

Sieving random iterative function systems

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It is known that backward iterations of independent copies of a contractive random Lipschitz function converge almost surely under mild assumptions. By a sieving (or thinning) procedure based on adding to the functions time and space components, it is possible to construct a scale invariant stochastic process. We study its distribution and paths properties. In particular, we show that it is càdlàg and has finite total variation. We also provide examples and analyse various properties of particular sieved iterative function systems including perpetuities and infinite Bernoulli convolutions, iterations of maximum, and random continued fractions.

Keywords: infinite Bernoulli convolutions; iteration; perpetuity; random Lipschitz function; scale invariant process; sieving; thinning

1. Introduction

Iteration is one of fundamental tools in mathematics going back to famous fixed point theorems for contractive mappings. In probabilistic setting, one often works with iterated independent identically distributed (i.i.d.) Lipschitz functions $(f_i)_{i \in \mathbb{N}}$ defined on a complete separable metric space and study the convergence of either backward $f_1(f_2(f_3(\cdots f_n(\cdot))))$ or forward $f_n(f_{n-1}(f_{n-2}(\cdots f_1(\cdot))))$ iterations as $n \rightarrow \infty$. An incomplete list of early works on random iterations include [8,10,12,13,26] and references therein. A comprehensive study of convergence regimes for contractive (a precise definition will be given below) iterated random functions goes back to the prominent paper by Diaconis and Freedman [11].

An important special case of iterated random affine mappings (called stochastic perpetuities) was studied in [4,19,20,24] and in many other works. The recent books [7,22] provide more comprehensive lists of further references. A particular instance of perpetuities, infinite Bernoulli convolutions, have been attracting enormous attention since 1930th, see, for example, [1,15,16,29,30,32].

We recall the main setting, restricting ourselves to the case of Lipschitz functions on the real line \mathbb{R} . Let \mathcal{G} be the space of Lipschitz functions $f : \mathbb{R} \mapsto \mathbb{R}$ endowed with the usual Lipschitz norm $\|f\|_{\text{Lip}} := |f(0)| + L_f$, where

$$L_f := \sup_{x, y \in \mathbb{R}, x \neq y} \frac{|f(y) - f(x)|}{|x - y|}$$

is the Lipschitz constant of $f \in \mathcal{G}$. The composition of functions $f \circ g$ defined by $(f \circ g)(x) := f(g(x))$ for $x \in \mathbb{R}$ endows \mathcal{G} with the semigroup structure and is continuous with respect to $\|\cdot\|_{\text{Lip}}$.

Equip \mathcal{G} with a probability measure ν on the Borel σ -algebra of \mathcal{G} . Since the composition operation is continuous, the composition of two \mathcal{G} -measurable functions is again \mathcal{G} -measurable. If f is a random function with distribution ν such that

$$K_f := \mathbf{E}L_f = \int_{\mathcal{G}} L_f \, d\nu(f) < \infty, \quad \mathbf{E} \log L_f = \int_{\mathcal{G}} \log L_f \, d\nu(f) < 0, \quad (1)$$

and

$$\mathbf{E}|f(z_0) - z_0| = \int_{\mathcal{G}} |f(z_0) - z_0| dv(f) < \infty \quad (2)$$

for some $z_0 \in \mathbb{R}$, then the sequence of backward iterations

$$Z_n := f_1 \circ \cdots \circ f_n(z_0) \quad (3)$$

converges almost surely as $n \rightarrow \infty$ and the limit Z_∞ does not depend on the choice of z_0 , see Theorem 1 and Proposition 1 in [11]. From this, one deduces that the sequence of forward iterations $f_n \circ \cdots \circ f_1(z_0)$ converges in distribution to Z_∞ , see Theorems 1.1 and 5.1 in [11]. Furthermore, the limiting random variable Z_∞ satisfies the stochastic fixed-point equation

$$Z_\infty \stackrel{d}{=} f(Z_\infty), \quad (4)$$

where f and Z_∞ on the right-hand side are independent.

Many important distributions appear as limits for random iterated functions. This work aims to extend this construction in order to come up with stochastic processes (in general, set-indexed) whose univariate distributions arise from iterations and joint distributions are related by leaving some iterations out. For instance, assume that each of the functions f_i is associated with a uniformly distributed random variable U_i and is deleted from the iteration chain in (3) if U_i exceeds a given number x . The limit of such iterations is a random variable $\zeta(x)$ whose distribution is the same as that of Z_∞ . The properties of $\zeta(x)$ considered a random function of x is the main subject of this paper.

As a preparation to a general construction of such stochastic processes presented in Section 2, we shall provide a few examples.

Example 1.1. Consider an infinite sequence $(Q_n)_{n \in \mathbb{N}}$ of independent copies of a random variable Q taking values 0 or 1 equally likely. For $\lambda \in (0, 1)$, the Bernoulli convolution

$$Z_\infty := \sum_{n=1}^{\infty} \lambda^{n-1} Q_n$$

results from the backward iteration of independent copies of the function $f(z) = \lambda z + Q$. Now consider a sequence $(U_n)_{n \in \mathbb{N}}$ of i.i.d. uniform random variables on $[0, 1]$ which is independent of $(Q_n)_{n \in \mathbb{N}}$. Put $T_k(x) := \sum_{j=1}^k \mathbb{1}_{\{U_j \leq x\}}$, $k \in \mathbb{N}$, $x \in (0, 1]$, where $\mathbb{1}_{\{x_i \in A\}}$ is the indicator of the event $\{x_i \in A\}$, and further $S_n(x) := \inf\{k \in \mathbb{N} : T_k(x) = n\}$, $n \in \mathbb{N}$, $x \in (0, 1]$. Let

$$\zeta(x) := \sum_{n=1}^{\infty} \lambda^{n-1} Q_{S_n(x)}, \quad x \in (0, 1].$$

This yields a stochastic process, whose univariate marginals are all distributed like Z_∞ . We will explore path properties of this process, show that for $\lambda \in (0, 1/2]$ it is Markov in both forward and reverse time and find its generating operator. It is well known that if $\lambda = 1/2$, then $\zeta(x)$ is uniformly distributed on $[0, 2]$ for every $x \in (0, 1]$. We show that the bivariate distributions are singular for some x close enough to 1, determine a bound on their Hausdorff dimension and calculate the local dimension on the set of binary rational points.

Example 1.2. Generalising the previous example, consider a sequence $(Z_n)_{n \in \mathbb{N}}$ of backward iterations of affine mappings $f_n(x) = M_n x + Q_n$, $n \in \mathbb{N}$, applied to the initial point $z_0 = 0$, where $(M_n, Q_n)_{n \in \mathbb{N}}$

are i.i.d. random vectors in \mathbb{R}^2 . A criterion for a.s. convergence of (Z_n) is known, see [19], Th. 2.1. In particular, by [19], Cor. 4.1, convergence takes place whenever $\mathbf{E} \log |M| \in (-\infty, 0)$, $\mathbf{E} \log^+ |Q| < \infty$, where $\log^+ x := \log(x \vee 1)$, and an additional nondegeneracy assumption, see formula (38) below, holds. Let $(U_n)_{n \in \mathbb{N}}$ be i.i.d. uniform random variables on $[0, 1]$ which are independent of $(M_n, Q_n)_{n \in \mathbb{N}}$. Consider a coupled family of processes

$$\zeta(x) := \sum_{n=1}^{\infty} \left(\prod_{k=1}^{n-1} M_k^{\mathbb{1}_{\{U_k \leq x\}}} \right) Q_n \mathbb{1}_{\{U_n \leq x\}}, \quad x \in (0, 1].$$

We establish the uniform convergence of partial sums of the above series to the limit $\zeta(x)$ and explore its path properties.

Example 1.3. Consider a continued fraction $W_n = \frac{1}{W_{n-1} + a_n}$ with (possibly i.i.d. random) coefficients $a_n > 0$, $n \in \mathbb{N}$. If $\sum a_n = \infty$ a.s., the continued fraction converges in distribution by the Stern–Stolz theorem, see Theorem 10 in [25]. Given once again a sequence $(U_n)_{n \in \mathbb{N}}$ of i.i.d. uniform random variables on $[0, 1]$ which is independent of $(a_n)_{n \in \mathbb{N}}$, we modify the continued fraction by letting

$$W_n(x) = \begin{cases} \frac{1}{W_{n-1}(x) + a_n}, & \text{if } U_n \leq x, \\ W_{n-1}(x), & \text{if } U_n > x, \end{cases} \quad x \in (0, 1].$$

Note that for every fixed $x \in (0, 1]$, $W_n(x)$ is the forward iteration of the mappings

$$f_{n,x}(z) = \frac{1}{a_n + z} \mathbb{1}_{\{U_n \leq x\}} + z \mathbb{1}_{\{U_n > x\}}, \quad z > 0.$$

The a.s. pointwise limits of the corresponding backward iterations $Z_n(x)$ as $n \rightarrow \infty$ constitute a stochastic process on $(0, 1]$. We show that this process has a finite total variation, and is Markov if $(a_n)_{n \in \mathbb{N}} \subset \mathbb{N}$.

Note that in all above examples we eliminate some iterations from the infinite sequence

$$Z_\infty = f_1 \circ f_2 \circ \cdots \circ f_n \circ \cdots$$

by replacing the corresponding functions with the identity mapping in a coupled manner. In Section 2, we suggest a sieving scheme for iterated functions, which is generated by an auxiliary Poisson point process. As a result, we are led to a set-indexed stochastic process whose univariate marginals are all the same and are distributed as the almost sure limit of Z_n in (3). By taking its values on the segments $[x, 1]$ with $x \in (0, 1]$, we obtain all constructions mentioned in the above examples as special cases.

The distributional properties of the set-indexed process are analysed in Section 3, in particular, it is shown that a variant of this process on the half-line is scale invariant. By restricting the process to a finite interval, it is possible to rephrase our sieving scheme as iteration in a functional space. With this idea, in Section 4 we use tools from the theory of empirical processes to establish the uniform convergence on some classes of sets. In Section 5, it is shown that the limiting process ζ is càdlàg and has a finite total variation on any bounded interval separated from zero. We also discuss integration with respect to ζ and integrability properties of ζ . Section 6 establishes the Markov property of the process, assuming a kind of a strong separation condition known in fractal geometry.

The most well-studied family of iterations are perpetuities, also known as autoregressive processes of the first order, see Example 1.2 above. The sieving scheme is applied to them in Section 7, where also

an important example of Bernoulli convolutions is considered, see Example 1.1. Section 8 outlines several other instances of iterations that provide new examples of self-similar stochastic processes. In the [Appendix](#), we collect some technical proofs and provide several further examples related to perpetuities.

2. Sieving scheme for iterated functions

Let \mathcal{X} be a complete separable metric space with its Borel σ -algebra $\mathcal{B}(\mathcal{X})$ and equipped with a σ -finite measure μ . Recall that \mathcal{G} is the family of Lipschitz functions on the real line with a probability measure ν satisfying (1) and (2).

Let $\mathbb{R}_+ := [0, \infty)$ be the positive half-line with the Lebesgue measure Leb . Consider a Poisson process \mathcal{P} on $\mathbb{R}_+ \times \mathcal{X} \times \mathcal{G}$ with intensity measure $\text{Leb} \otimes \mu \otimes \nu$. Note that in a triplet $(t, x, f) \in \mathcal{P}$ the function f may be considered as a mark of the point (t, x) , the marks of different points are independent and ν is the probability distribution of the typical mark denoted by f . For $A \in \mathcal{B}(\mathcal{X})$, denote by \mathcal{P}_A the intersection of \mathcal{P} with $\mathbb{R}_+ \times A \times \mathcal{G}$.

For a sequence $(f_n)_{n \in \mathbb{N}}$ of i.i.d. random Lipschitz functions, write

$$f^{k \uparrow n} = f_k \circ \cdots \circ f_n,$$

and $f^{k \uparrow \infty}$ for the almost sure limit of these iterations as $n \rightarrow \infty$ provided it exists. For $k > n$, we stipulate that $f^{k \uparrow n}$ is the identity function Id .

For each $A \in \mathcal{B}(\mathcal{X})$ with $\mu(A) \in (0, \infty)$, enumerate the points $\{(t_{k,A}, x_{k,A}, f_{k,A}) : x_{k,A} \in A, k \geq 1\}$ of \mathcal{P}_A , so that the first component is a.s. increasing, and define the *sieved backward iterations* of $(f_{k,A})_{k \in \mathbb{N}}$:

$$\zeta_t(A) := f_A^{1 \uparrow N_A(t)}(z_0) = f_{1,A} \circ \cdots \circ f_{N_A(t),A}(z_0), \quad t > 0, \quad (5)$$

where $z_0 \in \mathbb{R}$ is fixed and nonrandom, and

$$N_A(t) := \sup\{k \geq 1 : t_{k,A} \leq t, x_{k,A} \in A\}$$

with the convention $\sup \emptyset = 0$. Thus, $\zeta_t(A)$ is a finite backward composition of marks f_k for (t_k, x_k) from the rectangle $[0, t] \times A$. Equivalently, $\zeta_t(A)$ is the composition of functions $f_i \mathbb{1}_{\{x_i \in A\}} + \text{Id} \mathbb{1}_{\{x_i \notin A\}}$ for $t_i \leq t$ applied to the starting point z_0 . This equivalent interpretation makes transparent the “sieved” structure of $\zeta_t(A)$ which might seem disguised in the definition (5).

In what follows, we always assume that conditions (1) and (2) are in force. Then $\zeta_t(A)$ in (5) converges almost surely as $t \rightarrow \infty$. The limiting random element is denoted by $\zeta(A)$; it is a random set-indexed function defined on

$$\mathcal{B}_+(\mathcal{X}) := \{A \in \mathcal{B}(\mathcal{X}) : \mu(A) \in (0, \infty)\}.$$

Furthermore, $\zeta(A)$ does not depend on the choice of z_0 .

If \mathcal{X} is the half-line $\mathbb{R}_+ = [0, \infty)$ and μ is the Lebesgue measure, then we work with a Poisson process on $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathcal{G}$, and, for $A = [0, x]$ with $x > 0$, the random variable $\zeta_t(A)$ is the result of iterating the functions f_i ordered according to $t_i \leq t$ and such that $x_i \leq x$. In this case, we write $\zeta(x)$ as a shorthand for $\zeta([0, x])$, $x > 0$, and regard $(\zeta(x))_{x > 0}$ as a stochastic process on $(0, \infty)$. Note that by passing from $\zeta(x)$ to $\zeta(y)$, we sieve some iterations out if $y < x$ and insert additional ones if $y > x$.

3. Distributional properties

3.1. Finite-dimensional distributions and scale invariance

Recall that $\zeta(A)$ is defined for $A \in \mathcal{B}_+(\mathcal{X})$, that is, for Borel A such that $\mu(A) \in (0, \infty)$. Note the following simple facts.

Proposition 3.1. *The distribution of $\zeta(A)$ does not depend on $A \in \mathcal{B}_+(\mathcal{X})$ and $\zeta(A) \stackrel{d}{=} Z_\infty$. If $\mathbf{P}\{Z_\infty = 0\} < 1$, the set function ζ is not additive, and hence is not a measure on \mathcal{X} . If $A_1 \cap A_2 = \emptyset$ for $A_1, A_2 \in \mathcal{B}_+(\mathcal{X})$, then $\zeta(A_1)$ and $\zeta(A_2)$ are independent.*

Theorem 3.2. *Let $\phi : \mathcal{X} \mapsto \mathcal{X}$ be any measurable bijection such that $\mu(\phi^{-1}(A)) = c\mu(A)$ for a constant $c > 0$ and all $A \in \mathcal{B}_+(\mathcal{X})$. Then $\zeta(\phi(A))$ and $\zeta(A)$ share the same finite-dimensional distributions as functions of $A \in \mathcal{B}_+(\mathcal{X})$.*

Proof. By the transformation theorem for Poisson processes, the process with intensity measure $\mu(\phi^{-1}(A))$ can be obtained as $(\phi(x_i))_{i \in \mathbb{N}}$, where $(x_i)_{i \in \mathbb{N}}$ is the Poisson process with intensity μ . Thus, $\zeta(\phi(A))$, $A \in \mathcal{B}_+(\mathcal{X})$, coincides with the limiting set-indexed process obtained by using Poisson points from the process of intensity $c\text{Leb} \otimes \mu \otimes \nu$. This process is obtained from the original one by transform $t_i \mapsto c^{-1}t_i$, which does not change the order of the t_i s and so the limit in (5). \square

It is possible to describe two-dimensional distributions of the set function ζ as follows. Let A_1, A_2 be two sets from $\mathcal{B}_+(\mathcal{X})$. Consider the triplet (t_*, x_*, f_*) such that t_* is the smallest among all triplets (t_i, x_i, f_i) with $x_i \in A_1 \cup A_2$. Then

$$\begin{aligned} (\zeta(A_1), \zeta(A_2)) &\stackrel{d}{=} (f_*(\zeta(A_1)), f_*(\zeta(A_2)))\mathbb{1}_{\{x_* \in A_1 \cap A_2\}} \\ &\quad + (\zeta(A_1), f_*(\zeta(A_2)))\mathbb{1}_{\{x_* \in A_2 \setminus A_1\}} + (f_*(\zeta(A_1)), \zeta(A_2))\mathbb{1}_{\{x_* \in A_1 \setminus A_2\}}. \end{aligned} \quad (6)$$

A similar equation can be written for the joint distribution of $(\zeta(A_1), \zeta(A_2), \dots, \zeta(A_m))$ for any $A_1, \dots, A_m \in \mathcal{B}_+(\mathcal{X})$.

In the special case of $\mathcal{X} = \mathbb{R}_+$ with μ being the Lebesgue measure, Theorem 3.2 yields that the finite-dimensional distributions of $(\zeta(x))_{x>0}$ do not change after scaling of its argument by any positive constant, meaning that $(\zeta(x))_{x>0}$ is scale invariant. After the exponential change of time, the process $\tilde{\zeta}(s) := \zeta(e^s)$, $s \in \mathbb{R}$, is strictly stationary on \mathbb{R} .

3.2. Power moments

Using known results for perpetuities, it is easy to deduce the following statement.

Proposition 3.3. *Assume that, for some $p > 0$, we have $\mathbf{E}L_f^p < 1$ and $\mathbf{E}|f(z_0) - z_0|^p < \infty$. Then $\mathbf{E}|\zeta(A)|^p < \infty$ for all $A \in \mathcal{B}_+(\mathcal{X})$.*

Proof. For every $A \in \mathcal{B}_+(\mathcal{X})$, the random variable $\zeta(A)$ has the same distribution as Z_∞ . By the triangle inequality

$$|f_i(z) - z_0| \leq |f_i(z_0) - z_0| + L_{f_i}|z - z_0|, \quad z \in \mathbb{R}, \quad (7)$$

and, therefore,

$$|f^{1\uparrow n}(z_0) - z_0| \leq \sum_{k=1}^n |f_k(z_0) - z_0| \prod_{j=1}^{k-1} L_{f_j} \quad \text{a.s., } n \in \mathbb{N}.$$

Letting $n \rightarrow \infty$ yields

$$|Z_\infty - z_0|^p \leq \left(\sum_{k=1}^{\infty} |f_k(z_0) - z_0| \prod_{j=1}^{k-1} L_{f_j} \right)^p \quad \text{a.s.}$$

The term in the parentheses on the right-hand side is a perpetuity. The criterion for existence of power moments of perpetuities is given in [4], Th. 1.4. In particular, under our assumptions the right-hand side of the last display is finite. The proof is complete. \square

Remark 3.4. The inequality $\mathbf{E}|Z_\infty|^p < \infty$ is stated under weaker assumptions in Theorem 2.3(d) in [3]. However, in the claimed generality this result does not hold, see the corrigendum [2] for a correct form which is weaker than Proposition 3.3.

If the conditions of Proposition 3.3 hold for $p = 2$, then $\zeta(A)$ is square integrable for all $A \in \mathcal{B}_+(\mathcal{X})$, and (6) leads to an iterative equation for the second moments of ζ as

$$\begin{aligned} & \mu(A_1 \cup A_2) \mathbf{E}(\zeta(A_1)\zeta(A_2)) \\ &= \mu(A_1 \cap A_2) \mathbf{E}(f(\zeta(A_1))f(\zeta(A_2))) \\ & \quad + \mu(A_1 \setminus A_2) \mathbf{E}(f(\zeta(A_1))(\zeta(A_2))) + \mu(A_2 \setminus A_1) \mathbf{E}(\zeta(A_1)f(\zeta(A_2))), \end{aligned} \quad (8)$$

where f is a random element in \mathcal{G} with distribution ν independent of $\zeta(A_1)$ and $\zeta(A_2)$. For processes on the half-line, (8) becomes

$$y \mathbf{E}(\zeta(x)\zeta(y)) = x \mathbf{E}(f(\zeta(x))f(\zeta(y))) + (y-x) \mathbf{E}(\zeta(x)f(\zeta(y))), \quad 0 < x \leq y,$$

where f is independent of $\zeta(x)$ and $\zeta(y)$.

4. Iterations in a finite measure space

4.1. Iterations in a functional space

Assume that μ is not identically zero and finite on \mathcal{X} , that is, $\mu(\mathcal{X}) \in (0, \infty)$. Then, the construction of the limiting process can be done as follows. Let $(f_i)_{i \in \mathbb{N}}$ be a sequence of i.i.d. copies of f from \mathcal{G} distributed according to ν , and let $(U_i)_{i \in \mathbb{N}}$ be i.i.d. copies of a random element $U \in \mathcal{X}$ with distribution

$$\mathbf{P}\{U \in A\} = \frac{\mu(A)}{\mu(\mathcal{X})}, \quad A \in \mathcal{B}(\mathcal{X}). \quad (9)$$

Assume further that $(f_i)_{i \in \mathbb{N}}$ and $(U_i)_{i \in \mathbb{N}}$ are independent.

Let $A \in \mathcal{B}_+(\mathcal{X})$. Then f_i contributes to the iterations constituting $\zeta(A)$ if $U_i \in A$, and otherwise f_i is replaced by the identity map. In other words, we have the following identity

$$\zeta(A) = f_A^{1\uparrow\infty}(z_0), \quad A \in \mathcal{B}_+(\mathcal{X}),$$

where the limit $f_A^{1\uparrow\infty}(z_0)$ of $f_A^{1\uparrow n}(z_0)$, as $n \rightarrow \infty$, is understood in the a.s. sense, $z_0 \in \mathbb{R}$, and $f_A^{1\uparrow n}$ are backward iterations of i.i.d. copies of the function

$$f_A(\cdot) := f(\cdot)\mathbb{1}_{\{U \in A\}} + \text{Id}(\cdot)\mathbb{1}_{\{U \notin A\}}. \quad (10)$$

The set function $(\zeta(A))_{A \in \mathcal{B}_+(\mathcal{X})}$ is the solution of the following iterative distributional equation

$$(\zeta(A))_{A \in \mathcal{B}_+(\mathcal{X})} \stackrel{\text{f.d.}}{=} (f(\zeta(A))\mathbb{1}_{\{U \in A\}} + \zeta(A)\mathbb{1}_{\{U \notin A\}})_{A \in \mathcal{B}_+(\mathcal{X})}, \quad (11)$$

where f , U and ζ on the right-hand side are independent. Note that the Lipschitz constant of f_A is

$$L_{f_A} = L_f \mathbb{1}_{\{U \in A\}} + \mathbb{1}_{\{U \notin A\}},$$

hence,

$$\log L_{f_A} = (\log L_f)\mathbb{1}_{\{U \in A\}}.$$

Example 4.1. Assume that $\mathcal{X} = [0, 1]$ with the Lebesgue measure. Then U has the standard uniform distribution on $[0, 1]$, and

$$\zeta(x) = f_x^{1\uparrow\infty}(z_0), \quad x \in (0, 1],$$

where the a.s. limit does not depend on $z_0 \in \mathbb{R}$, and $f_x^{1\uparrow\infty}$ are iterations composed of i.i.d. copies of the function

$$f_x(\cdot) := f(\cdot)\mathbb{1}_{\{U \leq x\}} + \text{Id}(\cdot)\mathbb{1}_{\{U > x\}}.$$

4.2. Uniform convergence of sieved iterations

We now aim to prove the uniform convergence of iterations as functions of Borel set A by reducing the problem to the uniform convergence of empirical processes. Let \mathcal{A} be a subclass of Borel sets in \mathcal{X} . A finite set I of cardinality n is shattered by \mathcal{A} if each of its 2^n subsets can be obtained as $I \cap A$ for some $A \in \mathcal{A}$. The Vapnik–Červonenkis dimension of \mathcal{A} is the supremum of cardinalities n of all finite sets I in \mathcal{X} shattered by \mathcal{A} . The family \mathcal{A} is called a Vapnik–Červonenkis class if its Vapnik–Červonenkis dimension is finite. We refer to the classical book [31], see in particular Section 4.9 therein, for the details of the Vapnik–Červonenkis theory.

Theorem 4.2. *Let \mathcal{A} be a collection of Borel subsets of \mathcal{X} with $\mu(\mathcal{X}) < \infty$ such that \mathcal{A} is a Vapnik–Červonenkis class and $\inf_{A \in \mathcal{A}} \mu(A) > 0$. Then*

$$\sup_{A \in \mathcal{A}} |\zeta(A) - f_A^{1\uparrow n}(z_0)| \xrightarrow{\text{a.s.}} 0 \quad \text{as } n \rightarrow \infty. \quad (12)$$

Lemma 4.3. *Assume that a family \mathcal{A} satisfies conditions of Theorem 4.2. Further, let $(\xi_k)_{k \in \mathbb{N}}$ be a sequence of i.i.d. copies of an integrable random variable ξ such that $\mathbf{E}\xi < 0$, and let $(U_k)_{k \in \mathbb{N}}$ be a sequence of i.i.d. copies of the random element U with distribution (9), which is also independent of $(\xi_k)_{k \in \mathbb{N}}$. Then*

$$\sup_{A \in \mathcal{A}} \frac{\sum_{k=1}^n \xi_k \mathbb{1}_{\{U_k \in A\}}}{n} \xrightarrow{\text{a.s.}} \frac{\inf_{A \in \mathcal{A}} \mu(A)}{\mu(\mathcal{X})} \mathbf{E}\xi < 0 \quad \text{as } n \rightarrow \infty. \quad (13)$$

Proof. Define a random measure (or abstract empirical process) on $\mathcal{B}(\mathcal{X})$ by

$$S_n(A) := \sum_{k=1}^n \phi_{nk}(A), \quad A \in \mathcal{B}(\mathcal{X}),$$

where

$$\phi_{nk}(A) := \frac{1}{n} \xi_k \mathbb{1}_{\{U_k \in A\}}, \quad k = 1, \dots, n, A \in \mathcal{B}(\mathcal{X}).$$

Note that $\mathbf{E}S_n(A) = \mu(A)\mathbf{E}\xi/\mu(\mathcal{X})$ does not depend on n . For a function $\psi : \mathcal{A} \mapsto \mathbb{R}$, denote $\|\psi\| := \sup_{A \in \mathcal{A}} |\psi(A)|$.

Assume first that $|\xi_i| \leq c$ a.s. Then

$$\mathbf{E} \left(\sum_{k=1}^n \|\phi_{nk}\| \right) \leq c, \quad n \geq 1.$$

For $\delta > 0$,

$$\mathbf{E} \left(\sum_{k=1}^n \mathbb{1}_{\{\|\phi_{nk}\| > \delta\}} \|\phi_{nk}\| \right) \leq \frac{c}{n} \mathbf{E} \left(\sum_{k=1}^n \mathbb{1}_{\{|\xi_k| > n\delta\}} \right) = c\mathbf{P}\{|\xi_k| > n\delta\} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

By results from the theory of empirical processes (see [14], Ch. 6),

$$\mathbf{E}\|S_n - \mathbf{E}S_n\| \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (14)$$

The sequence $S_n(A) - \mathbf{E}S_n(A)$ is a reverse martingale for each A , and so $\|S_n - \mathbf{E}S_n\|$ is a reverse submartingale which is bounded, since $|S_n(A) - \mathbf{E}S_n(A)| \leq 2c$ for all A and n . Thus,

$$\|S_n - \mathbf{E}S_n\| \xrightarrow{\text{a.s.}} 0 \quad \text{as } n \rightarrow \infty.$$

For a not necessarily bounded ξ_k , decompose the random measure as

$$S_n(A) = S'_n(A) + S''_n(A) = \sum_{k=1}^n \phi_{nk}(A) \mathbb{1}_{\{|\xi_k| \leq c\}} + \sum_{k=1}^n \phi_{nk}(A) \mathbb{1}_{\{|\xi_k| > c\}}.$$

Then $\|S'_n - \mathbf{E}S'_n\| \rightarrow 0$ a.s. as $n \rightarrow \infty$ by the argument above applicable to the bounded ξ_k s. Furthermore,

$$\|S''_n - \mathbf{E}S''_n\| \leq \frac{1}{n} \sum_{k=1}^n |\xi_k| \mathbb{1}_{\{|\xi_k| > c\}} + \frac{1}{n} \sum_{k=1}^n \mathbf{E}(|\xi_k| \mathbb{1}_{\{|\xi_k| > c\}}).$$

The a.s. upper limit of the right-hand side can be made arbitrarily small by the choice of c . Therefore,

$$\sup_{A \in \mathcal{A}} \frac{\sum_{k=1}^n \xi_k \mathbb{1}_{\{U_k \in A\}}}{n} \xrightarrow{\text{a.s.}} \sup_{A \in \mathcal{A}} \frac{\mu(A)}{\mu(\mathcal{X})} \mathbf{E} \xi = \frac{\inf_{A \in \mathcal{A}} \mu(A)}{\mu(\mathcal{X})} \mathbf{E} \xi \quad \text{as } n \rightarrow \infty,$$

because $\mathbf{E} \xi < 0$. □

Remark 4.4. It is possible to impose weaker conditions that guarantee the uniform convergence of $S_n(A)$ over $A \in \mathcal{A}$. Let $N(\varepsilon, \mathcal{A}, \hat{\rho}_n)$ be the cardinality of the smallest ε -net in \mathcal{A} with respect to the random pseudometric

$$\hat{\rho}_n(A, A') := \sum_{k=1}^n |\phi_{nk}(A) - \phi_{nk}(A')| = \frac{1}{n} \sum_{k=1}^n |\xi_k| \mathbb{1}_{\{U_k \in A \Delta A'\}}.$$

Then the Vapnik–Červonenkis class assumption may be replaced by the assumption that $N(\varepsilon, \mathcal{A}, \hat{\rho}_n)$ converges to zero in probability as $n \rightarrow \infty$. A comprehensive treaty of uniform convergence for empirical processes can be found in [31], Ch. 3. In particular, a necessary and sufficient conditions are given in Theorem 3.5 on p. 101 therein.

Proof of Theorem 4.2. Let $(L_{f_{i,A}})_{i \in \mathbb{N}}$ be the Lipschitz constants of the i.i.d. copies $(f_{i,A})_{i \in \mathbb{N}}$ of the function f_A given by (10). By the Lipschitz property,

$$|\zeta(A) - f_A^{1 \uparrow n}(z_0)| \leq L_{f_{1,A}} \cdots L_{f_{n,A}} |f_A^{(n+1) \uparrow \infty}(z_0) - z_0| \quad (15)$$

for any $A \in \mathcal{B}_+(\mathcal{X})$. Moreover, for each $i \in \mathbb{N}$ and an arbitrary $z \in \mathbb{R}$, we have, similarly to (7),

$$\begin{aligned} |f_{i,A}(z) - z_0| &= |z \mathbb{1}_{\{U_i \notin A\}} + f_i(z) \mathbb{1}_{\{U_i \in A\}} - z_0| \\ &= |(z - z_0) \mathbb{1}_{\{U_i \notin A\}} - z_0 \mathbb{1}_{\{U_i \in A\}} + f_i(z_0) \mathbb{1}_{\{U_i \in A\}} + (f_i(z) - f_i(z_0)) \mathbb{1}_{\{U_i \in A\}}| \\ &\leq |f_i(z_0) - z_0| + (L_{f_i} \mathbb{1}_{\{U_i \in A\}} + \mathbb{1}_{\{U_i \notin A\}}) |z - z_0| =: Q_i + L_{f_{i,A}} |z - z_0|. \end{aligned}$$

Iterating the above inequality for $|f_{i,A}(z) - z_0|$ yields

$$|f_A^{(n+1) \uparrow \infty}(z_0) - z_0| \leq \sum_{i=n+1}^{\infty} Q_i \prod_{k=n+1}^{i-1} L_{f_{k,A}}. \quad (16)$$

Plugging this upper bound into (15), we obtain

$$|\zeta(A) - f_A^{1 \uparrow n}(z_0)| \leq \sum_{i=n+1}^{\infty} Q_i \prod_{k=1}^{i-1} L_{f_{k,A}} \leq \sum_{i=n+1}^{\infty} Q_i \left(\sup_{A \in \mathcal{A}} \prod_{k=1}^{i-1} L_{f_{k,A}} \right). \quad (17)$$

To show that the right-hand side of the above inequality converges to zero a.s., it suffices to check that

$$\sum_{i=1}^{\infty} Q_i \left(\sup_{A \in \mathcal{A}} \prod_{k=1}^{i-1} L_{f_{k,A}} \right) < \infty \quad \text{a.s.} \quad (18)$$

This follows from Cauchy's radical test. Indeed,

$$\begin{aligned} & \limsup_{i \rightarrow \infty} \frac{1}{i} \left(\log Q_i + \sup_{A \in \mathcal{A}} \sum_{k=1}^{i-1} \log L_{f_{k,A}} \right) \\ & \leq \limsup_{i \rightarrow \infty} \frac{1}{i} \log^+ Q_i + \limsup_{i \rightarrow \infty} \frac{1}{i} \sup_{A \in \mathcal{A}} \sum_{k=1}^{i-1} (\log L_{f_k}) \mathbb{1}_{\{U_k \in A\}} = \frac{\inf_{A \in \mathcal{A}} \mu(A)}{\mu(\mathcal{X})} \mathbf{E} \log L_f < 0, \end{aligned}$$

where the second lim sup was calculated in Lemma 4.3. The first lim sup is equal to zero by the Borel–Cantelli lemma and the fact that $\mathbf{E} \log^+ Q_1 = \mathbf{E} \log^+ |f_1(z_0) - z_0| < \infty$ by the assumption (2). Thus,

$$\limsup_{i \rightarrow \infty} \left(Q_i \left(\sup_{A \in \mathcal{A}} \prod_{k=1}^{i-1} L_{f_{k,A}} \right) \right)^{1/i} \leq \exp \left\{ \frac{\inf_{A \in \mathcal{A}} \mu(A)}{\mu(\mathcal{X})} \mathbf{E} \log L_f \right\} < 1, \quad (19)$$

and this completes the proof. \square

Since the Vapnik–Červonenkis dimension of a monotone family of sets is 2, we obtain the following result.

Corollary 4.5. *Let $\mathcal{A} = \{A_t, t \geq 0\}$ be a nondecreasing (respectively, nonincreasing) subfamily of $\mathcal{B}_+(\mathcal{X})$ of finite measure such that $\bigcup_t A_t$ (respectively, $\bigcap_t A_t$) is of finite positive measure. Then (12) holds.*

4.3. Uniform convergence for sieved iterations on the half-line

Let $\mathcal{X} = \mathbb{R}_+$ be the half-line with μ being the Lebesgue measure. Let us consider stochastic process $(\zeta(x))_{x>0} = (\zeta([0, x]))_{x>0}$. By Corollary 4.5, the iterations

$$\zeta_n(x) := f_{[0,x]}^{1 \uparrow n}(z_0), \quad n \geq 1, \quad (20)$$

converge a.s. to ζ uniformly over $x \in [a, b]$ for each $0 < a \leq b < \infty$. The following result establishes their uniform convergence in L^p .

Denote $\Phi(x) := \mathbf{E} L_f^x$, and let

$$\mathcal{I} := \{x > 0 : \Phi(x) < 1\}. \quad (21)$$

The set \mathcal{I} is not empty under assumption (1), because it contains all sufficiently small positive numbers. This follows from the following three relations: $\Phi(0) = 1$, $\Phi'(0) = \mathbf{E} \log L_f < 0$ and $\Phi(1) < \infty$.

Proposition 4.6. *For each $a > 0$ and $p \in \mathcal{I} \cap (0, 1]$,*

$$\mathbf{E} \sup_{x \in [a, 1]} |\zeta(x) - \zeta_n(x)|^p \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (22)$$

Proof. Repeat the arguments from the proof of Theorem 4.2, see (17), and use the subadditivity of the function $t \mapsto t^p$ to arrive at

$$\mathbf{E} \sup_{x \in [a, 1]} |\zeta(x) - \zeta_n(x)|^p \leq \sum_{i=n+1}^{\infty} (\mathbf{E} Q_i)^p \mathbf{E} \left(\sup_{x \in [a, 1]} \prod_{k=1}^{i-1} L_{f_{k,[0,x]}} \right)^p. \quad (23)$$

Fix $i \geq 2$. In order to calculate the last expectation, recall that $(t_k, x_k, f_k)_{k=1, \dots, i-1}$ is the enumeration of the first $i-1$ atoms of $\mathcal{P}_{[0,1]}$ ordered such that $t_1 < t_2 < \dots < t_{i-1}$. Let $x_{(i-1:1)} < \dots < x_{(i-1:i-1)}$ be the ordered points x_1, \dots, x_{i-1} , and let $f_{(i-1:k), [0,x]}$ be the corresponding functions. Note that

$$\prod_{k=1}^{i-1} L_{f_k, [0,x]} = \prod_{k=1}^{i-1} L_{f_{(i-1,k), [0,x]}} = \prod_{k=1}^{i-1} L_{f_{(i-1,k)}^{\mathbb{1}_{\{a < x_{(i-1,k)} \leq x\}}}} \prod_{k=1}^{i-1} L_{f_{(i-1,k)}^{\mathbb{1}_{\{x_{(i-1,k)} \leq a\}}}}, \quad (24)$$

The two factors on the right-hand side of (24) are independent by the Poisson property, and the second factor does not depend on x . Thus,

$$\mathbf{E} \left(\sup_{x \in [a,1]} \prod_{k=1}^{i-1} L_{f_k, [0,x]} \right)^p \leq \mathbf{E} \left(\sup_{x \in [a,1]} \prod_{k=1}^{i-1} L_{f_{(i-1,k)}^{\mathbb{1}_{\{a < x_{(i-1,k)} \leq x\}}} \right)^p \mathbf{E} \left(\prod_{k=1}^{i-1} L_{f_{(i-1,k)}^{\mathbb{1}_{\{x_{(i-1,k)} \leq a\}}} \right)^p.$$

Further, since (f_k) and (x_k) are independent,

$$\begin{aligned} \mathbf{E} \left(\sup_{x \in [a,1]} \prod_{k=1}^{i-1} L_{f_{(i-1,k)}^{\mathbb{1}_{\{a < x_{(i-1,k)} \leq x\}}} \right)^p &= \mathbf{E} \left(\sup_{x \in [a,1]} \prod_{k=1}^{i-1} L_{f_k^{\mathbb{1}_{\{a < x_{(i-1,k)} \leq x\}}} \right)^p \leq \mathbf{E} \left(\sup_{i \geq 1} \prod_{k=1}^{i-1} L_{f_k} \right)^p \\ &\leq \sum_{i=1}^{\infty} \mathbf{E} \left(\prod_{k=1}^{i-1} L_{f_k} \right)^p = \sum_{i=1}^{\infty} (\Phi(p))^{i-1} = \frac{1}{1 - \Phi(p)} < \infty. \end{aligned}$$

Therefore,

$$\begin{aligned} \mathbf{E} \left(\sup_{x \in [a,1]} \prod_{k=1}^{i-1} L_{f_k, [0,x]} \right)^p &\leq \frac{1}{1 - \Phi(p)} \mathbf{E} \left(\prod_{k=1}^{i-1} L_{f_{(i-1,k)}^{\mathbb{1}_{\{x_{(i-1,k)} \leq a\}}} \right)^p \\ &= \frac{1}{1 - \Phi(p)} \sum_{k=0}^{j-1} \binom{i-1}{k} \Phi(p)^k a^k (1-a)^{i-1-k} = \frac{(1 - (1 - \Phi(p))a)^{i-1}}{1 - \Phi(p)}, \end{aligned}$$

where the first equality holds by conditioning on the number of points x_1, x_2, \dots, x_{i-1} which fall in the interval $[0, a]$. Summarising, we see that the series on the right-hand side of (23) converges and (22) follows. \square

5. Path regularity properties

5.1. Set-indexed functions

As we have already mentioned, the set function ζ is not a measure, yet it is possible to show that it is a.s. continuous from below and from above.

Proposition 5.1. *Assume that $(A_m)_{m \in \mathbb{N}}$ is a nondecreasing sequence of sets from $\mathcal{B}_+(\mathcal{X})$ such that $\mu(A_\infty) < \infty$, where $A_\infty := \bigcup_{m=1}^{\infty} A_m$. Then*

$$\zeta(A_m) \xrightarrow{\text{a.s.}} \zeta(A_\infty) \quad \text{as } m \rightarrow \infty.$$

The same holds for a nonincreasing sequence $(A_m)_{m \in \mathbb{N}}$ from $\mathcal{B}_+(\mathcal{X})$ such that $0 < \mu(A_\infty)$ with $A_\infty := \bigcap_{m=1}^{\infty} A_m$.

Proof. Assume that $(A_m)_{m \in \mathbb{N}}$ is nondecreasing. For fixed $t > 0$, we can write

$$\begin{aligned} |\zeta(A_\infty) - \zeta(A_m)| &\leq |\zeta(A_\infty) - \zeta_t(A_\infty)| + |\zeta_t(A_\infty) - \zeta_t(A_m)| + |\zeta_t(A_m) - \zeta(A_m)| \\ &\leq |\zeta(A_\infty) - \zeta_t(A_\infty)| + |\zeta_t(A_\infty) - \zeta_t(A_m)| + \sup_{m \geq 1} |\zeta_t(A_m) - \zeta(A_m)|, \end{aligned}$$

where ζ_t was defined by (5). Letting $m \rightarrow \infty$ yields

$$\limsup_{m \rightarrow \infty} |\zeta(A_\infty) - \zeta(A_m)| \leq |\zeta(A_\infty) - \zeta_t(A_\infty)| + \sup_{m \geq 1} |\zeta_t(A_m) - \zeta(A_m)|,$$

since $N_{A_m}(t) \rightarrow N_{A_\infty}(t)$ a.s., and so $N_{A_n}(t) = N_{A_\infty}(t)$ a.s. for all sufficiently large m . Letting t go to infinity and applying Corollary 4.5 yield the desired statement. The proof for nonincreasing sequences is similar. \square

Remark 5.2. Proposition 5.1 also holds in the sense of L^p -convergence for $p \in \mathcal{I}$, see (21) for the definition of \mathcal{I} .

The recursive equation (6) makes it possible to obtain bounds on the increments of ζ . Let $A \subset B$ with $A, B \in \mathcal{B}_+(\mathcal{X})$, and let U be distributed in B according to the normalised μ , see formula (9) with $\mathcal{X} = B$. Then

$$\begin{aligned} |\zeta(B) - \zeta(A)| &\stackrel{d}{=} |(f(\zeta(B)) - f(\zeta(A)))\mathbb{1}_{\{U \in A\}} + (f(\zeta(A)) - \zeta(A))\mathbb{1}_{\{U \in B \setminus A\}}| \\ &\leq L_f |\zeta(B) - \zeta(A)| + |f(\zeta(A)) - \zeta(A)|\mathbb{1}_{\{U \in B \setminus A\}}, \end{aligned}$$

with independent ζ , f and U on the right-hand side. If $K_f = \mathbf{E}L_f < 1$ and (2) holds, then $\zeta(A)$ is integrable by Proposition 3.3. Since $\zeta(A)$ and $f(\zeta(A))$ share the same distribution, it holds

$$\mathbf{E}|\zeta(B) - \zeta(A)| \leq \frac{\mu(B) - \mu(A)}{(1 - K_f)\mu(B)} \mathbf{E}|f(\zeta(A)) - \zeta(A)|.$$

Note that the latter expectation does not depend on A , since the distribution of $\zeta(A)$ does not depend on A . A similar estimate can also be written for p -th moment, $p > 0$, assuming $\Phi(p) < 1$ and $\mathbf{E}|f(z_0) - z_0|^p < \infty$.

5.2. Stochastic processes on the half-line

In the special case of μ being the Lebesgue measure on $\mathcal{X} = \mathbb{R}_+$, we obtain the following result.

Proposition 5.3. *The process $(\zeta(x))_{x>0}$ is a.s. continuous at every fixed (nonrandom) $x > 0$; it has càdlàg paths and it is not pathwise continuous and is not pathwise monotone.*

Proof. Fix arbitrary $x > 0$. Let $(x_n)_{n \geq 1}$ and $(x'_n)_{n \geq 1}$ be sequences such that $x_n \downarrow x$ and $x'_n \uparrow x$ as $n \rightarrow \infty$. By Proposition 5.1, we obtain

$$\zeta([0, x_n]) \xrightarrow{\text{a.s.}} \zeta([0, x]) \quad \text{and} \quad \zeta([0, x'_n]) \xrightarrow{\text{a.s.}} \zeta([0, x]) \quad \text{as } n \rightarrow \infty.$$

Since $\zeta(x-) := \zeta([0, x]) = \zeta(x)$ a.s. we deduce the first property.

The process $\zeta_n(x)$ from (20) is càdlàg in x by construction, being a composition of càdlàg functions $f_{i,[0,x]}$. The uniform convergence established in Corollary 4.5 yields that $(\zeta(x))_{x>0}$ has càdlàg paths a.s. on $[a, b]$ for any $0 < a < b$, being the uniform limit of càdlàg functions.

To show that it is not pathwise continuous, consider the projection $\{(t_i, x_i) : i \geq 1\}$ of the Poisson process \mathcal{P} and take an atom¹ (t, x) such that the rectangle $[0, t] \times [0, x]$ does not contain other atoms of $\{(t_i, x_i) : i \geq 1\}$. Then ζ is not left-continuous at x . More precisely, for such (random) x we have $\zeta(x) = f(\zeta(x-))$, where f is the mark of the atom at (t, x) . Since the jump $f(\zeta) - \zeta$ has expectation zero, see (4), the process is not pathwise monotone. \square

Our next result concerns the total variation of ζ .

Theorem 5.4. *The total variation of the process $(\zeta(x))_{x>0}$ is a.s. finite on every interval $[a, b]$ with $a > 0$.*

Proof. Without loss of generality, we may consider an interval $[a, 1]$ for a fixed $a \in (0, 1)$. Fix an arbitrary partition $a = y_0 < y_1 < y_2 < \dots < y_m = 1$, and let

$$\tau_j = \tau(y_j, y_{j+1}) := \inf\{k \geq 1 : x_k \in (y_j, y_{j+1}]\}, \quad j = 0, 1, \dots, m-1,$$

be the index of the first point in \mathcal{P} such that the second coordinate of this point falls in $(y_j, y_{j+1}]$. Note that, for every fixed $i \in \mathbb{N}$,

$$\begin{aligned} \sum_{j=0}^{m-1} \mathbb{1}_{\{\tau_j \leq i\}} &= \sum_{j=0}^{m-1} \sum_{k=1}^i \mathbb{1}_{\{\tau_j = k\}} \leq \sum_{j=0}^{m-1} \sum_{k=1}^i \mathbb{1}_{\{x_k \in (y_j, y_{j+1}]\}} \\ &= \sum_{k=1}^i \sum_{j=0}^{m-1} \mathbb{1}_{\{x_k \in (y_j, y_{j+1}]\}} = \sum_{k=1}^i \mathbb{1}_{\{x_k \in (a, 1]\}} \leq i. \end{aligned} \quad (25)$$

Let us now consider the increments $\zeta(y_{j+1}) - \zeta(y_j)$ for $j = 0, \dots, m-1$. Write $(f_{i,y})_{i \in \mathbb{N}}$ for the i.i.d. copies of the function $f_y := f_{[0,y]}$ from (10). We have

$$\begin{aligned} |\zeta(y_{j+1}) - \zeta(y_j)| &\leq |\zeta(y_{j+1}) - f_{y_j}^{1 \uparrow (\tau_j - 1)}(z_0)| + |\zeta(y_j) - f_{y_j}^{1 \uparrow (\tau_j - 1)}(z_0)| \\ &= |\zeta(y_{j+1}) - f_{y_{j+1}}^{1 \uparrow (\tau_j - 1)}(z_0)| + |\zeta(y_j) - f_{y_j}^{1 \uparrow (\tau_j - 1)}(z_0)|, \end{aligned}$$

where the second equality holds because $f_{k,y_{j+1}} = f_{k,y_j}$ for $k < \tau_j$ by the definition of τ_j . By the Lipschitz property,

$$\begin{aligned} |\zeta(y_{j+1}) - \zeta(y_j)| &\leq L_{f_{1,y_{j+1}}} \cdots L_{f_{\tau_j-1,y_{j+1}}} |f_{y_{j+1}}^{\tau_j \uparrow \infty}(z_0) - z_0| \\ &\quad + L_{f_{1,y_j}} \cdots L_{f_{\tau_j-1,y_j}} |f_{y_j}^{\tau_j \uparrow \infty}(z_0) - z_0|, \end{aligned}$$

and, with the help of (16),

$$|\zeta(y_{j+1}) - \zeta(y_j)| \leq \sum_{i=\tau_j}^{\infty} Q_i \prod_{k=1}^{i-1} L_{f_{k,y_{j+1}}} + \sum_{i=\tau_j}^{\infty} Q_i \prod_{k=1}^{i-1} L_{f_{k,y_j}} \leq 2 \sum_{i=\tau_j}^{\infty} Q_i \left(\sup_{x \in [a,1]} \prod_{k=1}^{i-1} L_{f_{k,x}} \right), \quad (26)$$

¹Such atoms are usually called (lower) records of the point process.

where $Q_i := |f_i(z_0) - z_0|$ as in the proof of Theorem 4.2. Summing over $j = 0, \dots, m-1$ and subsequently taking the supremum over the set P of all partitions, we deduce

$$\begin{aligned} \sup_P \sum_{j=0}^{m-1} |\zeta(y_{j+1}) - \zeta(y_j)| &\leq \sup_P \sum_{i=1}^{\infty} Q_i \left(\sup_{x \in [a, 1]} \prod_{k=1}^{i-1} L_{f_{k,x}} \right) \sum_{j=0}^{m-1} \mathbb{1}_{\{\tau_j \leq i\}} \\ &\leq \sum_{i=1}^{\infty} i Q_i \left(\sup_{x \in [a, 1]} \prod_{k=1}^{i-1} L_{f_{k,x}} \right), \end{aligned} \quad (27)$$

where (25) yields the last bound. It remains to note that the series on the right-hand side converges a.s. by the Cauchy radical test using the same reasoning as in the proof of Theorem 4.2 in conjunction with a trivial observation $i^{1/i} \rightarrow 1$ as $i \rightarrow \infty$. The proof is complete. \square

Inequality (26) provides an upper a.s. bound on the increments of the process ζ in terms of the tail of a convergent series.

Denote by $V_p(\zeta; [a, b])$ the p -variation of ζ over the interval $[a, b]$. The next result demonstrates that the p -variation of ζ is integrable. Recall that the set \mathcal{I} was defined in (21).

Proposition 5.5. *For every $p \in \mathcal{I} \cap (0, 1]$ and $0 < a \leq b$, we have*

$$\mathbf{E}V_p(\zeta; [a, b]) \leq (\log(b/a)) \frac{2\mathbf{E}|Z_\infty|^p}{1 - \Phi(p)}. \quad (28)$$

Proof. First of all note that $\mathbf{E}|Z_\infty|^p$ is finite by Proposition 3.3. Further, it suffices to prove the statement for the interval $[a/b, 1]$ with $0 < a < b$. The scale invariance property then yields the desired result for the interval $[a, b]$. Thus, without loss of generality, assume that $b = 1$ and $a \in (0, 1)$.

The process $(\zeta(x))_{x \in (0, 1]}$ has a countable dense set of jumps occurring at points $(x_k)_{k \geq 1}$, where the enumeration $(t_k, x_k, f_k)_{k \geq 1}$ of the atoms of \mathcal{P} is such that $0 < t_1 < t_2 < t_3 < \dots$ and $x_k \in [0, 1]$ a.s.

The size of a jump at $x_k \in [0, 1]$, $k \in \mathbb{N}$, depends on the number of iterations applied before f_k . More precisely,

$$\zeta(x_k) - \zeta(x_{k-}) \stackrel{d}{=} f^{1 \uparrow (\theta_k + 1)}(Z_\infty) - f^{1 \uparrow \theta_k}(Z_\infty), \quad k \in \mathbb{N},$$

where $\theta_k := \sum_{j=1}^{k-1} \mathbb{1}_{\{x_j < x_k\}}$, and Z_∞ is independent of $(f_j)_{j \leq k}$ and distributed like $\zeta(x)$ for any x . Taking the expectation and using independence we obtain

$$\begin{aligned} \mathbf{E}|\zeta(x_k) - \zeta(x_{k-})|^p \mathbb{1}_{\{x_k > a\}} &\leq \mathbf{E}((\Phi(p))^{\theta_k} \mathbb{1}_{\{x_k > a\}}) \mathbf{E}|f(Z_\infty) - Z_\infty|^p \\ &\leq \mathbf{E}((\Phi(p))^{\theta_k} \mathbb{1}_{\{x_k > a\}}) 2\mathbf{E}|Z_\infty|^p, \end{aligned}$$

where (4) has been utilised on the last step. Since x_k has the uniform distribution on $[0, 1]$,

$$\begin{aligned} \mathbf{E}((\Phi(p))^{\theta_k} \mathbb{1}_{\{x_k > a\}}) &= \sum_{j=0}^{k-1} \binom{k-1}{j} (\Phi(p))^j \int_a^1 y^j (1-y)^{k-1-j} dy \\ &= \int_a^1 (1-y + y(\Phi(p)))^{k-1} dy = \frac{(1 - (1 - (\Phi(p)))a)^k - (\Phi(p))^k}{k(1 - (\Phi(p)))}. \end{aligned} \quad (29)$$

Finally, note that

$$\mathbf{E} \sum_{x_k \in [a, 1]} |\zeta(x_k) - \zeta(x_k-)|^p = (-\log a) \frac{2\mathbf{E}|Z_\infty|^p}{1 - \Phi(p)}. \quad \square$$

By the subadditivity of the function $t \mapsto t^p$, Proposition 5.5 implies that the total variation of ζ over $[a, b]$ is p -integrable with a bound on its p th moment given by the right-hand side of (28).

For each $\varepsilon > 0$, the set $J := \{x > 0 : |\zeta(x) - \zeta(x-)| \geq \varepsilon\}$ of jumps of size at least ε is a scale invariant point process on $(0, \infty)$. Indeed, since the p -variation of ζ is finite on any interval $[a, b]$ with $0 < a < b < \infty$ and sufficiently small $p > 0$, the number of points in $J \cap [a, b]$ is a.s. finite. For $c > 0$, cJ is the functional of $(\zeta(c^{-1}x))_{x>0}$, which coincides in distribution with $(\zeta(x))_{x>0}$, hence J is scale invariant.

5.3. Integration with respect to ζ

Since the process ζ has finite total variation, it is possible to integrate continuous functions with respect to ζ on intervals bounded from 0 in the sense of Riemann–Stieltjes integration. For arbitrary $0 < a < b < \infty$ and continuous $h : [a, b] \rightarrow \mathbb{R}$, we have

$$\int_{(a, b]} h(x) d\zeta(x) = \sum_{x_k \in (a, b]} h(x_k) (\zeta(x_k) - \zeta(x_k-)), \quad (30)$$

and the series on the right-hand side converges absolutely a.s.

The following proposition shows that one can also integrate over intervals $(0, b]$ provided h satisfies an additional integrability assumption.

Proposition 5.6. *Let $h : [0, 1] \mapsto \mathbb{R}$ be a continuous function such that $\int_0^1 |h(t)|^p t^{-1} dt < \infty$ for some $p \in \mathcal{I} \cap (0, 1]$. Then the limit $\int_{(0, 1]} h(x) d\zeta(x)$ of (30) as $a \downarrow 0$ exists a.s. and in L^p .*

Proof. It suffices to show that

$$\mathbf{E} \left(\sum_{x_k \in (0, 1]} |h(x_k)| |\zeta(x_k) - \zeta(x_k-)| \right)^p < \infty. \quad (31)$$

This immediately implies

$$\sum_{x_k \in (0, 1]} |h(x_k)| |\zeta(x_k) - \zeta(x_k-)| < \infty \quad \text{a.s.}$$

and, thus, by the dominated convergence

$$\sum_{x_k \in (a, 1]} h(x_k) (\zeta(x_k) - \zeta(x_k-)) \xrightarrow{\text{a.s.}} \sum_{x_k \in (0, 1]} h(x_k) (\zeta(x_k) - \zeta(x_k-)) \quad \text{as } a \downarrow 0.$$

Furthermore, (31) implies that

$$\left\| \int_{(a, 1]} h(x) d\zeta(x) - \int_{(0, 1]} h(x) d\zeta(x) \right\|_p \leq \mathbf{E} \left(\sum_{x_k \in (0, a]} |h(x_k)| |\zeta(x_k) - \zeta(x_k-)| \right)^p \rightarrow 0$$

as $a \downarrow 0$. To prove (31), note that subadditivity of $t \mapsto t^p$ yields

$$\mathbf{E} \left(\sum_{x_k \in (0,1]} |h(x_k)| |\zeta(x_k) - \zeta(x_k-)| \right)^p \leq \sum_{k=1}^{\infty} \mathbf{E} (|\zeta(x_k) - \zeta(x_k-)|^p \cdot |h(x_k)|^p).$$

Let us prove that the series on the right-hand side converges. Using the same calculations as in (29), we derive

$$\begin{aligned} \mathbf{E} (|\zeta(x_k) - \zeta(x_k-)|^p \cdot |h(x_k)|^p) &\leq 2\mathbf{E}|Z_\infty|^p \mathbf{E}((\Phi(p))^{\theta_k} |h(x_k)|^p) \\ &= 2\mathbf{E}|Z_\infty|^p \int_0^1 (1-y+y\Phi(p))^{k-1} |h(y)|^p dy. \end{aligned}$$

The right-hand side is summable to $\frac{2\mathbf{E}|Z_\infty|^p}{1-\Phi(p)} \int_0^1 |h(y)|^p y^{-1} dy$, which is finite by assumptions. Note that $\mathbf{E}|Z_\infty|^p$ is finite by Proposition 3.3, since $\Phi(p) < 1$ and $\mathbf{E}|f(z_0) - z_0|^p < \infty$ in view of (2). \square

Let us now turn to integration of ζ with respect to a continuous deterministic function h . The easiest way to define such integrals is via integration by parts, that is, for $0 < a < b$ and continuous $h : [a, b] \mapsto \mathbb{R}$, put

$$\begin{aligned} \int_{(a,b]} \zeta(x) dh(x) &:= \zeta(b)h(b) - \zeta(a)h(a) - \int_{(a,b]} h(x) d\zeta(x) \\ &= \zeta(b)h(b) - \zeta(a)h(a) - \sum_{x_k \in (a,b]} h(x_k)(\zeta(x_k) - \zeta(x_k-)). \end{aligned} \quad (32)$$

Under the additional integrability assumption, the definition can be extended to $a = 0$.

Theorem 5.7. *Under assumptions of Proposition 5.6 and assuming also that h is continuous, the integral $\int_{(a,b]} \zeta(x) dh(x)$ converges in L^p as $a \downarrow 0$ to a limit which is denoted by $\int_{(0,1]} h(x) d\zeta(x)$.*

Proof. In view of Proposition 5.6 and definition (32), it suffices to check that

$$\lim_{a \downarrow 0} |h(a)|^p \mathbf{E} |\zeta(a)|^p = 0.$$

But this follows immediately from $\lim_{a \downarrow 0} |h(a)| = |h(0)| = 0$ and $\mathbf{E} |\zeta(a)|^p = \mathbf{E} |\zeta(1)|^p < \infty$, where finiteness is secured by Proposition 3.3. \square

In particular, if $\Phi(p) < 1$ for some $p \in (0, 1]$ then $\int_0^1 \zeta(x) dx$ is well defined in L^p . Hence, $\int_0^1 \zeta(x) dx$ is well defined in L^p for all sufficiently small $p > 0$.

6. Markov property

Assume that knowledge of the value $y = f(x)$ allows one to recover in a unique way the value of the argument $x \in \mathbb{R}$ and also the realisation of the random function $f(\cdot)$ with the distribution ν . To get a better understanding of this assumption, suppose for a moment that ν is supported by a finite set of strictly monotone and strictly contractive functions $h_1, \dots, h_m \in \mathcal{G}$ which satisfy the *strong separation*

condition. The latter means that the unique attractor of the iterated function system $\{h_1, h_2, \dots, h_m\}$, that is the unique nonempty compact set \mathcal{K} such that

$$\mathcal{K} = \bigcup_{i=1}^m h_i(\mathcal{K}),$$

satisfies additionally $h_i(\mathcal{K}) \cap h_j(\mathcal{K}) = \emptyset$ for $i \neq j$. Thus, for every point $y \in \mathcal{K}$ we can find the unique index i such that $y \in h_i(\mathcal{K})$, hence, uniquely recover a deterministic function h_i , the realisation of f . Since h_1, \dots, h_m are assumed to be strictly monotone, it is further possible to find the unique $x \in \mathcal{K}$ such that $y = h_i(x) = f(x)$. Moreover, since the support of Z_∞ , the limit of iterations (3), is equal to \mathcal{K} , given the event $\{Z_\infty = z\}$ one can uniquely determine the full (deterministic) sequence $(g_n^z)_{n \in \mathbb{N}} \subset \mathcal{G}$, such that

$$z = (g^z)^{1 \uparrow \infty}(z_0).$$

A typical example of a random Lipschitz mapping satisfying the above recovery property are Bernoulli convolutions with $\lambda < 1/2$, see Example 1.1 and Section 7.3 below, for which

$$m = 2, \quad h_1(x) = \lambda x, \quad h_2(x) = \lambda x + 1, \quad \nu(\{h_1\}) = \nu(\{h_2\}) = 1/2.$$

An example where ν is not finitely supported, yet the corresponding random function f satisfies the recovery property, is given by random continued fractions with integer entries, see Example 8.3 below, in which

$$f(x) = \frac{1}{\xi + x}$$

with $\xi \in \mathbb{N}$ a.s. From the value $y = f(x)$ one can recover the function f by letting ξ be the integer part of $1/y$ and x the fractional part of $1/y$.

The aim of this section is to show that the process $(\zeta(x))_{x>0}$ generated by a random Lipschitz function f which satisfies the recovery property, is Markov and calculate the corresponding generators. Denote by $\mathfrak{F}_{[a,b]}$ the σ -algebra generated by $\zeta(x)$, $x \in [a, b]$, where $0 < a \leq b \leq \infty$. Write \mathfrak{F}_a for $\mathfrak{F}_{[a]}$. The recovery property implies that \mathfrak{F}_a is equal to the σ -algebra generated by the projection of $\mathcal{P}_{[0,a]}$ onto the last component \mathcal{G} .

Theorem 6.1. *Assume that each $z \in \text{supp } Z_\infty$ corresponds to a unique sequence $(g_n^z)_{n \in \mathbb{N}}$ from \mathcal{G} such that*

$$z = (g^z)^{1 \uparrow \infty}(z_0), \tag{33}$$

and, for all $n \in \mathbb{N}$, the mapping $z \mapsto g_n^z$ is measurable as a function from \mathbb{R} to \mathcal{G} . Then the process $(\zeta(x))_{x>0}$ is Markov both in forward and reverse time, that is, with respect to filtration $(\mathfrak{F}_{(0,x]})_{x>0}$ and $(\mathfrak{F}_{[x,\infty)})_{x>0}$, respectively.

Proof. Fix $x, u > 0$. Given $\{\zeta(x) = z\}$, we know the projection of $\mathcal{P}_{[0,x]}$ onto \mathcal{G} , which is the sequence $(g_n^z)_{n \in \mathbb{N}}$. The σ -algebra $\mathfrak{F}_{(0,x]}$ is generated by the family of sequences $(g_n^{\zeta(y)})_{n \in \mathbb{N}}$ with $y \leq x$. Note that $(g_n^{\zeta(y)})_{n \in \mathbb{N}}$ is a subsequence of $(g_n^{\zeta(y')})_{n \in \mathbb{N}}$ if $y \leq y'$.

Let $\kappa_0 := 0$ and put

$$\kappa_{k+1} := \min\{i > \kappa_k : x_i \leq x\}, \quad k \geq 0,$$

where $(t_k, x_k, f_k)_{k \in \mathbb{N}}$ is the enumeration of atoms of $\mathcal{P}_{[0, x+u]}$ such that $(t_k)_{k \in \mathbb{N}}$ is a.s. increasing. We have

$$\zeta(x+u) = \lim_{n \rightarrow \infty} f^{(1)} \circ f_{\kappa_1} \circ \cdots \circ f^{(n)} \circ f_{\kappa_n}(z_0), \quad (34)$$

where the limit is in the almost sure sense, $f_{\kappa_k} = g_k^{\zeta(x)}$, and

$$f^{(k)}(z) := f_{(x, x+u]}^{(\kappa_{k-1}+1) \uparrow (\kappa_k-1)}(z), \quad k \in \mathbb{N}.$$

Note that $\mathcal{P}_{(x, x+u]}$ is independent of $\mathcal{P}_{[0, x]}$ by the Poisson property and, in particular, $(\kappa_k - \kappa_{k-1})_{k \in \mathbb{N}}$ are i.i.d. with geometric distribution which are also independent of $\mathfrak{F}_{(0, x]}$. Hence, $\zeta(x+u)$ is determined by $\zeta(x)$ and $\mathcal{P}_{(x, x+u]}$, so that the conditional distribution of $\zeta(x+u)$ given $\mathfrak{F}_{(0, x]}$ coincides with the conditional distribution given \mathfrak{F}_x .

Let us now prove the Markov property in the reverse time. Given $\{\zeta(x) = z\}$ and $y \leq x$, we have

$$\zeta(y) = (g_y^z)^{1 \uparrow \infty}(z_0).$$

where $g_{j, y}^z$ is equal in distribution to g_j^z with probability y/x , is the identity function with probability $1 - y/x$ and the choices are mutually independent given $\{\zeta(x) = z\}$. That is to say, $\zeta(y)$ is a functional of $(g_n^z)_{n \in \mathbb{N}}$, and $\mathcal{P}_{[0, x]}$. Since $\zeta(x)$ determines the sequence $(g_n^z)_{k \in \mathbb{N}}$, and by the independence of Poisson processes, we have the Markov property in the reverse time. \square

The conditional distribution of $\zeta(x+u)$ given $\{\zeta(x) = z\}$ can be determined as follows. Let $(g_n^z)_{n \in \mathbb{N}}$ be a sequence recovered from $\{\zeta(x) = z\}$. In view of (34), $\zeta(x+u)$ for $u > 0$ can be derived by inserting between each consecutive pair of functions in the infinite iteration

$$z = \zeta(x) = g_1^z \circ g_2(z) \circ \cdots \circ g_n^z \circ \cdots,$$

an independent copy of a mapping $f^{(k)}$ composed of a geometric number of independent copies of f . The aforementioned geometric random variables take values in $\{0, 1, 2, \dots\}$, are independent and all have the same parameter $u/(x+u)$. Similarly, it is possible to determine the conditional distribution of $\zeta(y)$ given $\{\zeta(x) = z\}$ with $x \geq y$ by deleting each of the functions g_n^z independently of others with probability $1 - y/x$.

Maintaining assumptions of Theorem 6.1, we now aim at finding the generating operator of the time-homogeneous Markov process $\tilde{\zeta}(t) := \zeta(e^t)$, $t \in \mathbb{R}$. This generating operator in the forward time is defined as the limit

$$(\mathbf{A} \uparrow h)(z) := \lim_{\delta \downarrow 0} \frac{1}{\delta} [\mathbf{E}(h(\zeta(e^{t+\delta})) | \zeta(e^t) = z) - h(z)],$$

and in the reverse time as

$$(\mathbf{A} \downarrow h)(z) := \lim_{\delta \downarrow 0} \frac{1}{\delta} [\mathbf{E}(h(\zeta(e^{t-\delta})) | \zeta(e^t) = z) - h(z)]$$

for all functions h from their domains of definition.

We calculate the above generators under additional assumptions:

$$L_f \leq c_f \text{ for some deterministic constant } c_f < 1 \quad \text{and} \quad Z_\infty \text{ is compactly supported.} \quad (35)$$

The above assumption holds, for example, for Bernoulli convolutions.

Proposition 6.2. *Assume that f possesses the recovery property and (35) holds. Then, for $h \in C^1(\text{supp } Z_\infty)$, it holds*

$$(A_\uparrow h)(z) = \sum_{k=0}^{\infty} [\mathbf{E}h((g^z)^{1\uparrow k} \circ f \circ ((g^z)^{1\uparrow k})^{-1}(z)) - h(z)], \quad z \in \text{supp } Z_\infty, \quad (36)$$

and also

$$(A_\downarrow h)(z) = \sum_{k=0}^{\infty} [h((g^z)^{1\uparrow(k-1)} \circ (g_k^z)^{-1} \circ ((g^z)^{1\uparrow(k-1)})^{-1}(z)) - h(z)], \quad z \in \text{supp } Z_\infty, \quad (37)$$

where $(g_j^z)_{j \in \mathbb{N}}$ is a sequence of deterministic functions which is uniquely determined by $z \in \text{supp } Z_\infty$.

The proof of Proposition 6.2 is given in the [Appendix](#).

We close this section by noticing that the Markov property holds without the recovery property but with respect to a larger filtration generated by the Poisson process in horizontal strips.

7. Perpetuities

7.1. Moments and covariances

Let $f(z) = Mz + Q$, where (M, Q) is a random vector in \mathbb{R}^2 , so that $L_f = M$. The iterations of i.i.d. copies of the affine random mapping $z \mapsto Mz + Q$ are known as *perpetuities*. In order to avoid trivialities, we assume throughout this section that

$$\mathbf{P}\{Mx + Q = x\} < 1 \quad \text{for all } x \in \mathbb{R}. \quad (38)$$

Assume that $\mathbf{E}|M| < 1$ and Q is integrable. Then $\zeta(A)$ is integrable and

$$\mathbf{E}\zeta(A) = \frac{\mathbf{E}Q}{1 - \mathbf{E}M}.$$

If $\mathbf{E}M^2 < 1$ and $\mathbf{E}Q^2 < \infty$, then $\zeta(A)$ is square integrable for all $A \in \mathcal{B}_+(\mathcal{X})$, see [4] and Proposition 3.3, Th. 1.4, and the general expression for the covariance can be found from (8). Assuming additionally $\mathbf{E}Q = 0$ and independence of M and Q , we obtain

$$\mathbf{E}(\zeta(A_1)\zeta(A_2)) = \frac{\mathbf{E}Q^2\mu(A_1 \cap A_2)}{(1 - \mathbf{E}M)\mu(A_2 \cup A_1) + (\mathbf{E}M - \mathbf{E}M^2)\mu(A_1 \cap A_2)}.$$

From this, we deduce

$$\mathbf{E}\zeta(A)^2 = \frac{\mathbf{E}Q^2}{1 - \mathbf{E}M^2},$$

and

$$\mathbf{E}(\zeta(A_1) - \zeta(A_2))^2 = \frac{2\mathbf{E}Q^2(1 - \mathbf{E}M)\mu(A_1 \Delta A_2)}{(1 - \mathbf{E}M^2)((\mathbf{E}M - \mathbf{E}M^2)\mu(A_1 \cap A_2) + (1 - \mathbf{E}M)\mu(A_1 \cup A_2))}.$$

Thus, ζ is continuous in L^2 with respect to the convergence of its argument in measure.

From now on, assume that $\mathcal{X} = \mathbb{R}_+$, and $\zeta(x) = \zeta([0, x])$. Then

$$\mathbf{E}(\zeta(x)\zeta(y)) = \frac{x\mathbf{E}Q^2}{(1-\mathbf{E}M)y + (\mathbf{E}M - \mathbf{E}M^2)x}, \quad x \leq y.$$

By exponential change of time, we obtain the stationary process $\tilde{\zeta}(s) = \zeta(e^s)$, $s \in \mathbb{R}$, with covariance

$$\mathbf{E}(\tilde{\zeta}(0)\tilde{\zeta}(s)) = \frac{a}{ce^{|s|} + 1},$$

where $a > 0$ and $c > 1$. Note that the covariance is not differentiable at zero, so the process is not L^2 -differentiable.

If M and Q are independent, but Q is not centred, then

$$\mathbf{E}(\zeta(x)\zeta(y)) = \frac{1}{1-\mathbf{E}M} \cdot \frac{\mathbf{E}Q^2(1-\mathbf{E}M) + (2\mathbf{E}M-1)(\mathbf{E}Q)^2 + (\mathbf{E}Q)^2(y/x)}{(1-\mathbf{E}M)(y/x) + \mathbf{E}M - \mathbf{E}M^2}. \quad (39)$$

The covariance between $\zeta(x)$ and $\zeta(y)$ tends to $(\mathbf{E}Q)^2/(1-\mathbf{E}M)^2$ as $y/x \rightarrow \infty$, which, in particular means that *correlation* between $\zeta(x)$ and $\zeta(y)$ tends to 0, as $y/x \rightarrow \infty$.

7.2. The case of a finite interval

Now consider iterations of $f(z) = Mz + Q$ on the finite interval $(0, 1]$ as described in Section 4. Equation (11) can be written as

$$(\zeta(x))_{x \in (0, 1]} \stackrel{\text{f.d.}}{=} ((M\mathbb{1}_{\{U \leq x\}} + \mathbb{1}_{\{U > x\}})\zeta(x) + Q\mathbb{1}_{\{U \leq x\}})_{x \in (0, 1]}. \quad (40)$$

The process $\zeta(x)$ can be also expressed as the a.s. (pointwise) convergent functional series

$$\zeta(x) = \sum_{n=1}^{\infty} M'_{1,x} \cdots M'_{n-1,x} Q'_{n,x}, \quad x \in (0, 1], \quad (41)$$

where

$$M'_{n,x} := M_n \mathbb{1}_{\{U_n \leq x\}} + \mathbb{1}_{\{U_n > x\}} = M_n^{\mathbb{1}_{\{U_n \leq x\}}}, \quad Q'_{n,x} := Q_n \mathbb{1}_{\{U_n \leq x\}}, \quad n \geq 1, x \in (0, 1].$$

If $M = \lambda \in (0, 1)$ is fixed, then

$$\zeta(x) = \sum_{n=1}^{\infty} \lambda^{\mathbb{1}_{\{U_1 \leq x\}} + \cdots + \mathbb{1}_{\{U_{n-1} \leq x\}}} Q_n \mathbb{1}_{\{U_n \leq x\}} =: \sum_{n=1}^{\infty} \lambda^{T_{n-1}(x)} Q_n \mathbb{1}_{\{U_n \leq x\}}, \quad (42)$$

where $T_n(x) := n\widehat{F}_n(x)$, with $\widehat{F}_n(x)$ being the standard empirical distribution function for the sample $\{U_1, \dots, U_n\}$. Note that if Q_1 is Gaussian, then the sum also has a Gaussian distribution, that is, the univariate distributions of $\zeta(x)$ are Gaussian. Recall that the distribution of $\zeta(x)$ does not depend on $x > 0$.

Let us derive an alternative representation for $\zeta(x)$. Put

$$S_n(x) := \inf \left\{ k \in \mathbb{N} : \sum_{j=1}^k \mathbb{1}_{\{U_j \leq x\}} = n \right\} = \inf \{ k \in \mathbb{N} : T_k(x) = n \}, \quad n \in \mathbb{N},$$

and note that $\{T_{k-1}(x) = j, U_k \leq x\} = \{S_{j+1}(x) = k\}$ for all $j \geq 0$ and $k \in \mathbb{N}$. Thus,

$$\begin{aligned} \zeta(x) &= \sum_{n=1}^{\infty} \lambda^{T_{n-1}(x)} Q_n \mathbb{1}_{\{U_n \leq x\}} = \sum_{n=1}^{\infty} \sum_{j=0}^{\infty} \lambda^j \mathbb{1}_{\{T_{n-1}(x)=j\}} Q_n \mathbb{1}_{\{U_n \leq x\}} \\ &= \sum_{j=0}^{\infty} \lambda^j \sum_{n=1}^{\infty} Q_n \mathbb{1}_{\{S_{j+1}(x) = n\}} = \sum_{j=0}^{\infty} \lambda^j Q_{S_{j+1}(x)}. \end{aligned}$$

Summarising we derive the following representation

$$\zeta(x) = \sum_{j=1}^{\infty} \lambda^{j-1} Q_{S_j(x)}, \quad x \in (0, 1]. \quad (43)$$

Note that $S_n(x)$ is distributed as a sum of n independent geometric random variables on $\{1, 2, \dots\}$ with success probability x .

7.3. Bernoulli convolutions

If $M = \lambda \in (0, 1)$ and Q takes values 0 and 1 with equal probabilities, then $\zeta(x) = \zeta([0, x])$ is the Bernoulli convolution for each $x > 0$, see [15,16,30].² By (39),

$$\mathbf{E}(\zeta(x)\zeta(y)) = \frac{x+y}{4(1-\lambda)^2(y+\lambda x)}, \quad x \leq y. \quad (44)$$

If $\lambda < 1/2$, then the distribution of $\zeta(x)$ and the finite-dimensional distributions of the process ζ are singular. If $\lambda = 1/2$, then $\zeta(x)$ has the uniform distribution on $[0, 2]$ for all x . Let $\mu_{\text{BC},1/2}^{(x)}$ denote the joint distribution of $(\zeta(x), \zeta(1))$ for $\lambda = 1/2$. A sample from the distribution $\mu_{\text{BC},1/2}^{(0.8)}$ is shown on Figure 1, suggesting that the $\mu_{\text{BC},1/2}^{(x)}$ is singular for $x \in (0, 1)$.

The probability measure $\mu_{\text{BC},1/2}^{(x)}$ is the invariant measure for the affine iterated function system on \mathbb{R}^2 generated by $g_i(z) := \widehat{\mathbf{M}}_i z + \widehat{\mathbf{Q}}_i$, $i = 1, \dots, 4$, where

$$\widehat{\mathbf{M}}_1 = \widehat{\mathbf{M}}_2 := \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}, \quad \widehat{\mathbf{M}}_3 = \widehat{\mathbf{M}}_4 := \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix},$$

and

$$\widehat{\mathbf{Q}}_1 := (1, 1), \quad \widehat{\mathbf{Q}}_2 := (0, 0), \quad \widehat{\mathbf{Q}}_3 := (0, 1), \quad \widehat{\mathbf{Q}}_4 := (0, 0).$$

The corresponding probabilities

$$p_1 = p_2 := \frac{x}{2}, \quad p_3 = p_4 := \frac{1-x}{2} \quad (45)$$

determine a measure on $\{1, \dots, 4\}$ and then the product measure m on $\{1, \dots, 4\}^{\mathbb{Z}}$. Then $\mu_{\text{BC},1/2}^{(x)}$ is the image of m under the map

$$\{1, \dots, 4\}^{\mathbb{Z}^+} \ni (i_0, i_1, \dots) \mapsto \lim_{n \rightarrow \infty} (\widehat{\mathbf{Q}}_{i_0} + \widehat{\mathbf{M}}_{i_0} \widehat{\mathbf{Q}}_{i_1} + \dots + \widehat{\mathbf{M}}_{i_0} \dots \widehat{\mathbf{M}}_{i_{n-1}} \widehat{\mathbf{Q}}_{i_n}).$$

²It is often alternatively assumed that Q takes values 1 and -1 .

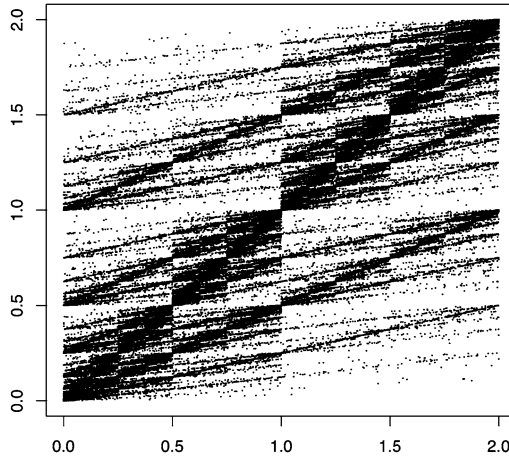


Figure 1. A simulated sample of values for $(\zeta(0.8), \zeta(1))$ for Bernoulli convolutions.

The above system of affine maps g_1, g_2, g_3, g_4 exhibits exact overlaps, for example, $g_1 \circ g_3 \circ g_3 \circ g_1 = g_3 \circ g_1 \circ g_1 \circ g_3$.

The top Lyapunov exponent is

$$\lambda_1(x) := \lim_{n \rightarrow \infty} \frac{1}{n} \log \|\widehat{\mathbf{M}}_{i_1} \cdots \widehat{\mathbf{M}}_{i_n}\| = -x \log 2,$$

where the limit holds for m -almost all sequences (i_1, i_2, \dots) by the strong law of large numbers. Since the top Lyapunov exponent is negative, the iterated function system is contracting on average. Noticing that m is ergodic, Theorem 1.2 in [17] applies and yields that $\mu_{\text{BC},1/2}^{(x)}$ is exact dimensional. By definition, this means that the limit

$$\dim_{\text{loc}}(\mu_{\text{BC},1/2}^{(x)}, z) := \lim_{r \downarrow 0} \frac{\log \mu_{\text{BC},1/2}^{(x)}(B_r(z))}{\log r}, \tag{46}$$

which defines the local dimension of $\mu_{\text{BC},1/2}^{(x)}$ at point z , exists and takes the same value for $\mu_{\text{BC},1/2}^{(x)}$ -almost all z . Moreover, this common value coincides with the Hausdorff dimension $\dim_{\text{H}} \mu_{\text{BC},1/2}^{(x)}$. Here $B_r(z)$ is the Euclidean ball of radius r centred at z . The dimension formula of Feng [17], Th. 1.3, applies in this case and yields that

$$\dim_{\text{H}} \mu_{\text{BC},1/2}^{(x)} = \frac{h_1 - h_0}{\lambda_1(x)} + \frac{h_2 - h_1}{\lambda_2}, \tag{47}$$

where $\lambda_2 = -\log 2$ is the second Lyapunov exponent, and h_0, h_1, h_2 are (conditional) entropies of the system. First,

$$h_0 = -x \log x - (1 - x) \log(1 - x) + \log 2 := I(x) + \log 2$$

is the unconditional entropy of the distribution (45). While the exact calculation of h_2 constitutes a hard combinatorial problem, h_1 can be determined by noticing that the first summand in (47) is equal to the dimension of the invariant measure for the iterative system on the line composed of the functions

$x/2 + 1, x/2, x, x$ with probabilities (45).³ This invariant measure is the uniform distribution on $[0, 2]$, hence, $h_1 = h_0 - x \log 2$. Finally, since $h_2 \geq 0$, we obtain

$$1 \leq \dim_{\text{H}} \mu_{\text{BC}, 1/2}^{(x)} \leq \min(2, 2 - x + I(x)/\log 2). \quad (48)$$

The upper bound alternatively arises from the calculation of the Lyapunov dimension of the iterative function system, see [23] and [21]. Note that the right-hand side of (48) is smaller than 2 if and only if $x \in (x^*, 1)$ where $x^* \approx 0.772908$ is the unique positive root of the equation $I(x) = x \log 2$. Thus, $\mu_{\text{BC}, 1/2}^{(x)}$ is singular for $x > x^*$. In particular, if $x = 0.8$, then the upper bound equals ≈ 1.92 , confirming singularity of the distribution corresponding to Figure 1. We conjecture that $\mu_{\text{BC}, 1/2}^{(x)}$ is singular for all $x \in (0, 1)$.

Theorem 7.1. *The local dimension of the distribution $\mu_{\text{BC}, 1/2}^{(x)}$ of $(\zeta(x), \zeta(1))$ in the Bernoulli convolution scheme with $\lambda = 1/2$ equals*

$$\dim_{\text{loc}}(\mu_{\text{BC}, 1/2}^{(x)}, z) = 2 - \frac{\log(1+x)}{\log 2} \quad (49)$$

for arbitrary $z := (z_1, z_2) \in [0, 2]^2$ with finite binary expansions such that the expansion of z_1 is a substring of the expansion of z_2 .

The proof is postponed to Appendix.

Since the binary rational points in $[0, 2]^2$ have $\mu_{\text{BC}, 1/2}^{(x)}$ -measure zero, Theorem 7.1 does not allow us to conclude that the dimension of μ is given by (49). We leave the stronger variant of this statement as a conjecture. Note that (49) complies with the bounds given in (48).

Conjecture 7.2. *We conjecture that $\dim_{\text{H}} \mu_{\text{BC}, 1/2}^{(x)} = 2 - \frac{\log(1+x)}{\log 2}$.*

The following result shows that the right-hand side of (49) provides a lower bound on the dimension of $\mu_{\text{BC}, 1/2}^{(x)}$.

Theorem 7.3. *The dimension of the distribution $\mu_{\text{BC}, 1/2}^{(x)}$ of $(\zeta(x), \zeta(1))$ in the Bernoulli convolution scheme with $\lambda = 1/2$ satisfies*

$$\dim_{\text{H}} \mu_{\text{BC}, 1/2}^{(x)} \geq 2 - \frac{\log(1+x)}{\log 2}. \quad (50)$$

We close the section on perpetuities by referring the reader to the last subsection of the Appendix where two further examples related to perpetuities are discussed in brief.

8. Other examples

Example 8.1. Assume that $f(x) \equiv Q$ for some random variable Q whose distribution we denote by \mathbf{P}_Q . Note that this case corresponds to a degenerate perpetuity with $M = 0$ a.s. Let us assume that \mathcal{X} is $[0, \infty)$ with μ being the Lebesgue measure. The process \mathcal{P} can be regarded as a marked

³The authors are grateful to D.-J. Feng for this argument.

Poisson process on $[0, \infty)^2$ with unit intensity and the marks being i.i.d. random variables (Q_k) with distribution \mathbf{P}_Q which are also independent of positions of the points in \mathcal{P} . Let $(t_k, x_k)_{k \in \mathbb{Z}}$ be the set of lower left records of \mathcal{P} such that (t_0, x_0) and (t_1, x_1) are separated by the bisectrix $x = t$. Using this notation, the process $(\zeta(x))_{x > 0}$ can be written as follows

$$\zeta(x) = Q_{\inf\{n \in \mathbb{Z}: x_n \leq x\}}, \quad x > 0.$$

In other words, $\zeta(x) = Q_i$ if $x \in (x_{i-1}, x_i]$, $i \in \mathbb{Z}$. The jump points $(x_k)_{k \in \mathbb{Z}}$ form a scale invariant Poisson point process, see, for example, [18], Prop. 2. After the exponential time change, we obtain a process $(\tilde{\zeta}(s))_{s \in \mathbb{R}} = (\zeta(e^s))_{s \in \mathbb{R}}$ that takes i.i.d. values distributed as Q between the points of a standard two-sided Poisson process on \mathbb{R} with unit intensity.

Example 8.2. Assume that $f(z) = \max(1, e^\xi z)$, where $\mathbf{E}\xi < 0$. Then $L_f = \min(1, e^\xi)$. The backward iterations converge a.s. to a random variable e^Y such that Y satisfies the Lindley equation $Y \stackrel{d}{=} \max(0, \xi + Y)$ from queuing theory. It is well known that Y is distributed as

$$\sup_{j \geq 0} \sum_{i=1}^j \xi_i,$$

where $(\xi_i)_{i \in \mathbb{N}}$ are i.i.d. copies of ξ . In other words, $\zeta(x)$ is the supremum of a random walk with negative drift. For the corresponding process $(\zeta(x))_{x \in (0, 1]}$ we have the representation

$$\zeta(x) = \sup_{j \geq 0} \sum_{i=1}^j \xi_i \mathbb{1}_{\{U_i \geq x\}}, \quad x \in (0, 1],$$

where $(U_i)_{i \in \mathbb{N}}$ are i.i.d. uniform on $[0, 1]$ which are also independent of $(\xi_j)_{j \in \mathbb{N}}$.

Example 8.3. Let $f(z) = 1/(z + \xi)$, where ξ is a positive random variable and $z \geq 0$. The iterations produce random continued fractions, see, for example, [27]. The Lipschitz constant of f is $L_f = \xi^{-2}$, so (1) and (2) are fulfilled if $\mathbf{E}\xi^{-2} < \infty$ and $\mathbf{E} \log \xi > 0$.

If ξ is Gamma distributed, then the backwards iterations converge almost surely, and the limit $\zeta(x)$ has the inverse Gaussian distribution. Therefore, one obtains a stochastic process whose all univariate marginals are inverse Gaussian.

If ξ takes values from \mathbb{N} , then it is possible to uniquely recover the sequence of iterations from the limit, so Theorem 6.1 yields the Markov property of the process $(\zeta(x))_{x > 0}$.

9. Concluding remarks

Most of the presented results (with appropriate amendements) hold for Lipschitz functions taking values in an arbitrary Polish space; in this case, one obtains set-indexed functions with values in this Polish space.

It is possible to amend the sieving construction in various ways. For instance, let \mathcal{P} be the Poisson process $\{(x_i, f_i)\}$ in \mathbb{R}^d marked by i.i.d. random Lipschitz functions satisfying (1) and (2). For each point $x \in \mathbb{R}^d$, order the points $(x_i)_{i \in \mathbb{N}}$ of the process according to their distance to x and take the backward iterations of the corresponding functions. This results in a random field indexed by \mathbb{R}^d whose one-dimensional distributions are all identical and which is also scale invariant.

For yet another alternative construction, let \mathcal{P} be the Poisson process $\{(s_i, t_i, f_i)\}$ in $\mathbb{R} \times \mathbb{R}_+ \times \mathcal{G}$. Fix $a > 0$, and for each $x \in \mathbb{R}$ consider the points (s_i, t_i) such that $|x - s_i| \leq at_i$. Order these points by increasing second coordinates t_i and let $\zeta(x)$ be the limit of the backwards iterations of the corresponding functions.

Finally, let us make a concluding remark that a different notion of probabilistic sieving related to so-called generalized leader-election procedures has been recently considered in [5] and [6].

Appendix

In the [Appendix](#), we collect the promised proofs and examples skipped in the main text.

Proof of Proposition 6.2

Let us prove (36), the proof of (37) is similar. Order all points of $\mathcal{P}_{[0, x+\delta]}$ according to their arrival times t_i . Formula (33) implies that, conditionally on $\{\zeta(e^t) = z\}$, the random variable $\zeta(e^{t+\delta})$ is distributed as the following a.s. limit

$$\lim_{n \rightarrow \infty} (g^z)^{1 \uparrow (\tau_1 - 1)} \circ f_1 \circ (g^z)^{\tau_1 \uparrow (\tau_2 - 1)} \circ f_2 \circ \dots \circ (g^z)^{\tau_{n-1} \uparrow (\tau_n - 1)} \circ f_n(z_0),$$

where

$$\tau_0 := 0, \quad \tau_n := \min\{i > \tau_{n-1} : x_i \in (e^t, e^{t+\delta}]\}, \quad n \in \mathbb{N},$$

and $(f_k)_{k \in \mathbb{N}}$ are i.i.d. copies of f which are independent of everything else. Note that $(\tau_n - \tau_{n-1})_{n \in \mathbb{N}}$ are i.i.d. and

$$\mathbf{P}\{\tau_1 = j\} = e^{-\delta(j-1)}(1 - e^{-\delta}), \quad j \in \mathbb{N}.$$

On the first step we show that it is possible to neglect f_2, f_3, \dots , which have been inserted after $g_{\tau_1}^z$, that is

$$\begin{aligned} & \lim_{\delta \downarrow 0} \frac{1}{\delta} \mathbf{E} \left(h \left((g^z)^{1 \uparrow (\tau_1 - 1)} \circ f_1 \circ (g^z)^{\tau_1 \uparrow (\tau_2 - 1)} \circ f_2 \circ \dots \circ (g^z)^{\tau_{n-1} \uparrow (\tau_n - 1)} \circ f_n \circ \dots (z_0) \right) \right. \\ & \left. - h \left((g^z)^{1 \uparrow (\tau_1 - 1)} \circ f_1 \circ (g^z)^{\tau_1 \uparrow \infty} (z_0) \right) \right) = 0. \end{aligned} \quad (51)$$

Using the fact that h' is continuous, hence, bounded on the compact set $\text{supp } Z_\infty$, we derive using the mean value theorem for differentiable functions

$$\begin{aligned} & \left| h \left((g^z)^{1 \uparrow (\tau_1 - 1)} \circ f_1 \circ (g^z)^{\tau_1 \uparrow (\tau_2 - 1)} \circ f_2 \circ \dots \circ (g^z)^{\tau_{n-1} \uparrow (\tau_n - 1)} \circ f_n \circ \dots (z_0) \right) \right. \\ & \left. - h \left((g^z)^{1 \uparrow (\tau_1 - 1)} \circ f_1 \circ (g^z)^{\tau_1 \uparrow \infty} (z_0) \right) \right| \leq \text{const} \cdot c_f^{\tau_2}. \end{aligned}$$

Since $\delta^{-1} \mathbf{E} c_f^{\tau_2} = \delta^{-1} (\mathbf{E} c_f^{\tau_1})^2 \rightarrow 0$ as $\delta \downarrow 0$, the generating operator is given by

$$\begin{aligned} (\mathbf{A}_\uparrow h)(z) &= \lim_{\delta \downarrow 0} \frac{1}{\delta} \left[\mathbf{E} h \left((g^z)^{1 \uparrow (\tau_1 - 1)} \circ f_1 \circ (g^z)^{\tau_1 \uparrow \infty} (z_0) \right) - h(z) \right] \\ &= \lim_{\delta \downarrow 0} \frac{1}{\delta} \left[\mathbf{E} h \left((g^z)^{1 \uparrow (\tau_1 - 1)} \circ f \circ \left((g^z)^{1 \uparrow (\tau_1 - 1)} \right)^{-1} (z) \right) - h(z) \right] \end{aligned}$$

$$= \lim_{\delta \downarrow 0} \frac{1}{\delta} \sum_{k=0}^{\infty} [\mathbf{E}h((g^z)^{1 \uparrow k} \circ f \circ ((g^z)^{1 \uparrow k})^{-1}(z)) - h(z)] e^{-\delta k} (1 - e^{-\delta}), \quad (52)$$

where for the second equality we have used that

$$(g^z)^{\tau_1 \uparrow \infty}(z_0) = ((g^z)^{1 \uparrow (\tau_1 - 1)})^{-1}(z).$$

Using the inequality

$$\mathbf{E}h((g^z)^{1 \uparrow k} \circ f \circ ((g^z)^{1 \uparrow k})^{-1}(z)) - h(z) \leq \text{const} \cdot c_f^k, \quad k \geq 0,$$

by the Lebesgue dominated convergence theorem we can swap the sum and the limit on the right-hand side of (52). This completes the proof of (36).

Markov processes generated by Bernoulli convolutions

As we have mentioned in Section 6, the process $(\zeta(x))_{x>0}$ generated by the mapping $f(x) = \lambda x + Q$ with $\lambda \in (0, 1/2)$ and Q equally likely taking the values 0 and 1, is Markov both in forward and reverse time.

In order to calculate its generating operator, note that each $z \in \text{supp } Z_{\infty} \subset [0, (1 - \lambda)^{-1}]$ corresponds to a sequence $(q_n^z)_{n \in \mathbb{N}}$ from $\{0, 1\}^{\mathbb{N}}$ such that $g_n^z(x) = \lambda x + q_n^z$ and

$$z = (g^z)^{1 \uparrow \infty}(z_0) = \sum_{k=1}^{\infty} \lambda^{k-1} q_k^z. \quad (53)$$

Direct calculations yield

$$g_1^z \circ \cdots \circ g_k^z \circ f \circ (g_k^z)^{-1} \cdots (g_1^z)^{-1}(x) = (1 - \lambda) \sum_{i=1}^k \lambda^{i-1} q_i^z + \lambda^k Q + \lambda x, \quad x \in \mathbb{R}.$$

By (36),

$$(\mathbf{A}_{\uparrow} h)(z) = \sum_{k=0}^{\infty} [\mathbf{E}h(-(1 - \lambda)\hat{z}_k + \lambda^k Q + z) - h(z)],$$

where Q equally likely takes values 0, 1 and

$$\hat{z}_k := \sum_{i=k+1}^{\infty} \lambda^{i-1} q_i^z, \quad k \geq 0.$$

If $h(z) = z$, then

$$(\mathbf{A}_{\uparrow} h)(z) = -(1 - \lambda) \sum_{k=0}^{\infty} \hat{z}_k + \frac{\mathbf{E}Q}{1 - \lambda} = -(1 - \lambda) \sum_{i=1}^{\infty} i \lambda^{i-1} q_i^z + \frac{\mathbf{E}Q}{1 - \lambda}.$$

A curious observation is that the sum in the last formula is the derivative of $\lambda \mapsto \lambda z(\lambda)$ in (53). The generating operator in the reverse time is given by

$$(A_{\downarrow} h)(z) = \sum_{k=0}^{\infty} [h(-(1-\lambda^{-1})\hat{z}_{k-1} - \lambda^{k-2}q_k^z + z) - h(z)].$$

Proof of Theorems 7.1 and 7.3

Proof. First of all, note that we may replace the Euclidean ball in (46) with the ℓ_{∞} -ball. Further, for any sequence $r_n \downarrow 0$, there exist a sequence $(k_n) \uparrow \infty$ of integers such that $2^{-k_n} \leq r_n < 2^{-k_n+1}$, and by the standard sandwich argument we see that it suffices to prove (46) along the sequence $r_k = 2^{-k}$ as $k \rightarrow \infty$.

Order the points $(t_i, x_i, f_i)_{i \in \mathbb{N}}$ of the point process $\mathcal{P}_{[0,1]}$ so that $t_1 \leq t_2 \leq \dots$. Recall the notation

$$T_n(x) = \sum_{j=1}^n \mathbb{1}_{\{x_j \leq x\}} \quad \text{and} \quad S_n(x) = \inf\{k \in \mathbb{N} : T_k(x) = n\}, \quad n \in \mathbb{N}.$$

Since x is fixed, in the following the argument x is omitted.

Let $z_1 = \sum_{k=1}^m \gamma_k/2^{k-1}$, $\gamma_m = 1$, and $z_2 = \sum_{k=1}^n \gamma'_k/2^{k-1}$, $\gamma'_n = 1$, be the binary expansions of z_1 and z_2 , respectively. By the assumption, $\gamma_1\gamma_2\dots\gamma_m$ is a substring of $\gamma'_1\gamma'_2\dots\gamma'_n$ and, in particular, $m \leq n$. Recalling the representation (43) for ζ , we can write

$$\zeta(x) = \sum_{n=1}^{\infty} \frac{Q_{S_n}}{2^{n-1}} \quad \text{and} \quad \zeta(1) = \sum_{n=1}^{\infty} \frac{Q_n}{2^{n-1}}.$$

For $k > \max(n, m)$, we have

$$\begin{aligned} \mu_{\text{BC},1/2}^{(x)}(z + [0, 2^{-k}]^2) &= \mathbf{P}\{\zeta(x) \in [z_1, z_1 + 2^{-k}], \zeta(1) \in [z_2, z_2 + 2^{-k}]\} \\ &= \mathbf{P}\{Q_1 = \gamma'_1, \dots, Q_n = \gamma'_n, Q_{n+1} = \dots = Q_{k+1} = 0, \\ &\quad Q_{S_1} = \gamma_1, \dots, Q_{S_m} = \gamma_m, Q_{S_{m+1}} = \dots = Q_{S_{k+1}} = 0\}. \end{aligned}$$

Denote the event under the last probability sign by A . Since we assume $\gamma_m = 1$, event A can occur only if $\{S_m \leq n\} \cup \{S_m > k+1\}$. We proceed by bounding $\mathbf{P}\{A, S_m > k+1\}$ as follows:

$$\begin{aligned} &\mathbf{P}\{A, S_m > k+1\} \\ &\leq \mathbf{P}\{S_m > k+1, Q_{n+1} = \dots = Q_{k+1} = 0, Q_{S_{m+1}} = \dots = Q_{S_{k+1}} = 0\} \\ &= \mathbf{P}\{S_m > k+1\} \left(\frac{1}{2}\right)^{k-n+1} \left(\frac{1}{2}\right)^{k-m+1} \leq m \mathbf{P}\left\{S_1 > \frac{k+1}{m}\right\} \left(\frac{1}{2}\right)^{k-n+1} \left(\frac{1}{2}\right)^{k-m+1} \\ &= m(1-x)^{(k+1)/m} \left(\frac{1}{2}\right)^{k-n+1} \left(\frac{1}{2}\right)^{k-m+1} = \mathcal{O}\left(\frac{(1-x)^{1/m}}{4}\right)^k \quad \text{as } k \rightarrow \infty. \end{aligned}$$

In order to calculate $\mathbf{P}\{A, S_m \leq n\}$, note that $\{S_m \leq n\} = \{T_n \geq m\}$. By definition, $S_{T_n} \leq n$ and $S_{T_n+1} > n$. Therefore,

$$\mathbf{P}\{A, S_m \leq n\} = \sum_{l=m}^n \sum_{j=l}^n \mathbf{P}\{S_m \leq n, T_n = l, S_l = j, S_{l+1} > n, B_{m,n}, C_{n,k}\}$$

where

$$B_{m,n} = \{Q_1 = \gamma'_1, \dots, Q_n = \gamma'_n, Q_{S_1} = \gamma_1, \dots, Q_{S_m} = \gamma_m, Q_{S_{m+1}} = \dots = Q_{S_{T_n}} = 0\},$$

$$C_{n,k} = \{Q_{n+1} = \dots = Q_{k+1} = 0, Q_{S_{T_n+1}} = \dots = Q_{S_{k+1}} = 0\}.$$

Note that $S_{l+i} = S_l + S'_i$, $i \geq 1$, where $(S'_i)_{i \in \mathbb{N}}$ is a distributional copy of the random walk $(S_i)_{i \in \mathbb{N}}$. Then

$$\begin{aligned} & \mathbf{P}\{S_m \leq n, T_n = l, S_l = j, S_{l+1} > n, B_{m,n}, C_{n,k}\} \\ &= \mathbf{P}\{C_{n,k}, S_{l+1} > n | T_n = l, S_l = j, B_{m,n}\} \mathbf{P}\{S_m \leq n, T_n = l, S_l = j, B_{m,n}\}, \end{aligned}$$

and further

$$\begin{aligned} & \mathbf{P}\{C_{n,k}, S_{l+1} > n | T_n = l, S_l = j, B_{m,n}\} \\ &= \mathbf{P}\{Q_{n+1} = \dots = Q_{k+1} = 0, Q_{j+S'_1} = \dots = Q_{j+S'_{k-l+1}} = 0, j + S'_1 > n\} \\ &= \mathbf{P}\{Q_{n+1} = \dots = Q_{k+1} = 0, Q_{n+S'_1} = \dots = Q_{n+S'_{k-l+1}} = 0\} \mathbf{P}\{j + S'_1 > n\}, \end{aligned}$$

where the last equality relies on the memoryless property of the geometrically distributed S'_i . Let N be binomially distributed $\text{Bin}(k - n + 1, x)$. Then

$$\begin{aligned} & \mathbf{P}\{Q_{n+1} = \dots = Q_{k+1} = Q_{n+S'_1} = \dots = Q_{n+S'_{k-l+1}} = 0\} \\ &= \left(\frac{1}{2}\right)^{k-n+1} \mathbf{E} \left(\frac{1}{2}\right)^{k-l+1-N} = \sum_{i=0}^{k-n+1} \binom{k-n+1}{i} x^i (1-x)^{k-n+1-i} \left(\frac{1}{2}\right)^{2k-n-l+2-i} \\ &= \left(\frac{1}{2}\right)^{k-l+1} \left(\frac{1+x}{2}\right)^{k-n+1}. \end{aligned}$$

Thus,

$$\begin{aligned} & \mathbf{P}\{A, S_m \leq n\} \\ &= \left(\frac{1}{2}\right)^k \left(\frac{1+x}{2}\right)^k \\ & \quad \times \sum_{l=m}^n \sum_{j=l}^n (1-x)^{n-j} \left(\frac{1}{2}\right)^{-l+1} \left(\frac{1+x}{2}\right)^{-n+1} \mathbf{P}\{S_m \leq n, T_n = l, S_l = j, S_{l+1} > n, B_{m,n}\}. \end{aligned}$$

Note that the double sum does not depend on k . Hence,

$$\mu_{\text{BC},1/2}^{(x)}(z + [0, 2^{-k}]^2) = \text{const} \left(\frac{1}{2}\right)^k \left(\frac{1+x}{2}\right)^k + \mathcal{O}\left(\frac{(1-x)^{1/m}}{4}\right)^k,$$

where the constant does not depend on k (but might depend on m and n). The same expression holds for $\mu_{\text{BC},1/2}^{(x)}(z + [-2^{-k}, 0]^2)$.

Furthermore,

$$\begin{aligned} & \mu_{\text{BC},1/2}^{(x)}(z + [-2^{-k}, 0] \times [0, 2^{-k}]) \\ &= \mathbf{P}\{Q_1 = \gamma'_1, \dots, Q_n = \gamma'_n, Q_{n+1} = \dots = Q_{k+1} = 0, \\ & \quad Q_{S_1} = \gamma_1, \dots, Q_{S_{m-1}} = \gamma_{m-1}, Q_{S_m} = 0, Q_{S_{m+1}} = \dots = Q_{S_{k+1}} = 1\}. \end{aligned}$$

The event under probability sign occurs only if $S_{m+1} \leq n$ or $S_{m+1} > k + 1$. By taking the double sum over $T_n = l$ and $S_l = j$ for $m + 1 \leq l \leq n$ and $l \leq j \leq n$ as above, we arrive at

$$\mu_{\text{BC},1/2}^{(x)}(z + [-2^{-k}, 0] \times [0, 2^{-k}]) \leq \text{const} \cdot \left(\frac{1-x}{4}\right)^k,$$

where the constant does not depend on k . Furthermore, $\mu_{\text{BC},1/2}^{(x)}(z + [0, -2^{-k}] \times [-2^{-k}, 0])$ is bounded by the same expression. Thus,

$$\mu_{\text{BC},1/2}^{(x)}(z + [-2^{-k}, 2^{-k}]^2) = c \left(\frac{1}{2}\right)^k \left(\frac{1+x}{2}\right)^k + \mathcal{O}\left(\frac{(1-x)^{1/m}}{4}\right)^k.$$

Finally, (46) yields (49). □

Proof of Theorem 7.3. All points $z := (z_1, z_2)$ in the support of $\mu_{\text{BC},1/2}^{(x)}$ can be represented as binary expansions $z_1 = \sum_{k=1}^{\infty} \gamma_k / 2^{k-1}$ and $z_2 = \sum_{k=1}^{\infty} \gamma'_k / 2^{k-1}$, where the sequences $\gamma := (\gamma_k)_{k \in \mathbb{N}}$ and $\gamma' := (\gamma'_k)_{k \in \mathbb{N}}$ in $\{0, 1\}^{\mathbb{N}}$ are such that γ is a subsequence of γ' . For almost all z , there is an infinite increasing sequence $(\tau_k)_{k \in \mathbb{N}}$ of natural numbers such that $\gamma_{\tau_k+1} = \gamma'_{\tau_k+1} = 0$ and $\gamma_{\tau_k+2} = \gamma'_{\tau_k+2} = 1$ for all $k \geq 1$. This follows from the Borel–Cantelli lemma applied to the sequence of independent events

$$B_n := \{Q_{Y_n} = 0, Q_{Y_{n+1}} = 1, Q_{S_{Y_n}} = 0, Q_{S_{Y_{n+1}}} = 1\}, \quad n \geq 1,$$

where $Y_1 = 1$, and $Y_{n+1} = S_{Y_n} + 1$, $n \geq 1$. Note that the sequence $(\tau_k)_{k \in \mathbb{N}}$ is not random, it is determined by the sequences γ and γ' . Given that $\mu_{\text{BC},1/2}^{(x)}$ is exact dimensional and the limit in (46) exists, it is possible to take the limit along $r_k = 2^{-\tau_k}$, $k \in \mathbb{N}$.

Consider $\tilde{z} := (\tilde{z}_1, \tilde{z}_2)$ with

$$\tilde{z}_1 := \sum_{j=1}^{\tau_k} 2^{-(j-1)} \gamma_j \quad \text{and} \quad \tilde{z}_2 := \sum_{j=1}^{\tau_k} 2^{-(j-1)} \gamma'_j.$$

Then

$$z + [-2^{-(\tau_k+1)}, 2^{-(\tau_k+1)}] \subset \tilde{z} + [0, 2^{-\tau_k}] \subset z + [-2^{-\tau_k}, 2^{-\tau_k}],$$

where we used that $\gamma_{\tau_k+2} = \gamma'_{\tau_k+2} = 1$. Therefore, it suffices to consider

$$\begin{aligned} & \mu_{\text{BC},1/2}^{(x)}(\tilde{z} + [0, 2^{-\tau_k}]) \\ &= \mathbf{P}\{Q_i = \gamma'_i, i = 1, \dots, \tau_k, \gamma'_j = \gamma_j, j = 1, \dots, T_{\tau_k}, Q_{S_l} = \gamma_{S_l}, l = T_{\tau_k} + 1, \dots, \tau_k\}. \end{aligned}$$

Note that $T_{\tau_k} := \sum_{j=1}^{\tau_k} \mathbb{1}_{\{x_j \leq x\}}$ has the binomial distribution $\text{Bin}(\tau_k, x)$, so that

$$\begin{aligned} \mu_{\text{BC},1/2}^{(x)}(\tilde{z} + [0, 2^{-\tau_k}]) &\leq \mathbf{P}\{Q_i = \gamma'_i, i = 1, \dots, \tau_k, Q_{S_l} = \gamma_{S_l}, l = T_{\tau_k} + 1, \dots, \tau_k\} \\ &= \left(\frac{1}{2}\right)^{\tau_k} \mathbf{E}\left(\frac{1}{2}\right)^{\tau_k - T_{\tau_k}} = \left(\frac{1}{2}\right)^{\tau_k} \left(\frac{1+x}{2}\right)^{T_{\tau_k}}. \end{aligned}$$

The conclusion follows from (46). □

Further examples related to perpetuities

Example A.1. Assume that Q is standard normal and $M = \lambda \in (0, 1)$ is constant. Then $\zeta(x)$, $x > 0$, has univariate Gaussian marginals, and its covariance is given by

$$\mathbf{E}(\zeta(x)\zeta(y)) = \frac{x}{(1-\lambda)(y+\lambda x)}.$$

By time change $x = e^s$, we arrive at a centred stationary process $\tilde{\zeta}(s)$, $s \in \mathbb{R}$, with univariate Gaussian marginals and covariance

$$\mathbf{E}(\tilde{\zeta}(0)\tilde{\zeta}(s)) = \frac{1}{(1-\lambda)(e^{|s|} + \lambda)}.$$

The bivariate distributions of this process are no longer Gaussian, see Figure 2.

Example A.2. Let $M = Q$ for the standard uniform Q . In this case, $\zeta(x)$ for each $x > 0$ follows the Dickman distribution, see, for example, [28]. While the obtained stochastic process has all univariate Dickman marginals, it does not have independent increments like the Dickman process constructed using the infinite divisibility property of the Dickman law, see [9]. In our case we have the representation

$$\zeta(x) = \sum_{n=1}^{\infty} Q_1^{\mathbb{1}_{\{U_1 \leq x\}}} \dots Q_{n-1}^{\mathbb{1}_{\{U_{n-1} \leq x\}}} Q_n \mathbb{1}_{\{U_n \leq x\}}.$$

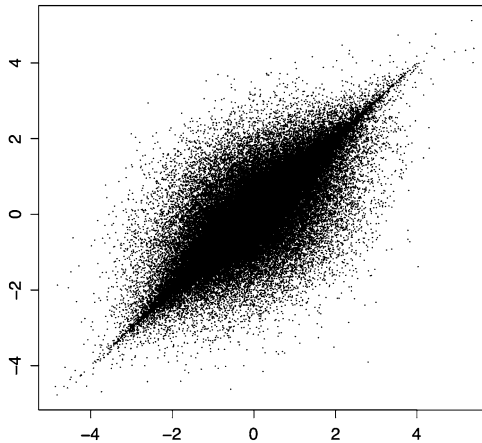


Figure 2. A simulation of $(\zeta(0.7), \zeta(1))$ for normal Q and $\lambda = 1/2$.

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References

- [1] Akiyama, S., Feng, D.-J., Kempton, T. and Persson, T. (2018). On the Hausdorff dimension of Bernoulli convolutions. *Int. Math. Res. Not.* rny209. <https://doi.org/10.1093/imrn/rny209>
- [2] Alsmeyer, G. and Fuh, C. (2002). Corrigendum to: "Limit theorems for iterated random functions by regenerative methods". *Stochastic Process. Appl.* **97** 341–345. MR1875338 [https://doi.org/10.1016/S0304-4149\(01\)00144-2](https://doi.org/10.1016/S0304-4149(01)00144-2)
- [3] Alsmeyer, G. and Fuh, C.-D. (2001). Limit theorems for iterated random functions by regenerative methods. *Stochastic Process. Appl.* **96** 123–142. MR1856683 [https://doi.org/10.1016/S0304-4149\(01\)00104-1](https://doi.org/10.1016/S0304-4149(01)00104-1)
- [4] Alsmeyer, G., Iksanov, A. and Rösler, U. (2009). On distributional properties of perpetuities. *J. Theoret. Probab.* **22** 666–682. MR2530108 <https://doi.org/10.1007/s10959-008-0156-8>
- [5] Alsmeyer, G., Kabluchko, Z. and Marynych, A. (2016). Leader election using random walks. *ALEA Lat. Am. J. Probab. Math. Stat.* **13** 1095–1122. MR3580090 <https://doi.org/10.30757/alea.v13-39>
- [6] Alsmeyer, G., Kabluchko, Z. and Marynych, A. (2017). A leader-election procedure using records. *Ann. Probab.* **45** 4348–4388. MR3737913 <https://doi.org/10.1214/16-AOP1167>
- [7] Buraczewski, D., Damek, E. and Mikosch, T. (2016). *Stochastic Models with Power-Law Tails: The Equation $X = AX + B$* . Springer Series in Operations Research and Financial Engineering. Cham: Springer. MR3497380 <https://doi.org/10.1007/978-3-319-29679-1>
- [8] Chamayou, J.-F. and Letac, G. (1991). Explicit stationary distributions for compositions of random functions and products of random matrices. *J. Theoret. Probab.* **4** 3–36. MR1088391 <https://doi.org/10.1007/BF01046992>
- [9] Covo, S. (2009). On approximations of small jumps of subordinators with particular emphasis on a Dickman-type limit. *J. Appl. Probab.* **46** 732–755. MR2562319 <https://doi.org/10.1239/jap/1253279849>
- [10] DeGroot, M.H. and Rao, M.M. (1963). Stochastic give-and-take. *J. Math. Anal. Appl.* **7** 489–498. MR0157423 [https://doi.org/10.1016/0022-247X\(63\)90071-4](https://doi.org/10.1016/0022-247X(63)90071-4)
- [11] Diaconis, P. and Freedman, D. (1999). Iterated random functions. *SIAM Rev.* **41** 45–76. MR1669737 <https://doi.org/10.1137/S0036144598338446>
- [12] Dubins, L.E. and Freedman, D.A. (1967). Random distribution functions. In *Proc. Fifth Berkeley Sympos. Math. Statist. and Probability (Berkeley, Calif., 1965/66)* 183–214. Berkeley, CA: Univ. California Press. MR0214109
- [13] Dufflo, M. (1997). *Random Iterative Models. Applications of Mathematics (New York)* **34**. Berlin: Springer. Translated from the 1990 French original by Stephen S. Wilson and revised by the author. MR1485774 <https://doi.org/10.1007/978-3-662-12880-0>
- [14] Dümbgen, L. (2017). Empirische Prozesse. Lecture Notes. Available at https://ilias.unibe.ch/goto_ilias3_unibe_cat_915738.html.
- [15] Erdős, P. (1939). On a family of symmetric Bernoulli convolutions. *Amer. J. Math.* **61** 974–976. MR0000311 <https://doi.org/10.2307/2371641>
- [16] Erdős, P. (1940). On the smoothness properties of a family of Bernoulli convolutions. *Amer. J. Math.* **62** 180–186. MR0000858 <https://doi.org/10.2307/2371446>
- [17] Feng, D.-J. (2019). Dimension of invariant measures for affine iterated function systems. Preprint available at <https://arxiv.org/abs/1901.01691>.

- [18] Gnedin, A.V. (2008). Corners and records of the Poisson process in quadrant. *Electron. Commun. Probab.* **13** 187–193. MR2399280 <https://doi.org/10.1214/ECP.v13-1351>
- [19] Goldie, C.M. and Maller, R.A. (2000). Stability of perpetuities. *Ann. Probab.* **28** 1195–1218. MR1797309 <https://doi.org/10.1214/aop/1019160331>
- [20] Grincevičius, A.K. (1981). A random difference equation. *Lith. Math. J.* **21** 302–306.
- [21] Hochman, M. and Rapaport, A. (2019). Hausdorff dimension of planar self-affine sets and measures with overlaps. Preprint available at <https://arxiv.org/abs/1904.09812>.
- [22] Iksanov, A. (2016). *Renewal Theory for Perturbed Random Walks and Similar Processes. Probability and Its Applications*. Cham: Birkhäuser/Springer. MR3585464 <https://doi.org/10.1007/978-3-319-49113-4>
- [23] Jordan, T., Pollicott, M. and Simon, K. (2007). Hausdorff dimension for randomly perturbed self affine attractors. *Comm. Math. Phys.* **270** 519–544. MR2276454 <https://doi.org/10.1007/s00220-006-0161-7>
- [24] Kesten, H. (1973). Random difference equations and renewal theory for products of random matrices. *Acta Math.* **131** 207–248. MR0440724 <https://doi.org/10.1007/BF02392040>
- [25] Khinchin, A.Ya. (1997). *Continued Fractions*, Russian ed. Mineola, NY: Dover. With a preface by B. V. Gnedenko, Reprint of the 1964 translation. MR1451873
- [26] Letac, G. (1986). A contraction principle for certain Markov chains and its applications. In *Random Matrices and Their Applications (Brunswick, Maine, 1984)*. *Contemp. Math.* **50** 263–273. Providence, RI: Amer. Math. Soc. MR0841098 <https://doi.org/10.1090/conm/050/841098>
- [27] Letac, G. and Seshadri, V. (1983). A characterization of the generalized inverse Gaussian distribution by continued fractions. *Z. Wahrsch. Verw. Gebiete* **62** 485–489. MR0690573 <https://doi.org/10.1007/BF00534200>
- [28] Penrose, M.D. and Wade, A.R. (2004). Random minimal directed spanning trees and Dickman-type distributions. *Adv. in Appl. Probab.* **36** 691–714. MR2079909 <https://doi.org/10.1239/aap/1093962229>
- [29] Peres, Y., Schlag, W. and Solomyak, B. (2000). Sixty years of Bernoulli convolutions. In *Fractal Geometry and Stochastics, II (Greifswald/Koserow, 1998)*. *Progress in Probability* **46** 39–65. Basel: Birkhäuser. MR1785620
- [30] Solomyak, B. (1995). On the random series $\sum \pm \lambda^n$ (an Erdős problem). *Ann. of Math. (2)* **142** 611–625. MR1356783 <https://doi.org/10.2307/2118556>
- [31] Vapnik, V.N. (1998). *Statistical Learning Theory. Adaptive and Learning Systems for Signal Processing, Communications, and Control*. New York: Wiley. A Wiley-Interscience Publication. MR1641250
- [32] Varjú, P.P. (2019). On the dimension of Bernoulli convolutions for all transcendental parameters. *Ann. of Math. (2)* **189** 1001–1011. MR3961088 <https://doi.org/10.4007/annals.2019.189.3.9>

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