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Fractional stochastic wave equation driven by a Gaussian noise rough in space

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In this article, we consider fractional stochastic wave equations on $\mathbb{R}$ driven by a multiplicative Gaussian noise which is white/colored in time and has the covariance of a fractional Brownian motion with Hurst parameter $H \in (\frac{1}{4}, \frac{1}{2})$ in space. We prove the existence and uniqueness of the mild Skorohod solution, establish lower and upper bounds for the pth moment of the solution for all $p \geq 2$, and obtain the Hölder continuity in time and space variables for the solution.

Keywords: fractional Brownian motion; Hölder continuity; intermittency; Malliavin calculus; Skorohod integral; stochastic wave equation

1. Introduction

Consider the following fractional stochastic wave equation (SWE) on $\mathbb{R}$

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2}(t,x) = -(-\Delta)^{\frac{\kappa}{2}} u(t,x) + u\dot{W}(t,x), \quad t > 0, \\ u(0,x) = 1, \qquad \dfrac{\partial u}{\partial t}(0,x) = 0, \end{cases} \tag{1.1}$$

with $\kappa \in (0,2]$, where $\dot{W}(t,x)$ is a Gaussian noise with covariance

$$\mathbb{E}\big[\dot{W}(t,x)\dot{W}(s,y)\big] = f_0(t-s)f(x-y).$$

In this article, we assume that the noise is rough in space, i.e., $f(x) = (|x|^{2H})''$ with $H \in (\frac{1}{4}, \frac{1}{2})$, where $(|x|^{2H})''$ means the second derivative of $|x|^{2H}$ in the sense of distribution. Note that for fractional Brownian motion B^H on $\mathbb{R}$ with Hurst parameter $H \in (0,1)$, its derivative (in the sense of distribution) $\dot{B}^H$ has the covariance $\mathbb{E}[\dot{B}^H(x)\dot{B}^H(y)] = H(2H-1)f(x-y)$. We also assume that the temporal covariance function $f_0(t)$ is either the Dirac delta function $\delta(t)$ or

a nonnegative and nonnegative-definite function such that $f_0(t) \sim |t|^{2H_0-2}$ with $H_0 \in (\frac{1}{2}, 1)$, that is, $c|t|^{2H_0-2} \le f_0(t) \le C|t|^{2H_0-2}$ for some constants $0 < c < C < \infty$.

The Itô-type probabilistic approach for stochastic partial differential equations (SPDEs) was established in [43], where Walsh introduced martingale measures and defined stochastic integrals with respect to the martingale measures, and then SPDEs driven by space-time white noise were investigated. Following Walsh's approach, SWEs on $\mathbb{R}^d$ with $d \le 2$ were studied, for instance in [19,36,37]. In [18], Dalang extended Walsh's stochastic integral and applied it to solve SPDEs whose Green's function is not a function but a Schwartz distribution. In particular, Dalang's theory is applicable to SWEs in d-dimension with $d \ge 3$, and we refer to [15,22,26] and the references therein for the study of SWEs in high dimensions.

For SPDEs driven by a multiplicative Gaussian noise which is colored in time (i.e., the temporal covariance f_0 is not the Dirac delta function), the probabilisitic approach based on martingale properties cannot be applied directly since the noise does not have martingale structure in time. An alternative approach is to apply Malliavin calculus to study the chaos expansion of the Skorohod solution, see, for instance, [27,29,30,41] for stochastic heat equations (SHEs) and [1,6] for SWEs.

In this article, we shall prove the existence and uniqueness of the mild Skorohod solution to (1.1) (Theorem 3.2), establish lower and upper bounds for the pth moment of the solution for all $p \ge 2$ (Proposition 4.1), and obtain the Hölder continuity for the solution in time and space variables (Proposition 5.1). In the following, we briefly describe some related recent development on SHEs and SWEs driven by multiplicative Gaussian noise.

Hu and Nualart [29] investigated SHEs driven by a multiplicative fractional Brownian sheet that is colored in time and white in space. Hu et al. [30] obtained Feynman-Kac formulae for solutions of SHEs driven by a fractional Brownian sheet, and used them to investigate the regularity of the solutions. The result in [30] then was extended to SHEs driven by a general Gaussian noise in Hu et al. [27] and to SHEs with the Laplacian operator being replaced by the infinitesimal generator of a symmetric Lévy process in Song [41]. The noise considered in the papers [27,29,30,41] is not "rough", as its spatial covariance corresponds to that of fractional Brownian motion with Hurst parameter $H \ge \frac{1}{2}$. The SHEs and SWEs on $\mathbb{R}$ driven by a Gaussian noise that is white in time and rough in space were investigated in Hu et al. [25] and in Balan et al. [3], respectively. Recently, Chen conducted a systematic investigation on SHEs with noise that is rough in space and/or in time in [11,12].

If the solution of a dynamic system with noise develops very high peaks, it is said that the system possesses the intermittency property. The concept of intermittency arose in physics, and in mathematics it is related to the long-term asymptotics of the moments of the solution. The intermittency property was studied, for instance, in [2,8,9,17,23,27,40] for heat equations, and in [2,16,20] for wave equations. In particular, precise long-term asymptotics for SHEs was obtained in [10,13,14,31,32], and the second order Lyapunov exponent for SWEs was obtained in [7].

For the Hölder continuity of SHEs driven by multiplicative Gaussian noise colored in time, we refer to [5,27,28,41] and the references therein. For SWEs with noise white in time, Hölder continuity of the solutions was studied in [19] for the spatial dimension $d = 2$, in [22,26] for $d = 3$, and in [15] for general dimensions; for SWEs with noise correlated in time and space, Hölder continuity was established in [1,6] for general dimensions.

Finally, we would like to make some comments on our results.

(a) Note that we require $H > \frac{1}{4}$ for SWEs in Theorem 3.2, which was also assumed in [31] for SHEs with rough spatial noise. Nevertheless, the approach used in the proof of Theorem 3.2 can be also applied to SHEs and relax the condition $H > \frac{1}{4}$ to $H_0 + H > \frac{3}{4}$ (see Remark 3.3).

(b) The rate of the bounds for the pth moments obtained in Proposition 4.1 is consistent with the known results in, for instance, [2,7]. The lower bound is relatively more difficult to establish. One of the obstacles is that the Fourier transform of the Green's function of the fractional wave equation is not a nonnegative function, and this issue is resolved by showing that the integral of the Fourier transform of the Green's function is positive (see Lemma A.5).

(c) The Hölder continuity obtained in Proposition 5.1 is consistent with the known results (e.g., [1], Theorem 5.1, [6], Proposition 8.3, and [15], Theorem 7.6) which dealt with SWEs driven by the noise that is not rough in space. The major difference/difficulty of obtaining the Hölder continuity for SWEs with rough spatial noise is the following. Denoting the spectral measure of the spatial covariance $\mu(d\xi) = \widehat{f}(\xi)\, d\xi$, the condition

$$\sup_{\eta\in\mathbb{R}} \int_{\mathbb{R}} \frac{1}{1+|\xi-\eta|^2} \mu(d\xi) < \infty \tag{1.2}$$

plays a critical role in obtaining the Hölder continuity of the solution when the spatial covariance f is a nonnegative, nonnegative definite, and locally integrable function (see, e.g., [1,6,15]). However, when $H < \frac{1}{2}$, the spatial covariance $(|x|^{2H})''$ is a genuine distribution (see, e.g., [33]), and condition (1.2) is indeed violated (see [3], Lemma A.1).

This article is organized as follows. In Section 2, some preliminary results on Malliavin calculus associated with the noise W are provided. In Section 3, the existence and uniqueness of the solution to (1.1) is obtained under proper conditions. In Section 4, we derive the lower and upper bounds for the pth moment of the solution for $p \geq 2$ and then deduce the weak intermittency. In Section 5, the Hölder continuity of the solution in time and space is obtained. Finally, some lemmas used in the preceding sections are gathered in the Appendix.

2. Preliminaries

In this section, we recall some preliminaries on Malliavin calculus associated with the Gaussian noise $\dot{W}$. We refer to [38] for more details.

Let $\mathcal{H}$ be the completion of the Schwarz space $\mathcal{S}(\mathbb{R}_+ \times \mathbb{R})$ under the inner product

$$\langle \varphi, \phi \rangle_{\mathcal{H}} = C_H \int_{\mathbb{R}_+^2} \int_{\mathbb{R}} f_0(r-s) \widehat{\varphi}(r,\xi) \overline{\widehat{\phi}(s,\xi)} \mu(d\xi)\, dr\, ds, \tag{2.1}$$

where

$$C_H = \frac{\Gamma(2H+1)\sin(\pi H)}{2\pi} \tag{2.2}$$

and $\mu(d\xi) = |\xi|^{1-2H} d\xi$ with $H \in (0, \frac{1}{2})$. Here, $\widehat{\varphi}$ is the Fourier transform of φ in the space, i.e., for $\varphi \in \mathcal{S}(\mathbb{R}_+ \times \mathbb{R})$,

$$\widehat{\varphi}(s,\xi) = \int_{\mathbb{R}} e^{-i\xi x} \varphi(s,x)\, dx.$$

In particular, if φ is a measurable function such that $\widehat{\varphi}$ is also a measurable function and

$$\int_{\mathbb{R}_+^2} \int_{\mathbb{R}} f_0(r-s) \big|\widehat{\varphi}(r,\xi)\big| \big|\widehat{\varphi}(s,\xi)\big| \mu(d\xi)\, dr\, ds < \infty,$$

then $\varphi \in \mathcal{H}$. Note that $\mathcal{H}$ may contain distributions rather than just measurable functions if $f_0(r-s) \sim |r-s|^{2H_0-2}$ for some $H_0 \in (\frac{1}{2}, 1)$ (see [33,39]).

In a complete probability space $(\Omega, \mathcal{F}, P)$, let $W = \{W(\varphi), \varphi \in \mathcal{H}\}$ be an isnormal Gaussian process with the covariance

$$\mathbb{E}\big[W(\varphi) W(\phi)\big] = \langle \varphi, \phi \rangle_{\mathcal{H}},$$

and we also denote

$$W(\varphi) = \int_{\mathbb{R}_+} \int_{\mathbb{R}} \varphi(t,x) W(dt, dx).$$

We also call $W(\varphi)$ the Wiener integral of φ with respect to W. In light of [39], Theorem 3.1, and [33], Proposition 4.1, the Gaussian family $\{W(\varphi), \varphi \in \mathcal{H}\}$ coincides with the linear expansion of the Gaussian family $\{W(t,x), (t,x) \in \mathbb{R}_+ \times \mathbb{R})\}$ with the covariance

$$\mathbb{E}\big[W(t,x) W(s,y)\big] = \frac{1}{2}\big(|x|^{2H} + |y|^{2H} - |x-y|^{2H}\big) \int_0^t \int_0^s f_0(r_1 - r_2)\, dr_1\, dr_2,$$

and in particular $W(t,x) = W(I_{[0,t]\times[0,x]})$ with the convention $I_{[0,t]\times[x,0]} = -I_{[0,t]\times[0,x]}$ for $x < 0$.

For the smooth and cylindrical random variables of the form $F = h(W(\varphi_1), \ldots, W(\varphi_n))$ with h being smooth and its partial derivatives having at most polynomial growth, the Malliavin derivative DF of F is the $\mathcal{H}$-valued random variable defined by

$$DF = \sum_{k=1}^{n} \frac{\partial h}{\partial x_k}\big(W(\varphi_1), \ldots, W(\varphi_n)\big)\varphi_k.$$

Noting that D is closable from $L^2(\Omega)$ to $L^2(\Omega; \mathcal{H})$, we define the Sobolev space $\mathbb{D}^{1,2}$ as the closure of the space of the smooth and cylindrical random variables under the norm

$$\|D\|_{1,2} = \big(\mathbb{E}\big[F^2\big] + \mathbb{E}\big[\|DF\|_{\mathcal{H}}^2\big]\big)^{\frac{1}{2}}.$$

The divergence operator δ, which is also known as the Skorohod integral, is the adjoint of the Malliavin derivative operator D defined by the duality

$$\mathbb{E}\big[F\delta(u)\big] = E\big[\langle DF, u\rangle_{\mathcal{H}}\big] \quad \forall F \in \mathbb{D}^{1,2}, \forall u \in \operatorname{Dom}\delta.$$

Here $\operatorname{Dom}\delta$ is the domain of the divergence operator δ, which is the space of the $\mathcal{H}$-valued random variables $u \in L^2(\Omega;\mathcal{H})$ such that $|E[\langle DF, u\rangle_{\mathcal{H}}]| \le c_F \|F\|_2$ with some constant c_F depending on F, for all $F \in \mathbb{D}^{1,2}$. Thus, for $u \in \operatorname{Dom}\delta$, $\delta(u) \in L^2(\Omega)$. In particular, $\mathbb{E}[\delta(u)] = 0$. We also use the following notation

$$\delta(u) = \int_{\mathbb{R}_+}\int_{\mathbb{R}} u(t,x)W(dt,dx), \quad u \in \operatorname{Dom}\delta.$$

Now we recall the Wiener chaos expansion. Let $\mathbb{H}_0 = \mathbb{R}$, and for any integer $n \ge 1$, let $\mathbb{H}_n$ be the closed linear subspace of $L^2(\Omega)$ containing the set of random variables $\{H_n(W(\varphi)), \varphi \in \mathcal{H}, \|\varphi\|_{\mathcal{H}} = 1\}$, where H_n is the nth Hermite polynomial, that is, $H_n(x) = (-1)^n e^{x^2}\frac{d^n}{dx^n}(e^{-x^2})$. Then $\mathbb{H}_n$ is called the nth Wiener chaos of W. Assuming $\mathcal{F}$ is the σ-field generated by $\{W(\varphi), \varphi \in \mathcal{H}\}$, we have the following Wiener chaos decomposition

$$L^2(\Omega, \mathcal{F}, P) = \bigoplus_{n=0}^{\infty} \mathbb{H}_n.$$

For $n \ge 1$, denote by $\mathcal{H}^{\otimes n}$ the nth tensor product of $\mathcal{H}$, and let $\widetilde{\mathcal{H}}^{\otimes n}$ be the symmetrization of $\mathcal{H}^{\otimes n}$. Then the mapping $I_n(h^{\otimes n}) = H_n(W(h))$ for any $h \in \mathcal{H}$ can be extended to a linear isometry between $\widetilde{\mathcal{H}}^{\otimes n}$ and the nth Wiener chaos $\mathbb{H}_n$. Thus, for any random variable $F \in L^2(\Omega, \mathcal{F}, P)$, it has the following unique Wiener chaos expansion in the sense of $L^2(\Omega)$,

$$F = \mathbb{E}[F] + \sum_{n=1}^{\infty} I_n(f_n) \quad \text{with } f_n \in \widetilde{\mathcal{H}}^{\otimes n}.$$

Throughout the paper, the generic constant C varies at different places.

3. Existence and uniqueness of the solution

In this section, we obtain the existence and uniqueness of the mild Skorohod solution to (1.1) under some conditions in Theorem 3.2, and we show that in Proposition 3.4 these conditions are also necessary if the noise is white in time.

Let $G_t(x)$ be the fundamental solution of the equation $\frac{\partial^2}{\partial t^2}u + (-\Delta)^{\frac{\kappa}{2}}u = 0$ on $\mathbb{R}^d$, then its Fourier transform in space $\widehat{G}_t(\xi)$ solves the following equation

$$\frac{\partial^2 \widehat{G}_t(\xi)}{\partial t^2} + |\xi|^{\kappa}\widehat{G}_t(\xi) = 0,$$

and it is given by (see [24], Section 2.2; [18], Example 6, and [42], Chapter 1, Section 7, for the case $\kappa = 2$)

$$\widehat{G}_t(\xi) = \frac{\sin(t|\xi|^{\kappa/2})}{|\xi|^{\kappa/2}}.$$

Recall that when $\kappa = 2$, the Green's function $G_t(x)$ is a measurable function for $d \le 2$,

$$G_t(x) = \begin{cases} \dfrac{1}{2} I_{\{|x|<t\}} & \text{if } d = 1, \\ \dfrac{1}{2\pi} \dfrac{1}{\sqrt{t^2 - |x|^2}} I_{\{|x|<t\}} & \text{if } d = 2, \end{cases}$$

$G_t(\cdot) = \frac{1}{4\pi t}\sigma_t$ for $d = 3$, where σ_t is the surface measure on the sphere $\{x \in \mathbb{R}^3; |x| = t\}$, and $G_t(\cdot)$ is a genuine distribution with compact support in $\mathbb{R}^d$ if $d \ge 3$. Note that when $\kappa \in (1, 2)$ and $d = 1$, the Green's function $G_t(x) \in L^2(\mathbb{R})$ for all $t \ge 0$ as its Fourier transform $\frac{\sin(t|\xi|^{\kappa/2})}{|\xi|^{\kappa/2}} \in L^2(\mathbb{R})$.

We consider the following filtration

$$\mathcal{F}_t = \sigma\big\{W(I_{[0,s]}\varphi), 0 \le s \le t, \varphi \in \mathcal{S}(\mathbb{R})\big\} \vee \mathcal{N},$$

where $\mathcal{N}$ denotes the collection of null sets.

Definition 3.1. An adapted random field $u = \{u(t, x), t \ge 0, x \in \mathbb{R}\}$ is a mild Skorohod solution to (1.1) if $\mathbb{E}[u^2(t, x)] < \infty$ for all $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$ and it satisfies the following integral equation

$$u(t, x) = 1 + \int_0^t \int_{\mathbb{R}} G_{t-s}(x - y)u(s, y)W(ds, dy), \tag{3.1}$$

where the integral on the right-hand side is a Skorohod integral.

Note that if $\mathbb{E}[|u(t, x)|^2] < \infty$, the solution has a unique Wiener chaos expansion

$$u(t, x) = \sum_{n=0}^{\infty} I_n\big(g_n(\cdot, t, x)\big)$$

with $g_n(\cdot, t, x) \in \widetilde{\mathcal{H}}^{\otimes n}$. Now assume that $u(t, x)$ is a mild Skorohod solution to (1.1). Let $\mathcal{P}_n$ be the set of permutations on $\{1, 2, \ldots, n\}$. Following the approach used in [29], Section 4.1, we get

$$g_n(s_1, \ldots, s_n, x_1, \ldots, x_n, t, x) = \frac{1}{n!} G_{t-s_{\rho(n)}}(x - x_{\rho(n)}) \cdots G_{s_{\rho(2)}-s_{\rho(1)}}(x_{\rho(2)} - x_{\rho(1)}), \tag{3.2}$$

where $\rho \in \mathcal{P}_n$ is the permutation such that $0 < s_{\rho(1)} < s_{\rho(2)} < \cdots < s_{\rho(n)} < t$. Thus, to prove the existence and uniqueness of the solution to (1.1) is equivalent to prove

$$\mathbb{E}\big[\big|u(t, x)\big|^2\big] = \sum_{n=0}^{\infty} n! \big\|g_n(\cdot, t, x)\big\|^2_{\mathcal{H}^{\otimes n}} < \infty. \tag{3.3}$$

Theorem 3.2. *Assume that $H_0 \in [\frac{1}{2}, 1)$, $H \in (\frac{1}{4}, \frac{1}{2})$ and $\kappa \in (3 - 4H, 2]$. Then there exists a unique square integrable mild Skorohod solution to* (1.1).

Proof. It suffices to prove (3.3). We use the notation $\boldsymbol{\xi} := (\xi_1, \dots, \xi_n)$ and similarly for $\mathbf{s}$, $\mathbf{r}$ and $\boldsymbol{\mu}(d\boldsymbol{\xi})$.

We first consider the case $H_0 \in (\frac{1}{2}, 1)$. Since we assume that $f_0(s) \sim |s|^{2H_0-2}$ for $H_0 \in (\frac{1}{2}, 1)$, throughout the rest of the article, we will simply assume $f_0(s) = |s|^{2H_0-2}$ in this case. Note that

$$\begin{aligned} &n!\big\|g_n(\cdot, t, x)\big\|^2_{\mathcal{H}^{\otimes n}} \\ &\quad = n! \int_{\mathbb{R}^n} \int_{[0,t]^{2n}} \mathcal{F}g_n(\mathbf{s}, \cdot, t, x)(\boldsymbol{\xi}) \overline{\mathcal{F}g_n(\mathbf{r}, \cdot, t, x)(\boldsymbol{\xi})} \prod_{j=1}^n |s_j - r_j|^{2H_0-2} \, d\mathbf{s}\, d\mathbf{r} \boldsymbol{\mu}(d\boldsymbol{\xi}) \end{aligned} \tag{3.4}$$

with

$$\mathcal{F}g_n(\mathbf{s}, \cdot, t, x)(\boldsymbol{\xi}) = \frac{1}{n!} e^{-ix(\xi_1 + \cdots + \xi_n)} \prod_{j=1}^n \frac{\sin((s_{\rho(j+1)} - s_{\rho(j)})|\xi_{\rho(1)} + \cdots + \xi_{\rho(j)}|^{\kappa/2})}{|\xi_{\rho(1)} + \cdots + \xi_{\rho(j)}|^{\kappa/2}},$$

where we use the convention $s_{\rho(n+1)} = t$. Thus, by Lemma B.3 in [2] (see also [35]) and a change of variables, we have

$$\begin{aligned} &n!\big\|g_n(\cdot, t, x)\big\|^2_{\mathcal{H}^{\otimes n}} \\ &\quad \le n! \bigg(\int_{[0,t]^n} \bigg(\int_{\mathbb{R}^n} \big|\mathcal{F}g_n(\mathbf{s}, \cdot, t, x)(\boldsymbol{\xi})\big|^2 \boldsymbol{\mu}(d\boldsymbol{\xi}) \bigg)^{\frac{1}{2H_0}} d\mathbf{s} \bigg)^{2H_0} \\ &\quad = \frac{1}{n!} \bigg(\int_{[0,t]^n} \bigg(\int_{\mathbb{R}^n} \prod_{j=1}^n \frac{\sin^2((s_{\rho(j+1)} - s_{\rho(j)})|\xi_{\rho(1)} + \cdots + \xi_{\rho(j)}|^{\kappa/2})}{|\xi_{\rho(1)} + \cdots + \xi_{\rho(j)}|^{\kappa}} \boldsymbol{\mu}(d\boldsymbol{\xi}) \bigg)^{\frac{1}{2H_0}} d\mathbf{s} \bigg)^{2H_0} \\ &\quad = (n!)^{2H_0-1} \bigg(\int_{[0,t]^n_<} \bigg(\int_{\mathbb{R}^n} \prod_{j=1}^n \frac{\sin^2((s_{j+1} - s_j)|\xi_1 + \cdots + \xi_j|^{\kappa/2})}{|\xi_1 + \cdots + \xi_j|^{\kappa}} \boldsymbol{\mu}(d\boldsymbol{\xi}) \bigg)^{\frac{1}{2H_0}} d\mathbf{s} \bigg)^{2H_0} \\ &\quad = (n!)^{2H_0-1} \bigg(\int_{[0,t]^n_<} \bigg(\int_{\mathbb{R}^n} \prod_{j=1}^n \frac{\sin^2((s_{j+1} - s_j)|\eta_j|^{\kappa/2})}{|\eta_j|^{\kappa}} |\eta_j - \eta_{j-1}|^{1-2H} \, d\boldsymbol{\eta} \bigg)^{\frac{1}{2H_0}} d\mathbf{s} \bigg)^{2H_0}, \end{aligned} \tag{3.5}$$

where $[0, t]^n_< = [0 = s_0 < s_1 < \cdots < s_n < s_{n+1} = t]$.

Let $\mathcal{A}_n$ be a subset of the index set $\{(\alpha_1, \dots, \alpha_n) \in \{0, 1, 2\}^n\}$ such that

$$x_1 \prod_{j=2}^n (x_j + x_{j-1}) = \sum_{\alpha \in \mathcal{A}_n} \prod_{j=1}^n x_j^{\alpha_j}.$$

Then $\#(\mathcal{A}_n) = 2^{n-1}$, and for each $\alpha \in \mathcal{A}_n$, we have the following properties: $\alpha_1 \in \{1, 2\}, \alpha_n \in \{0, 1\}, \alpha_2, \dots, \alpha_{n-1} \in \{0, 1, 2\}, \sum_{j=1}^n \alpha_j = n$ and $\alpha_j + \alpha_{j+1} \in \{1, 2, 3\}$ for $j = 1, 2, \dots, n-1$.

Hence, noting that $|a+b|^{1-2H} \le |a|^{1-2H} + |b|^{1-2H}$, we get

$$\prod_{j=1}^{n} |\eta_j - \eta_{j-1}|^{1-2H} = |\eta_1|^{1-2H} \prod_{j=2}^{n} |\eta_j - \eta_{j-1}|^{1-2H} \le \sum_{\alpha \in \mathcal{A}_n} \prod_{j=1}^{n} |\eta_j|^{(1-2H)\alpha_j}. \tag{3.6}$$

Using (3.6) and the fact that $(\sum x_m)^{\frac{1}{2H_0}} \le \sum x_m^{\frac{1}{2H_0}}$ for all $x_m \ge 0$, the estimation (3.5) now becomes

$$\begin{aligned}
&n! \big\| g_n(\cdot, t, x) \big\|^2_{\mathcal{H}^{\otimes n}} \\
&\le (n!)^{2H_0-1} \left(\int_{[0,t]^n_<} \left(\int_{\mathbb{R}^n} \prod_{j=1}^{n} \frac{\sin^2((s_{j+1}-s_j)|\eta_j|^{\kappa/2})}{|\eta_j|^{\kappa}} |\eta_j - \eta_{j-1}|^{1-2H} \, d\boldsymbol{\eta} \right)^{\frac{1}{2H_0}} d\mathbf{s} \right)^{2H_0} \\
&\le (n!)^{2H_0-1} \left(\int_{[0,t]^n_<} \left(\sum_{\alpha \in \mathcal{A}_n} \int_{\mathbb{R}^n} \prod_{j=1}^{n} \frac{\sin^2((s_{j+1}-s_j)|\eta_j|^{\kappa/2})}{|\eta_j|^{\kappa}} |\eta_j|^{\alpha_j(1-2H)} \, d\boldsymbol{\eta} \right)^{\frac{1}{2H_0}} d\mathbf{s} \right)^{2H_0} \\
&\le (n!)^{2H_0-1} \left(\int_{[0,t]^n_<} \sum_{\alpha \in \mathcal{A}_n} \left(\int_{\mathbb{R}^n} \prod_{j=1}^{n} \frac{\sin^2((s_{j+1}-s_j)|\eta_j|^{\kappa/2})}{|\eta_j|^{\kappa}} |\eta_j|^{\alpha_j(1-2H)} \, d\boldsymbol{\eta} \right)^{\frac{1}{2H_0}} d\mathbf{s} \right)^{2H_0} \\
&= (n!)^{2H_0-1} \left(\int_{[0,t]^n_<} \sum_{\alpha \in \mathcal{A}_n} \left(\frac{2}{\kappa} \right)^{\frac{n}{2H_0}} \prod_{j=1}^{n} (s_{j+1}-s_j)^{\frac{1}{2H_0}[2-\frac{2}{\kappa}-\frac{2}{\kappa}\alpha_j(1-2H)]} \right. \\
&\quad \left. \times \left(\int_{\mathbb{R}} \frac{\sin^2(\eta)}{\eta^2} |\eta|^{\frac{2}{\kappa}\alpha_j(1-2H)+\frac{2}{\kappa}-1} \, d\eta \right)^{\frac{1}{2H_0}} d\mathbf{s} \right)^{2H_0}.
\end{aligned} \tag{3.7}$$

It follows from Lemma A.4 and the condition $\kappa > 3-4H$ that for all $\alpha_j \in \{0,1,2\}$

$$\int_{\mathbb{R}} \frac{\sin^2(\eta)}{\eta^2} |\eta|^{\frac{2}{\kappa}\alpha_j(1-2H)+\frac{2}{\kappa}-1} \, d\eta < \infty.$$

Lemma A.4 is applicable here, since the condition $\kappa > 3-4H$ implies $\frac{2}{\kappa}\alpha_j(1-2H) + \frac{2}{\kappa} - 3 \in (-3,-1)$ for all $\alpha_j \in \{0,1,2\}$.

Therefore, one can find a positive constant C depending only on (κ, H_0, H) such that

$$\begin{aligned}
&n! \big\| g_n(\cdot, t, x) \big\|^2_{\mathcal{H}^{\otimes n}} \\
&\quad \le C^n (n!)^{2H_0-1} \left(\int_{[0,t]^n_<} \sum_{\alpha \in \mathcal{A}_n} \prod_{j=1}^{n} (s_{j+1}-s_j)^{\frac{1}{2H_0}[2-\frac{2}{\kappa}-\frac{2}{\kappa}\alpha_j(1-2H)]} d\mathbf{s} \right)^{2H_0}.
\end{aligned}$$

For each fixed $\alpha \in \mathcal{A}_n$, denote $\beta_j = \frac{1}{2H_0}[2 - \frac{2}{\kappa} - \frac{2}{\kappa}\alpha_j(1-2H)]$, $j = 1, \dots, n$, and $\beta = \sum_{j=1}^{n} \beta_j = \frac{n}{H_0}[(1-\frac{2}{\kappa}) + \frac{2H}{\kappa}]$. Note that $H_0 \in (\frac{1}{2}, 1)$, $H \in (\frac{1}{4}, \frac{1}{2})$ and $\kappa > 3-4H$ implies that

$\beta_j > 0$ and $\beta > 0$. By Lemma A.2, we have

$$\int_{[0,t]^n_<} \prod_{j=1}^n (s_{j+1} - s_j)^{\beta_j} \, d\mathbf{s} = \frac{\prod_{i=1}^n \Gamma(1+\beta_i) t^{n+\beta}}{\Gamma(n+1+\beta)}.$$

Therefore, from (A.1) in Lemma A.3 with $a = 1 + \frac{1}{H_0}[(1 - \frac{2}{\kappa}) + \frac{2H}{\kappa}]$ and $b = 1$, and the fact $\#(\mathcal{A}_n) = 2^{n-1}$, it follows that there exists some positive constant C such that,

$$\begin{aligned} & n! \big\| g_n(\cdot, t, x) \big\|^2_{\mathcal{H}^{\otimes n}} \\ & \quad \le C^n (n!)^{2H_0 - 1} \left(\frac{t^{n+\beta}}{\Gamma(n+1+\beta)} \right)^{2H_0} \\ & \quad = \frac{C^n (n!)^{2H_0-1} t^{n(2H_0 + 2[(1-\frac{2}{\kappa}) + \frac{2H}{\kappa}])}}{\Gamma(an+1)^{2H_0}} \sim \frac{C^n t^{n(2H_0 + 2[(1-\frac{2}{\kappa}) + \frac{2H}{\kappa}])}}{(n!)^{2[(1-\frac{2}{\kappa}) + \frac{2H}{\kappa}]+1} a^{2H_0 an + H_0} n^{H_0(1-a)}} \end{aligned} \tag{3.8}$$

when $n \to \infty$. Notice that there exists $\lambda > 1$ such that $\lambda^{-n} \le a^{2H_0 an + H_0} n^{H_0(1-a)} \le \lambda^n$ for all n. Hence, by (A.2) in Lemma A.3, there exists a positive constant C such that

$$\begin{aligned} \mathbb{E}\big[\big| u(t,x) \big|^2 \big] &= \sum_{n=0}^\infty n! \big\| g_n(\cdot, t, x) \big\|^2_{\mathcal{H}^{\otimes n}} \le \sum_{n=0}^\infty \frac{C^n t^{n(2H_0 + 2[(1-\frac{2}{\kappa}) + \frac{2H}{\kappa}])}}{(n!)^{2[(1-\frac{2}{\kappa}) + \frac{2H}{\kappa}]+1}} \\ &\le C \exp\big(C t^{\frac{2\kappa H_0 + 2(\kappa-2) + 4H}{3\kappa - 4 + 4H}} \big). \end{aligned} \tag{3.9}$$

Next, we consider the case $H_0 = \frac{1}{2}$, i.e., $f_0(t) = \delta(t)$, and we have

$$\begin{aligned} & n! \big\| g_n(\cdot, t, x) \big\|^2_{\mathcal{H}^{\otimes n}} \\ & \quad = n! \int_{\mathbb{R}^n} \int_{[0,t]^n} \big| \mathcal{F} g_n(\mathbf{s}, \cdot, t, x)(\boldsymbol{\xi}) \big|^2 \, d\mathbf{s} \boldsymbol{\mu}(d\boldsymbol{\xi}) \\ & \quad = \frac{1}{n!} \int_{[0,t]^n} \int_{\mathbb{R}^n} \prod_{j=1}^n \frac{\sin^2((s_{\rho(j+1)} - s_{\rho(j)}) |\xi_{\rho(1)} + \cdots + \xi_{\rho(j)}|^{\kappa/2})}{|\xi_{\rho(1)} + \cdots + \xi_{\rho(j)}|^\kappa} \prod_{j=1}^n |\xi|^{1-2H} d\boldsymbol{\xi} \, d\mathbf{s} \\ & \quad = \int_{[0,t]^n_<} \int_{\mathbb{R}^n} \prod_{j=1}^n \frac{\sin^2((s_{j+1} - s_j) |\xi_1 + \cdots + \xi_j|^{\kappa/2})}{|\xi_1 + \cdots + \xi_j|^\kappa} \prod_{j=1}^n |\xi|^{1-2H} d\boldsymbol{\xi} \, d\mathbf{s} \\ & \quad = \int_{[0,t]^n_<} \int_{\mathbb{R}^n} \prod_{j=1}^n \frac{\sin^2((s_{j+1} - s_j) |\eta_j|^{\kappa/2})}{|\eta_j|^\kappa} \prod_{j=1}^n |\eta_j - \eta_{j-1}|^{1-2H} \, d\boldsymbol{\eta} \, d\mathbf{s}. \end{aligned}$$

The last term in the above equation equals the right-hand side of (3.5) with $H_0 = \frac{1}{2}$. Analogue to the arguments in (3.6)–(3.9), we shall get the following estimation for the second moment

$$\mathbb{E}\big[\big|u(t,x)\big|^2\big] = \sum_{n=0}^{\infty} n! \big\|g_n(\cdot,t,x)\big\|^2_{\mathcal{H}^{\otimes n}} \le C\exp(Ct). \tag{3.10}$$

We complete the proof. □

Remark 3.3. For SHEs on $\mathbb{R}$ driven by a multiplicative Gaussian noise that is rough in space, the existence and uniqueness of the mild Skorohod solution was obtained in [25] for the noise white in time and in [31] for the noise colored in time. The condition $H > \frac{1}{4}$ was assumed in both [25] and [31]. However, the method used in the proof of the above theorem suggests that the condition can be reduced to $H_0 + H > \frac{3}{4}$, and this is consistent with the result in [11]. Indeed, for the following SHE on $\mathbb{R}$,

$$\begin{cases} \dfrac{\partial u^h}{\partial t}(t,x) = \dfrac{1}{2}\Delta u^h(t,x) + u^h \dot{W}(t,x), \quad t > 0, \\ u^h(0,x) = 1, \end{cases}$$

the Green's function is the heat kernel $G^h_t(x) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{|x|^2}{2t}}$. Consequently, the Wiener chaos expansion of the solution is

$$u^h(t,x) = \sum_{n=0}^{\infty} I_n\big(g^h_n(\cdot,t,x)\big),$$

where

$$g^h_n(s_1,\dots,s_n,x_1,\dots,x_n,t,x) = \frac{1}{n!} G^h_{t-s_{\rho(n)}}(x - x_{\rho(n)}) \cdots G^h_{s_{\rho(2)}-s_{\rho(1)}}(x_{\rho(2)} - x_{\rho(1)})$$

and

$$\mathcal{F}g^h_n(\mathbf{s},\cdot,t,x)(\boldsymbol{\xi}) = \frac{1}{n!} e^{-ix(\xi_1+\cdots+\xi_n)} \prod_{j=1}^{n} \exp\bigg[-\frac{1}{2}(s_{\rho(j+1)} - s_{\rho(j)})|\xi_{\rho(1)} + \cdots + \xi_{\rho(j)}|^2\bigg].$$

Now, the second moment of each chaos is

$$\begin{aligned} &n!\big\|g^h_n(\cdot,t,x)\big\|^2_{\mathcal{H}^{\otimes n}} \\ &\quad = n! \int_{\mathbb{R}^n} \int_{[0,t]^{2n}} \mathcal{F}g^h_n(\mathbf{s}\cdot,t,x)(\boldsymbol{\xi}) \overline{\mathcal{F}g^h_n(\mathbf{r},\cdot,t,x)(\boldsymbol{\xi})} \prod_{j=1}^{n} |s_j - r_j|^{2H_0-2} d\mathbf{s}\, d\mathbf{r} \boldsymbol{\mu}(d\boldsymbol{\xi}), \\ &\quad \le n! \bigg(\int_{[0,t]^n} \bigg(\int_{\mathbb{R}^n} \big|\mathcal{F}g^h_n(\mathbf{s},\cdot,t,x)(\boldsymbol{\xi})\big|^2 \boldsymbol{\mu}(d\boldsymbol{\xi}) \bigg)^{\frac{1}{2H_0}} d\mathbf{s} \bigg)^{2H_0} \end{aligned}$$

$$= \frac{1}{n!}\left(\int_{[0,t]^n}\left(\int_{\mathbb{R}^n}\prod_{j=1}^{n}\exp\bigl[-[s_{\rho(j+1)}-s_{\rho(j)}]|\xi_{\rho(1)}+\cdots+\xi_{\rho(j)}|^2\bigr]\boldsymbol{\mu}(d\boldsymbol{\xi})\right)^{\frac{1}{2H_0}} d\mathbf{s}\right)^{2H_0}$$

$$= (n!)^{2H_0-1}\left(\int_{[0,t]^n_<}\left(\int_{\mathbb{R}^n}\prod_{j=1}^{n}\exp\bigl[-[s_{j+1}-s_j]|\xi_1+\cdots+\xi_j|^2\bigr]\boldsymbol{\mu}(d\boldsymbol{\xi})\right)^{\frac{1}{2H_0}} d\mathbf{s}\right)^{2H_0}$$

$$= (n!)^{2H_0-1}\left(\int_{[0,t]^n_<}\left(\int_{\mathbb{R}^n}\prod_{j=1}^{n}\exp\bigl[-[s_{j+1}-s_j]|\eta_j|^2\bigr]|\eta_j-\eta_{j-1}|^{1-2H}\,d\boldsymbol{\eta}\right)^{\frac{1}{2H_0}} d\mathbf{s}\right)^{2H_0}.$$

Then, using similar argument as in the proof of Theorem 3.2 and Lemma A.1, we have

$$n!\bigl\|g_n^h(\cdot,t,x)\bigr\|^2_{\mathcal{H}^{\otimes n}}$$

$$\le (n!)^{2H_0-1}\left(\int_{[0,t]^n_<}\left(\sum_{\alpha\in\mathcal{A}_n}\int_{\mathbb{R}^n}\prod_{j=1}^{n}\exp\bigl[-(s_{j+1}-s_j)|\eta_j|^2\bigr]|\eta_j|^{(1-2H)\alpha_j}\,d\boldsymbol{\eta}\right)^{\frac{1}{2H_0}} d\mathbf{s}\right)^{2H_0}$$

$$\le C^n(n!)^{2H_0-1}\left(\int_{[0,t]^n_<}\sum_{\alpha\in\mathcal{A}_n}\prod_{j=1}^{n}(s_{j+1}-s_j)^{-\frac{1}{4H_0}[1+(1-2H)\alpha_j]}\,d\mathbf{s}\right)^{2H_0}.$$

For each fixed $\alpha\in\mathcal{A}_n$, denote $\beta_j=\frac{1}{4H_0}[1+(1-2H)\alpha_j]\in(0,1)$ noting that $H_0\in(\frac12,1)$, $H\in(0,\frac12)$ and $H_0+H>3/4$, and then $\beta=\sum_{j=1}^n\beta_j=\frac{n(1-H)}{2H_0}$. By Lemma A.2, we have

$$\int_{[0,t]^n_<}\prod_{j=1}^{n}(s_{j+1}-s_j)^{-\beta_j}\,d\mathbf{s}=\frac{\prod_{i=1}^n\Gamma(1-\beta_i)t^{n-\beta}}{\Gamma(n+1-\beta)}.$$

Therefore, since $\#(\mathcal{A}_n)=2^{n-1}$, there exists some positive constant C such that,

$$n!\bigl\|g_n^h(\cdot,t,x)\bigr\|^2_{\mathcal{H}^{\otimes n}}$$

$$\le C^n(n!)^{2H_0-1}\left(\frac{t^{n-\beta}}{\Gamma(n+1-\beta)}\right)^{2H_0}=C^n(n!)^{2H_0-1}\frac{1}{\Gamma(\frac{(2H_0+H-1)n}{2H_0}+1)^{2H_0}}t^{n(2H_0+H-1)}.$$

It follows from Lemma A.3 and a similar argument in dealing with (3.8) that there exists a positive constant C such that

$$\mathbb{E}\bigl[\bigl|u^h(t,x)\bigr|^2\bigr]=\sum_{n=0}^{\infty}n!\bigl\|g_n^h(\cdot,t,x)\bigr\|^2_{\mathcal{H}^{\otimes n}}\le C\exp\bigl(Ct^{\frac{2H_0+H-1}{H}}\bigr).$$

Proposition 3.4. *When $H_0 = \frac{1}{2}$ and $d = 1$, the condition $\kappa > 3 - 4H$ is also a necessary condition for the existence of the square integrable solutions to (1.1). When $H_0 = \frac{1}{2}$ and $H_j < \frac{1}{2}$, $j = 1, \dots, d$, the equation (1.1) has a solution only if $d = 1$.*

Proof. When $d = 1$, the L^2-norm of the second chaos of the solution is

$$\begin{aligned}
&\big\|g_2(\cdot,t,x)\big\|^2_{\mathcal{H}^{\otimes 2}} \\
&\quad = 2\int_{[0,t]^2_<}\int_{\mathbb{R}^2} \frac{\sin^2((s_2-s_1)|\eta_1|^{\kappa/2})}{|\eta_1|^{\kappa}} \frac{\sin^2((t-s_2)|\eta_2|^{\kappa/2})}{|\eta_2|^{\kappa}} |\eta_1|^{1-2H}|\eta_2-\eta_1|^{1-2H} d\boldsymbol{\eta}\, d\mathbf{s} \\
&\quad \geq \int_{[0,t]^2_<}\int_{\mathbb{R}^2_+} \frac{\sin^2((s_2-s_1)|\eta_1|^{\kappa/2})}{|\eta_1|^{\kappa}} \frac{\sin^2((t-s_2)|\eta_2|^{\kappa/2})}{|\eta_2|^{\kappa}} |\eta_1|^{1-2H}|\eta_2+\eta_1|^{1-2H} d\boldsymbol{\eta}\, d\mathbf{s} \\
&\quad \geq \int_{[0,t]^2_<}\int_{\mathbb{R}^2_+} \frac{\sin^2((s_2-s_1)|\eta_1|^{\kappa/2})}{|\eta_1|^{\kappa}} \frac{\sin^2((t-s_2)|\eta_2|^{\kappa/2})}{|\eta_2|^{\kappa}} |\eta_1|^{2(1-2H)} d\boldsymbol{\eta}\, d\mathbf{s},
\end{aligned}$$

where the last integral is infinity if $\kappa \leq 3 - 4H$ due to Lemma A.4.

For general dimension d, the above estimation becomes

$$\begin{aligned}
&\big\|g_2(\cdot,t,x)\big\|^2_{\mathcal{H}^{\otimes 2}} \\
&\quad = 2\int_{[0,t]^2_<}\int_{\mathbb{R}^{2d}} \frac{\sin^2((s_2-s_1)|\eta_1|^{\kappa/2})}{|\eta_1|^{\kappa}} \frac{\sin^2((t-s_2)|\eta_2|^{\kappa/2})}{|\eta_2|^{\kappa}} \prod_{j=1}^{d} |\eta_1^j|^{1-2H_j} |\eta_2^j-\eta_1^j|^{1-2H_j} d\boldsymbol{\eta}\, d\mathbf{s} \\
&\quad \geq \int_{[0,t]^2_<}\int_{\mathbb{R}^{2d}_+} \frac{\sin^2((s_2-s_1)|\eta_1|^{\kappa/2})}{|\eta_1|^{\kappa}} \frac{\sin^2((t-s_2)|\eta_2|^{\kappa/2})}{|\eta_2|^{\kappa}} \prod_{j=1}^{d} |\eta_1^j|^{1-2H_j} |\eta_2^j+\eta_1^j|^{1-2H_j} d\boldsymbol{\eta}\, d\mathbf{s} \\
&\quad \geq \int_{[0,t]^2_<}\int_{\mathbb{R}^{2d}_+} \frac{\sin^2((s_2-s_1)|\eta_1|^{\kappa/2})}{|\eta_1|^{\kappa}} \frac{\sin^2((t-s_2)|\eta_2|^{\kappa/2})}{|\eta_2|^{\kappa}} \prod_{j=1}^{d} |\eta_1^j|^{2(1-2H_j)} d\boldsymbol{\eta}\, d\mathbf{s}.
\end{aligned}$$

Now, by the change of variables

$$\begin{cases}
\eta_1^1 = r\cos(\theta_1), \\
\eta_1^2 = r\sin(\theta_1)\cos(\theta_2), \\
\eta_1^3 = r\sin(\theta_1)\sin(\theta_2)\cos(\theta_3), \\
\vdots \\
\eta_1^{d-1} = r\sin(\theta_1)\cdots\sin(\theta_{d-2})\cos(\theta_{d-1}), \\
\eta_1^{d} = r\sin(\theta_1)\cdots\sin(\theta_{d-2})\sin(\theta_{d-1}),
\end{cases}$$

we have

$$\int_{\mathbb{R}_+^d} \frac{\sin^2(|\eta_1|^{\kappa/2})}{|\eta_1|^{\kappa}} \prod_{j=1}^{d} |\eta_1^j|^{2(1-2H_j)}\, d\eta_1 = C_d \int_0^\infty \sin^2\big(r^{\kappa/2}\big) r^{\sum_{j=1}^d 2(1-2H_j)-\kappa+d-1}\, dr,$$

which by Lemma A.4 is infinite when $d > 1$ since $\sum_{j=1}^d 2(1-2H_j) - \kappa + d - 1 \geq -1$ for $H_j \in (0, \frac{1}{2})$. □

4. Moments of the solution and weak intermittency

In this section, we first obtain the lower bound and upper bound for the pth moment of the solution to (1.1) for $p \geq 2$, and then deduce the weak intermittency.

Proposition 4.1. *Under the conditions in Theorem 3.2, there exist $0 < C_1, C_2 < \infty$ such that for all $p \geq 2$*

$$C_1 \exp\big(C_1 t^{\frac{2\kappa H_0+2(\kappa-2)+4H}{3\kappa-4+4H}}\big) \leq \big\|u(t,x)\big\|_p \leq C_2 \exp\big(C_2 p^{\frac{\kappa}{3\kappa-4+4H}} t^{\frac{2\kappa H_0+2(\kappa-2)+4H}{3\kappa-4+4H}}\big) \tag{4.1}$$

and

$$\begin{aligned} C_1 &\leq \liminf_{t\to\infty} t^{-\frac{2\kappa H_0+2(\kappa-2)+4H}{3\kappa-4+4H}} \log\big\|u(t,x)\big\|_p \\ &\leq \limsup_{t\to\infty} t^{-\frac{2\kappa H_0+2(\kappa-2)+4H}{3\kappa-4+4H}} \log\big\|u(t,x)\big\|_p \leq C_2 p^{\frac{\kappa}{3\kappa-4+4H}}. \end{aligned} \tag{4.2}$$

In particular, $\frac{2\kappa H_0+2(\kappa-2)+4H}{3\kappa-4+4H} = 1$ if $H_0 = \frac{1}{2}$.

Proof. We shall prove (4.1) for $H_0 \in (\frac{1}{2}, 1)$. The proof for $H_0 = \frac{1}{2}$ is similar and thus omitted.

By (3.4), we have

$$\begin{aligned} &n!\big\|g_n(\cdot,t,x)\big\|^2_{\mathcal{H}^{\otimes n}} \\ &\quad = n! \int_{\mathbb{R}^n} \int_{[0,t]^{2n}} \mathcal{F}g_n(\mathbf{s},\cdot,t,x)(\boldsymbol{\xi}) \overline{\mathcal{F}g_n(\mathbf{r},\cdot,t,x)(\boldsymbol{\xi})} \prod_{j=1}^n |s_j - r_j|^{2H_0-2}\, d\mathbf{s}\, d\mathbf{r}\boldsymbol{\mu}(d\boldsymbol{\xi}) \\ &\quad = n! \int_{\mathbb{R}^n} \int_{([0,t]^n_<)^2} \prod_{j=1}^n \frac{\sin((s_{j+1}-s_j)|\xi_1+\cdots+\xi_j|^{\kappa/2})}{|\xi_1+\cdots+\xi_j|^{\kappa/2}} \\ &\qquad \times \prod_{j=1}^n \frac{\sin((r_{j+1}-r_j)|\xi_1+\cdots+\xi_j|^{\kappa/2})}{|\xi_1+\cdots+\xi_j|^{\kappa/2}} \prod_{j=1}^n |s_j - r_j|^{2H_0-2}\, d\mathbf{s}\, d\mathbf{r}\boldsymbol{\mu}(d\boldsymbol{\xi}) \end{aligned}$$

$$= n! \int_{\mathbb{R}^n} \left(\int_{([0,t]^n_<)^2} \prod_{j=1}^n \frac{\sin((s_{j+1}-s_j)|\eta_j|^{\kappa/2})}{|\eta_j|^{\kappa/2}} \frac{\sin((r_{j+1}-r_j)|\eta_j|^{\kappa/2})}{|\eta_j|^{\kappa/2}} \right.$$
$$\left. \times \prod_{j=1}^n |s_j - r_j|^{2H_0-2} \, d\mathbf{s}\, d\mathbf{r} \right) \prod_{j=1}^n |\eta_j - \eta_{j-1}|^{1-2H} \, d\boldsymbol{\eta}$$
$$\geq n! \int_{\mathbb{D}_n} \left(\int_{([0,t]^n_<)^2} \prod_{j=1}^n \frac{\sin((s_{j+1}-s_j)|\eta_j|^{\kappa/2})}{|\eta_j|^{\kappa/2}} \frac{\sin((r_{j+1}-r_j)|\eta_j|^{\kappa/2})}{|\eta_j|^{\kappa/2}} \right.$$
$$\left. \times \prod_{j=1}^n |s_j - r_j|^{2H_0-2} \, d\mathbf{s}\, d\mathbf{r} \right) \prod_{j=1}^n |\eta_j|^{1-2H} \, d\boldsymbol{\eta},$$

noting that in the last step we used the facts that the inner integral with respect to $d\mathbf{s}\, d\mathbf{r}$ is nonnegative and that $|\eta_j - \eta_{j-1}|^{1-2H} \geq |\eta_j|^{1-2H}$ on $\mathbb{D}_n$ with $\mathbb{D}_n = \{(\eta_1, \dots, \eta_n) \in \mathbb{R}^n : \eta_1 \geq 0, \eta_2 \leq 0, \eta_3 \geq 0, \eta_4 \leq 0, \dots\}$.

Now, we have

$$n! \big\| g_n(\cdot, t, x) \big\|^2_{\mathcal{H}^{\otimes n}}$$
$$\geq n! \int_{([0,t]^n_<)^2} \left(\int_{\mathbb{D}_n} \prod_{j=1}^n \sin\big((s_{j+1}-s_j)|\eta_j|^{\kappa/2}\big) \sin\big((r_{j+1}-r_j)|\eta_j|^{\kappa/2}\big) \prod_{j=1}^n |\eta_j|^{1-2H-\kappa} \, d\boldsymbol{\eta} \right)$$
$$\times \prod_{j=1}^n |s_j - r_j|^{2H_0-2} \, d\mathbf{s}\, d\mathbf{r}$$
$$= n! \int_{([0,t]^n_<)^2} \left(\prod_{j=1}^n \int_{\mathbb{R}_+} \sin\big((s_{j+1}-s_j)|\eta|^{\kappa/2}\big) \sin\big((r_{j+1}-r_j)|\eta|^{\kappa/2}\big) |\eta|^{1-2H-\kappa} \, d\eta \right)$$
$$\times \prod_{j=1}^n |s_j - r_j|^{2H_0-2} \, d\mathbf{s}\, d\mathbf{r}$$
$$\geq n! t^{n(2H_0-2)} \int_{\mathbb{R}^n_+} \left| \prod_{j=1}^n \int_{[0,t]^n_<} \prod_{j=1}^n \sin\big((s_{j+1}-s_j)|\eta_j|^{\kappa/2}\big) \, d\mathbf{s} \right|^2 \prod_{j=1}^n |\eta_j|^{1-2H-\kappa} \, d\boldsymbol{\eta}, \quad (4.3)$$

where the last step holds due to $|s_j - r_j| \leq t$ and the fact that the integral with respect to η is nonnegative by Lemma A.5.

Let

$$A_n(t) = \int_{\mathbb{R}^n_+} \left| \int_{[0,t]^n_<} \prod_{j=1}^n \sin\big((s_{j+1}-s_j)|\eta_j|^{\kappa/2}\big) \, d\mathbf{s} \right|^2 \prod_{j=1}^n |\eta_j|^{1-2H-\kappa} \, d\boldsymbol{\eta}. \quad (4.4)$$

Make the change of variables $s'_j = s_j/t$ and $\eta'_j = \eta_j t^{2/\kappa}$, and we have the scaling

$$A_n(t) = t^{4n(1-\frac{1-H}{\kappa})} A_n(1). \tag{4.5}$$

Now we estimate $\mathbb{E}[A_n(\tau)]$ where τ is an exponential random time with parameter 1. By Fubini's theorem and Jensen's inequality, we obtain

$$\begin{aligned}
&\mathbb{E}\big[A_n(\tau)\big] \\
&\quad = \int_0^\infty e^{-t} A_n(t)\,dt \\
&\quad = \int_{\mathbb{R}_+^n} \int_0^\infty e^{-t} \left| \int_{[0,t]^n_<} \prod_{j=1}^n \sin\big((s_{j+1}-s_j)|\eta_j|^{\kappa/2}\big)\,d\mathbf{s} \right|^2 dt \prod_{j=1}^n |\eta_j|^{1-2H-\kappa}\,d\boldsymbol{\eta} \\
&\quad \geq \int_{\mathbb{R}_+^n} \left| \int_0^\infty e^{-t} \int_{[0,t]^n_<} \prod_{j=1}^n \sin\big((s_{j+1}-s_j)|\eta_j|^{\kappa/2}\big)\,d\mathbf{s}\,dt \right|^2 \prod_{j=1}^n |\eta_j|^{1-2H-\kappa}\,d\boldsymbol{\eta}. \qquad (4.6)
\end{aligned}$$

Applying the change of variables $r_j = s_{j+1} - s_j$ for $j = 0, 1, \ldots, n$ with the convention $s_0 = 0$ and $s_{n+1} = t$ and using Lemma A.6, we have

$$\begin{aligned}
&\int_0^\infty e^{-t} \int_{[0,t]^n_<} \prod_{j=1}^n \sin\big((s_{j+1}-s_j)|\eta_j|^{\kappa/2}\big)\,d\mathbf{s}\,dt \\
&\quad = \int_{\mathbb{R}_+^{n+1}} e^{-(r_0+r_1+\cdots+r_n)} \prod_{j=1}^n \sin\big(r_j|\eta_j|^{\kappa/2}\big)\,dr_0\,dr_1\cdots dr_n = \prod_{j=1}^n \frac{|\eta_j|^{\kappa/2}}{1+|\eta_j|^\kappa}. \qquad (4.7)
\end{aligned}$$

Now, combining (4.6) and (4.7), we get

$$\mathbb{E}\big[A_n(\tau)\big] \geq \int_{\mathbb{R}_+^n} \prod_{j=1}^n \frac{|\eta_j|^{1-2H}}{(1+|\eta_j|^\kappa)^2}\,d\boldsymbol{\eta} = \left(\int_{\mathbb{R}_+} \frac{|\eta|^{1-2H}}{(1+|\eta|^\kappa)^2}\,d\eta \right)^n = c^n,$$

where $c = \int_{\mathbb{R}_+} \frac{|\eta|^{1-2H}}{(1+|\eta|^\kappa)^2}\,d\eta \in (0,\infty)$. Together with the scaling property (4.5), we have

$$c^n \leq \mathbb{E}\big[A_n(\tau)\big] = \mathbb{E}\big[\tau^{4n(1-\frac{1-H}{\kappa})}\big] A_n(1). \tag{4.8}$$

Therefore, it implies from (4.3), (4.4), (4.5), (4.8) and the fact $\mathbb{E}[\tau^x] = \Gamma(x+1)$ that

$$\begin{aligned}
n!\big\|g_n(\cdot,t,x)\big\|^2_{\mathcal{H}^{\otimes n}} &\geq n! t^{n(2H_0-2)} A_n(t) \\
&= n! t^{n(2H_0-2)} t^{4n(1-\frac{1-H}{\kappa})} A_n(1) \\
&\geq C^n n! t^{n(2H_0-2)} t^{4n(1-\frac{1-H}{\kappa})} \frac{1}{\mathbb{E}[\tau^{4n(1-\frac{1-H}{\kappa})}]}
\end{aligned}$$

$$= C^n t^{n(2H_0+2-\frac{4-4H}{\kappa})} \frac{n!}{\Gamma(4n(1-\frac{1-H}{\kappa})+1)}$$
$$\sim \frac{C^n t^{n(2H_0+2-\frac{4-4H}{\kappa})}}{(n!)^{3-\frac{4(1-H)}{\kappa}} a^{an+\frac{1}{2}} n^{\frac{1-a}{2}}},$$

where the last step follows from (A.1) in Lemma A.3 with $a = 4(1-\frac{1-H}{\kappa})$ and $b = 1$. Noting that there exists $\lambda > 1$ such that $\lambda^{-n} \le a^{an+\frac{1}{2}} n^{\frac{1-a}{2}} \le \lambda^n$, we obtain

$$n! \big\| g_n(\cdot, t, x) \big\|^2_{\mathcal{H}^{\otimes n}} \ge \frac{C^n t^{n(2H_0+2-\frac{4-4H}{\kappa})}}{(n!)^{3-\frac{4(1-H)}{\kappa}}}.$$

Therefore, applying (A.2) in Lemma A.3, we have

$$\big\| u(t,x) \big\|_p \ge \big\| u(t,x) \big\|_2 \ge \left(\sum_{n=0}^{\infty} \frac{C^n t^{n(2H_0+2-\frac{4-4H}{\kappa})}}{(n!)^{3-\frac{4(1-H)}{\kappa}}} \right)^{\frac{1}{2}} \ge C_1 \exp\big(C_1 t^{\frac{2\kappa H_0+2(\kappa-2)+4H}{3\kappa-4+4H}} \big)$$

for some $C_1 > 0$.

For the upper bound, noting that $\| I_n(g_n) \|_p \le (p-1)^{\frac{n}{2}} \| I_n(g_n) \|_2$ (see the last line on Page 62 in [38]), by Minkowski's inequality and similar arguments in (3.8)–(3.9) we have

$$\begin{aligned} \| u \|_p &\le \sum_{n=0}^{\infty} \big\| I_n(g_n) \big\|_p \le \sum_{n=0}^{\infty} (p-1)^{\frac{n}{2}} \big\| I_n(g_n) \big\|_2 \\ &\le \sum_{n=0}^{\infty} \frac{(p-1)^{\frac{n}{2}} C^{\frac{n}{2}} t^{n(H_0+(1-\frac{2}{\kappa})+\frac{2H}{\kappa})}}{(n!)^{[(1-\frac{2}{\kappa})+\frac{2H}{\kappa}]+\frac{1}{2}}} \\ &\le C \exp\big(C p^{\frac{\kappa}{3\kappa-4+4H}} t^{\frac{2\kappa H_0+2(\kappa-2)+4H}{3\kappa-4+4H}} \big), \end{aligned}$$

and hence, we get

$$\| u \|_p \le C_2 \exp\big(C_2 p^{\frac{\kappa}{3\kappa-4+4H}} t^{\frac{2\kappa H_0+2(\kappa-2)+4H}{3\kappa-4+4H}} \big)$$

for some $C_2 > 0$. □

Remark 4.2. Note that the lower and upper bounds (lower bounds, resp.) for the pth moment were studied in [20] (in [2], resp.) based on the probabilistic representation for the second moment of the solution to wave equations obtained in [21].

Recall that the lower Lyapunov exponent $L_l(p)$ and the upper Lyapunov exponent $L_u(p)$ of order $p \ge 2$ of the solution $u(t,x)$ are defined, respectively, by

$$L_l(p) = \liminf_{t\to\infty} \frac{1}{R(t)} \inf_{x\in\mathbb{R}} \log \mathbb{E}\big[\big| u(t,x) \big|^p \big]$$

and

$$L_u(p) = \limsup_{t\to\infty} \frac{1}{R(t)} \sup_{x\in\mathbb{R}} \log \mathbb{E}\big[|u(t,x)|^p\big]$$

for some positive function $R(t)$. If $L_l(2) > 0$ and $L_u(p) < \infty$ for all $p \geq 0$, we say that $u(t,x)$ possesses the weak intermittency. Heuristically speaking, if a process $u(t,x)$ is weakly intermittent, it concentrates on a few of very high peaks (see, e.g., [34] and the references therein). Taking $R(t) = t^{\frac{2\kappa H_0+2(\kappa-2)+4H}{3\kappa-4+4H}}$, the proposition below follows directly from Proposition 4.1.

Proposition 4.3. *Under the conditions in Theorem 3.2, the solution $u(t,x)$ to (1.1) is weakly intermittent.*

5. Hölder continuity

In this section, the Hölder continuity in time and space for the solution $u(t,x)$ to the SWE (1.1) is obtained in Proposition 5.1. The result is consistent with the Hölder continuity for SWEs with noise that is not rough in space obtained in [1,6], and [15]. Note that exponents of Hölder continuity in both time and space are independent of the temporal covariance function $f_0(t)$. Similar phenomenon occurs also for SHEs (see [5], Theorem 3.2).

Proposition 5.1. *Assume the same conditions as in Theorem 3.2. Then on any set $[0,T]\times\mathbb{M}$ where $\mathbb{M}\subset\mathbb{R}$ is a compact set, $u(t,x)$ has a modification which is θ_1-Hölder continuous in time for all $\theta_1 \in (0, 1-\frac{2}{\kappa}+\frac{2H}{\kappa})$ and θ_2-Hölder continuous in space for all $\theta_2 \in (0, H+\frac{\kappa}{2}-1)$.*

In particular, when $\kappa = 2$, the solution has a version that is θ-Höder continuous both in time and in space for all $\theta \in (0,H)$.

Proof. First, we show the Hölder continuity in space. Noting that $\|I_n(g_n)\|_p \leq (p-1)^{\frac{n}{2}} \times \|I_n(g_n)\|_2$, by Minkowski's inequality, we have

$$\begin{aligned}
&\big\|u(t,x+z)-u(t,x)\big\|_p \\
&\quad\leq \sum_{n=1}^{\infty}(p-1)^{\frac{n}{2}}(n!)^{\frac{1}{2}}\big\|g_n(\cdot,t,x+z)-g_n(\cdot,t,x)\big\|_{\mathcal{H}^{\otimes n}} \\
&\quad= \sum_{n=1}^{\infty}(p-1)^{\frac{n}{2}}\Bigg(n!\int_{\mathbb{R}^n}\int_{[0,t]^{2n}}\mathcal{F}\big[g_n(\mathbf{s},\cdot,t,x+z)(\boldsymbol{\xi})-g_n(\mathbf{s},\cdot,t,x)(\boldsymbol{\xi})\big] \\
&\qquad\times\overline{\mathcal{F}\big[g_n(\mathbf{r},\cdot,t,x+z)(\boldsymbol{\xi})-g_n(\mathbf{r},\cdot,t,x)(\boldsymbol{\xi})\big]}\prod_{j=1}^{n}|s_j-r_j|^{2H_0-2}\,d\mathbf{s}\,d\mathbf{r}\boldsymbol{\mu}(d\boldsymbol{\xi})\Bigg)^{\frac{1}{2}} \\
&\quad\leq \sum_{n=1}^{\infty}(p-1)^{\frac{n}{2}}(n!)^{H_0-\frac{1}{2}}\Bigg(\int_{[0,t]^n_<}\Bigg[\int_{\mathbb{R}^n}\prod_{j=1}^{n}\frac{\sin^2((s_{j+1}-s_j)|\xi_1+\cdots+\xi_j|^{\kappa/2})}{|\xi_1+\cdots+\xi_j|^{\kappa}}|\xi_j|^{1-2H}
\end{aligned}$$

$$\times \left|1-e^{-iz(\xi_1+\cdots+\xi_n)}\right|^2 d\boldsymbol{\xi}\Bigg]^{\frac{1}{2H_0}} d\mathbf{s}\Bigg)^{H_0}$$

$$=\sum_{n=1}^{\infty}(p-1)^{\frac{n}{2}}(n!)^{H_0-\frac{1}{2}}\Bigg(\int_{[0,t]^n_<}\Bigg[\int_{\mathbb{R}^n}\prod_{j=1}^{n}\frac{\sin^2((s_{j+1}-s_j)|\eta_j|^{\kappa/2})}{|\eta_j|^{\kappa}}|\eta_j-\eta_{j-1}|^{1-2H}$$

$$\times \left|1-e^{-iz\eta_n}\right|d\boldsymbol{\eta}\Bigg]^{\frac{1}{2H_0}} d\mathbf{s}\Bigg)^{H_0}.$$

Now, by changing of variables, we have

$$\int_{[0,t]^n_<}\Bigg[\int_{\mathbb{R}^n}\prod_{j=1}^{n}\frac{\sin^2((s_{j+1}-s_j)|\eta_j|^{\kappa/2})}{|\eta_j|^{\kappa}}|\eta_j-\eta_{j-1}|^{1-2H}\left|1-e^{-iz\eta_n}\right|d\boldsymbol{\eta}\Bigg]^{\frac{1}{2H_0}} d\mathbf{s}\,d\mathbf{t}$$

$$\le\int_{[0,t]^n_<}\Bigg[\sum_{\alpha\in\mathcal{A}_n}\int_{\mathbb{R}^n}\prod_{j=1}^{n}\frac{\sin^2((s_{j+1}-s_j)|\eta_j|^{\kappa/2})}{|\eta_j|^{\kappa}}|\eta_j|^{\alpha_j(1-2H)}\big(|z\eta_n|\wedge 2\big)d\boldsymbol{\eta}\Bigg]^{\frac{1}{2H_0}} d\mathbf{s}\,d\mathbf{t}$$

$$\le\int_{[0,t]^n_<}\sum_{\alpha\in\mathcal{A}_n}\left(\frac{2}{\kappa}\right)^{\frac{n}{2H_0}}\Bigg[\prod_{j=1}^{n-1}(s_{j+1}-s_j)^{\frac{1}{2H_0}[2-\frac{2}{\kappa}-\frac{2}{\kappa}\alpha_j(1-2H)]}$$

$$\times\left(\int_{\mathbb{R}}\frac{\sin^2(\eta)}{\eta^2}|\eta|^{\frac{2}{\kappa}\alpha_j(1-2H)+\frac{2}{\kappa}-1}d\eta\right)^{\frac{1}{2H_0}}$$

$$\times\left(\int_{\mathbb{R}}\frac{\sin^2((t-s_n)y)}{y^2}|y|^{\frac{2}{\kappa}\alpha_n(1-2H)+\frac{2}{\kappa}-1}\big(\big(|z||y|^{\frac{2}{\kappa}}\big)\wedge 2\big)\,dy\right)^{\frac{1}{2H_0}}\Bigg]d\mathbf{s}. \tag{5.1}$$

Recall that $\alpha_n\in\{0,1\}$, and by Lemma A.7 with $\lambda=\frac{2}{\kappa}$, $\beta=\frac{2}{\kappa}(\alpha_n(1-2H)+1)-1$, $\gamma=0$ for $\alpha_n=1$ and $\gamma=\frac{1-2H}{\kappa}$ for $\alpha_n=0$, we have

$$\int_{\mathbb{R}}\frac{\sin^2((t-s_n)y)}{y^2}|y|^{\frac{2}{\kappa}\alpha_n(1-2H)+\frac{2}{\kappa}-1}\big(\big(|z||y|^{\frac{2}{\kappa}}\big)\wedge 2\big)\,dy\le C\big(1\vee(t-s_n)^{\frac{2(1-2H)}{\kappa}}\big)|z|^{2H+\kappa-2}$$

$$\le C|z|^{2H+\kappa-2}.$$

Thus, there exists a positive constant C depending only on (p,H_0,H,T) such that

$$\big\|u(t,x+z)-u(t,x)\big\|_p$$

$$\le\sum_{n=1}^{\infty}C^n(n!)^{H_0-\frac{1}{2}}\Bigg(\int_{[0,t]^n_<}\sum_{\alpha\in\mathcal{A}_n}\prod_{j=1}^{n-1}(s_{j+1}-s_j)^{\frac{1}{2H_0}[2-\frac{2}{\kappa}-\frac{2}{\kappa}\alpha_j(1-2H)]}d\mathbf{s}\Bigg)^{H_0}.$$

When $n = 1$, the integral on the right-hand side of the above inequality equals t.

When $n \geq 2$, for each fixed $\alpha \in \mathcal{A}_n$, denote $\beta_j = \frac{1}{2H_0}[2 - \frac{2}{\kappa} - \frac{2}{\kappa}\alpha_j(1-2H)]$, $j = 1, \ldots, n$. Noting that $\alpha_n \in \{0, 1\}$ and $\sum_{j=1}^{n} \alpha_j = n$, we have

$$\sum_{j=1}^{n-1} \beta_j = \begin{cases} \dfrac{(\kappa - 2 + 2H)n}{H_0\kappa} - \dfrac{\kappa - 2 + 2H}{H_0\kappa} & \text{when } \alpha_n = 1, \\ \dfrac{(\kappa - 2 + 2H)n}{H_0\kappa} - \dfrac{\kappa - 1}{H_0\kappa} & \text{when } \alpha_n = 0. \end{cases} \tag{5.2}$$

Using Lemma A.2, we obtain

$$\begin{aligned} \int_{[0,t]^n_<} \prod_{j=1}^{n-1} (s_{j+1} - s_j)^{\beta_j} \, d\mathbf{s} &= \int_0^t \int_{0<s_1<\cdots<s_n} \prod_{j=1}^{n-1} (s_{j+1} - s_j)^{\beta_j} \, d\mathbf{s} \\ &= \frac{\prod_{j=1}^{n-1} \Gamma(1+\beta_j)}{\Gamma(\beta_1 + \cdots + \beta_{n-1} + n)} \int_0^t s_n^{\beta_1 + \cdots + \beta_{n-1} + n - 1} ds_n \\ &= \frac{\prod_{j=1}^{n-1} \Gamma(1+\beta_j)}{(\beta_1 + \cdots + \beta_{n-1} + n)\Gamma(\beta_1 + \cdots + \beta_{n-1} + n)} t^{\beta_1 + \cdots + \beta_{n-1} + n} \\ &= \frac{\prod_{j=1}^{n-1} \Gamma(1+\beta_j)}{\Gamma(\beta_1 + \cdots + \beta_{n-1} + n + 1)} t^{\beta_1 + \cdots + \beta_{n-1} + n}. \end{aligned}$$

Then applying (A.1) in Lemma A.3 with $a = 1 + \frac{\kappa - 2 + 2H}{H_0\kappa} \in (1, 2)$ and either $b = 1 - \frac{\kappa - 2 + 2H}{H_0\kappa} \in (0, 1)$ or $b = 1 - \frac{\kappa - 1}{H_0\kappa} \in [0, 1)$, we have

$$\begin{aligned} &\int_{[0,t]^n_<} \prod_{j=1}^{n-1} (s_{j+1} - s_j)^{\beta_j} \, d\mathbf{s} \\ &\leq \frac{C^{n-1}}{\Gamma(\beta_1 + \cdots + \beta_{n-1} + n + 1)} t^{\beta_1 + \cdots + \beta_{n-1} + n} = \frac{C^{n-1}}{\Gamma(an + b)} t^{a(n-1) + \frac{\kappa - 2 + 2H}{H_0\kappa} + b} \\ &\sim \frac{C^{n-1}}{(n!)^a a^{an + b - \frac{1}{2}} n^{b - \frac{1}{2} - \frac{a}{2}}} t^{a(n-1) + \frac{\kappa - 2 + 2H}{H_0\kappa} + b} \leq \frac{C^{n-1} T^{\frac{\kappa - 2 + 2H}{H_0\kappa} + b} t^{a(n-1)}}{(n!)^a}. \end{aligned}$$

Therefore, there exists a positive constant C depending only on (p, H_0, H, T) such that

$$\begin{aligned} \big\| u(t, x+z) - u(t, x) \big\|_p &\leq |z|^{H + \frac{\kappa}{2} - 1} \left((p-1)^{\frac{1}{2}} Ct + \sum_{n=2}^{\infty} \frac{(p-1)^{\frac{n}{2}} C^{n-1} t^{\frac{(H_0\kappa + \kappa - 2 + 2H)(n-1)}{\kappa}}}{(n!)^{\frac{3\kappa - 4 + 4H}{2\kappa}}} \right) \\ &\leq C|z|^{H + \frac{\kappa}{2} - 1} \sum_{n=0}^{\infty} \frac{C^n t^{\frac{(H_0\kappa + \kappa - 2 + 2H)n}{\kappa}}}{(n!)^{\frac{3\kappa - 4 + 4H}{2\kappa}}} \leq C|z|^{H + \frac{\kappa}{2} - 1}. \end{aligned} \tag{5.3}$$

Then the θ_2-Hölder continuity for $\theta_2 \in (0, H + \frac{\kappa}{2} - 1)$ follows from the Kolmogorov's continuity criterion.

Now we consider the Hölder continuity in time.

$$\begin{aligned}\big\|u(t+h,x)-u(t,x)\big\|_p &\le \sum_{n=1}^{\infty}(p-1)^{\frac{n}{2}}\big\|g_n(\cdot,t+h,x)-g_n(\cdot,t,x)\big\|_{\mathcal{H}^{\otimes n}}\\ &\le \sum_{n=1}^{\infty}(p-1)^{\frac{n}{2}}(n!)^{\frac{1}{2}}\big[\sqrt{A_n(t,h)}+\sqrt{B_n(t,h)}\big], \qquad (5.4)\end{aligned}$$

where

$$A_n(t,h)=\big\|g_n(\cdot,t+h,x)I_{[0,t]^n}-g_n(\cdot,t,x)\big\|^2_{\mathcal{H}^{\otimes n}}$$

and

$$B_n(t,h)=\big\|g_n(\cdot,t+h,x)I_{[0,t+h]^n\setminus[0,t]^n}\big\|^2_{\mathcal{H}^{\otimes n}}.$$

For $A_n(t,h)$, we have

$$\begin{aligned}A_n(t,h)&=\big\|g_n(\cdot,t+h,x)I_{[0,t]^n}-g_n(\cdot,t,x)\big\|^2_{\mathcal{H}^{\otimes n}}\\ &=\int_{\mathbb{R}^n}\int_{[0,t]^{2n}}\mathcal{F}\big[g_n(\mathbf{s},\cdot,t+h,x)(\boldsymbol{\xi})-g_n(\mathbf{s},\cdot,t,x)(\boldsymbol{\xi})\big]\\ &\quad\times\overline{\mathcal{F}\big[g_n(\mathbf{r},\cdot,t+h,x)(\boldsymbol{\xi})-g_n(\mathbf{r},\cdot,t,x)(\boldsymbol{\xi})\big]}\prod_{j=1}^{n}|s_j-r_j|^{2H_0-2}d\mathbf{s}\,d\mathbf{r}\boldsymbol{\mu}(d\boldsymbol{\xi})\\ &\le(n!)^{2H_0-2}\bigg(\int_{[0,t]^n_<}\bigg[\int_{\mathbb{R}^n}\prod_{j=1}^{n-1}\frac{\sin^2((s_{j+1}-s_j)|\xi_1+\cdots+\xi_j|^{\kappa/2})}{|\xi_1+\cdots+\xi_j|^{\kappa}}\prod_{j=1}^{n}|\xi_j|^{1-2H}\\ &\quad\times\bigg|\frac{\sin((t+h-s_n)|\xi_1+\cdots+\xi_n|^{\kappa/2})}{|\xi_1+\cdots+\xi_n|^{\kappa/2}}-\frac{\sin((t-s_n)|\xi_1+\cdots+\xi_n|^{\kappa/2})}{|\xi_1+\cdots+\xi_n|^{\kappa/2}}\bigg|^2d\boldsymbol{\xi}\bigg]^{\frac{1}{2H_0}}d\mathbf{s}\bigg)^{2H_0}\\ &=(n!)^{2H_0-2}\bigg(\int_{[0,t]^n_<}\bigg[\int_{\mathbb{R}^n}\bigg(\prod_{j=1}^{n-1}\frac{\sin^2((s_{j+1}-s_j)|\eta_j|^{\kappa/2})}{|\eta_j|^{\kappa}}\bigg)\prod_{j=1}^{n}|\eta_j-\eta_{j-1}|^{1-2H}\\ &\quad\times\frac{|\sin((t+h-s_n)|\eta_n|^{\kappa/2})-\sin((t-s_n)|\eta_n|^{\kappa/2})|^2}{|\eta_n|^{\kappa}}d\boldsymbol{\eta}\bigg]^{\frac{1}{2H_0}}d\mathbf{s}\bigg)^{2H_0}\\ &\le(n!)^{2H_0-2}\bigg(\int_{[0,t]^n_<}\sum_{\alpha\in\mathcal{A}_n}\bigg[\int_{\mathbb{R}^n}\bigg(\prod_{j=1}^{n-1}\frac{\sin^2((s_{j+1}-s_j)|\eta_j|^{\kappa/2})}{|\eta_j|^{\kappa}}\bigg)\prod_{j=1}^{n}|\eta_j|^{\alpha_j(1-2H)}\\ &\quad\times\frac{C_\gamma^2((|h|^{2\gamma}|\eta_n|^{\gamma\kappa})\wedge(|h|^2|\eta_n|^{\kappa}))}{|\eta_n|^{\kappa}}d\boldsymbol{\eta}\bigg]^{\frac{1}{2H_0}}d\mathbf{s}\bigg)^{2H_0},\end{aligned}$$

where the last step follows from Lemma A.8 with $\gamma \in (0, 1 - \frac{2}{\kappa} + \frac{2H}{\kappa})$. Note that when $\gamma \in (0, 1 - \frac{2}{\kappa} + \frac{2H}{\kappa})$,

$$\int_{\mathbb{R}} \frac{((|h|^{2\gamma}|\eta_n|^{\gamma\kappa}) \wedge (|h|^2|\eta_n|^{\kappa}))}{|\eta_n|^{\kappa}} |\eta_n|^{\alpha_n(1-2H)}\, d\eta_n \le C\big(|h|^2 + |h|^{2\gamma}\big) \quad \text{for } \alpha_n \in \{0, 1\}$$

and following the approach in the analysis of (5.1)–(5.3), we can show that

$$\sum_{n=1}^{\infty} (p-1)^{\frac{n}{2}} (n!)^{\frac{1}{2}} \sqrt{A_n(t,h)} \le C\big(|h| + |h|^{\gamma}\big) \le C|h|^{\gamma} \tag{5.5}$$

for $\gamma \in (0, 1 - \frac{2}{\kappa} + \frac{2H}{\kappa})$ with C depending on $(p, \kappa, H_0, H, T, \mathbb{M}, \gamma)$.

Now we consider the term $B_n(t,h)$. Denote $E_{t,h} = [0, t+h]^n \setminus [0,t]^n$, and then

$$E_{t,h} = \bigcup_{\rho \in \mathcal{P}_n} \big\{(s_1, \dots, s_n) : s_{\rho(1)} \le s_{\rho(2)} \le \cdots \le s_{\rho(n)}, t < s_{\rho(n)} \le t+h\big\}.$$

Therefore, we have

$$\begin{aligned}
B_n(t,h) &= \big\| g_n(\cdot, t+h, x) I_{[0,t+h]^n \setminus [0,t]^n} \big\|^2_{\mathcal{H}^{\otimes n}} \\
&= \int_{\mathbb{R}^n} \int_{[0,t+h]^{2n}} \mathcal{F}\big[g_n(\mathbf{s}, \cdot, t+h, x)(\boldsymbol{\xi})\big] \overline{\mathcal{F}\big[g_n(\mathbf{r}, \cdot, t+h, x)(\boldsymbol{\xi})\big]} \\
&\quad \times I_{E_{t,h}}(\mathbf{s}) I_{E_{t,h}}(\mathbf{r}) \prod_{j=1}^{n} |s_j - r_j|^{2H_0 - 2}\, d\mathbf{s}\, d\mathbf{r} \boldsymbol{\mu}(d\boldsymbol{\xi}) \\
&\le (n!)^{-2} \Bigg(\sum_{\rho \in \mathcal{P}_n} \int_t^{t+h} \int_{0 < s_{\rho(1)} < s_{\rho(2)} < \cdots < s_{\rho(n)}} \Bigg[\int_{\mathbb{R}^n} \prod_{j=1}^{n} \frac{\sin^2((s_{\rho(j+1)} - s_{\rho(j)})|\xi_{\rho(1)} + \cdots + \xi_{\rho(j)}|^{\kappa/2})}{|\xi_{\rho(1)} + \cdots + \xi_{\rho(j)}|^{\kappa}} \\
&\quad \times \prod_{j=1}^{n} |\xi_j|^{1-2H} d\boldsymbol{\xi} \Bigg]^{\frac{1}{2H_0}} d\mathbf{s} \Bigg)^{2H_0} \\
&= (n!)^{-2} \Bigg(n! \int_t^{t+h} \int_{0 < s_1 < s_2 < \cdots < s_n} \Bigg[\int_{\mathbb{R}^n} \prod_{j=1}^{n} \frac{\sin^2((s_{j+1} - s_j)|\xi_1 + \cdots + \xi_j|^{\kappa/2})}{|\xi_1 + \cdots + \xi_j|^{\kappa}} \\
&\quad \times \prod_{j=1}^{n} |\xi_j|^{1-2H} d\boldsymbol{\xi} \Bigg]^{\frac{1}{2H_0}} d\mathbf{s} \Bigg)^{2H_0} \\
&= (n!)^{2H_0 - 2} \Bigg(\int_t^{t+h} \int_{0 < s_1 < s_2 < \cdots < s_n} \sum_{\alpha \in \mathcal{A}_n} \Big(\frac{2}{\kappa}\Big)^{\frac{n}{2H_0}} \prod_{j=1}^{n} (s_{j+1} - s_j)^{\frac{1}{2H_0}[2 - \frac{2}{\kappa} - \frac{2}{\kappa}\alpha_j(1-2H)]} \\
&\quad \times \Big(\int_{\mathbb{R}} \frac{\sin^2(\eta)}{\eta^2} |\eta|^{\frac{2}{\kappa}\alpha_j(1-2H) + \frac{2}{\kappa} - 1} d\eta \Big)^{\frac{1}{2H_0}} d\mathbf{s} \Bigg)^{2H_0}
\end{aligned}$$

where by convention $s_{j+1} = t + h$.

Note that in the case $n = 1$, we have

$$B_n(t,h) \le \frac{2}{\kappa}\left(\int_{\mathbb{R}} \frac{\sin^2(\eta)}{\eta^2}|\eta|^{\frac{2}{\kappa}(1-2H)+\frac{2}{\kappa}-1}\,d\eta\right)\frac{h^{2-\frac{2}{\kappa}-\frac{2}{\kappa}(1-2H)+2H_0}}{(\beta+1)^{2H_0}}$$

with $\beta = \frac{1}{2H_0}[2-\frac{2}{\kappa}-\frac{2}{\kappa}(1-2H)]$, and in the case $n \ge 2$, it is easy to see from (5.2) that $\beta_1+\cdots+\beta_{n-1}+n-1>0$ and hence by Lemma A.2, we have

$$\begin{aligned}
&\int_t^{t+h}\int_{0<s_1<s_2<\cdots<s_n}\prod_{j=1}^n (s_{j+1}-s_j)^{\beta_j}\,d\mathbf{s}\\
&\quad= \frac{\prod_{j=1}^{n-1}\Gamma(1+\beta_j)}{\Gamma(\beta_1+\cdots+\beta_{n-1}+n)}\int_t^{t+h} s_n^{\beta_1+\cdots+\beta_{n-1}+n-1}(t+h-s_n)^{\beta_n}ds_n\\
&\quad\le T^{\beta_1+\cdots+\beta_{n-1}+n-1}\frac{\prod_{j=1}^{n-1}\Gamma(1+\beta_j)}{\Gamma(\beta_1+\cdots+\beta_{n-1}+n)}\frac{1}{\beta_n+1}h^{\beta_n+1}\\
&\quad= T^{\beta_1+\cdots+\beta_{n-1}+n-1}\frac{(\beta_1+\cdots+\beta_{n-1}+n)\prod_{j=1}^{n-1}\Gamma(1+\beta_j)}{\Gamma(\beta_1+\cdots+\beta_{n-1}+n+1)}\frac{1}{\beta_n+1}h^{\beta_n+1},
\end{aligned}$$

where $\beta_j = \frac{1}{2H_0}[2-\frac{2}{\kappa}-\frac{2}{\kappa}\alpha_j(1-2H)]$. Now similar to the calculus below the equation (5.2) and recalling that $\alpha_n \in \{0,1\}$, we can show that

$$\sum_{n\ge 0}(p-1)^{\frac{n}{2}}(n!)^{\frac{1}{2}}\sqrt{B_n(t,h)} \le C\big(|h|^{1-\frac{1}{\kappa}+H_0}+|h|^{1-\frac{2}{\kappa}+\frac{2H}{\kappa}+H_0}\big) \tag{5.6}$$

with C depending on (p, H_0, H, T).

Finally, combining inequalities (5.4), (5.5) and (5.6), for $|h| \le 1$, we have

$$\big\|u(t+h,x)-u(t,x)\big\|_p \le C|h|^{\theta_1}$$

for any $\theta_1 \in (0, 1-\frac{2}{\kappa}+\frac{2H}{\kappa})$ where C is a constant depending only on $(p,\kappa,H_0,H,T,\mathbb{M},\theta_1)$, and the Hölder continuity in space follows from the Kolmogorov's continuity criterion.

The proof is concluded. □

Appendix

In this section, we collect the lemmas that were used in the preceding sections. Some of the proofs are obvious and hence omitted.

Lemma A.1. *For $a>0$ and $\theta>-1$,*

$$\int_{\mathbb{R}}\exp\big(-ax^2\big)|x|^{\theta}\,dx = a^{-\frac{1}{2}(1+\theta)}\int_{\mathbb{R}}\exp\big(-x^2\big)|x|^{\theta}\,dx.$$

Lemma A.2. *Suppose $\alpha_i \in (-1, \infty)$, $i = 1, \dots, n$ and let $\alpha = \alpha_1 + \cdots + \alpha_n$. Then*

$$\int_{[0<r_1<\cdots<r_n<r_{n+1}=t]} \prod_{i=1}^{n} (r_{i+1} - r_i)^{\alpha_i} \, dr_1 \dots dr_n = \frac{\prod_{i=1}^{n} \Gamma(\alpha_i + 1) t^{\alpha+n}}{\Gamma(\alpha + n + 1)},$$

where $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} \, dt$ is the Gamma function.

Lemma A.3. *For any $a > 0$ and $b \in [0, 1]$, we have*

$$\lim_{n\to\infty} \frac{\Gamma(an + b)}{(n!)^a a^{an+b-\frac{1}{2}} n^{b-\frac{1}{2}-\frac{a}{2}}} = 1 \tag{A.1}$$

and

$$c_1 \exp\bigl(c_2 x^{\frac{1}{a}}\bigr) \le \sum_{n=0}^{\infty} \frac{x^n}{(n!)^a} \le C_1 \exp\bigl(C_2 x^{\frac{1}{a}}\bigr) \quad \forall x > 0, \tag{A.2}$$

where $c_1 > 0$, $c_2 > 0$, $C_1 > 0$ and $C_2 > 0$ are some constants depending on a.

Proof. The proof of (A.1) follows from Stirling's formula (see also (68) in [2] which is (A.1) in the case of $b = 1$). See Lemma A. 1 in [2] for the upper bound in (A.2) and Lemma 5.2 in [4] for the lower bound in (A.2). □

Lemma A.4.

$$\int_0^\infty \sin^2(x) x^{-\alpha} \, dx < \infty$$

if and only if $\alpha \in (1, 3)$.

Proof. The sufficiency is obvious. The necessity follows from the estimation

$$\begin{aligned} \int_0^\infty \sin^2(x) x^{-\alpha} \, dx &\ge \int_0^{\frac{\pi}{4}} \frac{\sin^2(x)}{x^2} x^{2-\alpha} \, dx + \sum_{n=0}^{\infty} \int_{(n+1/4)\pi}^{(n+3/4)\pi} \sin^2(x) x^{-\alpha} \, dx \\ &\ge \int_0^{\frac{\pi}{4}} \frac{\sin^2(x)}{x^2} x^{2-\alpha} \, dx + \frac{1}{4} \pi^{1-\alpha} \sum_{n=0}^{\infty} (n + 3/4)^{-\alpha}. \end{aligned}$$

The proof is completed. □

Lemma A.5. *For $H \in (0, 1)$ and $r, s > 0$,*

$$C_H \int_{\mathbb{R}} \sin\bigl(r|\eta|\bigr) \sin\bigl(s|\eta|\bigr) |\eta|^{-1-2H} \, d\eta = \frac{1}{4} \bigl(|r + s|^{2H} - |r - s|^{2H}\bigr)$$

with C_H given by (2.2). In particular, this integral is positive.

Proof. Let $\mathbb{X}^H$ be the Hilbert space associated with fractional Brownian motion $\{B^H(x), x \in \mathbb{R}\}$ with Hurst parameter $H \in (0,1)$, that is, it is the linear expansion of indicator functions under the inner product

$$\langle I_{[0,x]}, I_{[0,y]} \rangle_{\mathbb{X}^H} = \frac{1}{2}\big(|x|^{2H} + |y|^{2H} - |x-y|^{2H}\big).$$

Using the convention $I_{[0,x]} = -I_{[x,0]}$ for $x < 0$, the linear mapping $B^H : I_{[0,x]} \to B^H(x)$ extends to a linear isometry between $\mathbb{X}^H$ and the Gaussian space $\{B^H(\varphi), \varphi \in \mathbb{X}^H\}$ spanned by B^H. Furthermore, for $\varphi, \phi \in \mathbb{X}^H$, we have (see [33])

$$\langle \varphi, \phi \rangle_{\mathbb{X}^H} = C_H \int_{\mathbb{R}} \widehat{\varphi}(\eta)\overline{\widehat{\phi}(\eta)}|\eta|^{1-2H}\,d\eta.$$

Now, noting that $(\mathcal{F}I_{[-r,r]}(\cdot))(\xi) = \overline{(\mathcal{F}I_{[-r,r]}(\cdot))(\xi)} = \frac{2\sin(r|\xi|)}{|\xi|}$, we have

$$\begin{aligned}
&C_H \int_{\mathbb{R}} \sin\big(r|\eta|\big)\sin\big(s|\eta|\big)|\eta|^{-1-2H}\,d\eta \\
&\quad = \frac{1}{4}C_H \int_{\mathbb{R}} (\mathcal{F}I_{[|x|\le r]})(\eta)(\mathcal{F}I_{[|x|\le s]})(\eta)|\eta|^{1-2H}\,d\eta \\
&\quad = \frac{1}{4}\big\langle I_{[-r,r]}(\cdot), I_{[-s,s]}(\cdot)\big\rangle_{\mathbb{X}^H} \\
&\quad = \frac{1}{4}\mathbb{E}\big[B^H\big(I_{[-r,r]}(\cdot)\big)B^H\big(I_{[-s,s]}(\cdot)\big)\big] \\
&\quad = \frac{1}{4}\mathbb{E}\big[\big(B^H(r) - B^H(-r)\big)\big(B^H(s) - B^H(-s)\big)\big] \\
&\quad = \frac{1}{4}\big(|r+s|^{2H} - |r-s|^{2H}\big).
\end{aligned}$$

We complete the proof. □

Lemma A.6. *For any $a \ge 0$, we have*

$$\int_0^\infty e^{-t}\sin(at)\,dt = \frac{a}{1+a^2}.$$

Proof. The result follows by integration by parts. □

Lemma A.7. *Let a and b be two positive constants, and $\lambda \in [1,\infty)$, $\beta \in [0,1)$, $\gamma \in [0,1]$ such that $1 - \lambda < \beta + 2\gamma < 1$. Then we have*

$$\int_{\mathbb{R}} \sin^2(ax)|x|^{-2+\beta}\big(\big(b|x|^\lambda\big) \wedge 2\big)\,dx \le Ca^{2\gamma}b^{\frac{1}{\lambda}(1-\beta-2\gamma)},$$

where

$$C = C_{\lambda,\beta,\gamma} = \max\left\{\int_{|y|\le 2} |y|^{\frac{1}{\lambda}(-1+\beta+2\gamma)}\,dy, \int_{|y|>2} |y|^{\frac{1}{\lambda}(-1+\beta+2\gamma)-1}\,dy\right\}.$$

Proof. We write

$$\begin{aligned}&\int_{\mathbb{R}} \sin^2(ax)|x|^{-2+\beta}\big(\big(b|x|^{\lambda}\big)\wedge 2\big)\,dx\\ &\quad= \int_{b|x|^{\lambda}\le 2} \sin^2(ax)|x|^{-2+\beta}b|x|^{\lambda}\,dx + 2\int_{b|x|^{\lambda}>2} \sin^2(ax)|x|^{-2+\beta}\,dx.\end{aligned}$$

Noting that $|\sin(x)| \le |x|^{\gamma}$ for $\gamma \in [0,1]$, we have

$$\begin{aligned}\int_{b|x|^{\lambda}\le 2} \sin^2(ax)|x|^{-2+\beta}b|x|^{\lambda}\,dx &\le \int_{b|x|^{\lambda}\le 2} |ax|^{2\gamma}|x|^{-2+\beta}b|x|^{\lambda}\,dx\\ &= \frac{1}{\lambda}a^{2\gamma}b^{\frac{1}{\lambda}(1-\beta-2\gamma)}\int_{|y|\le 2} |y|^{\frac{1}{\lambda}(-1+\beta+2\gamma)}\,dy,\end{aligned}$$

and

$$\begin{aligned}\int_{b|x|^{\lambda}>2} \sin^2(ax)|x|^{-2+\beta}\,dx &\le \int_{b|x|^{\lambda}>2} |ax|^{2\gamma}|x|^{-2+\beta}\,dx\\ &= \frac{1}{\lambda}a^{2\gamma}b^{\frac{1}{\lambda}(1-\beta-2\gamma)}\int_{|y|>2} |y|^{\frac{1}{\lambda}(-1+\beta+2\gamma)-1}\,dy.\end{aligned}$$

Thus, the proof is concluded. □

Lemma A.8. *For any $t, h \in \mathbb{R}$ and $\gamma \in (0,1]$, there exists a constant C_γ such that*

$$\big|\sin\big((t+h)x\big) - \sin(tx)\big| \le C_\gamma |hx|^{\gamma}.$$

In particular,

$$\big|\sin\big((t+h)x\big) - \sin(tx)\big| \le C_\gamma\big(|hx|^{\gamma}\wedge |hx|\big).$$

Proof. By the mean value theorem and the fact of $y \le \frac{2^{1-\gamma}}{\gamma}y^{\gamma}$ for $y \in [0,2]$ and $\gamma \in (0,1]$, we have

$$\big|\sin\big((t+h)x\big) - \sin(tx)\big| \le |hx|\big|\cos(sx)\big| \le |hx|$$

and

$$\big|\sin\big((t+h)x\big) - \sin(tx)\big| \le \frac{2^{1-\gamma}}{\gamma}\big|\sin\big((t+h)x\big) - \sin(tx)\big|^{\gamma} = C_\gamma|hx|^{\gamma}\big|\cos(sx)\big|^{\gamma} \le C_\gamma|hx|^{\gamma},$$

where s is a number between t and $t+h$. The desired results can be obtained. □

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