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Stratonovich stochastic differential equation with irregular coefficients: Girsanov's example revisited

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In this paper, we study the Stratonovich stochastic differential equation $\mathrm{d}X = |X|^\alpha \circ \mathrm{d}B$, $\alpha \in (-1, 1)$, which has been introduced by Cherstvy et al. (*New J. Phys.* **15** (2013) 083039) in the context of analysis of anomalous diffusions in heterogeneous media. We determine its weak and strong solutions, which are homogeneous strong Markov processes spending zero time at 0: for $\alpha \in (0, 1)$, these solutions have the form

$$X_t^\theta = \big((1-\alpha)B_t^\theta\big)^{1/(1-\alpha)},$$

where B^θ is the θ-skew Brownian motion driven by B and starting at $\frac{1}{1-\alpha}(X_0)^{1-\alpha}$, $\theta \in [-1, 1]$, and $(x)^\gamma = |x|^\gamma \operatorname{sign} x$; for $\alpha \in (-1, 0]$, only the case $\theta = 0$ is possible. The central part of the paper consists in the proof of the existence of a quadratic covariation $[f(B^\theta), B]$ for a locally square integrable function f and is based on the time-reversion technique for Markovian diffusions.

Keywords: generalized Itô's formula; Girsanov's example; heterogeneous diffusion process; local time; non-uniqueness; singular stochastic differential equation; skew Brownian motion; Stratonovich integral; time reversion

1. Introduction

Girsanov [16] considered the Itô stochastic differential equation

$$Y_t = Y_0 + \int_0^t |Y_s|^\alpha \,\mathrm{d}B_s, \quad t \geq 0, \tag{1.1}$$

driven by a standard Brownian motion B as an example of an SDE with non-unique solution. In particular, it was shown that for $\alpha \in (0, 1/2)$, equation (1.1) has infinitely many continuous strong Markov (weak) solutions as well as non-homogeneous Markov solutions; non-Markovian solutions can also be constructed.

Since then, equation (1.1) serves as a benchmark example of various peculiar effects which come to light when one weakens the standard regularity assumptions on the coefficients of an SDE.

The proof of weak uniqueness for $\alpha \geq 1/2$ and non-uniqueness for $\alpha \in (0, 1/2)$ with the help of random time change was given by McKean [23], §3.10b, whereas a construction of an uncountable set of weak solutions can be found in Engelbert and Schmidt [11], Example 3.3. The existence and uniqueness of a strong solution for $\alpha \in [1/2, 1]$ was established by Zvonkin [32], Theorem 4.

Furthermore for $\alpha \in (0, 1/2)$, it was shown in Engelbert and Schmidt [11], Theorem 5.2, that for every initial value $Y_0 \in \mathbb{R}$, there is a weak solution to (1.1) that spends zero time at 0 (the so-called fundamental solution) and the law of such a solution is unique. Path-wise uniqueness among those solutions to (1.1) that spend zero time at 0, and existence of a strong solution was proven by Bass et al. [4], Theorem 1.2.

An analogue of (1.1) with the *Stratonovich integral*

$$X_t = X_0 + \int_0^t |X_s|^\alpha \circ \mathrm{d}B_s \tag{1.2}$$

was recently introduced in the physical literature by Cherstvy et al. [8] under the name *heterogeneous diffusion process*. The authors studied the autocorrelation function of this process analytically and investigated its sub- and super-diffusive behaviour with the help of numerical simulations. A similar system with an additional linear drift was considered earlier by Denisov and Horsthemke [9].

In this paper, we will further investigate equation (1.2) with $\alpha \in (-1, 1)$. It turns out that equation (1.2) has properties quite different from its Itô counterpart. Let us first make some observations about it. The only problematic point of the diffusion coefficient $\sigma(x) = |x|^\alpha$ is $x = 0$. For $\alpha \in (0, 1)$, the Lipschitz continuity fails at this point, and for $\alpha \in (-1, 0)$ even the continuity and boundedness. However, one can easily solve (1.2) locally for initial points $X_0 \neq 0$.

Indeed, assume for definiteness that $X_0 > 0$. For any $\varepsilon > 0$, using the properties of the Stratonovich integral, we see that the process given by

$$X_t^0 = \big((1-\alpha)B_t + X_0^{1-\alpha}\big)^{\frac{1}{1-\alpha}}. \tag{1.3}$$

solves equation (1.2) until the time $\tau_\varepsilon = \inf\{t \geq 0\colon X_t = \varepsilon\}$. Moreover, the solution is unique until τ_ε. Consequently, the formula (1.3) defines a unique strong solution until the time $\tau_0 = \inf\{t \geq 0\colon X_t^0 = 0\}$ when the process first hits zero. It is clear that extending the solution by zero value beyond τ_0 gives a strong solution. However, the uniqueness fails: as it is shown in Section 3, the formula (1.3) defines a strong solution (called a benchmark solution), if we understand the right-hand side as a signed power function, i.e. $(x)^{\frac{1}{1-\alpha}} = |x|^{\frac{1}{1-\alpha}} \operatorname{sign} x$. In Section 3, we also construct some non-Markov solutions of (1.2).

The next question is whether, as for the Itô equation, uniqueness holds within the class of solutions spending zero time at 0; this question is addressed in Section 4. For $\alpha \in (0, 1)$, this question is answered negatively: in Theorem 4.5 we show that equation (1.2) has also 'skew' solutions

$$X_t^\theta = \big|(1-\alpha)B_t^\theta\big|^{\frac{1}{1-\alpha}} \cdot \operatorname{sign}\big((1-\alpha)B_t^\theta\big),$$

where for $\theta \in [-1, 1]$, B^θ is the skew Brownian motion, which solves the stochastic differential equation $B_t^\theta = \frac{1}{1-\alpha}(X_0)^{1-\alpha} + B_t + \theta L_t(B^\theta)$, L being the symmetric local time at 0. Moreover, we show all solutions, which are homogeneous strong Markov processes and which spend zero time at 0, have this form.

In Section 5, we propose an explanation for a diversity of strong solutions to (1.2) and discuss further questions regarding the equation. Sections 6 and 7 contain the proofs of results concerning weak solutions to equation (1.2), Section 8 contains the proof of existence of the bracket $[f(B^\theta), B]$, and Section 9 is devoted to the proof of the main result concerning strong solutions.

Several proofs are either repetitions of known ideas or routine calculations, so they are omitted from the article and given in detail in supplemental article (Pavlyukevich and Shevchenko [24]).

2. Preliminaries and conventions

Throughout the article, we work on a stochastic basis $(\Omega, \mathscr{F}, \mathbb{F}, \mathbf{P})$, that is, a complete probability space with a filtration $\mathbb{F} = (\mathscr{F}_t)_{t\ge 0}$ satisfying the standard assumptions. The process $B = (B_t)_{t\ge 0}$ is a standard continuous Brownian motion on this stochastic basis.

First, we briefly recall definitions related to stochastic integration. More details may be found in Protter [26].

The main mode of convergence considered here is the uniform convergence on compacts in probability (the u.c.p. convergence for short): a sequence $X^n = (X^n_t)_{t\ge 0}$, $n \ge 1$, of stochastic processes converges to $X = (X_t)_{t\ge 0}$ in u.c.p. if for any $t \ge 0$

$$\sup_{s\in[0,t]} \left|X^n_s - X_s\right| \xrightarrow{\mathbf{P}} 0, \quad n \to \infty.$$

Let a sequence of partitions $D_n = \{0 = t^n_0 < t^n_1 < t^n_2 < \cdots\} = \{0 = t_0 < t_1 < t_2 < \cdots\}$ of $[0, \infty)$ be such that for each $t \ge 0$ the number of points in each interval $[0, t]$ is finite, and $\|D_n\| := \sup_{k\ge 1} |t^n_k - t^n_{k-1}| \to 0$ as $n \to \infty$. A continuous process X has quadratic variation $[X]$ if the limit

$$[X]_t := \lim_{n\to\infty} \sum_{t_k\in D_n, t_k<t} (X_{t_{k+1}} - X_{t_k})^2$$

exists in the u.c.p. sense. Similarly, the quadratic covariation $[X, Y]$ of two continuous processes X and Y is defined as a limit in u.c.p.

$$[X, Y]_t := \lim_{n\to\infty} \sum_{t_k\in D_n, t_k<t} (X_{t_{k+1}} - X_{t_k})(Y_{t_{k+1}} - Y_{t_k}).$$

When X and Y are semimartingales, the quadratic variations $[X]$, $[Y]$ and the quadratic covariation $[X, Y]$ exist, moreover, they have bounded variation on any finite interval.

Further, we define the Itô (forward) integral as a limit in u.c.p.

$$\int_0^t X_s \,\mathrm{d}Y_s = \lim_{n\to\infty} \sum_{t_k\in D_n, t_k<t} X_{t_k}(Y_{t_{k+1}} - Y_{t_k})$$

and the Stratonovich (symmetric) integral as limit in u.c.p.

$$\begin{aligned}\int_0^t X_s \circ \mathrm{d}Y_s &= \int_0^t X_s \,\mathrm{d}Y_s + \frac{1}{2}[X, Y]_t \\ &= \lim_{n\to\infty} \sum_{t_k\in D_n, t_k<t} \frac{1}{2}(X_{t_{k+1}} + X_{t_k})(Y_{t_{k+1}} - Y_{t_k}),\end{aligned}$$

provided that both the Itô integral and the quadratic variation exist. Again, when both X and Y are continuous semimartingales, both integrals exists, and the convergence holds in u.c.p. There is an alternative approach to Stratonovich stochastic integration, developed in Russo and Vallois [27–29], which allows integration with respect to non-semimartingales and non-Markov processes like fractional Brownian motion.

For a process X, by $L_t(X)$ we denote the symmetric local time at zero defined as the limit in probability

$$L_t(X) = \lim_{\varepsilon\downarrow 0} \frac{1}{2\varepsilon} \int_0^t \mathbb{I}_{[-\varepsilon,\varepsilon]}(X_s)\,\mathrm{d}s. \tag{2.1}$$

In this paper, we define

$$\operatorname{sign} x = \begin{cases} -1, & x < 0, \\ 0, & x = 0, \\ 1, & x > 0, \end{cases}$$

and for any $\alpha \in \mathbb{R}$ we set

$$|x|^\alpha = \begin{cases} |x|^\alpha, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

With this notation, for example, for $\alpha = 0$ we have $|x|^0 = \mathbb{I}(x \neq 0)$. We also denote

$$(x)^\alpha = |x|^\alpha \operatorname{sign} x.$$

Throughout the article, C will be used to denote a generic constant, whose value is not important and may change between lines.

3. Benchmark solution

Now we turn to equation (1.2). The concept of strong solution is defined in a standard manner.

Definition 3.1. A strong solution to (1.2) is a continuous stochastic process X such that

1. X is adapted to the augmented natural filtration of B;
2. for any $t \geq 0$, the Itô integral $\int_0^t |X_s|^\alpha \,\mathrm{d}B_s$ and the quadratic covariation $[|X|^\alpha, B]_t$ exist;
3. for any $t \geq 0$, equation (1.2) holds $\mathbf{P}$-a.s.

Define the *benchmark solution* to equation (1.2) by

$$X_t^0 = \big((1-\alpha)B_t + (X_0)^{1-\alpha}\big)^{\frac{1}{1-\alpha}}. \tag{3.1}$$

The following change of variable result is crucial for proving that it solves equation (1.2).

Theorem 3.2 (Föllmer et al. [14], Theorem 4.1). *Let F be absolutely continuous with locally square integrable derivative f. Then*

$$F(B_t) = F(B_0) + \int_0^t f(B_s)\,\mathrm{d}B_s + \frac{1}{2}\big[f(B), B\big]_t.$$

Theorem 3.3. *For $\alpha \in (-1, 1)$ and $X_0 \in \mathbb{R}$, the process X^0 given by (3.1) is a strong solution to (1.2).*

Proof. For each $X_0 \in \mathbb{R}$, we note that the function

$$F(x) = \big((1-\alpha)x - (X_0)^{1-\alpha}\big)^{\frac{1}{1-\alpha}}$$

is absolutely continuous for $\alpha \in (-1, 1)$ and its derivative

$$f(x) = \big|(1-\alpha)x - (X_0)^{1-\alpha}\big|^{\frac{\alpha}{1-\alpha}} = \big|F(x)\big|^\alpha$$

is locally square integrable, so by Theorem 3.2 the process $X^0 = F(B)$ satisfies (1.2). □

As explained in the introduction, the benchmark solution is a unique strong solution until the time

$$\tau_0 = \inf\{t \geq 0\colon X_t^0 = 0\} = \inf_{t\geq 0}\left\{t \geq 0\colon B_t = -\frac{(X_0)^{1-\alpha}}{1-\alpha}\right\}, \tag{3.2}$$

when it first hits 0. However, the uniqueness fails after this time. Namely, with the help of Theorem 3.2 one can easily construct other strong solutions; the proofs are the same as for X^0 and therefore omitted. One example is the solution stopped at 0.

Theorem 3.4. *For $\alpha \in (-1, 1)$ and $X_0 \in \mathbb{R}$, the process*

$$X_t' = \big((1-\alpha)B_t + (X_0)^{1-\alpha}\big)^{\frac{1}{1-\alpha}}\mathbb{I}_{t\leq\tau_0},$$

where τ_0 is given by (3.2), is a strong solution to (1.2).

Both X^0 and X' possess the strong Markov property, thanks to that of B. One can also construct an uncountable family of non-Markov solutions. Namely, for any $A, B > 0$, set

$$F_{A,B}(x) = \begin{cases} -|x+A|^{\frac{1}{1-\alpha}}, & x < -A, \\ 0, & -A \leq x \leq B, \\ |x-B|^{\frac{1}{1-\alpha}}, & x > B, \end{cases}$$

and

$$X_t^{A,B} = F_{A,B}\big((1-\alpha)B_t + (X_0)^{1-\alpha}\big)$$

which equals to zero as long as $(1-\alpha)B_t + (X_0)^{1-\alpha} \in [-A, B]$.

Theorem 3.5. *For $\alpha \in (-1, 1)$ and $X_0 \in \mathbb{R}$, the process $X^{A,B}$ is a strong solution to* (1.2).

4. Solutions spending zero time at 0

The property of spending zero time at 0 is known to be crucial to guarantee uniqueness, see, for example, Beck [5] for the deterministic differential equations and Bass et al. [4], Aryasova and Pilipenko [2] for the stochastic case. We will need also a concept of a weak solution to (1.2).

Definition 4.1. A weak solution of (1.2) is a pair $(\widetilde{X}, \widetilde{B})$ of adapted continuous processes on a stochastic basis $(\widetilde{\Omega}, \widetilde{\mathscr{F}}, \widetilde{\mathbb{F}}, \widetilde{\mathbf{P}})$ such that

1. $\widetilde{B}$ is a standard Brownian motion on $(\widetilde{\Omega}, \widetilde{\mathscr{F}}, \widetilde{\mathbb{F}}, \widetilde{\mathbf{P}})$;
2. for any $t \geq 0$, the Itô integral $\int_0^t |\widetilde{X}_s|^\alpha \,\mathrm{d}\widetilde{B}_s$ and the quadratic covariation $[|\widetilde{X}|^\alpha, \widetilde{B}]_t$ exist;
3. for any $t \geq 0$,

$$\widetilde{X}_t = X_0 + \int_0^t |\widetilde{X}_s|^\alpha \circ \mathrm{d}\widetilde{B}_s$$

holds **P**-a.s.

Definition 4.2. A process X is said to spend zero time at 0 if for each $t \geq 0$

$$\int_0^t \mathbb{I}_{\{0\}}(X_s)\,\mathrm{d}s = 0 \quad \mathbf{P}\text{-a.s.}$$

In order to define solutions to (1.2), different from the benchmark solution (3.1) and spending zero time at 0, recall the notion of skew Brownian motion.

For $\theta \in [-1, 1]$, the skew Brownian motion $B^\theta = B^\theta(x)$ starting at $x \in \mathbb{R}$ is the unique solution of the SDE

$$B_t^\theta = x + B_t + \theta L_t\big(B^\theta\big), \tag{4.1}$$

where $L(B^\theta)$ is the symmetric local time of B^θ at 0; see Harrison and Shepp [17] and Lejay [21], Section 5.

Roughly speaking, the process B^θ behaves like a standard Brownian motion outside of zero. At zero its decides to evolve in the positive or negative directions independently of the past (the strong Markov property) with the “flipping” probabilities $\beta_\pm = \frac{1\pm\theta}{2}$. For the initial value $x = 0$ and $\theta = 0$, $B^0 \equiv B$, and for $\theta = \pm 1$, the solution to the equation (4.1) is a reflected Brownian motion starting at zero and can be written explicitly, namely

$$B^1 = \Big(B^1_t = B_t - \min_{s\le t} B_s\Big)_{t\ge 0} \overset{\mathrm{d}}{=} \big(|B_t|\big)_{t\ge 0},$$

$$B^{-1} = \Big(B^{-1}_t = B_t - \max_{s\le t} B_s\Big)_{t\ge 0} \overset{\mathrm{d}}{=} \big(-|B_t|\big)_{t\ge 0}.$$

A complete account on the properties of the skew Brownian motion can be found in Lejay [21].

First, we describe the law of the absolute value of a weak solution that spends zero time at 0.

Theorem 4.3. *Let $\alpha \in (-1,1)$, and let X be a weak solution of (1.2) started at X_0 such that X spends zero time at 0. Then the law of the process $Z = (\frac{1}{1-\alpha}|X_t|^{1-\alpha})_{t\ge 0}$ coincides with the law of a reflected Brownian motion started at $\frac{1}{1-\alpha}|X_0|^{1-\alpha}$.*

Having the law of $|X|$ in hand, we will describe all possible laws of the solution X itself. Essentially, we look for a process which behaves as a Brownian motion outside of 0 and spends zero time at 0. It is the skew Brownian motion B^θ which comes to mind first as an example of a process different from B and $|B|$ which satisfies these conditions.

The skew Brownian motion is a homogeneous strong Markov process however it is not the unique process whose absolute value is distributed like a reflected Brownian motion.

Indeed, one can construct the so-called variably skewed Brownian motion with a variable skewness parameter $\theta\colon \mathbb{R} \to (-1,1)$ as a solution to the SDE

$$B^\Theta_t = x + B_t + \Theta\big(L_t\big(B^\Theta\big)\big), \quad t \ge 0, x \in \mathbb{R},$$

where $\Theta(x) = \int_0^x \theta(y)\,\mathrm{d}y$. This process with $|B^\Theta| \overset{\mathrm{d}}{=} |B|$ (see Barlow et al. [3], Lemma 2.1); however, if θ is non-constant, B^Θ is not Markov as the skewness parameter depends on the value of local time.

On the other hand, Étoré and Martinez [13] showed that the *inhomogeneous skew Brownian motion* which is a unique strong solution of the SDE

$$B^\beta_t = x + B_t + \int_0^t \beta(s)\,\mathrm{d}L_s\big(B^\beta\big), \quad t \ge 0,$$

with a deterministic Borel function $\beta\colon [0,\infty) \to [-1,1]$, is an inhomogeneous strong Markov process, and $|B^\beta| \overset{\mathrm{d}}{=} |B|$ for $x = 0$ (see also Weinryb [31]).

To exclude these processes, we restrict ourselves to the case of homogeneous strong Markov solutions.

Theorem 4.4. *Let $\alpha \in (-1,1)$, and let $(\widetilde{X}, \widetilde{B})$ be a weak solution of (1.2) such that $\widetilde{X}$ is a homogeneous strong Markov process spending zero time at 0. Then there exists $\theta \in [-1,1]$ such that*

$$\widetilde{X} = \big((1-\alpha)\widetilde{B}^\theta\big)^{\frac{1}{1-\alpha}} \tag{4.2}$$

with a θ-skew Brownian motion $\widetilde{B}^\theta$, which solves

$$\widetilde{B}^\theta_t = \frac{1}{1-\alpha}(X_0)^{1-\alpha} + \widetilde{B}_t + \theta L_t\big(\widetilde{B}^\theta\big), \quad t \geq 0. \tag{4.3}$$

Moreover, $\widetilde{X}$ is also a strong solution to

$$\mathrm{d}\widetilde{X}_t = |\widetilde{X}_t|^\alpha \circ \mathrm{d}\widetilde{B}_t, \quad t \geq 0. \tag{4.4}$$

For $\theta = 1$, the skew Brownian motion B^1 is a non-negative reflected Brownian motion. Aryasova and Pilipenko [2] studied non-negative solutions of a singular SDE written in the weak form. By Aryasova and Pilipenko [2], Theorem 1, there exists a strong solution to equation (1.2) (in the weak form) with initial condition $X_0 \geq 0$ spending zero time at the point 0 and the strong uniqueness holds in the class of solutions spending zero time at 0. Of course it is equal to the solution X^1 which can be determined explicitly as

$$X^1_t = \Big((1-\alpha)B_t + X_0^{1-\alpha} + \Big((1-\alpha)\min_{s\leq t} B_s + X_0^{1-\alpha}\Big)_-\Big)^{\frac{1}{1-\alpha}}, \quad t \geq 0,$$

where $x_- = -\min(x, 0)$ denotes the negative part of x.

Finally, we show the existence of strong solutions different from the benchmark solution (3.1) and characterize all solutions which are homogeneous strong Markov processes spending zero time at 0.

Theorem 4.5.

1. *Let $\alpha \in (0, 1)$ and $\theta \in [-1, 1]$. Let $X_0 \in \mathbb{R}$ and let B^θ be the unique strong solution of the SDE*

$$B^\theta_t = \frac{1}{1-\alpha}(X_0)^{1-\alpha} + B_t + \theta L_t\big(B^\theta\big), \quad t \geq 0.$$

Then

$$X^\theta_t = \big((1-\alpha)B^\theta_t\big)^{\frac{1}{1-\alpha}} \tag{4.5}$$

is a strong solution of (1.2) which is a homogeneous strong Markov process spending zero time at 0.

Moreover, X^θ is the unique strong solution of (1.2) which is a homogeneous strong Markov process spending zero time at 0 and such that

$$\mathbf{P}\big(X^\theta_t \geq 0 \mid X_0 = 0\big) = \beta_+ = \frac{1+\theta}{2}, \quad t > 0.$$

2. *Let $\alpha \in (-1, 0]$. Then the benchmark solution $X^0_t = ((1-\alpha)B_t + (X_0)^{1-\alpha})^{\frac{1}{1-\alpha}}$ is the unique strong solution of (1.2) which is a homogeneous strong Markov process spending zero time at 0.*

Remark 4.6. By Theorem 4.4, a similar uniqueness result also holds for weak solutions.

The explicit form of the solutions (4.5) allows to study their long time behaviour easily. Setting for simplicity $X_0 = 0$, we recall the transition probability density of the skew Brownian motion (see, e.g., Lejay [21], Eq. (17)) and find the mean square displacement

$$\mathrm{Var}\big(X_t^\theta\big) = t^{\frac{1}{1-\alpha}} \cdot \big(2(1-\alpha)^2\big)^{\frac{1}{1-\alpha}} \left[\pi^{-\frac{1}{2}}\Gamma\left(\frac{3-\alpha}{2(1-\alpha)}\right) - \theta^2\Gamma\left(\frac{2-\alpha}{2(1-\alpha)}\right)^2\right]. \tag{4.6}$$

Hence, X^θ demonstrates the diffusive behaviour $\mathrm{Var}\, X_t^\theta \sim t$ for $\alpha = 0$, as the diffusion coefficient is a.e. constant. For $\alpha \in (0, 1)$, the growing diffusion coefficient leads to the superdiffusion; for $\alpha \in (-1, 0)$, the diffusion coefficient decreases to zero at infinity, hence we have a subdiffusion. Such behaviour was recovered in Cherstvy et al. [8], where one can find a discussion on the physical interpretation.

The crucial part of the proof of Theorem 4.5 is the existence of the quadratic variation $[|X^\theta|^\alpha, B]$ which follows from the following theorem which is interesting on its own.

Theorem 4.7. *Let $f \in L^2_{\mathrm{loc}}(\mathbb{R})$ and let the θ-skew Brownian motion B^θ, $\theta \in (-1, 1)$, be the unique strong solution of the SDE (4.1). Then the quadratic variation*

$$\big[f\big(B^\theta\big), B\big]_t = \lim_{n\to\infty} \sum_{t_k \in D_n, t_k < t} \big(f\big(B^\theta_{t_k}\big) - f\big(B^\theta_{t_{k-1}}\big)\big)(B_{t_k} - B_{t_{k-1}})$$

exists as a limit in u.c.p. Moreover, let $\{h_m\}_{m\ge 1}$ be a sequence of continuous functions such that for each $A > 0$

$$\lim_{m\to\infty} \int_{-A}^{A} \big|h_m(x) - f(x)\big|^2 \,\mathrm{d}x = 0. \tag{4.7}$$

Then

$$\big[h_m\big(B^\theta\big), B\big]_t \to \big[f\big(B^\theta\big), B\big]_t, \quad m \to \infty,$$

in u.c.p.

The proof of this Theorem uses the approach by [14]. For $\theta \in (-1, 1) \setminus \{0\}$ it is combined with the time reversal technique from [18,19].

5. On the relation between the Stratonovich and Itô equations

Recall that a Stratonovich SDE $\mathrm{d}X = f(X) \circ \mathrm{d}B$ with a smooth function f can be rewritten in the Itô form as $\mathrm{d}X = f(X)\,\mathrm{d}B + \frac{1}{2} f(X) f'(X)\,\mathrm{d}t$ (see, e.g., Protter [26], Chapter 5). Although for $\alpha \in (-1, 1)$ the function $x \mapsto |x|^\alpha$ is not smooth, let us *formally* write the Stratonovich SDE (1.2) as an Itô SDE with irregular/singular coefficients

$$X_t = X_0 + \int_0^t |X_s|^\alpha \,\mathrm{d}B_s + \frac{\alpha}{2} \int_0^t (X_s)^{2\alpha-1} \,\mathrm{d}s. \tag{5.1}$$

Let us check whether the process X^θ satisfies this equation. For definiteness, we set $X_0 = 0$.

In order to be able to substitute X^θ into (5.1) we have to guarantee that the both summands of the SDE (5.1) are well defined. Hence, for the existence of the Itô integral we need

$$\int_0^t \left|X_s^\theta\right|^{2\alpha} \mathrm{d}s \overset{\mathrm{d}}{=} \int_0^t |B_s|^{\frac{2\alpha}{1-\alpha}} \mathrm{d}s < \infty \quad \text{a.s.,} \tag{5.2}$$

and for the existence of the drift term we need

$$\int_0^t \left|X_s^\theta\right|^{2\alpha-1} \mathrm{d}s \overset{\mathrm{d}}{=} \int_0^t |B_s|^{\frac{2\alpha-1}{1-\alpha}} \mathrm{d}s < \infty \quad \text{a.s.} \tag{5.3}$$

The Engelbert–Schmidt zero-one law (Engelbert and Schmidt [12], Theorem 1) implies that for a Borel function $\Phi\colon \mathbb{R} \to [0, +\infty]$

$$\mathbf{P}\left(\int_0^t \Phi(B_s)\,\mathrm{d}s < \infty, \forall t \geq 0\right) = 1 \quad \Leftrightarrow \quad \Phi \in L^1_{\mathrm{loc}}(\mathbb{R}),$$

and hence (5.2) is satisfied for all $\alpha > -1$, and the drift term (5.3) exists for $\alpha > 0$. This indicates that X^θ is a solution of (5.1) for $\theta \in [-1, 1]$ and $\alpha \in (0, 1)$.

To extend the existence result to $\alpha \in (-1, 0]$, we will consider the drift term in the principal value sense:

$$\text{v.p.} \int_0^t (B_s)^{\frac{2\alpha-1}{1-\alpha}} \mathrm{d}s := \lim_{\varepsilon \downarrow 0} \int_0^t (B_s)^{\frac{2\alpha-1}{1-\alpha}} \cdot \mathbb{I}\big(|B_s| > \varepsilon\big)\,\mathrm{d}s. \tag{5.4}$$

The principal value definition is intrinsically based on the symmetry of the Brownian motion and hence excludes the skew cases $\theta \neq 0$. Necessary and sufficient conditions for the existence of Brownian principal value integrals are given in Cherny [6], Theorem 3.1, page 352. In particular, the integral (5.4) is finite if and only if $\alpha > -1$.

This yields that for $\alpha \in (-1, 0]$, X^0 is the solution of the Itô SDE

$$X_t = X_0 + \int_0^t |X_s|^\alpha \,\mathrm{d}B_s + \frac{\alpha}{2} \cdot \text{v.p.} \int_0^t (X_s)^{2\alpha-1} \mathrm{d}s. \tag{5.5}$$

In their book, Cherny and Engelbert [7] consider singular SDEs in the sense of existence of the Lebesgue integrals (5.2) and (5.3). It follows from Cherny and Engelbert [7], Chapter 5, that for $\alpha \leq 0$ the SDE (5.1) has a unique solution which sticks to 0 after hitting it. This behaviour seems to contradict the fact that the benchmark solution X^0 is a solution to the Stratonovich equation which spends zero time at 0 for $\alpha \in (-1, 1)$ due to Theorem 3.3. This contradiction is resolved by taking into account the fact that for $\alpha \in (-1, 0]$ the noise-induced drift has to be understood in the *principal value* sense (5.4) and not as a Lebesque integral.

These observations lead to the following theorem.

Theorem 5.1.

1. *For* $\alpha \in (0, 1)$ *and* $\theta \in [-1, 1]$, *the process* X^θ *given by* (4.5) *is a strong solution of the Itô SDE* (5.1).
2. *For* $\alpha \in (-1, 0]$, *the process* X^0 *given by* (3.1) *is a strong solution of the Itô SDE* (5.5).

6. Proof of Theorem 4.3

We use the following characterization of the reflected Brownian motion, see Varadhan [30].

Proposition 6.1. *Let* $(\Omega, \mathscr{F}, \mathbb{F}, \mathbf{P})$ *be a filtered probability space. A continuous non-negative stochastic process* Z *is a reflected Brownian motion started at* x *if and only if*

1. $Z_0 = x$ *a.s.*;
2. Z *behaves locally like a Brownian motion on* $(0,\infty)$, *i.e. for any bounded smooth function* $f\colon [0,\infty)\to\mathbb{R}$ *such that* $f(x)=0$ *for* $x\in[0,\delta]$ *for some* $\delta=\delta(f)>0$, *the process*
$$f(Z_t)-f(x)-\frac{1}{2}\int_0^t f''(Z_s)\,\mathrm{d}s$$
is a martingale;
3. Z *spends a zero time at* 0, *that is*,
$$\mathbf{E}\int_0^\infty \mathbb{I}_{\{0\}}(Z_s)\,\mathrm{d}s=0.$$

For the proof of Theorem 4.3, we will need a variant of the change of variables formula for functions vanishing on a neighborhood of the irregular point of the SDE.

Lemma 6.2. *Let* $\varphi\in C^1(\mathbb{R}\backslash\{0\})$ *and let* $(\widetilde{X},\widetilde{B})$ *be a weak solution of the SDE*
$$X_t = x+\int_0^t \varphi(X_s)\circ\mathrm{d}B_s := x+\int_0^t\varphi(X_s)\,\mathrm{d}B_s+\frac{1}{2}\bigl[\varphi(X),B\bigr]_t.$$
Then for any $g\in C^2(\mathbb{R})$ *which vanishes on a neighborhood of zero we have*
$$\begin{aligned}g(\widetilde{X}_t)&=g(X_0)+\int_0^t g'(\widetilde{X}_s)\varphi(\widetilde{X}_s)\,\mathrm{d}\widetilde{B}_s\\&\quad+\frac{1}{2}\int_0^t\varphi(\widetilde{X}_s)\bigl(g''(\widetilde{X}_s)\varphi(\widetilde{X}_s)+g'(\widetilde{X}_s)\varphi'(\widetilde{X}_s)\bigr)\,\mathrm{d}s.\end{aligned}\tag{6.1}$$

The proof of this lemma essentially follows the lines of the proof of the classical Itô formula for Itô processes and is given in Pavlyukevich and Shevchenko [24], Section A.

Eventually, we prove Theorem 4.3. Let $(\widetilde{X},\widetilde{B})$ be a weak solution of the SDE (1.2) spending zero time at 0. We consider the process
$$Z_t=\frac{1}{1-\alpha}|\widetilde{X}_t|^{1-\alpha},\quad t\geq 0,$$
which starts at $Z_0=\frac{1}{1-\alpha}|X_0|^{1-\alpha}$ and also spends zero time at 0.

Let $f \in C_b^2(\mathbb{R}_+)$ be zero on a neighborhood of 0. The function $g(x) = f(\frac{1}{1-\alpha}|x|^{1-\alpha})$ is also twice continuously differentiable, bounded, and is zero on a neighborhood of 0, and

$$g'(x) = f'(z)(x)^{-\alpha},$$

$$g''(x) = f''(z)|x|^{-2\alpha} - \alpha f'(z)|x|^{-\alpha-1}, \quad z = \frac{1}{1-\alpha}|x|^{1-\alpha}.$$

Then Lemma 6.2 immediately yields

$$\begin{aligned} f(Z_t) &= g(\widetilde{X}_t) \\ &= g(X_0) + \int_0^t g'(\widetilde{X}_s)|\widetilde{X}_s|^{\alpha}\,\mathrm{d}\widetilde{B}_s \\ &\quad + \frac{1}{2}\int_0^t \bigl(g''(\widetilde{X}_s)|\widetilde{X}_s|^{2\alpha} + \alpha g'(\widetilde{X}_s)(\widetilde{X}_s)^{2\alpha-1}\bigr)\,\mathrm{d}s \\ &= f(Z_0) + \int_0^t f'(Z_s)\operatorname{sign}\widetilde{X}_s\,\mathrm{d}\widetilde{B}_s + \frac{1}{2}\int_0^t f''(Z_s)\,\mathrm{d}s, \end{aligned}$$

so that the process

$$t \mapsto f(Z_t) - f(Z_0) - \frac{1}{2}\int_0^t f''(Z_s)\,\mathrm{d}s \tag{6.2}$$

is a martingale.

7. Proof of the Theorem 4.4

Let $(\widetilde{X}, \widetilde{B})$ be a weak solution of the SDE (1.2) spending zero time at 0. Then, by Theorem 4.3,

$$\frac{1}{1-\alpha}|\widetilde{X}_t|^{1-\alpha} \stackrel{\mathrm{d}}{=} \left|W - \frac{X_0}{1-\alpha}\right|$$

for some standard Brownian motion W, that is, is a reflected Brownian motion starting at $\frac{|X_0|}{1-\alpha}$. We first establish (4.2).

Proposition 7.1. *Let Y be a continuous homogeneous strong Markov process starting at $y \in \mathbb{R}$ such that $|Y| \stackrel{\mathrm{d}}{=} |W - y|$, W being a standard Brownian motion. Then there is $\theta \in [-1, 1]$ such that $Y \stackrel{\mathrm{d}}{=} B^\theta$, where B^θ is the θ-skew Brownian motion starting at y.*

Proof. Since for any $\theta \in [-1, 1]$, $Y \stackrel{\mathrm{d}}{=} W + y \stackrel{\mathrm{d}}{=} B^\theta$ before the first hitting time of 0, it is sufficient to consider the case of the initial starting point $y = 0$.

Denote for $a < 0 < b$

$$\tau_{(a,b)} = \inf\bigl\{t \geq 0\colon Y_t \notin (a, b)\bigr\}$$

and show that the probability

$$p_+(\varepsilon) = \mathbf{P}(Y_{\tau_{(-\varepsilon,\varepsilon)}} = \varepsilon \mid Y_0 = 0), \quad \varepsilon > 0,$$

does not depend on ε.

Indeed, if $p_+(\varepsilon) = 0$ or $p_+(\varepsilon) = 1$ for all $\varepsilon > 0$, then the statement holds true.

Assume that there is $\varepsilon > 0$ such that $p_+(\varepsilon) = \beta_+ \in (0, 1)$.

Let $0 < \varepsilon < \varepsilon'$, then

$$\begin{aligned} p_+(\varepsilon') &= \mathbf{P}\big(Y_{\tau_{(-\varepsilon',\varepsilon')}} = \varepsilon' \mid Y_{\tau_{(-\varepsilon,\varepsilon)}} = \varepsilon, Y_0 = 0\big) p_+(\varepsilon) \\ &\quad + \mathbf{P}\big(Y_{\tau_{(-\varepsilon',\varepsilon')}} = \varepsilon' \mid Y_{\tau_{(-\varepsilon,\varepsilon)}} = -\varepsilon, Y_0 = 0\big)\big(1 - p_+(\varepsilon)\big) \\ &= \mathbf{P}\big(Y_{\tau_{(-\varepsilon',\varepsilon')}} = \varepsilon' \mid Y_0 = \varepsilon\big) p_+(\varepsilon) + \mathbf{P}\big(Y_{\tau_{(-\varepsilon',\varepsilon)}} = \varepsilon' \mid Y_0 = -\varepsilon\big)\big(1 - p_+(\varepsilon)\big). \end{aligned}$$

Since $\mathrm{Law}(Y_t; 0 \le t \le \tau(0, \varepsilon') \mid Y_0 = \varepsilon) = \mathrm{Law}(B_t; 0 \le t \le \tau(0, \varepsilon') \mid B_0 = \varepsilon)$ we get by virtue of the gambler ruin problem for Brownian motion that

$$\begin{aligned} \mathbf{P}\big(Y_{\tau_{(0,\varepsilon')}} = \varepsilon' \mid Y_0 = \varepsilon\big) &= \mathbf{P}\big(B_{\tau_{(0,\varepsilon')}} = \varepsilon' \mid B_0 = \varepsilon\big) = \frac{\varepsilon}{\varepsilon'}, \\ \mathbf{P}(Y_{\tau_{(0,\varepsilon')}} = 0 \mid Y_0 = \varepsilon) &= \mathbf{P}(B_{\tau_{(0,\varepsilon')}} = 0 \mid B_0 = \varepsilon) = 1 - \frac{\varepsilon}{\varepsilon'}, \end{aligned}$$

and hence

$$\begin{aligned} \mathbf{P}\big(Y_{\tau_{(-\varepsilon',\varepsilon')}} = \varepsilon' \mid Y_0 = \varepsilon\big) &= \mathbf{P}\big(Y_{\tau_{(0,\varepsilon')}} = \varepsilon' \mid Y_0 = \varepsilon\big) + \mathbf{P}(Y_{\tau_{(0,\varepsilon')}} = 0 \mid Y_0 = \varepsilon) p_+\big(\varepsilon'\big) \\ &= \frac{\varepsilon}{\varepsilon'} + \left(1 - \frac{\varepsilon}{\varepsilon'}\right) p_+\big(\varepsilon'\big). \end{aligned}$$

Analogously

$$\mathbf{P}\big(Y_{\tau_{(-\varepsilon',\varepsilon')}} = \varepsilon' \mid Y_0 = -\varepsilon\big) = \mathbf{P}(Y_{\tau_{(-\varepsilon',0)}} = 0 \mid Y_0 = -\varepsilon) p_+\big(\varepsilon'\big) = \left(1 - \frac{\varepsilon}{\varepsilon'}\right) p_+\big(\varepsilon'\big).$$

Hence, we obtain that

$$p_+\big(\varepsilon'\big) = \frac{\varepsilon}{\varepsilon'} p_+(\varepsilon) + \left(1 - \frac{\varepsilon}{\varepsilon'}\right) p_+\big(\varepsilon'\big) p_+(\varepsilon) + \left(1 - \frac{\varepsilon}{\varepsilon'}\right) p_+\big(\varepsilon'\big)\big(1 - p_+(\varepsilon)\big) = p_+(\varepsilon) = \beta_+.$$

Let now $0 < \varepsilon' < \varepsilon$. Due to the continuity of the paths of Y, $p_+(\varepsilon') > 0$, so repeating the previous argument with ε and ε' interchanged we eventually obtain that $p_+(\varepsilon) = \beta_+$ for all $\varepsilon > 0$.

Since Y is a continuous strong Markov process its law is uniquely determined by the Dynkin characteristic operator

$$\mathfrak{A} f(x) := \lim_{U \downarrow x} \frac{\mathbf{E}_x f(Y_{\tau(U)}) - f(x)}{\mathbf{E}_x \tau(U)},$$

where U is a bounded open interval containing x, see Dynkin [10], Chapter 5 §3.

Choosing $U = U_\varepsilon = (x-\varepsilon, x+\varepsilon)$, a straightforward calculation yields that for $x \neq 0$ and f being twice continuously differentiable at x

$$\mathfrak{A} f(x) = \lim_{\varepsilon \downarrow 0} \frac{\frac{1}{2} f(x+\varepsilon) + \frac{1}{2} f(x-\varepsilon) - f(x)}{\varepsilon^2} = \frac{1}{2} f''(x).$$

For $x = 0$ the limit

$$\mathfrak{A} f(0) = \lim_{\varepsilon \downarrow 0} \frac{\beta_+ f(\varepsilon) + (1-\beta_+) f(-\varepsilon) - f(0)}{\varepsilon^2} = \frac{1}{2}\big(\beta_+ f''(0+) + (1-\beta_+) f''(0-)\big)$$

exists for any continuous f such that $\beta_+ f'(0+) = (1-\beta_+) f'(0-)$ and $f''(0+)$ and $f''(0-)$ exist and $f''(0+) = f''(0-)$.

Hence $\mathfrak{A}$ coincides with the generator of the θ-skew Brownian motion with $\theta = 2\beta_+ - 1$ (see Lejay [21]). □

By Proposition 7.1 and Theorem 4.3, the process $\widetilde{B}^\theta = \frac{1}{1-\alpha}(\widetilde{X})^{1-\alpha}$ is a θ-skew Brownian motion with some $\theta \in [-1, 1]$, starting at $B_0^\theta = \frac{1}{1-\alpha}(X_0)^{1-\alpha}$. Equivalently,

$$\widehat{B}_t = \frac{1}{1-\alpha}(X_0)^{1-\alpha} + \widetilde{B}_t^\theta - \theta L_t\big(\widetilde{B}^\theta\big) \tag{7.1}$$

is a standard Wiener process. Comparing with (4.3), we need to show that $\widehat{B} = \widetilde{B}$. By the results of Harrison and Shepp [17], $\widetilde{B}$ is adapted to the filtration generated by $\widehat{B}$. Now we want to show that $\widehat{B} = \widetilde{B}$ a.s. By Lemma 6.2, for any function $g \in C^2(\mathbb{R})$, vanishing on a neighbourhood of 0, and for any $t > 0$ it holds

$$\begin{aligned} g(\widetilde{X}_t) = g(X_0) &+ \int_0^t g'(\widetilde{X}_s)|\widetilde{X}_s|^\alpha \,\mathrm{d}\widetilde{B}_s \\ &+ \frac{1}{2}\int_0^t |\widetilde{X}_s|^\alpha \big(g''(\widetilde{X}_s)|\widetilde{X}_s|^\alpha + \alpha g'(\widetilde{X}_s)|\widetilde{X}_s|^{\alpha-1}\big)\,\mathrm{d}s. \end{aligned}$$

A similar formula, but with $\widehat{B}$ in place of $\widetilde{B}$, holds thanks to the equality $\widetilde{X} = ((1-\alpha)\widetilde{B}^\theta)^{\frac{1}{1-\alpha}}$, (7.1) and the usual Itô formula for semimartingales applied to $\widetilde{B}^\theta$. Indeed,

$$\begin{aligned} g(\widetilde{X}_t) &= g\Big(\big((1-\alpha)\widetilde{B}_t^\theta\big)^{\frac{1}{1-\alpha}}\Big) \\ &= g(X_0) + \int_0^t g'(\widetilde{X}_s)|\widetilde{X}_s|^\alpha \,\mathrm{d}\widehat{B}_s + \theta \int_0^t g'(\widetilde{X}_s)|\widetilde{X}_s|^\alpha \,\mathrm{d}L_s\big(\widetilde{B}^\theta\big) \\ &\quad + \frac{1}{2}\int_0^t |\widetilde{X}_s|^\alpha \big(g''(\widetilde{X}_s)|\widetilde{X}_s|^\alpha + \alpha g'(\widetilde{X}_s)|\widetilde{X}_s|^{\alpha-1}\big)\,\mathrm{d}s \end{aligned}$$

$$
\begin{aligned}
&= g(X_0) + \int_0^t g'(\widetilde{X}_s)|\widetilde{X}_s|^\alpha \, \mathrm{d}\widehat{B}_s \\
&\quad + \frac{1}{2}\int_0^t |\widetilde{X}_s|^\alpha \big(g''(\widetilde{X}_s)|\widetilde{X}_s|^\alpha + \alpha g'(\widetilde{X}_s)|\widetilde{X}_s|^{\alpha-1}\big)\, \mathrm{d}s.
\end{aligned}
$$

Consequently,

$$
\int_0^t g'(\widetilde{X}_s)|\widetilde{X}_s|^\alpha \, \mathrm{d}\widetilde{B}_s = \int_0^t g'(\widetilde{X}_s)|\widetilde{X}_s|^\alpha \, \mathrm{d}\widehat{B}_s.
$$

Now taking a sequence of non-negative functions $g_n \in C^2(\mathbb{R})$, $n \geq 1$, vanishing in some neighborhood of 0 and such that $g_n \geq 0$ $g_n'(x)|x|^\alpha \uparrow 1$, $x \neq 0$, $n \to \infty$, we get

$$
\int_0^t \mathbb{I}_{\widetilde{X}_t \neq 0} \, \mathrm{d}\widetilde{B}_t = \int_0^t \mathbb{I}_{\widetilde{X}_t \neq 0} \, \mathrm{d}\widehat{B}_t
$$

a.s. Since $\mathbb{I}_{\widetilde{X}_t \neq 0} = 1$ a.e. by assumption, it follows that $\widetilde{B}_t = \widehat{B}_t$ a.s. As a result, $\widetilde{B}^\theta$ is adapted to the augmented filtration of $\widetilde{B}$, so in view of (4.2), the same is true for $\widetilde{X}$. Since $\widetilde{X}$ also satisfies (4.4) by definition, it is a strong solution.

8. Proof of Theorem 4.7

Let $\theta \in (-1, 1) \setminus \{0\}$; the case of $\theta = 0$ is covered by [14]. Define

$$
\sigma(x) = \frac{2}{1 + \theta \operatorname{sign} x} \quad \text{and} \quad \beta(x) = \frac{1}{\sigma(x)}.
$$

Let Y^θ be the unique strong solution of the SDE

$$
Y_t^\theta = u + \int_0^t \sigma\big(Y_s^\theta\big)\, \mathrm{d}B_s, \quad u \in \mathbb{R}, \tag{8.1}
$$

and consider the two-dimensional Markov process (Y^θ, B) with the law

$$
\mathbf{P}_{u,w} := \operatorname{Law}\big(\big(Y^\theta, B\big) \mid Y_0^\theta = u, B_0 = w\big).
$$

The process Y^θ is called an oscillating Brownian motion, see e.g. Keilson and Wellner [20], Lejay and Pigato [22].

The skew Brownian motion with parameter θ, starting from $w_0 \in \mathbb{R}$ and driven by a Brownian motion B is the unique strong solution to the following stochastic differential equation

$$
B_t^\theta = w_0 + (B_t - w) + \theta L_t\big(B^\theta\big). \tag{8.2}
$$

Further, define the functions

$$r(x) = \frac{x}{\sigma(x)} = x\beta(x) = \begin{cases} \dfrac{1+\theta}{2}x, & x \geq 0, \\ \dfrac{1-\theta}{2}x, & x < 0, \end{cases} \qquad s(x) = x\sigma(x) = \frac{x}{\beta(x)},$$

then $s(r(x)) \equiv x$. The application of the Itô–Tanaka formula (compare with Lejay [21], Section 5.2) yields

$$\begin{aligned} r\big(Y_t^\theta\big) &= r(u) + B_t - w + \frac{\theta}{2} L_t\big(Y_t^\theta\big) \\ &= r(u) + B_t - w + \theta L_t\big(B^\theta\big) = r(u) - w_0 + B_t^\theta. \end{aligned} \tag{8.3}$$

In the following lemma, we will use the functional dependence (8.3) of the processes (Y^θ, B) and $(B^\theta, L(B^\theta))$ to determine the marginal density of the pair (Y_t^θ, B_t).

Lemma 8.1. *For $\theta \in (-1, 1) \setminus \{0\}$, $t > 0$, the joint distribution of Y_t^θ and B_t given $Y_0^\theta = u$, $B_0 = w$ is*

$$\begin{aligned} \mathbf{P}_{u,w}\big(Y_t^\theta \in \mathrm{d}y, B_t \in \mathrm{d}z\big) &= \frac{2\beta^2(y)}{\theta^2\sqrt{2\pi t^3}}\big(2y\beta^2(y) - \kappa u - z + w\big) \\ &\quad \times \exp\bigg(-\frac{1}{2\theta^2 t}\big(2y\beta^2(y) - \kappa u - z + w\big)^2\bigg)\,\mathrm{d}z\,\mathrm{d}y, \end{aligned} \tag{8.4}$$

where $\kappa = \frac{1}{2}(1-\theta^2)$, if $\theta^{-1}(r(y) - r(u) - z + w) > 0$, and

$$\begin{aligned} &\mathbf{P}_{u,w}\big(Y_t^\theta \in \mathrm{d}y, B_t = w + r(y) - r(u)\big) \\ &\quad = \frac{\beta(u)}{|\theta|\sqrt{2\pi t}}\big(\mathrm{e}^{-\frac{(r(y)-r(u))^2}{2t}} - \mathrm{e}^{-\frac{(r(y)+r(u))^2}{2t}}\big) \cdot \mathbb{I}_{uy>0}\,\mathrm{d}y. \end{aligned} \tag{8.5}$$

In particular, the joint density of (Y_t^θ, B_t) provided that $Y_0 = u = 0$, $B_0 = w = 0$ is

$$p(t, y, z) = \frac{2\beta^2(y)}{\theta^2\sqrt{2\pi t^3}}\big(2y\beta^2(y) - z\big)\exp\bigg(-\frac{1}{2\theta^2 t}\big(2y\beta^2(y) - z\big)^2\bigg) \cdot \mathbb{I}_{\theta^{-1}(r(y)-z)>0}. \tag{8.6}$$

Proof. The joint distribution of B_t^θ and $L_t(B^\theta)$, $t > 0$, is well known and can be found, for example, in Appuhamillage et al. [1], Étoré and Martinez [13], Gairat and Shcherbakov [15]:

$$\begin{aligned} &\mathbf{P}\big(B_t^\theta \in \mathrm{d}b, L_t\big(B^\theta\big) \in \mathrm{d}l \mid B_0^\theta = w_0\big) \\ &\quad = 2\beta_+ \cdot \mathbb{I}_{b\geq 0}\frac{l + |w_0| + |b|}{\sqrt{2\pi t^3}}\mathrm{e}^{-\frac{(l+|w_0|+|b|)^2}{2t}}\,\mathrm{d}b\,\mathrm{d}l \end{aligned}$$

$$
\begin{aligned}
&+ 2\beta_- \cdot \mathbb{I}_{b<0} \frac{l+|w_0|+|b|}{\sqrt{2\pi t^3}} \mathrm{e}^{-\frac{(l+|w_0|+|b|)^2}{2t}} \,\mathrm{d}b\,\mathrm{d}l \\
&= 2\beta(b) \cdot \frac{l+|w_0|+|b|}{\sqrt{2\pi t^3}} \mathrm{e}^{-\frac{(l+|w_0|+|b|)^2}{2t}} \,\mathrm{d}b\,\mathrm{d}l, \qquad b \in \mathbb{R}, l > 0,
\end{aligned} \tag{8.7}
$$

where in the last equality we redefined the value of the density at $b = 0$ for convenience.

Recall now that

$$
\begin{aligned}
B_t^\theta &= r\left(Y_t^\theta\right) - r(u) + w_0, \\
L_t\left(B^\theta\right) &= \frac{1}{\theta}\left(r\left(Y_t^\theta\right) - r(u) - B_t + w\right).
\end{aligned}
$$

The initial condition of $u = Y_0^\theta$ given, let us fix $w_0 = r(u)$, so that $B_t^\theta = r(Y_t^\theta)$ for $t \geq 0$. Then the change of variables $b = b(y,z)$, $l = l(y,z)$,

$$
b = r(y), \qquad l = \frac{1}{\theta}\left(r(y) - r(u) - z + w\right), \tag{8.8}
$$

yields

$$
\begin{aligned}
l + |w_0| + |b| &= \frac{1}{\theta}\left(r(y) - r(u) - z + w\right) + \left|r(u)\right| + \left|r(y)\right| \\
&= \frac{r(y) + \theta|r(y)| - r(u) + \theta|r(u)| - z + w}{\theta} \\
&= \frac{r(y)(1 + \theta \operatorname{sign} y) - r(u)(1 - \theta \operatorname{sign} u) - z + w}{\theta} \\
&= \frac{2y\beta^2(y)}{\theta} - \frac{u(1-\theta^2)}{2\theta} - \frac{z-w}{\theta},
\end{aligned} \tag{8.9}
$$

where we made use of the relation $|r(y)| = r(y) \operatorname{sign} y$. For $y, u \neq 0$, the Jacobian for the change of variables $(y,z) \to (b,l)$ is given by (8.8) and its determinant equal

$$
J = \begin{pmatrix} \beta(y) & 0 \\ \frac{\beta(y)}{\theta} & -\frac{1}{\theta} \end{pmatrix}, \qquad |\det J| = \frac{\beta(y)}{|\theta|},
$$

whence, noting that $\sigma(b) = \sigma(y)$, we get (8.4). Similarly, we have (8.5). The remaining formula (8.6) follows by plugging in $w = 0$. □

From now on, we assume without loss of generality that $\theta \in (0,1)$. Let all the processes under consideration will be started at zero, $u = w = w_0 = 0$, so that $Y^\theta = B^\theta / \beta(B^\theta)$.

Note that for any $t > 0$

$$
\begin{aligned}
r\left(Y_t^\theta\right) - B_t &= B_t^\theta - B_t = \theta L_t\left(B^\theta\right) > 0, \\
2Y_t^\theta \beta^2\left(Y_t^\theta\right) - B_t &= 2\beta\left(B_t^\theta\right) B_t^\theta - B_t = \left(1 + \theta \operatorname{sign} B_t^\theta\right) B_t^\theta - B_t \\
&= B_t^\theta - B_t + \theta B_t^\theta \operatorname{sign} B_t^\theta = \theta L_t\left(B^\theta\right) + \theta\left|B_t^\theta\right| > 0.
\end{aligned} \tag{8.10}
$$

Our aim now is to prove a generalized Itô formula for B^θ, in the spirit of Föllmer et al. [14]. Towards this end, on a fixed time interval $[0, T]$ we first establish a stochastic differential equation for the time-reversed pair $(\bar{Y}_t^\theta, \bar{B}_t) = (Y_{T-t}^\theta, B_{T-t})$, which is interesting by its own. We follow the method developed by Haussmann and Pardoux [18,19] for Markovian diffusions. For $y \neq 0$ and $z < r(y)$, $s \in [0, T)$ define the functions

$$\bar{b}^y(s, y, z) = \frac{1}{p(T-s, y, z)}\Big(\sigma^2(y) \cdot \frac{\partial p}{\partial y}(T-s, y, z) + \sigma(y) \cdot \frac{\partial p}{\partial z}(T-s, y, z)\Big),$$
$$\bar{b}^z(s, y, z) = \frac{1}{p(T-s, y, z)}\Big(\sigma(y) \cdot \frac{\partial p}{\partial y}(T-s, y, z) + \frac{\partial p}{\partial z}(T-s, y, z)\Big) = \frac{\bar{b}^y(s, y, z)}{\sigma(y)},$$

and set $\bar{b}^y(s, 0, z) = \bar{b}^z(s, 0, z) = \bar{b}^y(T, y, z) = \bar{b}^z(T, y, z) = 0$. Noting that

$$\frac{\partial p}{\partial z}(T-s, y, z) = \frac{2\beta^2(y)}{\theta^2\sqrt{2\pi(T-s)^3}} \exp\Big(-\frac{(2y\beta^2(y)-z)^2}{2\theta^2(T-s)}\Big)\Big(-1 + \frac{(2y\beta^2(y)-z)^2}{\theta^2(T-s)}\Big),$$
$$z < r(y),$$

and $\frac{\partial p}{\partial z}(T-s, y, z) = -2\beta^2(y)\frac{\partial p}{\partial y}(T-s, y, z)$, we get

$$\begin{aligned}\bar{b}^y(s, y, z) &= \frac{2\beta(y)-1}{\beta(y)}\Big(\frac{1}{2y\beta^2(y)-z} - \frac{2y\beta^2(y)-z}{\theta^2(T-s)}\Big)\\ &= \frac{\theta \operatorname{sign} y}{\beta(y)} \cdot \Big(\frac{1}{2y\beta^2(y)-z} - \frac{2y\beta^2(y)-z}{\theta^2(T-s)}\Big), \quad z < r(y).\end{aligned} \tag{8.11}$$

Proposition 8.2. *Let for $\theta \in (-1, 1) \setminus \{0\}$, B^θ be a solution of (8.2) started at 0. Then for any $T > 0$, $(\bar{Y}_t^\theta, \bar{B}_t) = (Y_{T-t}^\theta, B_{T-t})$ is a weak solution to the stochastic differential equation*

$$\begin{aligned}\bar{Y}_t^\theta &= Y_T^\theta + \int_0^t \bar{b}^y\big(s, \bar{Y}_s^\theta, \bar{B}_s\big)\,\mathrm{d}s + \int_0^t \sigma\big(\bar{Y}_s^\theta\big)\,\mathrm{d}\bar{W}_s,\\ \bar{B}_t &= B_T + \int_0^t \bar{b}^z\big(s, \bar{Y}_s^\theta, \bar{B}_s\big)\,\mathrm{d}s + \bar{W}_t, \quad t \in [0, T],\end{aligned} \tag{8.12}$$

$\bar{W}$ being a standard Brownian motion.

Remark 8.3.

(a) Thanks to (8.10), the coefficients of (8.12) are well defined.
(b) Equation (8.12) is very degenerate, and $\bar{Y}^\theta$ and $\bar{B}$ evolve proportionally whenever $\bar{Y}_t^\theta \neq 0$. This, however, will not hinder our analysis.

Proof. As above, we continue to assume without loss of generality that $\theta \in (0,1)$ and $T = 1$. We need to show that $(\bar{Y}^\theta, \bar{B})$ is a solution to the martingale problem with the generator

$$\bar{\mathcal{L}}_t f(y,z) = \Big(\bar{b}^y(t,y,z) \cdot \frac{\partial}{\partial y} + \bar{b}^y(t,y,z) \cdot \frac{\partial}{\partial y} \\ + \frac{\sigma(y)^2}{2} \cdot \frac{\partial^2}{\partial y^2} + \sigma(y) \cdot \frac{\partial^2}{\partial y \partial z} + \frac{1}{2} \cdot \frac{\partial^2}{\partial z^2}\Big) f(y,z).$$

Thanks to (8.10), it is enough to establish

$$\mathbf{E}\Big[\Big(f\big(\bar{Y}^\theta_t, \bar{B}_t\big) - f\big(\bar{Y}^\theta_s, \bar{B}_s\big) - \int_s^t \bar{\mathcal{L}}_u f\big(\bar{Y}^\theta_u, \bar{B}_u\big)\,\mathrm{d}u\Big) \cdot g\big(\bar{Y}^\theta_s, \bar{B}_s\big)\Big] = 0$$

for any $0 \le s < t < 1$ and functions $f, g \in C^\infty(\mathbb{R}^2)$ having compact support inside the domain $D := \{(y,z) \in \mathbb{R}^2 \colon r(y) - z > 0\}$. Equivalently,

$$\mathbf{E}\Big[\Big(f\big(Y^\theta_t, B_t\big) - f\big(Y^\theta_s, B_s\big) + \int_s^t \bar{\mathcal{L}}_{T-u} f\big(Y^\theta_u, B_u\big)\,\mathrm{d}u\Big) \cdot g\big(Y^\theta_t, B_t\big)\Big] = 0$$

for any fixed $0 < s < t \le 1$. Define for $(y,z) \in \mathbb{R}^2$

$$v(s,y,z) = \mathbf{E}\big[g\big(Y^\theta_t, B_t\big) \mid \big(Y^\theta_s, B_s\big) = (y,z)\big].$$

It is proved in Pavlyukevich and Shevchenko [24], Section B, that v solves the partial differential equation

$$\Big(\frac{\partial}{\partial s} + \mathcal{L}\Big) v(s,y,z) = 0,$$

where

$$\mathcal{L} f(y,z) = \Big(\frac{\sigma(y)^2}{2} \cdot \frac{\partial^2}{\partial y^2} + \sigma(y) \cdot \frac{\partial^2}{\partial y \partial z} + \frac{1}{2} \cdot \frac{\partial^2}{\partial z^2}\Big) f(y,z). \tag{8.13}$$

Denote by $\langle \cdot, \cdot \rangle$ the scalar product in $L^2(\mathbb{R}^2)$. Write

$$\begin{aligned}
&\mathbf{E}\big[f\big(Y^\theta_t, B_t\big) g\big(Y^\theta_t, B_t\big)\big] - \mathbf{E}\big[f\big(Y^\theta_s, B_s\big) g\big(Y^\theta_t, B_t\big)\big] \\
&\quad = \mathbf{E}\big[f\big(Y^\theta_t, B_t\big) v\big(t, Y^\theta_t, B_t\big)\big] - \mathbf{E}\big[f\big(Y^\theta_s, B_s\big) v\big(s, Y^\theta_s, B_s\big)\big] \\
&\quad = \big\langle f p(t), v(t)\big\rangle - \big\langle f p(s), v(s)\big\rangle \\
&\quad = \int_s^t \Big\langle f \frac{\partial}{\partial u} p(u), v(u)\Big\rangle \mathrm{d}u + \int_s^t \Big\langle f p(u), \frac{\partial}{\partial u} v(u)\Big\rangle \mathrm{d}u \\
&\quad = \int_s^t \Big\langle f \frac{\partial}{\partial u} p(u), v(u)\Big\rangle \mathrm{d}u - \int_s^t \big\langle f p(u), \mathcal{L} v(u)\big\rangle \mathrm{d}u.
\end{aligned}$$

Using Pavlyukevich and Shevchenko [24], (B.2), we get

$$\langle fp(u), \mathcal{L}v(u)\rangle = \langle v(u), \mathcal{L}\big(fp(u)\big)\rangle.$$

Write

$$p(u, y, z) = \frac{2\beta^2(y)}{\theta}\varphi_u'\left(\frac{z - 2y\beta^2(y)}{\theta}\right)\mathbb{I}_{z<r(y)}, \tag{8.14}$$

where $\varphi_t(x) = \frac{1}{\sqrt{2\pi t}}\mathrm{e}^{-\frac{x^2}{2t}}$ is the standard Gaussian density. Now we have

$$\begin{aligned}\mathcal{L}\big(fp(u)\big)(y, z) &= f(y, z)\cdot\mathcal{L}p(u, y, z) + p(u, y, z)\cdot\mathcal{L}f(y, z)\\ &\quad + \left(\sigma(y)\cdot\frac{\partial}{\partial y}f(y, z) + \frac{\partial}{\partial z}f(y, z)\right)\left(\sigma(y)\cdot\frac{\partial}{\partial y}p(u, y, z) + \frac{\partial}{\partial z}p(u, y, z)\right).\end{aligned}$$

From (8.14), for $y \neq 0$, $z < r(y)$,

$$\begin{aligned}\mathcal{L}p(u, y, z) &= \left(\frac{4\beta^4(y)}{\theta^3} - \frac{4\beta^3(y)}{\theta^3} + \frac{\beta^2(y)}{\theta^3}\right)\cdot\varphi_u'''\left(\frac{z - 2y\beta^2(y)}{\theta}\right)\\ &= \frac{\beta^2(y)}{\theta^3}\cdot\big(2\beta(y) - 1\big)^2\cdot\varphi_u'''\left(\frac{z - 2y\beta^2(y)}{\theta}\right) = \frac{\beta^2(y)}{\theta}\cdot\varphi_u'''\left(\frac{z - 2y\beta^2(y)}{\theta}\right).\end{aligned}$$

On the other hand, since $\frac{\partial}{\partial t}\varphi_t' = \frac{1}{2}\varphi_t'''$, we get

$$\mathcal{L}p(u, y, z) = \frac{\partial}{\partial u}p(u, y, z).$$

Further, denote

$$h(u, y, z) = \sigma(y)\cdot\frac{\partial}{\partial y}p(u, y, z) + \frac{\partial}{\partial z}p(u, y, z).$$

Then we have

$$\langle v(u), \mathcal{L}\big(fp(u)\big)\rangle = \left\langle v(u), f\frac{\partial}{\partial u}p(u)\right\rangle + \langle v(u), p(u)\mathcal{L}f\rangle + \left\langle v(u), h(u)\left(\sigma(y)\cdot\frac{\partial}{\partial y}f + \frac{\partial}{\partial z}f\right)\right\rangle.$$

Observe that

$$\begin{aligned}\langle v(u), p(u)\mathcal{L}f\rangle &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} v(u, y, z)p(u, y, z)\mathcal{L}f(y, z)\,\mathrm{d}z\,\mathrm{d}y\\ &= \mathbf{E}\big[v\big(u, Y_u^\theta, B_u\big)\mathcal{L}f\big(Y_u^\theta, B_u\big)\big] = \mathbf{E}\big[\mathbf{E}\big[g\big(Y_t^\theta, B_t\big)\mid Y_u^\theta, B_u\big]\mathcal{L}f\big(Y_u^\theta, B_u\big)\big]\\ &= \mathbf{E}\big[g\big(Y_t^\theta, B_t\big)\mathcal{L}f\big(Y_u^\theta, B_u\big)\big]\end{aligned}$$

and similarly

$$
\begin{aligned}
&\Big\langle v(u), h(u)\cdot\Big(\sigma(y)\cdot\frac{\partial}{\partial y}f+\frac{\partial}{\partial z}f\Big)\Big\rangle\\
&\quad=\Big\langle v(u), p(u)\cdot\frac{h(u)}{p(u)}\cdot\Big(\sigma(y)\cdot\frac{\partial}{\partial y}f+\frac{\partial}{\partial z}f\Big)\Big\rangle\\
&\quad=\Big\langle v(u), p(u)\Big(\bar{b}^y(T-u)\frac{\partial}{\partial y}f+\bar{b}^z(T-u)\frac{\partial}{\partial z}f\Big)\Big\rangle\\
&\quad=\mathbf{E}\Big[g\big(Y_t^\theta, B_t\big)\Big(\bar{b}^y(T-u, Y_u, B_u)\frac{\partial}{\partial y}f\big(Y_u^\theta, B_u\big)+\bar{b}^z(T-u, Y_u, B_u)\frac{\partial}{\partial z}f\big(Y_u^\theta, B_u\big)\Big)\Big].
\end{aligned}
$$

Collecting everything,

$$
\begin{aligned}
&\mathbf{E}\big[f\big(Y_t^\theta, B_t\big)g\big(Y_t^\theta, B_t\big)\big]-\mathbf{E}\big[f\big(Y_s^\theta, B_s\big)g\big(Y_t^\theta, B_t\big)\big]\\
&\quad=-\mathbf{E}\Big[g\big(Y_t^\theta, B_t\big)\int_s^t \mathcal{L}f\big(Y_u^\theta, B_u\big)\Big]\\
&\qquad-\mathbf{E}\Big[g\big(Y_t^\theta, B_t\big)\int_s^t\Big(\bar{b}^y\big(T-u, Y_u^\theta, B_u\big)\frac{\partial}{\partial y}f\big(Y_u^\theta, B_u\big)\\
&\qquad+\bar{b}^z\big(T-u, Y_u^\theta, B_u\big)\frac{\partial}{\partial z}f\big(Y_u^\theta, B_u\big)\Big)\,\mathrm{d}u\Big]\\
&\quad=-\mathbf{E}\Big[g\big(Y_t^\theta, B_t\big)\int_s^t \bar{\mathcal{L}}_{T-u}f\big(Y_u^\theta, B_u\big)\,\mathrm{d}u\Big],
\end{aligned}
$$

as required. □

Consider a sequence of partitions D_n of the form $0=t_0^n<t_1^n<\cdots<t_n^n=T$ with $|D_n|=\max_{1\le k\le n}|t_k^n-t_{k-1}^n|\to 0$, $n\to\infty$ (we will often omit the superscript n).

Proof of Theorem 4.7. Note that $f(B_t^\theta)=f(r(Y_t^\theta))=g(Y_t^\theta)$ with $g\in L^2_{\mathrm{loc}}(\mathbb{R})$, so it suffices to establish a similar statement for Y^θ. The rest of proof goes similarly to Föllmer et al. [14].

First note that by usual localization argument, we can assume that $g\in L^2(\mathbb{R})$. Also for a continuous function h, the quadratic variation

$$
\big[h\big(Y^\theta\big), B\big]_t=\lim_{n\to\infty}\sum_{t_k\in D_n, t_k<t}\big(h\big(Y_{t_k}^\theta\big)-h\big(Y_{t_{k-1}}^\theta\big)\big)(B_{t_k}-B_{t_{k-1}})
$$

exists as a limit in u.c.p. Indeed, since B is a semimartingale, and $h(Y^\theta)$ is an adapted continuous process, then by Protter [26], Theorem 21, page 64,

$$
\lim_{n\to\infty}\sum_{t_k\in D_n, t_k<t}h\big(Y_{t_{k-1}}^\theta\big)(B_{t_k}-B_{t_{k-1}})=\int_0^t h\big(Y_s^\theta\big)\,\mathrm{d}B_s
$$

in u.c.p. The time-reversed process is also a semimartingale, so arguing as in Föllmer et al. [14], we have the convergence to the backward integral

$$\lim_{n\to\infty}\sum_{t_k\in D_n,t_k<t} h\big(Y^\theta_{t_k}\big)(B_{t_k}-B_{t_{k-1}})=\int_0^t h\big(Y^\theta_s\big)\,\mathrm{d}^*B_s$$

in u.c.p. Therefore, we obtain

$$\big[h\big(Y^\theta\big),B\big]_t=\int_0^t h\big(Y^\theta_s\big)\,\mathrm{d}^*B_s-\int_0^t h\big(Y^\theta_s\big)\,\mathrm{d}B_s.$$

Fix some $T>0$ and let now $\{h_m\}_{m\ge 1}$ be a sequence of continuous functions, such that $h_m\to g$ in $L^2(\mathbb{R})$ as $m\to\infty$. Denote

$$\begin{aligned}I_m(t)&:=\int_0^t \big(h_m\big(Y^\theta_s\big)-g\big(Y^\theta_s\big)\big)\,\mathrm{d}B_s,\\ S_{n,m}(t)&:=\sum_{t_k\in D_n,t_k\le t}\big(h_m\big(Y^\theta_{t_{k-1}}\big)-g\big(Y^\theta_{t_{k-1}}\big)\big)(B_{t_k}-B_{t_{k-1}}).\end{aligned}$$

Since B^θ is a skew Brownian motion starting from 0, its density is $\mathbf{P}(B^\theta_t\in\mathrm{d}b)=\frac{\beta(b)}{\sqrt{2\pi t}}\mathrm{e}^{-\frac{b^2}{2t}}\,\mathrm{d}b$, $b\in\mathbb{R}$. Then the density of Y^θ_t satisfies $p(t,y)=\frac{\beta^2(y)}{\sqrt{2\pi t}}\mathrm{e}^{-\frac{r^2(y)}{2t}}\le\frac{1}{\sqrt{t}}$, so we can estimate, using the Doob inequality that

$$\begin{aligned}\mathbf{E}\sup_{t\in[0,T]}I_m^2(t)&\le 4\int_0^T\mathbf{E}\big[\big(h_m\big(Y^\theta_t\big)-g\big(Y^\theta_t\big)\big)^2\big]\,\mathrm{d}t\\ &=4\int_0^T\int_{-\infty}^{\infty}\big(h_m(y)-g(y)\big)^2p(t,y)\,\mathrm{d}y\,\mathrm{d}t\\ &\le 4\|h_m-g\|^2_{L^2(\mathbb{R})}\cdot\int_0^T\frac{\mathrm{d}t}{\sqrt{t}}\le C\|h_m-g\|^2_{L^2(\mathbb{R})}.\end{aligned}$$

Similarly,

$$\begin{aligned}\mathbf{E}\sup_{t\in[0,T]}S^2_{n,m}(t)&=\mathbf{E}\sum_{t_k\in D_n}\big(h_m\big(Y^\theta_{t_{k-1}}\big)-g\big(Y^\theta_{t_{k-1}}\big)\big)^2(t_k-t_{k-1})\\ &\le C\sum_{t_k\in D_n,k>1}\frac{t_k-t_{k-1}}{\sqrt{t_{k-1}}}\|h_m-g\|^2_{L^2(\mathbb{R})},\end{aligned}$$

whence

$$\limsup_{n\to\infty}\mathbf{E}\sup_{t\in[0,T]}S^2_{n,m}(t)\le C\|h_m-g\|^2_{L^2(\mathbb{R})}.$$

As a result, we get $\sup_{t\in[0,T]} |I_m(t)| \xrightarrow{\mathbf{P}} 0$, $m \to \infty$, and for any $\varepsilon > 0$

$$\lim_{m\to\infty} \limsup_{n\to\infty} \mathbf{P}\Big(\sup_{t\in[0,T]} \big|S_{n,m}(t)\big| > \varepsilon\Big) = 0.$$

Hence, using that

$$\sum_{t_k\in D_n, t_k<t} h_m\big(Y^\theta_{t_{k-1}}\big)(B_{t_k} - B_{t_{k-1}}) \to \int_0^t h_m\big(Y^\theta_s\big)\,\mathrm{d}B_s, \quad n\to\infty,$$

uniformly on $[0, T]$ in probability, we get that

$$\sum_{t_k\in D_n, t_k<t} g\big(Y^\theta_{t_{k-1}}\big)(B_{t_k} - B_{t_{k-1}}) \to \int_0^t g\big(Y^\theta_s\big)\,\mathrm{d}B_s, \quad n\to\infty,$$

uniformly on $[0, T]$ in probability.

Further, recall that the time-reversed process $(\bar{Y}^\theta_t, \bar{B}_t) = (Y^\theta_{T-t}, B_{T-t})$ satisfies (8.12) in the weak sense. As far as the convergence in probability is concerned, we can safely assume that $(\bar{Y}^\theta, \bar{B})$ satisfies (8.12) with the same Brownian motion $\bar{W}$. Then we can write

$$\int_0^t g\big(Y^\theta_s\big)\,\mathrm{d}^*B_s = \int_{T-t}^T g\big(\bar{Y}^\theta_s\big)\,\mathrm{d}\bar{B}_s = \int_{T-t}^T g\big(\bar{Y}^\theta_s\big)\,\mathrm{d}\bar{W}_s + \int_{T-t}^T g\big(\bar{Y}^\theta_s\big)\bar{b}^y\big(s, \bar{Y}^\theta_s, \bar{B}_s\big)\,\mathrm{d}s.$$

Arguing as above, we have

$$\begin{aligned}
&\sum_{t_k\in D_n, t_k<t} g\big(Y^\theta_{t_k}\big)(\bar{W}_{T-t_k} - \bar{W}_{T-t_{k-1}})\\
&\quad= \sum_{t_k\in D_n, t_k<t} g\big(\bar{Y}^\theta_{T-t_k}\big)(\bar{W}_{T-t_k} - \bar{W}_{T-t_{k-1}}) \to \int_{T-t}^T g\big(\bar{Y}^\theta_s\big)\,\mathrm{d}\bar{W}_s, \quad n\to\infty,
\end{aligned}$$

uniformly on $[0, T]$ in probability. It remains to show that

$$\begin{aligned}
&\sum_{t_k\in D_n, t_k<t} g\big(Y^\theta_{t_k}\big)\int_{t_{k-1}}^{t_k} b^y\big(T-s, Y^\theta_s, B_s\big)\,\mathrm{d}s\\
&\quad= \sum_{t_k\in D_n, t_k<t} g\big(\bar{Y}^\theta_{T-t_k}\big)\int_{T-t_{k-1}}^{T-t_k} b^y\big(s, \bar{Y}^\theta_s, \bar{B}_s\big)\,\mathrm{d}s\\
&\quad\to \int_{T-t}^T g\big(\bar{Y}^\theta_{T-s}\big)b^y\big(s, \bar{Y}^\theta_s, \bar{B}_s\big)\,\mathrm{d}s = \int_0^t g\big(Y^\theta_s\big)b^y\big(T-s, Y^\theta_s, B_s\big)\,\mathrm{d}s, \quad n\to\infty, \qquad (8.15)
\end{aligned}$$

uniformly on $[0, T]$ in probability.

We will first establish an estimate for $\int_0^T |g(Y_s^\theta)b^y(T-s, Y_s^\theta, B_s)|\,\mathrm{d}s$ with $g \in L^2(\mathbb{R})$. From (8.11), using the Cauchy–Schwarz inequality, we have

$$\begin{aligned}
&\int_0^T \mathbf{E}\big|g\big(Y_s^\theta\big)b^y\big(T-s, Y_s^\theta, B_s\big)\big|\,\mathrm{d}s \\
&\quad\le C\int_0^T \mathbf{E}\big|g\big(Y_s^\theta\big)\big|\bigg(\frac{1}{2Y_s^\theta\beta^2(Y_s^\theta)-B_s} + \frac{2Y_s^\theta\beta^2(Y_s^\theta)-B_s}{s}\bigg)\,\mathrm{d}s \\
&\quad\le C\int_0^T \bigg[\mathbf{E}\big|g\big(Y_s^\theta\big)\big|^2\cdot\mathbf{E}\bigg(\frac{1}{(2Y_s^\theta\beta^2(Y_s^\theta)-B_s)^2} + \frac{(2Y_s^\theta\beta^2(Y_s^\theta)-B_s)^2}{s^2}\bigg)\bigg]^{\frac{1}{2}}\,\mathrm{d}s.
\end{aligned}$$

As before, from the estimate $p(s, Y_s^\theta) \le \frac{C}{\sqrt{s}}$ it follows that

$$\mathbf{E}\big|g\big(Y_s^\theta\big)\big|^2 \le \frac{C}{\sqrt{s}}\|g\|^2_{L^2(\mathbb{R})}.$$

Further, using (8.10) and (8.7) with $w_0 = 0$, we get

$$\begin{aligned}
\mathbf{E}\frac{1}{(2Y_s^\theta\beta^2(Y_s^\theta)-B_s)^2} &= \mathbf{E}\frac{1}{\theta^2(B_s^\theta + L_s(B^\theta))^2} \\
&= \frac{1}{\sqrt{2\pi s^3}}\int_0^\infty\int_{-\infty}^\infty \frac{\beta(b)}{l+|b|}\mathrm{e}^{-\frac{(l+|b|)^2}{2s}}\,\mathrm{d}b\,\mathrm{d}l \\
&\le \frac{C}{\sqrt{s^3}}\int_0^\infty\int_0^\infty \frac{1}{l+b}\mathrm{e}^{-\frac{(l+b)^2}{2s}}\,\mathrm{d}b\,\mathrm{d}l \\
&\le \frac{C}{\sqrt{s^3}}\int_0^\infty \mathrm{e}^{-\frac{z^2}{2s}}\,\mathrm{d}z \le \frac{C}{s}.
\end{aligned}$$

Similarly,

$$\mathbf{E}\big(2Y_s^\theta\beta^2\big(Y_s^\theta\big)-B_s\big)^2 \le \frac{C}{\sqrt{s^3}}\int_0^\infty\int_0^\infty (l+b)^3\mathrm{e}^{-\frac{(l+b)^2}{2s}}\,\mathrm{d}l\,\mathrm{d}b \le \frac{C}{\sqrt{s^3}}\int_0^\infty z^4\mathrm{e}^{-\frac{z^2}{2s}}\,\mathrm{d}z \le Cs.$$

Therefore,

$$\begin{aligned}
&\int_0^T \mathbf{E}\big|g\big(Y_s^\theta\big)b^y\big(T-s, Y_s^\theta, B_s\big)\big|\,\mathrm{d}s \\
&\quad\le C\|g\|_{L^2(\mathbb{R})}\int_0^T s^{-\frac{3}{4}}\,\mathrm{d}s \le C\|g\|_{L^2(\mathbb{R})}. \qquad (8.16)
\end{aligned}$$

Using similar estimates, we get

$$\begin{aligned}
&\sum_{t_k\in D_n, t_k<t}\int_{t_{k-1}}^{t_k}\mathbf{E}\big|g\big(Y_{t_k}^\theta\big)b^y\big(T-s,Y_s^\theta,B_s\big)\big|\,\mathrm{d}s\\
&\quad\le\sum_{t_k\in D_n, t_k<t}\int_{t_{k-1}}^{t_k}\big(\mathbf{E}\big|g\big(Y_{t_k}^\theta\big)\big|^2\cdot\mathbf{E}\big|b^y\big(T-s,Y_s^\theta,B_s\big)\big|^2\big)^{\frac12}\,\mathrm{d}s\\
&\quad\le C\|g\|_{L^2(\mathbb{R})}\sum_{t_k\in D_n, t_k<t}\int_{t_{k-1}}^{t_k}t_k^{-1/4}s^{-1/2}\,\mathrm{d}s\\
&\quad\le C\|g\|_{L^2(\mathbb{R})}\int_0^T s^{-3/4}\,\mathrm{d}s\le C\|g\|_{L^2(\mathbb{R})}.
\end{aligned}\tag{8.17}$$

If $h\in C(\mathbb{R})$, then

$$\delta_n=\max_{t_k\in D_n}\sup_{s\in[t_{k-1},t_k]}\big|h\big(Y_{t_k}^\theta\big)-h\big(Y_s^\theta\big)\big|\to 0,\quad n\to\infty,$$

almost surely, and we can estimate

$$\begin{aligned}
&\bigg|\sum_{t_k\in D_n, t_k<t}h\big(Y_{t_k}^\theta\big)\int_{t_{k-1}}^{t_k}b^y\big(T-s,Y_s^\theta,B_s\big)\,\mathrm{d}s-\int_0^t h\big(Y_s^\theta\big)b^y\big(T-s,Y_s^\theta,B_s\big)\,\mathrm{d}s\bigg|\\
&\quad\le\delta_n\int_0^T\big|b^y\big(T-s,Y_s^\theta,B_s\big)\big|\,\mathrm{d}s.
\end{aligned}$$

Similarly to the calculations above,

$$\int_0^T\mathbf{E}\big|b^y\big(T-s,Y_s^\theta,B_s\big)\big|\,\mathrm{d}s\le\int_0^T\big(\mathbf{E}\big|b^y\big(T-s,Y_s^\theta,B_s\big)\big|^2\big)^{\frac12}\,\mathrm{d}s\le C\int_0^T s^{-\frac12}\,\mathrm{d}s\le C,$$

so that $\int_0^T|b^y(T-s,Y_s^\theta,B_s)|\,\mathrm{d}s$ is bounded in probability. Therefore, for $h\in C(\mathbb{R})$,

$$\sum_{t_k\in D_n, t_k<t}h\big(Y_{t_k}^\theta\big)\int_{t_{k-1}}^{t_k}b^y\big(T-s,Y_s^\theta,B_s\big)\,\mathrm{d}s\to\int_0^t h\big(Y_s^\theta\big)b^y\big(T-s,Y_s^\theta,B_s\big)\,\mathrm{d}s,\quad n\to\infty,$$

uniformly on $[0,T]$ in probability. Hence, taking, as before, a sequence $h_m\in C(\mathbb{R})$ converging to g in $L^2(\mathbb{R})$ and using (8.16) and (8.17), we arrive at (8.15). Combined with our previous findings, this leads to

$$\sum_{t_k\in D_n, t_k<t}\big(f\big(B_{t_k}^\theta\big)-f\big(B_{t_{k-1}}^\theta\big)\big)(B_{t_k}-B_{t_{k-1}})\to\int_0^t f\big(B_s^\theta\big)\,\mathrm{d}B_s-\int_0^t f\big(B_s^\theta\big)\,\mathrm{d}^*B_s,\quad n\to\infty,$$

uniformly on $[0,T]$ in probability. Since $T>0$ is arbitrary, this means precisely that the desired u.c.p. convergence holds. □

9. Proof of Theorem 4.5

For definiteness, we set $X_0 = 0$. Let us first show that X^θ is a strong solution to (1.2). For $\theta = 0$ and $\alpha \in (-1, 1)$, the statement follows directly from the Itô formula proven in Föllmer et al. [14]. Let $\theta \in (-1, 1) \setminus \{0\}$ and $\alpha \in (0, 1)$. If $h \in C^1(\mathbb{R})$, $h(0) = 0$, $H(x) = \int_0^x h(y)\,\mathrm{d}y$, then by the usual Itô formula for semimartingales (see, e.g., Protter [26], Theorem II.32), we have

$$H\big(B_t^\theta\big) = H(0) + \int_0^t h\big(B_s^\theta\big)\,\mathrm{d}B_s + \theta \int_0^t h\big(B_s^\theta\big)\,\mathrm{d}L\big(B^\theta\big) + \frac{1}{2}\big[h\big(B^\theta\big), B\big]_t + \frac{\theta}{2}\big[h\big(B^\theta\big), L\big(B^\theta\big)\big]_t,$$

where the decomposition of quadratic variation into the sum holds true since both $[h(B^\theta), B^\theta]$ and $[h(B^\theta), B]$ exist as u.c.p. limits. Furthermore since $h(0) = 0$, the quadratic variation $[h(B^\theta), L(B^\theta)]$ and the integral w.r.t. $L(B^\theta)$ vanish a.s., so that we obtain the equality

$$H\big(B_t^\theta\big) = H(0) + \int_0^t h\big(B_s^\theta\big)\,\mathrm{d}B_s + \frac{1}{2}\big[h\big(B^\theta\big), B\big]_t.$$

Taking a sequence $\{h_m\}$ of C^1-functions such that, $h_m(0) = 0$, $h_m(x) = |(1-\alpha)x|^\alpha$ for $|x| \geq 1$ and $\sup_{x\in[0,1]} |h_m(x) - (1-\alpha)|x|^\alpha| \to 0$, $m \to \infty$, we utilize the Itô isometry and Theorem 4.7 to get the desired result.

For $\theta = \pm 1$ and $\alpha \in (0, 1)$, the proof goes similarly with the help of [24], Proposition C.2, which uses the results of [25].

Concerning the uniqueness, by Theorem 4.4, any strong solution must be given by (4.5) with some $\theta \in [-1, 1]$. So it remains to show that for $\theta \neq 0$ and $\alpha \in (-1, 0]$, X^θ is not a solution of the SDE.

Let $\alpha = 0$. Clearly,

$$\int_0^t \mathbb{I}\big(B_s^\theta \neq 0\big)\,\mathrm{d}B_s = B_t \quad \text{a.s.}$$

However

$$\big[\mathbb{I}\big(B_\cdot^\theta \neq 0\big), B\big] \equiv 0 \quad \text{a.s.}$$

since $h(x) = \mathbb{I}(x \neq 0)$ can be approximated by $h_m(x) \equiv 1$ in $L^2(\mathbb{R})$, and $[1, B] \equiv 0$. Hence,

$$\int_0^t \mathbb{I}\big(B_s^\theta \neq 0\big) \circ \mathrm{d}B_s = B_t \neq X_t^\theta = B_t + \theta L_t\big(B^\theta\big).$$

For $\alpha \in (-1, 0)$ and $\theta \in (-1, 1) \setminus \{0\}$, the Stratonovich integral w.r.t. B is well defined as the sum

$$\int_0^t \big|B_s^\theta\big|^{\frac{\alpha}{1-\alpha}} \circ \mathrm{d}B_s = \int_0^t \big|B_s^\theta\big|^{\frac{\alpha}{1-\alpha}}\,\mathrm{d}B_s + \frac{1}{2}\big[\big|B^\theta\big|^{\frac{\alpha}{1-\alpha}}, B\big]_t.$$

Choosing again a sequence of C^1-functions $\{h_n\}$ such that

$$h_m(x) \equiv \left|(1-\alpha)x\right|^{\frac{\alpha}{1-\alpha}}, \quad |x| \geq 1,$$
$$\left\|h_m(\cdot) - \left|(1-\alpha)(\cdot)\right|^{\frac{\alpha}{1-\alpha}}\right\|_{L^2(\mathbb{R})} \to 0,$$

we obtain that

$$H_m(x) = \int_0^x h_m(y)\,\mathrm{d}y \to H(x) = \big((1-\alpha)x\big)^{\frac{1}{1-\alpha}}$$

uniformly on $\mathbb{R}$, so that we can apply the standard Itô formula to obtain

$$\begin{aligned} H_m\big(B_t^\theta\big) &= H_m(0) + \int_0^t h_m\big(B_s^\theta\big)\,\mathrm{d}B_s^\theta + \frac{1}{2}\big[h_m\big(B^\theta\big), B^\theta\big]_t \\ &= H_m(0) + \int_0^t h_m\big(B_s^\theta\big) \circ \mathrm{d}B_s + \theta \int_0^t h_m\big(B_s^\theta\big) \circ \mathrm{d}L_s\big(B^\theta\big). \end{aligned}$$

Passing to the limit as $m \to \infty$, we observe that $H_m(B_t^\theta) \to H(B_t^\theta) = X_t^\theta$ as well as

$$\begin{aligned} \int_0^t h_m\big(B_s^\theta\big) \circ \mathrm{d}B_s &= \int_0^t h_m\big(B_s^\theta\big)\,\mathrm{d}B_s + \frac{1}{2}\big[h_m\big(B^\theta\big), B\big]_t \\ &\to \int_0^t h\big(B_s^\theta\big)\,\mathrm{d}B_s + \frac{1}{2}\big[h\big(B^\theta\big), B\big]_t = \int_0^t h\big(B_s^\theta\big) \circ \mathrm{d}B_s = \int_0^t \big|X_s^\theta\big|^\alpha \circ \mathrm{d}B_s \end{aligned}$$

by the Itô isometry and Theorem 4.7. However it is easy to see, for example, by the monotone convergence (if we choose h_n monotonically increasing) that

$$\begin{aligned} \int_0^t h\big(B_s^\theta\big) \circ \mathrm{d}L_s &= \lim_{m\to\infty} \int_0^t h_m\big(B_s^\theta\big) \circ \mathrm{d}L_s \\ &= (1-\alpha)^{\frac{\alpha}{1-\alpha}} \lim_{m\to\infty} \lim_{n\to\infty} \sum_{t_k<t} \frac{h_m(B_{t_k}^\theta) + h_m(B_{t_{k-1}}^\theta)}{2} (L_{t_k} - L_{t_{k-1}}) = +\infty, \end{aligned}$$

so that the SDE (1.2) is not satisfied unless $\theta = 0$.

Note that the Riemann-Stieltjes integral w.r.t. L does not exist since the points of increase of L coincide with the points of discontinuity of $|B^\theta|^{\frac{\alpha}{1-\alpha}}$.

For $\alpha \in (-1, 0)$ and $\theta = \pm 1$, the quadratic variation $[|B^\theta|^{\frac{\alpha}{1-\alpha}}, B]$, and hence the Stratonovich integral w.r.t. B, is not well defined in view of [24], Proposition C.2.

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Supplementary Material

Supplement: Additional derivations (DOI: 10.3150/19-BEJ1161SUPP; .pdf). We provide technical proofs omitted from the main body of the article.

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