$ is always drawn uniformly at random. Furthermore, we investigate the situation of 2 and 5 disjoint bumps within $[0, 1]$.

The finite sample type I error of both Φ_n^a and Φ_n^d in all three sample size situations are shown in Figure 2 versus the correlation parameter $\rho \in \{-0.99, -0.98, \dots, 0.99\}$. We find that Φ_n^d is somewhat conservative, which is clearly due to the usage of the union bound in deriving the critical value in (4.4), see the proof of Theorem 2.2. Opposed, Φ_n^a is conservative for $\rho > 0$, and liberal for $\rho < 0$. This is clearly due to the simplified critical value in (4.3), which is only asymptotically valid, and furthermore the employed asymptotics depend on ρ due to the result by Ibragimov and Linnik [22], see also Section 6.1.3 of the Supplementary Material [13]. However, it seems that already in the small sample size regime our asymptotic results provide a very good approximation for both tests.

Next we computed the finite sample type II error in all three sample size situations for $\rho \in \{-0.99, -0.98, \dots, 0.99\}$ and $\Delta_n \in \{0.01, 0.02, \dots, 0.5\}$. The corresponding results are shown in Figures 3–5. We also depict the contour line of equation (4.2) for a comparison and find a remarkably good agreement with the contour lines of the power function already in the small sample regime, which strongly supports the finite sample validity of our asymptotic theory. Finally, we conclude that detection becomes easier for a larger number of bumps.

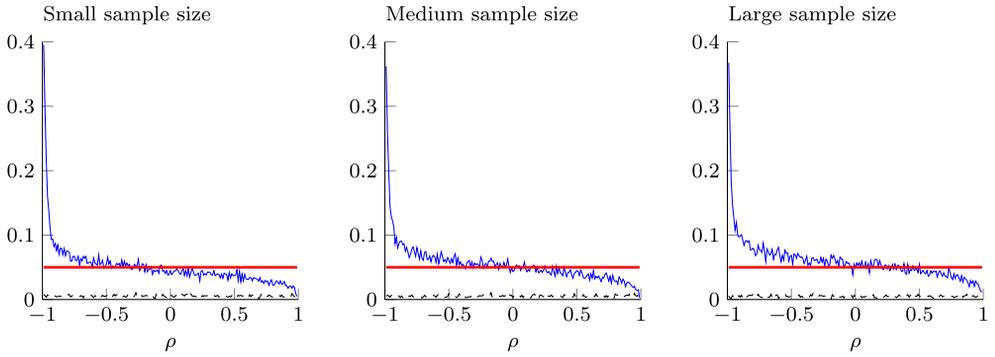


Figure 2. Type I error of the tests Φ_n^a (blue line) and Φ_n^d (dashed line) for the AR(1) case vs. ρ (x -axis) simulated by 2000 Monte Carlo simulations with the nominal type I error $\alpha = 0.05$ (red line) in three different situations: small sample size $\lambda_n = 0.1$, $n = 829$ (left), medium sample size $\lambda_n = 0.05$, $n = 2157$ (middle) and large sample size $\lambda_n = 0.025$, $n = 5312$ (right).

5. Proofs

Several useful results from various sources that we are going to use in our proofs can be found in the Supplementary Material [13]. The proof of Theorem 2.1 will then be split into three parts.

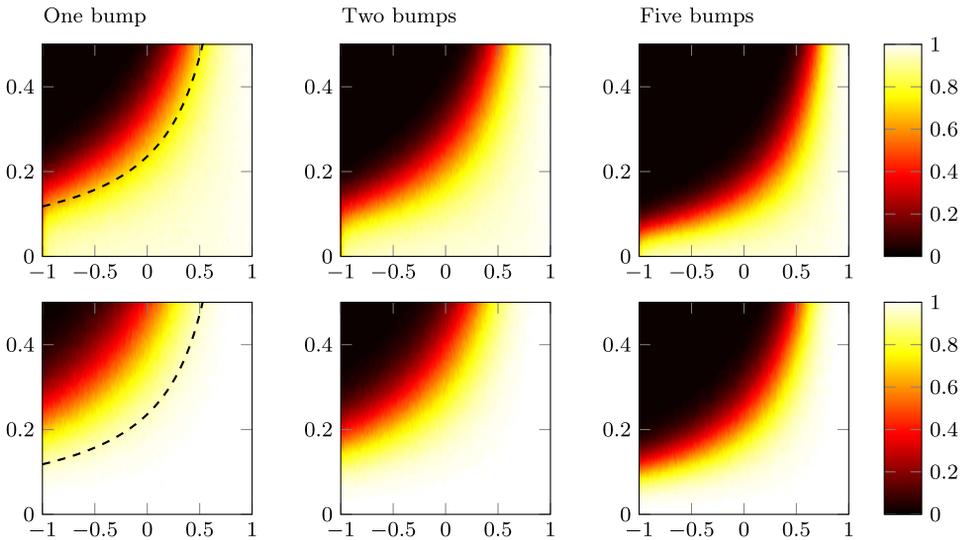


Figure 3. Type II error in the small sample regime $\lambda_n = 0.1$, $n = 829$ (top row: Φ_n^a ; bottom row: Φ_n^d) for the AR(1) case for ρ (x -axis) vs. Δ_n (y -axis) with one bump (left column) together with the contour line of the detection boundary equation (4.2), two bumps (middle column) and five bumps (right column), each simulated by 2000 Monte Carlo simulations.

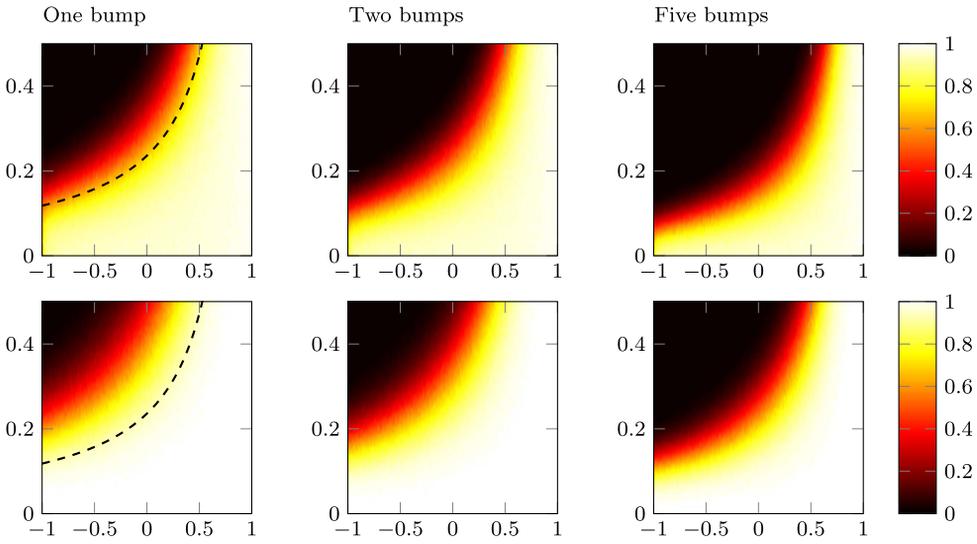


Figure 4. Type II error in the medium sample regime $\lambda_n = 0.05$, $n = 2157$ (top row: Φ_n^a ; bottom row: Φ_n^d) for the AR(1) case for ρ (x -axis) vs. Δ_n (y -axis) with one bump (left column) together with the contour line of the detection boundary equation (4.2), two bumps (middle column) and five bumps (right column), each simulated by 2000 Monte Carlo simulations.

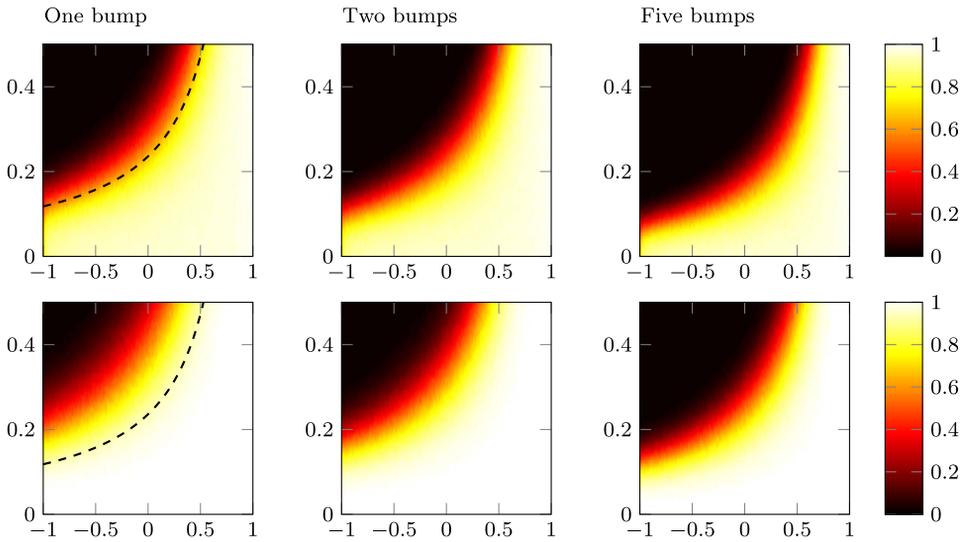


Figure 5. Type II error in the large sample regime $\lambda_n = 0.025$, $n = 5312$ (top row: Φ_n^a ; bottom row: Φ_n^d) for the AR(1) case for ρ (x -axis) vs. Δ_n (y -axis) with one bump (left column) together with the contour line of the detection boundary equation (4.2), two bumps (middle column) and five bumps (right column), each simulated by 2000 Monte Carlo simulations.

We will provide asymptotic upper and lower bounds in Sections 5.1.1 and 5.1.2, respectively. The lower bound result will in fact hold for a wider class of covariance matrices than those allowed by Assumption 2. Finally, in Section 5.1.3, this will be used to show that the upper and lower bound coincide asymptotically in the setting of Theorem 2.1, and this will yield the desired result. All remaining proofs can be found in Section 6 of the Supplementary Material [13].

5.1. Proofs for Section 2

5.1.1. Upper detection bound

For $I \in \mathcal{I}(\lambda_n)$ we define

$$\sigma_n(I) := \mathbf{1}_I^T \Sigma_n \mathbf{1}_I.$$

Lemma 5.1 (Upper detection bound). Fix $\alpha \in (0, 1)$, consider the testing problem (1.3) and suppose that Assumption 1 and Assumption 2 hold. In addition, assume that the sequence $(\Sigma_n)_{n \in \mathbb{N}}$ of covariance matrices satisfies

$$\Delta_n \inf_{I \in \mathcal{I}(\lambda_n)} \frac{\lfloor n\lambda_n \rfloor}{\sqrt{\sigma_n(I)}} \gtrsim (\sqrt{2} + \tilde{\varepsilon}_n) \sqrt{-\log \lambda_n}, \tag{5.1}$$

as $n \rightarrow \infty$, where $(\tilde{\varepsilon}_n)_{n \in \mathbb{N}}$ is a sequence of real numbers that satisfies $\tilde{\varepsilon}_n \rightarrow 0$ and $\tilde{\varepsilon}_n \sqrt{-\log \lambda_n} - \sqrt{\log \log n} \rightarrow \infty$ as $n \rightarrow \infty$.

Then the sequence of level α tests $(\Phi_n^a)_{n \in \mathbb{N}}$ as in (2.1) with suitably chosen $c_{\alpha,n}$ satisfies $\bar{\alpha}(\Phi_n^a) \leq \alpha$ for all $n \in \mathbb{N}$ and $\limsup_{n \rightarrow \infty} \beta(\Phi_n^a) \leq \alpha$.

Proof. Define the test statistic

$$T_n(Y) = \sup_{I \in \mathcal{I}(\lambda_n)} \frac{|\mathbf{1}_I^T Y|}{\sqrt{\sigma_n(I)}},$$

and recall that $\Phi_n^a(Y) := \mathbb{1}\{T_n(Y) > c_{\alpha,n}\}$, for some threshold $c_{\alpha,n}$ to be determined later.

We begin by noting that although the random process

$$\left(\frac{|\mathbf{1}_I^T Y|}{\sqrt{\sigma_n(I)}} \right)_{I \in \mathcal{I}(\lambda_n)}$$

has an infinite index set, it only takes finitely many different values. Thus, we can find a finite representative system $\mathcal{I}_{\text{fin}}(\lambda_n) \subseteq \mathcal{I}(\lambda_n)$, such that for any $I \in \mathcal{I}(\lambda_n)$ there is $I' \in \mathcal{I}_{\text{fin}}(\lambda_n)$, such that

$$\frac{|\mathbf{1}_I^T Y|}{\sqrt{\sigma_n(I)}} = \frac{|\mathbf{1}_{I'}^T Y|}{\sqrt{\sigma_n(I')}},$$

which implies that

$$T_n(Y) = \sup_{I \in \mathcal{I}_{\text{fin}}(\lambda_n)} \frac{|\mathbf{1}_I^T Y|}{\sqrt{\sigma_n(I)}}$$

i.e. $T_n(Y)$ can be written as the supremum over the absolute values of a Gaussian process with a finite index set. Let M_n such that $\mathbb{E}_0 T_n(Y) \leq M_n$. Then, for any $\lambda > 0$, it follows that

$$\begin{aligned} \mathbb{P}_0(T_n(Y) > \lambda + M_n) &\leq \mathbb{P}_0(T_n(Y) - \mathbb{E}_0 T_n(Y) > \lambda) \\ &\leq \mathbb{P}_0\left(\left| \sup_{I \in \mathcal{I}_{\text{fin}}(\lambda_n)} \frac{|\mathbf{1}_I^T Y|}{\sqrt{\sigma_n(I)}} - \mathbb{E}_0 \sup_{I \in \mathcal{I}_{\text{fin}}(\lambda_n)} \frac{|\mathbf{1}_I^T Y|}{\sqrt{\sigma_n(I)}} \right| > \lambda\right) \leq 2e^{-\frac{\lambda^2}{2}}, \end{aligned}$$

where the last inequality follows the results of Talagrand [38] and can be found in Theorem 2.1.20 of [16]. Thus, if we let

$$c_{\alpha,n} = \sqrt{2 \log \frac{2}{\alpha}} + M_n,$$

Φ_n^a has level α for any n .

In order to find a suitable bound M_n we consider an even coarser finite subset of $\mathcal{I}(\lambda_n)$. Let

$$\mathcal{C}_n = \left\{ \left[\frac{k}{n}, \frac{k}{n} + \lambda_n \right) : 1 \leq k \leq \lfloor n(1 - \lambda_n) \rfloor \right\} \subseteq \mathcal{I}(\lambda_n).$$

Clearly, $\#\mathcal{C}_n = \lfloor n(1 - \lambda_n) \rfloor \leq n < \infty$. For any $I \in \mathcal{I}(\lambda_n)$ there is $I' \in \mathcal{C}_n$, such that 1_I differs from $1_{I'}$ in at most one entry. Thus, it is easy to see that

$$T_n(Y) \leq \sup_{I \in \mathcal{C}_n} \frac{|\mathbf{1}_I^T Y|}{\sqrt{\sigma_n(I)}} + \frac{\sup_{1 \leq i \leq n} |Y_{i,n}|}{\inf_{I \in \mathcal{I}(\lambda_n)} \sqrt{\sigma_n(I)}},$$

Thus, we can set

$$M_n = \tilde{M}_n + \kappa_n,$$

where

$$\tilde{M}_n = \mathbb{E}_0 \sup_{I \in \mathcal{C}_n} \frac{|\mathbf{1}_I^T Y|}{\sqrt{\sigma_n(I)}}$$

and

$$\kappa_n = \mathbb{E}_0 \frac{\sup_{1 \leq i \leq n} |Y_{i,n}|}{\inf_{I \in \mathcal{I}(\lambda_n)} \sqrt{\sigma_n(I)}}.$$

The latter term is easy to handle: We have

$$\mathbb{E}_0 \sup_{1 \leq i \leq n} |Y_{i,n}| \leq \sqrt{2 f_0 \log(2n)},$$

since $Y_{i,n}$ has variance f_0 for any n and $1 \leq i \leq n$, and

$$\inf_{I \in \mathcal{I}(\lambda_n)} \sigma_n(I) = n \lambda_n f(0) (1 + o(1))$$

by Theorem 18.2.1 of Ibragimov and Linnik [22]. Thus,

$$\kappa_n = O\left(\sqrt{\frac{\log n}{n\lambda_n}}\right),$$

and thus, $\kappa_n \rightarrow 0$ by Assumption 1. The next part of the proof will be devoted to computing \tilde{M}_n . Note that under H_0 , we have

$$\frac{\mathbf{1}_I^T Y}{\sqrt{\sigma_n(I)}} \sim \mathcal{N}(0, 1)$$

for any $I \in \mathcal{C}_n$. For any $I, I' \in \mathcal{C}_n$, we have

$$\left| \frac{\mathbf{1}_I^T Y}{\sqrt{\sigma_n(I)}} - \frac{\mathbf{1}_{I'}^T Y}{\sqrt{\sigma_n(I')}} \right| = \left| \left(\frac{\mathbf{1}_I^T}{\sqrt{\sigma_n(I)}} - \frac{\mathbf{1}_{I'}^T}{\sqrt{\sigma_n(I')}} \right) Y \right| = \left(2 - 2 \frac{\mathbf{1}_I^T \Sigma_n \mathbf{1}_{I'}}{\sqrt{\sigma_n(I)\sigma_n(I')}} \right)^{\frac{1}{2}} |Z_{I,I'}|$$

for some random variable $Z_{I,I'} \sim \mathcal{N}(0, 1)$. Note that the system $\{Z_{I,I'} : I, I' \in \mathcal{C}_n\}$ is not necessarily independent. Let

$$d_n(I, I') := \left(2 - 2 \frac{\mathbf{1}_I^T \Sigma_n \mathbf{1}_{I'}}{\sigma_n(I)} \right)^{\frac{1}{2}}.$$

Since Σ_n is a Toeplitz matrix, it follows that $\sigma_n(I) = \sigma_n(I')$ for any $I, I' \in \mathcal{C}_n$, and thus, $d_n(I, I') = d_n(I', I)$. Since Σ_n is also positive definite, it is then easy to see that d_n is a metric on \mathcal{C}_n .

Now let $\mathcal{E}_n \subseteq \mathcal{C}_n$ be an η_n -net for (\mathcal{C}_n, d_n) , i.e. for any $I \in \mathcal{C}_n$ there is $J \in \mathcal{E}_n$, such that

$$d_n(I, J) \leq \eta_n.$$

For any $I \in \mathcal{C}_n$ and $J \in \mathcal{E}_n$ we have

$$\frac{|\mathbf{1}_I^T Y|}{\sqrt{\sigma_n(I)}} \leq \left| \frac{\mathbf{1}_I^T Y}{\sqrt{\sigma_n(I)}} - \frac{\mathbf{1}_J^T Y}{\sqrt{\sigma_n(J)}} \right| + \frac{|\mathbf{1}_J^T Y|}{\sqrt{\sigma_n(J)}},$$

and thus,

$$\begin{aligned} \sup_{I \in \mathcal{C}_n} \frac{|\mathbf{1}_I^T Y|}{\sqrt{\sigma_n(I)}} &\leq \sup_{I \in \mathcal{C}_n} \inf_{J \in \mathcal{E}_n} \left| \frac{\mathbf{1}_I^T Y}{\sqrt{\sigma_n(I)}} - \frac{\mathbf{1}_J^T Y}{\sqrt{\sigma_n(J)}} \right| + \sup_{J \in \mathcal{E}_n} \frac{|\mathbf{1}_J^T Y|}{\sqrt{\sigma_n(J)}} \\ &= \sup_{I \in \mathcal{C}_n} \inf_{J \in \mathcal{E}_n} d_n(I, J) |Z_{I,J}| + \sup_{J \in \mathcal{E}_n} \frac{|\mathbf{1}_J^T Y|}{\sqrt{\sigma_n(J)}} \\ &\leq \eta_n \sup_{I \in \mathcal{C}_n} \inf_{J \in \mathcal{E}_n} |Z_{I,J}| + \sup_{J \in \mathcal{E}_n} \frac{|\mathbf{1}_J^T Y|}{\sqrt{\sigma_n(J)}}. \end{aligned}$$

It follows that

$$\begin{aligned} \tilde{M}_n &\leq \eta_n \sqrt{2 \log(2n)} + \sqrt{2 \log(2N(C_n, d_n, \eta_n))} \\ &\leq \eta_n \sqrt{2 \log n} + \sqrt{2 \log N(C_n, d_n, \eta_n)} + (1 + \eta_n) \sqrt{2 \log 2}, \end{aligned}$$

where $N(C_n, d_n, \eta_n)$ is the η_n -covering number of (C_n, d_n) . Now let $I, I' \in C_n, I \neq I'$, with $d_H(I, I') \leq \frac{\lambda_n}{\log n}$, where d_H denotes the Hausdorff metric on the set of subintervals of $[0, 1]$ (with respect to the euclidean distance on $[0, 1]$), i.e. $d_H(I, I') = |\inf I - \inf I'|$. In addition, we assume that $\inf I < \inf I'$ without loss of generality. Then

$$\begin{aligned} \mathbf{1}_I^T \Sigma_n \mathbf{1}_{I'} &= (\mathbf{1}_{I \cap I'}^T + \mathbf{1}_{I \setminus I'}^T) \Sigma_n (\mathbf{1}_{I \cap I'} + \mathbf{1}_{I' \setminus I}) \\ &= \mathbf{1}_{I \cap I'}^T \Sigma_n \mathbf{1}_{I \cap I'} + \mathbf{1}_{I \setminus I'}^T \Sigma_n \mathbf{1}_{I \cap I'} + \mathbf{1}_{I \cap I'}^T \Sigma_n \mathbf{1}_{I' \setminus I} + \mathbf{1}_{I \setminus I'}^T \Sigma_n \mathbf{1}_{I' \setminus I}. \end{aligned}$$

Due to Σ_n being symmetric and Toeplitz, we have $\mathbf{1}_{I \cap I'}^T \Sigma_n \mathbf{1}_{I' \setminus I} = \mathbf{1}_{I \cap I'}^T \Sigma_n \mathbf{1}_{I \setminus I'}$, and thus,

$$\mathbf{1}_I^T \Sigma_n \mathbf{1}_{I'} = \mathbf{1}_I^T \Sigma_n \mathbf{1}_I - \mathbf{1}_{I \setminus I'}^T \Sigma_n \mathbf{1}_{I \setminus I'} + \mathbf{1}_{I \setminus I'}^T \Sigma_n \mathbf{1}_{I' \setminus I}.$$

It follows that

$$\begin{aligned} d_n^2(I, I') &= 2 - 2(\mathbf{1}_I^T \Sigma_n \mathbf{1}_I)^{-1} [\mathbf{1}_I^T \Sigma_n \mathbf{1}_I - \mathbf{1}_{I \setminus I'}^T \Sigma_n \mathbf{1}_{I \setminus I'} + \mathbf{1}_{I \setminus I'}^T \Sigma_n \mathbf{1}_{I' \setminus I}] \\ &= 2(\mathbf{1}_I^T \Sigma_n \mathbf{1}_I)^{-1} [\mathbf{1}_{I \setminus I'}^T \Sigma_n \mathbf{1}_{I \setminus I'} - \mathbf{1}_{I \setminus I'}^T \Sigma_n \mathbf{1}_{I' \setminus I}]. \end{aligned}$$

Since $\mathbf{1}_{I \setminus I'}^T \Sigma_n \mathbf{1}_{I' \setminus I}$ is the sum over a submatrix with $r_n = n|\inf I - \inf I'|$ rows, and its lower left entry is $f_{\lfloor n\lambda_n \rfloor - 1 - r_n}$, we find the trivial bound

$$|\mathbf{1}_{I \setminus I'}^T \Sigma_n \mathbf{1}_{I' \setminus I}| \leq \frac{n\lambda_n}{\log n} \sum_{h=\lfloor n\lambda_n(1-1/\log n) \rfloor - 1}^{\infty} M(1 + |h|)^{-1-\kappa} = o\left(\frac{n\lambda_n}{\log n}\right),$$

as $n \rightarrow \infty$. We use Theorem 18.2.1 of Ibragimov and Linnik [22] (see also Section 6.1.3 of the Supplementary Material [13]) to find $\mathbf{1}_I^T \Sigma_n \mathbf{1}_I = n\lambda_n f(0)(1 + o(1))$ and

$$\mathbf{1}_{I \setminus I'}^T \Sigma_n \mathbf{1}_{I \setminus I'} \leq f(0) \frac{n\lambda_n}{\log n} (1 + o(1)),$$

as $n \rightarrow \infty$. This yields

$$d_n(I, I') \leq \sqrt{\frac{2}{\log n}} + \zeta_n,$$

where $\zeta_n = o((\log n)^{-\frac{1}{2}})$. This implies that for any $I \in C_n$ and for large enough n we have the inclusion

$$\left\{ I' \in C_n : d_H(I, I') \leq \frac{\lambda_n}{\log n} \right\} \subseteq \left\{ I' \in C_n : d_n(I, I') \leq \sqrt{\frac{2}{\log n}} + \zeta_n \right\}.$$

Thus, if we choose $\eta_n = \sqrt{\frac{2}{\log n}} + \zeta_n$, this yields the bound

$$N(C_n, d_n, \eta_n) \leq \frac{\log n}{2\lambda_n},$$

and consequently,

$$\tilde{M}_n \leq 2 + \zeta_n \sqrt{2 \log n} + \sqrt{2 \log \frac{\log n}{2\lambda_n}} + \left(1 + \zeta_n + \sqrt{\frac{2}{\log n}}\right) \sqrt{2 \log 2}.$$

Thus, if we choose

$$\begin{aligned} c_{\alpha,n} &= 2 + \zeta_n \sqrt{2 \log n} + \sqrt{2 \log \frac{2}{\alpha}} + \sqrt{2 \log \frac{\log n}{2\lambda_n}} + \kappa_n \\ &\quad + \left(1 + \zeta_n + \sqrt{\frac{2}{\log n}}\right) \sqrt{2 \log 2}, \end{aligned} \tag{5.2}$$

the test Φ_n^a will have level α for all $n \in \mathbb{N}$. Note that $\zeta_n \sqrt{2 \log n} = o(1)$ as $n \rightarrow \infty$. Concerning the type II error of the test Φ_n^a , recall that, under H_1 , i.e. if $Y \sim \mathcal{N}(\delta \mathbf{1}_I, \Sigma_n)$ for some δ with $|\delta| > \Delta_n$, and $I \in \mathcal{I}(\lambda_n)$, we have

$$\frac{\mathbf{1}_{I'}^T Y}{\sqrt{\sigma_n(I')}} \sim \mathcal{N}\left(\frac{\delta \mathbf{1}_{I'}^T \mathbf{1}_I}{\sqrt{\sigma_n(I')}}, 1\right)$$

for all $I' \in \mathcal{I}(\lambda_n)$. For n large enough, it follows from plugging in (5.1) and (5.2), that

$$\begin{aligned} \bar{\beta}(\Phi_n^a) &= \sup_{I \in \mathcal{I}(\lambda_n)} \sup_{|\delta| \geq \Delta_n} \mathbb{P}_{I,\delta}[\Phi_n^a(Y) = 0] \\ &= \sup_{I \in \mathcal{I}(\lambda_n)} \sup_{|\delta| \geq \Delta_n} \mathbb{P}\left[\sup_{I' \in \mathcal{I}(\lambda_n)} \left|Z_{I'} + \frac{\delta \mathbf{1}_{I'}^T \mathbf{1}_I}{\sqrt{\sigma_n(I')}}\right| \leq c_{\alpha,n}\right] \\ &\leq \sup_{I \in \mathcal{I}(\lambda_n)} \sup_{|\delta| \geq \Delta_n} \mathbb{P}\left[\left|Z_I + \frac{\delta \mathbf{1}_I^T \mathbf{1}_I}{\sqrt{\sigma_n(I)}}\right| \leq c_{\alpha,n}\right] \\ &\leq \sup_{I \in \mathcal{I}(\lambda_n)} \sup_{|\delta| \geq \Delta_n} \mathbb{P}\left[|\delta| \frac{\mathbf{1}_I^T \mathbf{1}_I}{\sqrt{\sigma_n(I)}} - |Z_I| \leq c_{\alpha,n}\right] \\ &\leq \mathbb{P}\left[|Z| > \Delta_n \inf_{I \in \mathcal{I}(\lambda_n)} \frac{\mathbf{1}_I^T \mathbf{1}_I}{\sqrt{\sigma_n(I)}} - c_{\alpha,n}\right], \end{aligned}$$

where $(Z_I)_{I \in \mathcal{I}(\lambda_n)}$ and Z are (not necessarily independent) standard Gaussian random variables. Plugging in (5.1), we have

$$\begin{aligned} \Delta_n \inf_{I \in \mathcal{I}(\lambda_n)} \frac{\mathbf{1}_I^T \mathbf{1}_I}{\sqrt{\sigma_n(I)}} - c_{\alpha,n} &\geq \tilde{\varepsilon}_n \sqrt{\log \frac{1}{\lambda_n}} - 2 - \zeta_n \sqrt{2 \log n} - \sqrt{2 \log \frac{2}{\alpha}} - \sqrt{2 \log \frac{\log n}{2}} - \kappa_n \\ &\quad - \left(1 + \zeta_n + \sqrt{\frac{2}{\log n}}\right) \sqrt{2 \log 2} \end{aligned}$$

for n large enough. Since $\tilde{\varepsilon}_n \sqrt{-\log \lambda_n} - \sqrt{\log \log n} \rightarrow \infty$ by assumption and $\kappa_n = o(1)$ and $\zeta_n \sqrt{2 \log n} = o(1)$ as $n \rightarrow \infty$, it follows that the right-hand side diverges to ∞ . This finishes the proof. \square

5.1.2. Lower detection bound

We start by giving some technicalities on LR-statistics required throughout the paper at several places. As λ_n and Σ_n are known, the likelihood ratio $L_{\delta,I} = L_{\delta,I}(Y)$ between the distributions of Y under H_0 and $H_{\delta,I}^n$ is given by

$$L_{I,\delta} = \exp \left[\delta \mathbf{1}_I^T \Sigma_n^{-1} Y - \frac{1}{2} \delta^2 \mathbf{1}_I^T \Sigma_n^{-1} \mathbf{1}_I \right].$$

Note that, under H_0 , the likelihood ratio $L_{\delta,I}$ follows a log-normal distribution, i.e.

$$\log L_{I,\delta} = \delta \mathbf{1}_I^T \Sigma_n^{-1} Y - \frac{1}{2} \delta^2 \mathbf{1}_I^T \Sigma_n^{-1} \mathbf{1}_I \stackrel{H_0}{\sim} \mathcal{N}_1 \left(-\frac{1}{2} \delta^2 \tilde{\sigma}_n(I), \delta^2 \tilde{\sigma}_n(I) \right),$$

where

$$\tilde{\sigma}_n(I) := \mathbf{1}_I^T \Sigma_n^{-1} \mathbf{1}_I.$$

Note that, under H_0 , for $I, I' \in \mathcal{C}_n$ and $\delta \in \mathbb{R}$, we have $\mathbb{E} L_{I,\delta} = 1$, $\text{Var} L_{I,\delta} = \exp(\delta^2 \tilde{\sigma}_n(I)) - 1$ and $\text{Cov}(L_{I,\delta}, L_{I',\delta}) = \exp(\delta^2 \mathbf{1}_I^T \Sigma_n^{-1} \mathbf{1}_{I'}) - 1$. Finally, let

$$\mathcal{I}^0 := \{[(k-1)\lambda_n, k\lambda_n) : 1 \leq k \leq \lfloor \lambda_n^{-1} \rfloor\} \subseteq \mathcal{I}(\lambda_n)$$

be a system of non-overlapping intervals of length λ_n as defined in (2.2).

Lemma 5.2 (Lower detection bound). Fix $\alpha \in (0, 1)$, and suppose that (1.5) holds. Let $(\Sigma_n)_{n \in \mathbb{N}}$ be a sequence of covariance matrices, such that

$$\Delta_n \sup_{I \in \mathcal{I}(\lambda_n)} \sqrt{\tilde{\sigma}_n(I)} \lesssim (\sqrt{2} - \varepsilon_n) \sqrt{-\log \lambda_n}, \tag{5.3}$$

where $(\varepsilon_n)_{n \in \mathbb{N}}$ is a sequence that satisfies $\varepsilon_n \rightarrow 0$ and $\varepsilon_n \sqrt{-\log \lambda_n} \rightarrow \infty$ as $n \rightarrow \infty$. In addition, assume that for some $m \in \mathbb{N}_0$

$$\lim_{n \rightarrow \infty} \frac{1}{\lfloor \lambda_n^{-1} \rfloor^2} \sum_{\substack{I, I' \in \mathcal{I}^0 \\ n | \inf I - \inf I' | > m}} \text{Cov}(L_{I, \Delta_n}, L_{I', \Delta_n}) = 0, \tag{5.4}$$

as $n \rightarrow \infty$.

Then any sequence of tests $(\Phi_n)_{n \in \mathbb{N}}$ with $\limsup_{n \rightarrow \infty} \bar{\alpha}(\Phi_n) \leq \alpha$ will obey $\limsup_{n \rightarrow \infty} \bar{\beta}(\Phi_n) \geq 1 - \alpha$, i.e. the bump is asymptotically undetectable.

Proof. We employ the same strategy as in the proof of Theorem 3.1(a) of Dümbgen and Spokoiny [11]. We bound the type II error of any given test by an expression that does not depend on the test anymore, and then employ an appropriate L^1 -law of large numbers for dependent arrays of random variables.

For any sequence of tests Φ_n with asymptotic level α under H_0 we have

$$\begin{aligned} \bar{\beta}(\Phi_n) &= \sup_{I \in \mathcal{I}(\lambda_n)} \sup_{|\delta| \geq \Delta_n} \mathbb{E}_{I, \delta}[1 - \Phi_n(Y)] \geq \sup_{I \in \mathcal{I}^0} \sup_{|\delta| \geq \Delta_n} \mathbb{E}_{I, \delta}[1 - \Phi_n(Y)] \\ &\geq \frac{1}{\lfloor \lambda_n^{-1} \rfloor} \sum_{I \in \mathcal{I}^0} \sup_{|\delta| \geq \Delta_n} \mathbb{E}_{I, \delta}[1 - \Phi_n(Y)] \geq 1 - \frac{1}{\lfloor \lambda_n^{-1} \rfloor} \sum_{I \in \mathcal{I}^0} \mathbb{E}_{I, \Delta_n}[\Phi_n(Y)] \\ &\geq 1 - \frac{1}{\lfloor \lambda_n^{-1} \rfloor} \sum_{I \in \mathcal{I}^0} \mathbb{E}_0 \left[\Phi_n(Y) \frac{d\mathbb{P}_{I, \Delta_n}}{d\mathbb{P}_0} - \Phi_n(Y) \right] - \alpha + o(1) \\ &= 1 - \mathbb{E}_0 \left[\left(\frac{1}{\lfloor \lambda_n^{-1} \rfloor} \sum_{I \in \mathcal{I}^0} L_{I, \Delta_n} - 1 \right) \Phi_n(Y) \right] - \alpha + o(1) \\ &\geq 1 - \alpha - \mathbb{E}_0 \left| \frac{1}{\lfloor \lambda_n^{-1} \rfloor} \sum_{I \in \mathcal{I}^0} L_{I, \Delta_n} - 1 \right| + o(1). \end{aligned}$$

Next, we show that the array $\{L_{\Delta_n, I} : I \in \mathcal{I}^0, n \in \mathbb{N}\}$ is h -integrable with exponent 1 (see Definition 6.1 in the Supplementary Material [13] or Definition 1.5 in Sung, Lisawadi and Volodin[37]), i.e. we show that

$$\begin{aligned} \sup_{n \in \mathbb{N}} \frac{1}{\lfloor \lambda_n^{-1} \rfloor} \sum_{I \in \mathcal{I}^0} \mathbb{E}_0[|L_{I, \Delta_n}|] &< \infty \quad \text{and} \\ \lim_{n \rightarrow \infty} \frac{1}{\lfloor \lambda_n^{-1} \rfloor} \sum_{I \in \mathcal{I}^0} \mathbb{E}_0[|L_{I, \Delta_n}| \mathbf{1}\{|L_{I, \Delta_n}| > h(n)\}] &= 0, \end{aligned} \tag{5.5}$$

where $h(n) = \lfloor \lambda_n^{-1} \rfloor^{\frac{1}{2}(1+\varepsilon_n)(\sqrt{2}-\varepsilon_n)^2}$. Since $\mathbb{E}_0|L_{I, \Delta_n}| = 1$ for all $n \in \mathbb{N}$ and $I \in \mathcal{I}^0$, the first condition is satisfied.

Further, if n large is enough, we have

$$\begin{aligned} \frac{1}{\lfloor \lambda_n^{-1} \rfloor} \sum_{I \in \mathcal{I}^0} \mathbb{E}_0[L_{I, \Delta_n} \mathbf{1}\{L_{I, \Delta_n} > h(n)\}] &\leq \sup_{I \in \mathcal{I}^0} \mathbb{E}_0[L_{I, \Delta_n} \mathbf{1}\{L_{I, \Delta_n} > h(n)\}] \\ &= \sup_{I \in \mathcal{I}^0} \mathbb{P}\left(Z \leq \frac{\frac{1}{2} \Delta_n^2 \tilde{\sigma}_n(I) - \log h(n)}{\Delta_n \sqrt{\tilde{\sigma}_n(I)}}\right) \\ &\leq \mathbb{P}\left(Z \leq \sup_{I \in \mathcal{I}^0} \frac{1}{2} \Delta_n \sqrt{\tilde{\sigma}_n(I)} - \frac{\log h(n)}{\sup_{I \in \mathcal{I}^0} \Delta_n \sqrt{\tilde{\sigma}_n(I)}}\right) \\ &\stackrel{(a)}{\leq} \mathbb{P}(Z \leq -\varepsilon_n(\sqrt{2} - \varepsilon_n)\sqrt{-\log \lambda_n}), \end{aligned}$$

where Z is a standard Gaussian random variable. The inequality (a) follows immediately from (5.3) and the definition of $h(n)$. The claim follows from the assumption that $\lim_{n \rightarrow \infty} \varepsilon_n \times \sqrt{-\log \lambda_n} = \infty$ as $n \rightarrow \infty$.

Then, given that (5.4) and (5.5) hold, it follows from an L^1 -law of large numbers for dependent arrays (see Theorem 6.2 in the Supplementary Material [13] or Theorem 3.2 of Wang and Hu [40]), that

$$\mathbb{E}_0 \left| \frac{1}{\lfloor \lambda_n^{-1} \rfloor} \sum_{I \in \mathcal{I}^0} L_{\Delta_n, I} - 1 \right| \rightarrow 0, \tag{5.6}$$

as $n \rightarrow \infty$, which finishes the proof. □

5.1.3. Proof of Theorem 2.1

In the setting described in Theorem 2.1 the noise vector ξ_n in model (1.1) is given by n consecutive realizations of a stationary centered Gaussian process with the square summable autocovariance function $\gamma(h)$, $h \in \mathbb{Z}$ and the spectral density f . We suppose that Assumption 2 is satisfied, i.e. the autocovariance of ξ_n has a polynomial decay. In terms of Σ_n , this means

$$|\Sigma_n(i, j)| \leq C(1 + |i - j|)^{-(1+\kappa)}$$

for $1 \leq i, j \leq n$ and some constants $C > 0$ and $\kappa > 0$.

In order to apply Lemma 5.2 in such a setting, first, we need to examine the asymptotic behavior of the coefficients $\tilde{\sigma}_n(I)$, and second, we need to verify that condition (5.4) is satisfied under the lower detection boundary condition (5.3) and Assumption 2.

For the setting of Theorem 2.1, we will do the former in Lemma 5.3 and the latter in Lemma 5.4.

Lemma 5.3. *If Assumption 2 holds, then for any $I \in \mathcal{I}(\lambda_n)$, it follows that*

$$\tilde{\sigma}_n(I) = \frac{n\lambda_n}{f(0)}(1 + o(1)),$$

as $n \rightarrow \infty$.

The proof of Lemma 5.3 is very similar to the proof of Proposition C.1 in Keshavarz, Scott and Nguyen [26], and can be found in the Supplementary Material [13].

Lemma 5.4. *If Assumption 2 holds, and given that*

$$\Delta_n \sup_{I \in \mathcal{I}(\lambda_n)} \sqrt{\tilde{\sigma}_n(I)} \lesssim (\sqrt{2} - \varepsilon_n) \sqrt{-\log \lambda_n} \tag{5.7}$$

for a sequence $(\varepsilon_n)_{n \in \mathbb{N}}$ satisfying $\varepsilon_n \rightarrow 0$ and $\varepsilon_n \sqrt{-\log \lambda_n} \rightarrow \infty$ as $n \rightarrow \infty$, then condition (5.4) holds with $m = 1$, i.e.

$$\lim_{n \rightarrow \infty} \lambda_n^2 \sum_{\substack{I, I' \in \mathcal{I}^0 \\ n|\inf I - \inf I'| > 1}} \exp(\Delta_n^2 \mathbf{1}_I^T \Sigma_n^{-1} \mathbf{1}_{I'}) - 1 = 0.$$

Proof. For $I, I' \in \mathcal{I}^0$ with $n|\inf I - \inf I'| > 1$. Write

$$\begin{aligned} \exp(\Delta_n^2 \mathbf{1}_I^T \Sigma_n^{-1} \mathbf{1}_{I'}) - 1 &= \sum_{p=1}^{\infty} \frac{1}{p!} [\Delta_n^2 \mathbf{1}_I^T \Sigma_n^{-1} \mathbf{1}_{I'}]^p \\ &= \sum_{p=1}^{\infty} \frac{1}{p!} \left[\frac{1}{2} \Delta_n^2 \sqrt{\tilde{\sigma}_n(I) \tilde{\sigma}_n(I')} \right]^p \left[2 \frac{\mathbf{1}_I^T \Sigma_n^{-1} \mathbf{1}_{I'}}{\sqrt{\tilde{\sigma}_n(I) \tilde{\sigma}_n(I')}} \right]^p. \end{aligned}$$

If $n\lambda_n$ is an integer, the latter term $\mathbf{1}_I^T \Sigma_n^{-1} \mathbf{1}_{I'}$ is the sum over a square submatrix of Σ_n^{-1} , and if $n\lambda_n$ is not an integer, then the number of non-zero entries of $\mathbf{1}_I$ and $\mathbf{1}_{I'}$ cannot differ by more than 1. From Lemma A.1 of [19] (see also Section 6.1.2 in the Supplementary Material [13]), it trivially follows that

$$\begin{aligned} |\mathbf{1}_I^T \Sigma_n^{-1} \mathbf{1}_{I'}| &\leq C' \lceil n\lambda_n \rceil \sum_{t=1}^{\lceil n\lambda_n \rceil} (n|\inf I - \inf I'| \lfloor n\lambda_n \rfloor + t)^{-(1+\kappa)} \\ &\leq C' \lceil n\lambda_n \rceil \sum_{t=1}^{\lceil n\lambda_n \rceil} (\lfloor n\lambda_n \rfloor)^{-(1+\kappa)} = o(n\lambda_n). \end{aligned}$$

From Lemma 5.3, we know that $\sqrt{\tilde{\sigma}_n(I) \tilde{\sigma}_n(I')} = \frac{n\lambda_n}{f(0)} (1 + o(1))$ as $n \rightarrow \infty$, and thus, it follows that $\sqrt{\tilde{\sigma}_n(I) \tilde{\sigma}_n(I')}^{-1} \mathbf{1}_I^T \Sigma_n^{-1} \mathbf{1}_{I'} \rightarrow 0$ as $n \rightarrow \infty$. Hence, for n large enough, we have

$$\begin{aligned} &\left| \sum_{p=1}^{\infty} \frac{1}{p!} \left[\frac{1}{2} \Delta_n^2 \sqrt{\tilde{\sigma}_n(I) \tilde{\sigma}_n(I')} \right]^p \left[2 \frac{\mathbf{1}_I^T \Sigma_n^{-1} \mathbf{1}_{I'}}{\sqrt{\tilde{\sigma}_n(I) \tilde{\sigma}_n(I')}} \right]^p \right| \\ &\leq 2 \frac{|\mathbf{1}_I^T \Sigma_n^{-1} \mathbf{1}_{I'}|}{\sqrt{\tilde{\sigma}_n(I) \tilde{\sigma}_n(I')}} \exp \left[\frac{1}{2} \Delta_n^2 \sqrt{\tilde{\sigma}_n(I) \tilde{\sigma}_n(I')} \right]. \end{aligned} \tag{5.8}$$

Note that from the lower detection boundary condition (5.7) it immediately follows that

$$\exp\left[\frac{1}{2}\Delta_n^2\sqrt{\tilde{\sigma}_n(I)\tilde{\sigma}_n(I')}\right] \leq \lambda_n^{-\frac{1}{2}(\sqrt{2}-\varepsilon_n)^2} \leq \lambda_n^{-1} \quad (5.9)$$

for n large enough. Applying Lemma A.1 of [19] again, it follows that $|\Sigma_n^{-1}(i, j)| \leq C(1 + |i - j|)^{-(1+\kappa)}$ for some $C > 0$. Let Φ_n be the $n \times n$ -matrix with entries $\Phi_n(i, j) = C(1 + |i - j|)^{-(1+\kappa)}$, and let $\Phi(v) = \sum_{h=-\infty}^{\infty} C(1 + |i - j|)^{-(1+\kappa)} e^{-2\pi i h v}$. Then

$$\sum_{\substack{I, I' \in \mathcal{I}^0 \\ n|\inf I - \inf I'| > 1}} |\mathbf{1}_I^T \Sigma_n^{-1} \mathbf{1}_{I'}| \leq \sum_{\substack{I, I' \in \mathcal{I}^0 \\ I \neq I'}} \mathbf{1}_I^T \Phi_n \mathbf{1}_{I'} \leq \sum_{i, j=1}^n \Phi_n(i, j) - \sum_{I \in \mathcal{I}^0} \mathbf{1}_I^T \Phi_n \mathbf{1}_I \stackrel{(a)}{=} o(n), \quad (5.10)$$

where (a) follows from Theorem 18.2.1 of Ibragimov and Linnik [22], since it yields that $\sum_{i, j=1}^n \Phi_n(i, j) = n\Phi(0) + o(n)$ and $\mathbf{1}_I^T \Phi_n \mathbf{1}_I = n\lambda_n\Phi(0) + o(n\lambda_n)$ for any $I \in \mathcal{I}^0$.

Finally, combining (5.8), (5.9) and (5.10), and once again using that $\sqrt{\tilde{\sigma}_n(I)\tilde{\sigma}_n(I')} = \frac{n\lambda_n}{f(0)}(1 + o(1))$ as $n \rightarrow \infty$, we find

$$\sum_{\substack{I, I' \in \mathcal{I}^0 \\ n|\inf I - \inf I'| > 1}} 2 \frac{|\mathbf{1}_I^T \Sigma_n^{-1} \mathbf{1}_{I'}|}{\sqrt{\tilde{\sigma}_n(I)\tilde{\sigma}_n(I')}} \exp\left[\frac{1}{2}\Delta_n^2\sqrt{\tilde{\sigma}_n(I)\tilde{\sigma}_n(I')}\right] = o\left(\frac{1}{\lambda_n^2}\right),$$

which concludes the proof. \square

Since Lemma 5.4 guarantees that Lemma 5.2 can be applied in the setting of Theorem 2.1, the proof of the latter now follows immediately from Lemmas 5.1 and 5.2.

Proof of Theorem 2.1. The two Lemmas 5.1 and 5.2 yield that the asymptotic detection boundary is (in terms of Δ_n) given by

$$(\sqrt{2} - \varepsilon_n) \sqrt{\frac{-\log \lambda_n}{n\lambda_n}} \sup_{I \in \mathcal{I}(\lambda_n)} \sqrt{\frac{n\lambda_n}{\tilde{\sigma}_n(I)}} \lesssim \Delta_n \lesssim (\sqrt{2} + \tilde{\varepsilon}_n) \sqrt{\frac{-\log \lambda_n}{n\lambda_n}} \inf_{I \in \mathcal{I}(\lambda_n)} \sqrt{\frac{\sigma_n(I)}{n\lambda_n}}, \quad (5.11)$$

as $n \rightarrow \infty$. For any $I \in \mathcal{I}(\lambda_n)$, it follows from Theorem 18.2.1 of Ibragimov and Linnik [22] (see Section 6.1.3 of the Supplementary Material [13]) that $\sigma_n(I) = n\lambda_n f(0)(1 + o(1))$, and Lemma 5.3 yields

$$\tilde{\sigma}_n(I) = \frac{n\lambda_n}{f(0)}(1 + o(1)), \quad n \rightarrow \infty.$$

Plugging this into (5.11) finishes the proof. \square

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Supplementary Material

Supplementary material and additional proofs (DOI: [10.3150/20-BEJ1226SUPP](https://doi.org/10.3150/20-BEJ1226SUPP); .pdf). In this material [13] we provide all remaining proofs and recall several results that are necessary for our theory.

References

- [1] Aue, A. and Horváth, L. (2013). Structural breaks in time series. *J. Time Series Anal.* **34** 1–16. MR3008012 <https://doi.org/10.1111/j.1467-9892.2012.00819.x>
- [2] Baraud, Y. (2002). Non-asymptotic minimax rates of testing in signal detection. *Bernoulli* **8** 577–606. MR1935648
- [3] Boysen, L., Kempe, A., Liebscher, V., Munk, A. and Wittich, O. (2009). Consistencies and rates of convergence of jump-penalized least squares estimators. *Ann. Statist.* **37** 157–183. MR2488348 <https://doi.org/10.1214/07-AOS558>
- [4] Brockwell, P.J. and Davis, R.A. (2006). *Time Series: Theory and Methods*. Springer Series in Statistics. New York: Springer. MR2839251
- [5] Brodsky, B.E. and Darkhovsky, B.S. (1993). *Nonparametric Methods in Change-Point Problems. Mathematics and Its Applications* **243**. Dordrecht: Kluwer Academic. MR1228205 <https://doi.org/10.1007/978-94-015-8163-9>
- [6] Carlstein, E., Müller, H.-G. and Siegmund, D., eds. (1994). *Change-Point Problems. Institute of Mathematical Statistics Lecture Notes—Monograph Series* **23**. Hayward, CA: IMS. MR1477909
- [7] Chakar, S., Lebarbier, E., Lévy-Leduc, C. and Robin, S. (2017). A robust approach for estimating change-points in the mean of an AR(1) process. *Bernoulli* **23** 1408–1447. MR3606770 <https://doi.org/10.3150/15-BEJ782>
- [8] Chan, H.P. and Walther, G. (2013). Detection with the scan and the average likelihood ratio. *Statist. Sinica* **23** 409–428. MR3076173
- [9] Csörgő, M. and Horváth, L. (1997). *Limit Theorems in Change-Point Analysis*. Wiley Series in Probability and Statistics. Chichester: Wiley. MR2743035
- [10] Dette, H., Schöler, T. and Vetter, M. (2018). Multiscale change point detection for dependent data. Preprint. Available at arXiv:1811.05956.
- [11] Dümbgen, L. and Spokoiny, V.G. (2001). Multiscale testing of qualitative hypotheses. *Ann. Statist.* **29** 124–152. MR1833961 <https://doi.org/10.1214/aos/996986504>
- [12] Dümbgen, L. and Walther, G. (2008). Multiscale inference about a density. *Ann. Statist.* **36** 1758–1785. MR2435455 <https://doi.org/10.1214/07-AOS521>
- [13] Enikeeva, F., Munk, A., Pohlmann, M. and Werner, F. (2020). Supplement to “Dependent bump detection in the presence of dependency: Does it ease or does it load?” <https://doi.org/10.3150/20-BEJ1226SUPP>
- [14] Enikeeva, F., Munk, A. and Werner, F. (2018). Bump detection in heterogeneous Gaussian regression. *Bernoulli* **24** 1266–1306. MR3706794 <https://doi.org/10.3150/16-BEJ899>
- [15] Frick, K., Munk, A. and Sieling, H. (2014). Multiscale change point inference. *J. R. Stat. Soc. Ser. B. Stat. Methodol.* **76** 495–580. MR3210728 <https://doi.org/10.1111/rssb.12047>

- [16] Giné, E. and Nickl, R. (2016). *Mathematical Foundations of Infinite-Dimensional Statistical Models*. Cambridge Series in Statistical and Probabilistic Mathematics **40**. New York: Cambridge Univ. Press. MR3588285 <https://doi.org/10.1017/CBO9781107337862>
- [17] Goldenshluger, A., Juditsky, A. and Nemirovski, A. (2015). Hypothesis testing by convex optimization. *Electron. J. Stat.* **9** 1645–1712. MR3379005 <https://doi.org/10.1214/15-EJS1054>
- [18] Gut, A. (1992). The weak law of large numbers for arrays. *Statist. Probab. Lett.* **14** 49–52. MR1172289 [https://doi.org/10.1016/0167-7152\(92\)90209-N](https://doi.org/10.1016/0167-7152(92)90209-N)
- [19] Hall, P. and Jin, J. (2010). Innovated higher criticism for detecting sparse signals in correlated noise. *Ann. Statist.* **38** 1686–1732. MR2662357 <https://doi.org/10.1214/09-AOS764>
- [20] Hotz, T., Schütte, O.M., Sieling, H., Polupanow, T., Diederichsen, U., Steinem, C. and Munk, A. (2013). Idealizing ion channel recordings by a jump segmentation multiresolution filter. *IEEE Trans. Nanobiosci.* **12** 376–386. <https://doi.org/10.1109/TNB.2013.2284063>
- [21] Ibragimov, I.A. and Has'minskii, R.Z. (2013). *Statistical Estimation: Asymptotic Theory*. Berlin: Springer.
- [22] Ibragimov, I.A. and Linnik, Yu.V. (1971). *Independent and Stationary Sequences of Random Variables*. Groningen: Wolters-Noordhoff Publishing. MR0322926
- [23] Ingster, Yu.I. (1993). Asymptotically minimax hypothesis testing for nonparametric alternatives. II. *Math. Methods Statist.* **2** 171–189. MR1257983
- [24] Ingster, Yu.I. and Suslina, I.A. (2003). *Nonparametric Goodness-of-Fit Testing Under Gaussian Models. Lecture Notes in Statistics* **169**. New York: Springer. MR1991446 <https://doi.org/10.1007/978-0-387-21580-8>
- [25] Jeng, X.J., Cai, T.T. and Li, H. (2010). Optimal sparse segment identification with application in copy number variation analysis. *J. Amer. Statist. Assoc.* **105** 1156–1166. MR2752611 <https://doi.org/10.1198/jasa.2010.tm10083>
- [26] Keshavarz, H., Scott, C. and Nguyen, X.L. (2015). Optimal change point detection in Gaussian processes. Preprint. Available at [arXiv:1506.01338](https://arxiv.org/abs/1506.01338).
- [27] Keshavarz, H., Scott, C. and Nguyen, X. (2018). Optimal change point detection in Gaussian processes. *J. Statist. Plann. Inference* **193** 151–178. MR3713470 <https://doi.org/10.1016/j.jspi.2017.09.003>
- [28] Krivobokova, T., Briones, R., Hub, J.S., Munk, A. and de Groot, B.L. (2012). Partial least-squares functional mode analysis: Application to the membrane proteins AQP1, Aqy1, and CLC-ec1. *Biophys. J.* **103** 786–796.
- [29] Neher, E. and Sakmann, B. (1995). *Single-Channel Recording*. New York: Plenum Press.
- [30] Ordóñez Cabrera, M. and Volodin, A.I. (2005). Mean convergence theorems and weak laws of large numbers for weighted sums of random variables under a condition of weighted integrability. *J. Math. Anal. Appl.* **305** 644–658. MR2131528 <https://doi.org/10.1016/j.jmaa.2004.12.025>
- [31] Pein, F., Sieling, H. and Munk, A. (2017). Heterogeneous change point inference. *J. R. Stat. Soc. Ser. B. Stat. Methodol.* **79** 1207–1227. MR3689315 <https://doi.org/10.1111/rssb.12202>
- [32] Pein, F., Tecuapetla-Gomez, I., Schutte, O.M., Steinem, C. and Munk, A. (2018). Fully automatic multiresolution idealization for filtered ion channel recordings: Flickering event detection. *IEEE Trans. Nanobiosci.* **17** 300–320. <https://doi.org/10.1109/TNB.2018.2845126>
- [33] Shen, A. and Volodin, A. (2017). Weak and strong laws of large numbers for arrays of rowwise END random variables and their applications. *Metrika* **80** 605–625. MR3716607 <https://doi.org/10.1007/s00184-017-0618-z>
- [34] Siddiqui, M.M. (1958). On the inversion of the sample covariance matrix in a stationary autoregressive process. *Ann. Math. Stat.* **29** 585–588. MR0095568 <https://doi.org/10.1214/aoms/1177706636>
- [35] Siegmund, D. (1985). *Sequential Analysis: Tests and Confidence Intervals*. Springer Series in Statistics. New York: Springer. MR0799155 <https://doi.org/10.1007/978-1-4757-1862-1>

- [36] Singer, M., Krivobokova, T. and Munk, A. (2017). Kernel partial least squares for stationary data. *J. Mach. Learn. Res.* **18** Art. ID 123. [MR3725462](#)
- [37] Sung, S.H., Lisawadi, S. and Volodin, A. (2008). Weak laws of large numbers for arrays under a condition of uniform integrability. *J. Korean Math. Soc.* **45** 289–300. [MR2375136](#) <https://doi.org/10.4134/JKMS.2008.45.1.289>
- [38] Talagrand, M. (2005). *The Generic Chaining: Upper and Lower Bounds of Stochastic Processes*. *Springer Monographs in Mathematics*. Berlin: Springer. [MR2133757](#)
- [39] Tsybakov, A.B. (2009). *Introduction to Nonparametric Estimation*. *Springer Series in Statistics*. New York: Springer. [MR2724359](#) <https://doi.org/10.1007/b13794>
- [40] Wang, X. and Hu, S. (2014). Weak laws of large numbers for arrays of dependent random variables. *Stochastics* **86** 759–775. [MR3241006](#) <https://doi.org/10.1080/17442508.2013.879140>

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