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A unified approach to coupling SDEs driven by Lévy noise and some applications

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We present a general method to construct couplings of stochastic differential equations driven by Lévy noise in terms of coupling operators. This approach covers both coupling by reflection and refined basic coupling which are often discussed in the literature. As applications, we prove regularity results for the transition semigroups and obtain successful couplings for the solutions to stochastic differential equations driven by additive Lévy noise.

Keywords: coupling by reflection; coupling operator; Lévy process; optimal coupling; refined basic coupling; successful coupling

1. Introduction

Coupling methods are well-known powerful probabilistic tools, see [24,38] for an extensive overview. They can be efficiently used in order to prove ergodicity of Markov processes (see, e.g., the book [10] or the article [1] for the convergence of Markov processes to stationary distributions), and to show regularity properties of Markov semigroups – we refer, for instance, to [13, 31] for gradient estimates of diffusion semigroups. Coupling techniques have also been applied in a broad variety of contexts, including Markov chain Monte Carlo (MCMC) algorithms (see [16,34] and references therein), interacting particle systems (see [23]) or optimal transportation (cf. [32,39]), to mention but a few examples.

We call $\widetilde{X} = (\widetilde{X}_t^1, \widetilde{X}_t^2)_{t\ge 0}$ a Markov(ian) coupling for the Markov process $X := (X_t)_{t\ge 0}$, if

(i) each of the three processes $\widetilde{X}$, $\widetilde{X}^1 := (\widetilde{X}_t^1)_{t\ge 0}$ and $\widetilde{X}^2 := (\widetilde{X}_t^2)_{t\ge 0}$ is a Markov process with respect to the filtration generated by the pair $\widetilde{X}^1$ and $\widetilde{X}^2$;
(ii) the processes $\widetilde{X}^1$ and $\widetilde{X}^2$ have the same transition semigroup as X.

We call the coupling successful, if the paths of the processes $\widetilde{X}^1$ and $\widetilde{X}^2$ meet in finite time.

The main contribution of this paper is the study of Markovian couplings for stochastic differential equations driven by Lévy noise from the point of view of coupling operators. This is one of the first attempts to study, in a systematic way, coupling of jump processes which are not Lévy processes. This new approach allows us to unify the construction reflection coupling and refined

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basic coupling for Lévy processes; both are non-trivial. We will apply coupling techniques to prove regularity results for the transition semigroups and construct successful couplings of the solutions to stochastic differential equations driven by additive Lévy noise.

Let us briefly recall some definitions related to couplings; our main reference is [10], Chapter 2.

Definition 1.1. Let μ^k, $k=1,2$, be probability measures on $(\mathbb{R}^d, \mathscr{B}(\mathbb{R}^d))$. Any probability measure $\widetilde{\mu}$ on $(\mathbb{R}^{2d}, \mathscr{B}(\mathbb{R}^{2d}))$ is called a *coupling* of μ^1 and μ^2, if the measures μ^k are the marginal probabilities of $\widetilde{\mu}$, i.e. if

$$\widetilde{\mu}\big(A \times \mathbb{R}^d\big) = \mu^1(A), \qquad \widetilde{\mu}\big(\mathbb{R}^d \times A\big) = \mu^2(A), \quad A \in \mathscr{B}\big(\mathbb{R}^d\big).$$

In probabilistic terms, coupling amounts to construct a random variable $\widetilde{X} = (\widetilde{X}^1, \widetilde{X}^2)$ such that the components $\widetilde{X}^k$ are distributed like X^k.

Clearly, the product measure $\widetilde{\mu} = \mu^1 \times \mu^2$, is a coupling of μ^1 and μ^2; for obvious reasons this is often called *independent coupling*. This means, in particular, that there always exists a coupling of μ^1 and μ^2. Similarly, we can define a coupling process of two stochastic processes $(X_t^i)_{t\geq 0}$ in terms of their distributions at each fixed time t and for fixed initial points. Of course, for given marginal Markov processes, the resulting family of pairs $(\widetilde{X}_{1,t}, \widetilde{X}_{2,t})_{t\geq 0}$ may not be a Markov process; for example, the maximal coupling introduced by Griffeath [18] for time-discrete Markov processes is non-Markovian. In this paper, we will restrict ourselves to Markovian couplings.

Definition 1.2. Let X^k, $k=1,2$, be Markov processes with values in $(\mathbb{R}^d, \mathscr{B}(\mathbb{R}^d))$ and transition semigroups $(P_t^k)_{t\geq 0}$. A *Markov coupling* of X^1 and X^2 is a Markov process $\widetilde{X}$ on $(\mathbb{R}^{2d}, \mathscr{B}(\mathbb{R}^{2d}))$ whose transition semigroup $(\widetilde{P}_t)_{t\geq 0}$ satisfies the following condition

$$\widetilde{P}_t(f \otimes \mathbb{1})(x,y) = P_t^1 f(x), \qquad \widetilde{P}_t(\mathbb{1} \otimes f)(x,y) = P_t^2 f(y), \quad t \geq 0,\, f \in B_b\big(\mathbb{R}^d\big).$$

As usual, $f \otimes g(x,y) := f(x)g(y)$ denotes the tensor product of two functions $f, g : \mathbb{R}^d \to \mathbb{R}$, and $B_b(\mathbb{R}^d)$ is the set of all bounded Borel measurable functions on $\mathbb{R}^d$.

Probabilistically, this definition means that $\widetilde{X}$ and its coordinate processes $\widetilde{X}^k$ are Markov processes such that $\widetilde{X}^k$ and X^k have the same finite-dimensional distributions.

The semigroup description of coupling immediately leads to a coupling version for the infinitesimal generators. In the following definition, $\mathscr{D}$ denotes a suitable domain (of an extension) of the operator L. We do not insist that $(L, \mathscr{D})$ is a closed operator.

Definition 1.3. Let $L_k : \mathscr{D}_k \to B(\mathbb{R})$, $\mathscr{D}_k \subset B(\mathbb{R}^d)$, $k=1,2$, be the infinitesimal generators of two Markov processes. A linear operator $\widetilde{L} : \widetilde{\mathscr{D}} \to B(\mathbb{R})$, $\widetilde{\mathscr{D}} \subset B(\mathbb{R}^{2d})$ is called a *coupling operator* with marginals L_1 and L_2, if $\mathbb{1} \otimes \mathscr{D}_2 \cup \mathscr{D}_1 \otimes \mathbb{1} \subset \widetilde{\mathscr{D}}$ and

$$\widetilde{L}(f \otimes \mathbb{1})(x,y) = L_1 f(x), \qquad \widetilde{L}(\mathbb{1} \otimes f)(x,y) = L_2 f(y), \quad x, y \in \mathbb{R}^d,\, f \in C_b^2\big(\mathbb{R}^d\big).$$

For example, if the Markov processes are Feller processes such that the test functions $C_c^\infty(\mathbb{R}^d)$ are contained in the domain of L_k (in the sense of Feller operators and semigroup theory, that is, such that $(L_k, \mathcal{D}(L_k))$ is a closed operator), then we can take $\mathcal{D}_1 = \mathcal{D}_2 = C_b(\mathbb{R}^d)$ and $\widetilde{\mathcal{D}} = C_b(\mathbb{R}^{2d})$, see, for example, [7].

In many applications, one has $(P_t^1)_{t\ge 0} = (P_t^2)_{t\ge 0}$, hence $L_1 = L_2$, and we will make this assumption from now on. In this case, we simply call the process $\tilde{X}$ in Definition 1.2 a Markov coupling of the Markov process X^1 (or X^2).

While there are many publications on the coupling of diffusion processes, see for instance [4, 11,13,19,25,31] and the references mentioned in these tracts – in particular, the papers [11,31] use the notion of coupling operators –, only few papers consider the coupling of Lévy processes with jumps. The first systematic investigations on coupling of Lévy processes are [7,35,36] and [6], Chapter 6.2, which use the structure of Lévy processes in an essential way. Only recently, there have been some developments on the coupling of non-Lévy jump processes, which arise as solutions to stochastic differential equations (SDEs) driven by Lévy noises, see [26–28,41]. Coupling of jump processes is – compared to the diffusion case – still in its infancy. The starting point of the present paper is [10], Open Problem 2.19, page 26]: "What should be the representation of Markovian coupling operators for Lévy processes?"

We are going to investigate d-dimensional SDEs driven by an additive pure jump Lévy noise

$$dX_t = b(X_t)\,dt + dZ_t, \quad X_0 = x \in \mathbb{R}^d, \tag{1}$$

where $b : \mathbb{R}^d \to \mathbb{R}^d$ is a measurable function and $Z = (Z_t)_{t\ge 0}$ is a pure jump Lévy process on $\mathbb{R}^d$. We assume that the SDE (1) has a unique strong solution $X = (X_t)_{t\ge 0}$. This holds, for example, if b satisfies the local Lipschitz and linear growth conditions, see [20], Chapter IV.9, or if b is Hölder continuous and Z a Lévy process satisfying some moment condition for the Lévy measure at zero and at infinity and such that its transition semigroups enjoy certain regularity properties, see, for example, [12,21,30,42]. It is straightforward to see that the generator of X is given by

$$Lf(x) = \langle \nabla f(x), b(x)\rangle + \int_{\mathbb{R}^d} \big[f(x+u) - f(x) - \langle \nabla f(x), u\rangle \mathbb{1}_{(0,1)}(|u|)\big]\nu(du), \tag{2}$$

where ν is the Lévy measure of the pure jump Lévy process Z.

We are mainly interested in two questions: *(i)* we want to find a uniform formulation for coupling of the SDE (1) – this serves as a model case for more general SDEs with multiplicative noise, see Section 5.2; this can be done using the concept of coupling operators. Note that this covers *all* currently known non-trivial couplings for Lévy processes: coupling by reflection, refined basic coupling and coupling from the point view of optimal transport introduced in [28]. The new representation via coupling operators also presents a new understanding of [10], Open Problem 2.19, page 26. *(ii)* we want to establish new regularity results for the transition semigroups and the coupling property (i.e., the fact that the coupling is successful) of the solution to the SDE (1). These results are also new for Lévy processes and they illustrate the power of the coupling and coupling operator method when applied to Lévy processes. Lastly, we discuss the "optimality"of the three known couplings mentioned above. The notion of optimality can sometimes be deceptive, since its use and meaning depends on various applications of coupling techniques. Our point of view partly coincides with Chen's notion of optimality as in [9]. The readers may want to consult [8,16] for further discussions on optimal couplings with regards to coupling rate or coupling time. Let us finally point out that couplings for SDEs with Lévy noise turned out to be extremely useful for constructing numerical schemes, see [14,15].

Our paper is structured in the following way: In the next section, we study properties of Markov coupling operators for operators of the form (2), and then we provide a new construction of a coupling operator, see (6). In Section 3, we derive three types of coupling processes for the SDE (1) from the coupling operator (6). In particular, we establish the existence of the associated coupling process as strong solution of a suitable SDE. Section 4 is devoted to applications of our coupling techniques, including the regularity of the semigroups and the coupling property for the SDE (1). In the last section, we consider the optimality of three coupling operators mentioned in Section 3. A possible extension to SDEs with multiplicative Lévy noises are also discussed there.

Notation. Most of our notation is standard or self-explanatory. Lévy measures $\nu(du)$ and Lévy kernels $\nu(x, du)$ are, as usual, defined on $\mathbb{R}^d \setminus \{0\}$; for simplicity we will not make this explicit in our notation and keep writing $\int_{\mathbb{R}^d} \dots \nu(du)$ etc. By $a \wedge b$, we denote the minimum of a and b, and agree that "$\wedge$", when combined with "$+$" or "$-$", takes precedence over these operations, that is, $a \pm a \wedge b = a \pm (a \wedge b)$.

2. Coupling operators for SDEs with additive Lévy noise

Recall that the infinitesimal generator L of the SDE (1) is given by (2). The purpose of this section is to obtain a general formula for the coupling operator of the operator L.

Assume, for a moment, that A is the generator of a Feller process such that the test functions $C_c^\infty(\mathbb{R}^d)$ are contained in the domain $\mathfrak{D}(A)$. It is well known, cf. [6], that $\overline{C_c^\infty(\mathbb{R}^d)}^{\|\cdot\|_\infty} \subset D(A)$ and Af, $f \in C_c^\infty(\mathbb{R}^d)$, is necessarily of the form

$$
\begin{aligned}
Af(x) &= \langle \nabla f(x), b(x) \rangle + \frac{1}{2} \operatorname{div} Q(x) \nabla f(x) \\
&\quad + \int_{\mathbb{R}^d} \big[f(x+u) - f(x) - \langle \nabla f(x), u \rangle \mathbb{1}_{(0,1)}(|u|) \big] \nu(x, du);
\end{aligned}
$$

here, $(b(x), Q(x), \nu(x, du))$ is for every fixed $x \in \mathbb{R}^d$ a Lévy triplet, that is, $b(x) \in \mathbb{R}^d$, $Q(x) \in \mathbb{R}^{d\times d}$ is positive semidefinite, $\int_{\mathbb{R}^d} [1 \wedge |u|^2] \nu(x, du) < \infty$ and all expressions are measurable and locally bounded in x.

Therefore, the following *Ansatz* provides a natural candidate for a coupling operator related to (2): for any $f \in C_b^2(\mathbb{R}^{2d})$,

$$
\begin{aligned}
\widetilde{L} f(x, y) &= \langle \nabla_x f(x, y), b(x) \rangle + \langle \nabla_y f(x, y), b(y) \rangle \\
&\quad + \int_{\mathbb{R}^d \times \mathbb{R}^d} \big[f(x+u, y+v) - f(x, y) - \langle \nabla_x f(x, y), u \rangle \mathbb{1}_{(0,1)}(|u|) \\
&\quad - \langle \nabla_y f(x, y), v \rangle \mathbb{1}_{(0,1)}(|v|) \big] \widetilde{\nu}(x, y, du, dv),
\end{aligned}
\tag{3}
$$

where $\nabla_x f(x,y)$ and $\nabla_y f(x,y)$ denote the gradient of $f(x,y)$ with respect to x and y, and $\widetilde{\nu}(x,y,du,dv)$ is a Lévy-type kernel, that is, a measure on $\mathbb{R}^{2d}\setminus\{0\}$ satisfying

$$\int_{\mathbb{R}^{2d}} \big[1\wedge\big(|u|^2+|v|^2\big)\big]\widetilde{\nu}(x,y,du,dv)<\infty,\quad x,y\in\mathbb{R}^d. \tag{4}$$

Lemma 2.1. *The operator $\widetilde{L}$ defined by* (3) *is a coupling operator with marginal operator L of the form* (2) *if, and only if, $\widetilde{\nu}(x,y,du,dv)$ satisfies for all $A,B\in\mathscr{B}(\mathbb{R}^d\setminus\{0\})$ and $x,y\in\mathbb{R}^d$ the following conditions*

$$\widetilde{\nu}\big(x,y,A\times\mathbb{R}^d\big)=\nu(A),\qquad \widetilde{\nu}\big(x,y,\mathbb{R}^d\times B\big)=\nu(B). \tag{5}$$

Proof. By (3), we have for any $f\in C_b^2(\mathbb{R}^d)$,

$$\begin{aligned}\widetilde{L}(f\otimes\mathbb{1})(x,y)&=\big\langle\nabla f(x),b(x)\big\rangle\\&\quad+\int_{\mathbb{R}^d\times\mathbb{R}^d}\big[f(x+u)-f(x)-\big\langle\nabla f(x),u\big\rangle\mathbb{1}_{(0,1)}\big(|u|\big)\big]\widetilde{\nu}(x,y,du,dv).\end{aligned}$$

Let $f\in C_c^2(\mathbb{R}^d\setminus\{0\})$. We have

$$\widetilde{L}(f\otimes\mathbb{1})(0,y)=\int_{\mathbb{R}^d\times\mathbb{R}^d} f(u)\widetilde{\nu}(x,y,du,dv)\quad\text{and}\quad Lf(0)=\int_{\mathbb{R}^d} f(u)\nu(du).$$

Since $\widetilde{L}(f\otimes\mathbb{1})=Lf$, we get the first equality in (5) since the family $C_c^2(\mathbb{R}^d\setminus\{0\})$ is measure-determining on $\mathbb{R}^d\setminus\{0\}$. The second equality follows in a similar way.

Let us show that (5) is also sufficient. For any $f\in C_b^2(\mathbb{R}^d)$ and $x\in\mathbb{R}^d$, set $F_x(u):=f(x+u)-f(x)-\langle\nabla f(x),u\rangle\mathbb{1}_{(0,1)}(|u|)$. By definition, $F_x(u)\in C_b(\mathbb{R}^d)$ with $F_x(0)=0$. Thus, (5) along with a standard approximation argument yields

$$\int_{\mathbb{R}^d\times\mathbb{R}^d}F_x(u)\widetilde{\nu}(x,y,du,dv)=\int_{\mathbb{R}^d}F_x(u)\nu(du).$$

Similarly, we get in the other coordinate direction

$$\int_{\mathbb{R}^d\times\mathbb{R}^d}F_y(v)\widetilde{\nu}(x,y,du,dv)=\int_{\mathbb{R}^d}F_y(v)\nu(dv).$$

Hence, $\widetilde{L}$ defined by (3) is a coupling operator with marginal operator L. □

The condition (5) for a coupling operator – it is equivalent to the fact that for any $x,y\in\mathbb{R}^d$ the marginals of the kernel $\widetilde{\nu}(x,y,du,dv)$ coincide with the Lévy measure ν – is stronger than the requirement (4) for general Lévy-type operators. This means that the class of Lévy-type coupling operators is smaller than the class of Lévy-type operators – but to-date we are not aware of a structural characterization of general Lévy-type coupling operators, and we have to restrict ourselves to concrete examples.

To proceed, we need some further notation. For any bi-measurable function $f:\mathbb{R}^d\to\mathbb{R}^d$ and $A\in\mathcal{B}(\mathbb{R}^d)$, we define

$$(\nu\circ f)(A)=\nu\big(f(A)\big)\quad\text{and}\quad \mu_{\nu,f}=\nu\wedge(\nu\circ f);$$

the minimum of two measures ν_1 and ν_2 on $(\mathbb{R}^d,\mathcal{B}(\mathbb{R}^d))$ is defined as $\nu_1\wedge\nu_2=\nu_1-(\nu_1-\nu_2)^+$ where $(\nu_1-\nu_2)^\pm$ are the positive resp. negative parts of the Hahn–Jordan decomposition of the signed measure $\nu_1-\nu_2$. For any $1\le i<n+1\le\infty$, let ν_i be a nonnegative measure on $(\mathbb{R}^d,\mathcal{B}(\mathbb{R}^d))$ such that $\sum_{i=1}^n\nu_i\le\nu$, and $\Psi_i:\mathbb{R}^d\to\mathbb{R}^d$ a bijective and continuous mapping, that is, Ψ_i is invertible and continuous satisfying $\Psi_i(\mathbb{R}^d)=\mathbb{R}^d$. In particular, Ψ_i is bi-measurable from $\mathbb{R}^d$ to $\mathbb{R}^d$. For any $f\in C_b^2(\mathbb{R}^d\times\mathbb{R}^d)$ and $x,y\in\mathbb{R}^d$, we set

$$\begin{aligned}\widetilde{L}f(x,y)&=\big\langle\nabla_x f(x,y),b(x)\big\rangle+\big\langle\nabla_y f(x,y),b(y)\big\rangle\\&\quad+\sum_{i=1}^n\int_{\mathbb{R}^d}\big[f\big(x+z,y+\Psi_i(z)\big)-f(x,y)-\big\langle\nabla_x f(x,y),z\big\rangle\mathbb{1}_{(0,1)}\big(|z|\big)\\&\quad-\big\langle\nabla_y f(x,y),\Psi_i(z)\big\rangle\mathbb{1}_{(0,1)}\big(\big|\Psi_i(z)\big|\big)\big]\mu_{\nu_i,\Psi_i}(dz)\\&\quad+\int_{\mathbb{R}^d}\big[f(x+z,y+z)-f(x,y)-\big\langle\nabla_x f(x,y),z\big\rangle\mathbb{1}_{(0,1)}\big(|z|\big)\\&\quad-\big\langle\nabla_y f(x,y),z\big\rangle\mathbb{1}_{(0,1)}\big(|z|\big)\big]\left(\nu-\sum_{i=1}^n\mu_{\nu_i,\Psi_i}\right)(dz).\end{aligned}\tag{6}$$

Proposition 2.2. *If*

$$\sum_{i=1}^n\mu_{\nu_i,\Psi_i}=\sum_{i=1}^n\mu_{\nu_i,\Psi_i^{-1}},\tag{7}$$

then, the operator $\widetilde{L}$ defined by (6) is a coupling operator with marginal operator L given by (2).

Proof. Set

$$\widetilde{\nu}(x,y,du,dv)=\sum_{i=1}^n\mu_{\nu_i,\Psi_i}(du)\delta_{\Psi_i(u)}(dv)+\left(\nu-\sum_{i=1}^n\mu_{\nu_i,\Psi_i}\right)(du)\delta_u(dv).$$

The operator $\widetilde{L}$ defined by (6) is of the form (3) with the Lévy type kernel $\widetilde{\nu}(x,y,du,dv)$ shown above. It is clear that we have $\widetilde{\nu}(x,y,A\times\mathbb{R}^d)=\nu(A)$ for any $x,y\in\mathbb{R}^d$ and $A\in\mathcal{B}(\mathbb{R}^d\setminus\{0\})$. On the other hand, we have

$$\big(\mu_{\nu_i,\Psi_i}\circ\Psi_i^{-1}\big)(A)=\mu_{\nu_i,\Psi_i^{-1}}(A)\quad\text{for all } A\in\mathcal{B}\big(\mathbb{R}^d\setminus\{0\}\big)\text{ and }1\le i<n+1.$$

Together with (7) this yields that for $x,y\in\mathbb{R}^d$ and $B\in\mathcal{B}(\mathbb{R}^d\setminus\{0\})$, $\widetilde{\nu}(x,y,\mathbb{R}^d\times B)=\nu(B)$. The claim follows from Lemma 2.1. $\square$

The coupling operator $\widetilde{L}$ defined by (6) can be uniquely described by the drift $b(x)$ and the following jump system

$$(x,y) \longmapsto \begin{cases} \big(x+z, y+\Psi_i(z)\big), & \mu_{\nu_i,\Psi_i}(dz) \text{ for } 1 \le i < n+1; \\ (x+z, y+z), & \left(\nu - \sum_{i=1}^{n} \mu_{\nu_i,\Psi_i}\right)(dz). \end{cases} \tag{8}$$

We will adopt this description throughout the rest of the paper.

Remark 2.3. The coupling (8) has a very intuitive interpretation. Recall that the Lévy measure $\nu(dz)$ appearing in the formula (2) for the generator L stands for the frequency of jumps of height z; since ν does not depend on the last position x before a jump occurs, the measure $\nu(dz)$ governs jumps from $x \mapsto x+z$.

In a coupling, both marginal processes have to jump, and in the literature there are a few different possibilities of couplings for jump processes. For example,

synchronous coupling (or march coupling) means that $(x,y) \mapsto (x+z, y+z)$ with frequency $\nu(dz)$,

basic coupling means that $(x,y) \mapsto (x+z, x+z) = (x+z, y+(z+x-y))$, with frequency $\nu(dz)$,

see [10], pages 22–23.

The coupling proposed in (8), assigns to the first component x the full jump height $x \mapsto x+z$ with frequency $\nu(dz)$, while at the same time the second component is treated differently: with frequency $\mu_{\nu_i,\Psi_i}(dz)$ it is sent to $y \mapsto y+\Psi_i(z)$ while with the remaining frequency $(\nu - \sum_{i=1}^n \mu_{\nu_i,\Psi_i})(dz)$ we have a synchronous coupling $y \mapsto y+z$. If the jump of the first margin from $x \mapsto x+z$ happens with frequency ν_i, $\mu_{\nu_i,\Psi_i} = \nu_i \wedge (\nu_i \circ \Psi_i)$ is the biggest possible frequency for the jump of the coupling from $(x,y) \mapsto (x+z, y+\Psi_i(z))$ to guarantee the marginal condition of the coupling, that is, the fact that we still get a coupling. The maps $\Psi_i : \mathbb{R}^d \to \mathbb{R}^d$ are bijective and continuous; if $\Psi_i(z) = z$, then we recover the synchronous coupling, if $\Psi_i(z) = z+x-y$, then we are in the setting of the basic coupling.

It is clear from (8) that the total mass of the first component jumping from x to $x+z$ is $\nu(dz)$, and (7) is a natural condition to ensure that the total mass for the second component is also ν. We stress that Ψ_i may depend on $x, y \in \mathbb{R}^d$.

Remark 2.4. In most applications, one needs a pathwise realization of the coupling in form of a Markov process, that is a $2d$-dimensional Markov process $(X_t, Y_t)_{t\ge 0}$ such that $(X_t - X_0)_{t\ge 0}$ and $(Y_t - Y_0)_{t\ge 0}$ are Markov processes with infinitesimal generator L. Clearly, if $(X_t, Y_t)_{t\ge 0}$ exists, then the coupling operator $\widetilde{L}$ (with the generator L as marginal operator) is indeed the infinitesimal generator of $(X_t, Y_t)_{t\ge 0}$. The converse is more of a problem: from the mere definition of a coupling operator $\widetilde{L}$ we cannot immediately deduce the existence of an associated Markov process – we refer to [6] for an exhaustive discussion of the existence of processes generated by Lévy-type operators.

In general, one needs a further argument to deduce the existence of a coupling process $(X_t, Y_t)_{t\geq 0}$. For diffusions, the well-posedness of the associated martingale problem is the method of choice, see [11], Sections 2 and 3, and [31], Section 2, see also [40], Section 3.1, and [41], Section 2.2, for the Lévy case.

In the present context, all processes are given by SDEs, so it is more natural to require the existence of a strong solution to the SDE, see e.g. [22,27].

3. Explicit coupling processes for SDEs with additive Lévy noise via coupling operators

In this section, we will establish three different kinds of coupling processes for the SDE (1) by making full use of the coupling operator constructed in the previous section. In the literature, these three – in general highly non-trivial – couplings are treated in different settings; it is, therefore, surprising that we can handle them in a single framework based on the coupling operator (6).

3.1. Coupling by reflection: Rotationally symmetric Lévy noise

Assume that $Z = (Z_t)_{t\geq 0}$ is a pure jump rotationally symmetric Lévy process with Lévy measure ν. For any $x, y, z \in \mathbb{R}^d$, we write

$$R_{x,y}(z) := \begin{cases} z - \dfrac{2\langle x-y, z\rangle}{|x-y|^2}(x-y), & \text{if } x \neq y, \\ z, & \text{if } x = y, \end{cases} \tag{9}$$

for the reflection at the hyperplane orthogonal to $x-y$. Obviously, $R_{x,y}(z) = R_{y,x}(z)$, $R_{x,y}(z) = R^{-1}_{x,y}(z)$, $|R_{x,y}(z)| = |z|$ and $R_{x,y}(z) - (x-y) = R_{x,y}(z) + R_{x,y}(x-y) = R_{x,y}(z+x-y)$.

If we set in (8) $n = 1$, $\Psi_1(z) = R_{x,y}(z)$ and $\nu_1(dz) = \mathbb{1}_{\{|z|<\eta|x-y|\}}\nu(dz)$ for some fixed $\eta \in (0, \infty]$, we get

$$(x, y) \longmapsto \begin{cases} \big(x+z, y+R_{x,y}(z)\big), & \mathbb{1}_{\{|z|\leq\eta|x-y|\}}\nu(dz); \\ (x+z, y+z), & \mathbb{1}_{\{|z|>\eta|x-y|\}}\nu(dz). \end{cases} \tag{10}$$

Since ν is rotationally symmetric, ν_1 is invariant under the transformation $R_{x,y}(z) \rightsquigarrow z$, as $R_{x,y}(z) = R^{-1}_{x,y}(z)$ and $|R_{x,y}(z)| = |z|$. This shows $\nu_1 \circ \Psi_1 = \nu_1 \circ \Psi_1^{-1} = \nu_1$ which means that (7) is satisfied. Thus, according to Proposition 2.2, the jump system (10) determines a coupling operator $\widetilde{L}$.

Let us briefly verify the existence of a $2d$-dimensional coupling process which is generated by the coupling operator $\widetilde{L}$ given by (10). By the Lévy-Itô decomposition, there exists a Poisson random measure $N(dt, dz)$ such that

$$dZ_t = \int_{\{|z|\geq 1\}} zN(dt, dz) + \int_{\{|z|<1\}} z\widetilde{N}(dt, dz),$$

where $\widetilde{N}(dt,dz) = N(dt,dz) - dt\nu(dz)$ is the compensated Poisson random measure. To keep notation simple, we set $\breve{N}(dt,dz) = \mathbb{1}_{(0,1)}(|z|)\widetilde{N}(dt,dz) + \mathbb{1}_{[1,\infty)}(|z|)N(dt,dz)$, and so

$$dZ_t = \int z\breve{N}(dt,dz).$$

Consider the following system of SDEs on $\mathbb{R}^{2d}$:

$$\begin{cases} dX_t = b(X_t)\,dt + \int z\breve{N}(dt,dz), & t > 0, \\ dY_t = b(Y_t)\,dt + \int_{\{|z|<\eta|X_t-Y_t|\}} R_{X_{t-},Y_{t-}}(z)\breve{N}(dt,dz) & \\ \qquad + \int_{\{|z|\ge\eta|X_t-Y_t|\}} z\breve{N}(dt,dz), & t > 0. \end{cases} \tag{11}$$

For any $x, y, z \in \mathbb{R}^d$ with $x \neq y$, we have

$$\left(\mathrm{id}_d - \frac{2}{|x-y|^2}(x-y)(x-y)^\top\right)z = R_{x,y}(z),$$

where id_d denotes the $d \times d$ identity matrix. Thus, for any fixed $z \in \mathbb{R}^d$, $(x,y) \mapsto R_{x,y}(z)$ is locally Lipschitz continuous on $\{(x,y) \in \mathbb{R}^{2d} : x \neq y\}$. If we assume, in addition, that the drift term b is Lipschitz continuous, then the SDE (11) has a unique strong solution (X_t, Y_t) up to τ, where τ is the coupling time defined by

$$\tau := \inf\{t > 0 : X_t = Y_t\}, \tag{12}$$

see also the discussion in the proof of Proposition 3.1 below. Since, by assumption, the first SDE in (11) has a unique strong solution $(X_t)_{t\ge0}$, it is natural to identify the solution of (11) with (X_t, X_t) for all $t \ge \tau$. By Itô's formula, we can easily verify that the generator of $(X_t, Y_t)_{t\ge0}$ is the coupling operator $\widetilde{L}$ given by (10). Originally, the coupling operator $\widetilde{L}$ having the jump system (10) with $\eta = \frac{1}{2}$, appears in [41]; for the existence of the associated process, the martingale problem for the operator $\widetilde{L}$ was used. The present approach via SDEs (11) is new. We believe that this approach will be useful in applications, and it is more natural from the viewpoint of the coupling by reflection for diffusions which will be described below.

Recall that coupling by reflection for SDEs driven by an additive Brownian motion $B = (B_t)_{t\ge0}$ can be realized through the following system of SDEs, cf. [11,25]:

$$\begin{cases} dX_t = b(X_t)\,dt + dB_t, \\ dY_t = b(Y_t)\,dt + \left(\mathrm{id}_d - 2e_te_t^\top\right)dB_t, \end{cases}$$

where

$$e_t := |X_t - Y_t|^{-1}(X_t - Y_t),$$

$e_t^\top$ is the transpose of e_t, and τ is defined as (12). In particular, we have $(X_t, Y_t) = (X_t, Y_t)\mathbb{1}_{\{t<\tau\}} + (X_t, X_t)\mathbb{1}_{\{t\geq\tau\}}$. For any $t > 0$ the matrix $A_t = \mathrm{id}_d - 2e_t e_t^\top$ is an orthogonal matrix and, by the Lévy characterization of Brownian motion, the process $B^\#$ defined by $B_t^\# := A_t B_t$, $t > 0$, is also a Brownian motion. We can use a similar idea to construct the corresponding coupling for the SDE (1): If $Z = (Z_t)_{t\geq 0}$ is a rotationally symmetric pure jump Lévy process, then the process $Z^\#$ defined by $Z_t^\# := A_t Z_t$, $t > 0$, is again a rotationally symmetric pure jump Lévy process which has the same distribution as $(Z_t)_{t\geq 0}$. Indeed, let $L^\#$ be the generator of the process $Z^\#$. For any $f \in C_b^2(\mathbb{R}^d)$, we know

$$\begin{aligned} L^\# f(x) &= \int \big(f\big(x + R_{x,y}(z)\big) - f(x) - \big\langle \nabla f(x), R_{x,y}(z)\big\rangle \mathbb{1}_{(0,1)}\big(\big|R_{x,y}(z)\big|\big)\big)\nu(dz) \\ &= \int \big(f(x+z) - f(x) - \big\langle \nabla f(x), z\big\rangle \mathbb{1}_{(0,1)}\big(|z|\big)\big)\nu(dz) = Lf(x), \end{aligned}$$

where we use the fact that ν is invariant under the change of variables $R_{x,y}(z) \rightsquigarrow z$ due to the rotational symmetry of the process Z. This shows that $Z^\#$ is a pure-jump Lévy process with the same Lévy measure as Z, hence Z and $Z^\#$ coincide in law. In particular, the associated coupling process can be constructed using the SDE (11) with $\eta = \infty$. This is the reason why we call (10) *coupling by reflection*.

The construction with $\eta = \infty$ is not always the best choice. In contrast to the diffusion case, this is due to the fact that the above construction allows for a situation that two jump processes – even if they are already close – suddenly jump far apart. In order to apply coupling by reflection, we have to choose the parameter η carefully. If $\eta = \frac{1}{2}$, the coupling time is almost surely finite, that is, the coupling is successful, for a large class of rotationally symmetric Lévy processes, including symmetric α-stable processes, see Theorem 4.5 below; this is similar to the reflection coupling of Brownian motion. Note that the above argument still works if Z is of the form $Z = Z' + Z''$ where Z', Z'' are independent Lévy processes and Z'' is rotationally symmetric, see, for example, [41].

3.2. Refined basic coupling: General Lévy noise

In this part, we will show that the refined basic coupling for a general Lévy noise in [27] can be directly deduced from the coupling operator (6). Although the idea of the construction in [27] is also based on the coupling operator, the starting point of [27] comes from the notion of *basic coupling* introduced by M.-F. Chen [10], Example 2.10, when studying Markov q-processes. The idea behind the basic coupling is to force the two marginal processes to jump to the same point with the biggest possible rate. In the Lévy case, the biggest jump rate takes the form $\mu_{y-x}(dz) := [\nu \wedge (\delta_{y-x} * \nu)](dz)$, where ν is the Lévy measure and $x \neq y$ are the positions of the two marginal processes immediately before the jump.

Let ν be the Lévy measure of the Lévy process $Z = (Z_t)_{t\geq 0}$. Note that the construction of coupling in this section does not require any further (e.g., geometric) assumptions on the Lévy measure. For any $\kappa > 0$ and $x, y \in \mathbb{R}^d$, we define

$$(x-y)_\kappa := \left(1 \wedge \frac{\kappa}{|x-y|}\right)(x-y), \quad \left(\frac{1}{\infty} := 0\right).$$

The following *refined basic coupling* was introduced in [27], Section 2, for the first time:

$$(x, y) \longmapsto \begin{cases} \big(x+z, y+z+(x-y)_\kappa\big), & \frac{1}{2}\mu_{(y-x)_\kappa}(dz); \\ \big(x+z, y+z+(y-x)_\kappa\big), & \frac{1}{2}\mu_{(x-y)_\kappa}(dz); \\ (x+z, y+z), & \Big(\nu - \frac{1}{2}\mu_{(y-x)_\kappa} - \frac{1}{2}\mu_{(x-y)_\kappa}\Big)(dz). \end{cases} \tag{13}$$

Obviously, (13) is the same as (8) if $n = 2$, $\Psi_1(z) = z + (x-y)_\kappa$, $\Psi_2(z) = z + (y-x)_\kappa$ and $\nu_1 = \nu_2 = \frac{1}{2}\nu$. Since $\Psi_1^{-1}(z) = \Psi_2(z)$, (7) holds true, and so (13) yields a coupling operator.

Let us briefly discuss some properties of the refined basic coupling (13).

If $|x-y| \le \kappa$, then the refined basic coupling becomes

$$(x, y) \longmapsto \begin{cases} \big(x+z, y+z+(x-y)\big), & \frac{1}{2}\mu_{y-x}(dz); \\ \big(x+z, y+z+(y-x)\big), & \frac{1}{2}\mu_{x-y}(dz); \\ (x+z, y+z), & \Big(\nu - \frac{1}{2}\mu_{y-x} - \frac{1}{2}\mu_{x-y}\Big)(dz). \end{cases} \tag{14}$$

In the first row of (14), the distance of the two marginals decreases from $|x-y|$ to $|(x+z)-(y+z+(x-y))| = 0$, and this reflects the idea of the basic coupling – but only with half of the maximum common jump intensity from x to $x+z$ and y to $y+z+(x-y)$. In the second row of (14) the distance is doubled after jumping, with the remaining half of the maximum common jump intensity, while we couple the remaining mass synchronously as indicated in the third row of (14).

If $|x-y| > \kappa$, then the first row of (13) shows that the distance after the jump is $|x-y| - \kappa$. Therefore, the parameter κ is the threshold to determine whether the marginal processes jump to the same point, or become slightly closer to each other. This is a technical point, but is crucial for our argument to make the coupling (13) efficient for Lévy jump processes with bounded (finite-range) jumps.

Using the technique from [27], Section 2.3, we can construct the coupling process associated with the refined basic coupling. In a first step, we extend the Poisson random measure N from $\mathbb{R}_+ \times \mathbb{R}^d$ to $\mathbb{R}_+ \times \mathbb{R}^d \times [0,1]$ by adjoining an independent uniformly distributed random component

$$N(ds, dz, du) = \sum_{0<r\le s:\Delta Z_r \ne 0} \delta_{(r,\Delta Z_r)}(ds, dz)\mathbb{1}_{[0,1]}(u)\, du,$$

$$\widetilde{N}(ds, dz, du) = N(ds, dz, du) - ds\,\nu(dz)\, du,$$

and we set

$$\breve{N}(ds, dz, du) = \mathbb{1}_{[1,\infty)\times[0,1]}\big(|z|, u\big)N(ds, dz, du) + \mathbb{1}_{(0,1)\times[0,1]}\big(|z|, u\big)\widetilde{N}(ds, dz, du),$$

$$Z_t = \int_0^t \int_{\mathbb{R}^d\times[0,1]} z\breve{N}(ds, dz, du).$$

We are going to use the adjoined random variable to define a random threshold which determines whether the processes X and Y move towards each other or extend their distance. For this, we need the following control function ρ:

$$\rho(x,z) = \frac{\nu \wedge (\delta_x * \nu)(dz)}{\nu(dz)} \in [0,1], \quad x, z \in \mathbb{R}^d.$$

Recall that $(x)_\kappa = (1 \wedge (\kappa|x|^{-1}))x$ for any $x \neq 0$. Set $U_t = X_t - Y_t$,

$$V_t(z,u) = (U_t)_\kappa \big(\mathbb{1}_{[0,\frac{1}{2}\rho((-U_t)_\kappa,z)]}(u) - \mathbb{1}_{(\frac{1}{2}\rho((-U_t)_\kappa,z),\frac{1}{2}[\rho((-U_t)_\kappa,z)+\rho((U_t)_\kappa,z)]]}(u)\big)$$

and

$$dL_t^\# = \int_{\mathbb{R}^d\times[0,1]} V_{t-}(z,u) N(dt,dz,du).$$

Consider for any $x, y \in \mathbb{R}^d$ with $x \neq y$ the following system of SDEs:

$$\begin{cases} dX_t = b(X_t)\,dt + dZ_t, & X_0 = x; \\ dY_t = b(Y_t)\,dt + dZ_t + dL_t^\#, & Y_0 = y. \end{cases} \tag{15}$$

It is shown in [27], Propositions 2.2 and 2.3, that the system (15) has a unique strong solution which is a non-explosive coupling process $(X_t, Y_t)_{t\ge 0}$ of the SDE (1). Moreover, the generator of $(X_t, Y_t)_{t\ge 0}$ is the refined basic coupling operator constructed above, and $X_t = Y_t$ for any $t \ge \tau$, where $\tau = \inf\{t \ge 0 : X_t = Y_t\}$ is the coupling time of the process $(X_t, Y_t)_{t\ge 0}$. Note that for $x \neq 0$,

$$\mu_x\big(\mathbb{R}^d\big) \le \int_{\{|z|\le|x|/2\}} \delta_x * \nu(dz) + \int_{\{|z|>|x|/2\}} \nu(dz) \le 2\int_{\{|z|\ge|x|/2\}} \nu(dz) < \infty, \tag{16}$$

that is, μ_x is a finite measure on $(\mathbb{R}^d, \mathscr{B}(\mathbb{R}^d))$ for any $x \neq 0$.

3.3. Coupling vs. optimal transport: Rotationally symmetric Lévy noise

In this section, we discuss the coupling of SDEs driven by rotationally symmetric Lévy noise which was introduced in [28], Sections 2.1 and 2.2; the motivation of [28] was McCann's solution of the optimal transport problem for concave cost functions in $\mathbb{R}$, cf. [29]. In contrast to [28], we will use our coupling operator approach. This has the advantage that we can explicitly state the SDE – see (18) – which admits a unique strong solution whose generator is the coupling operator; [28], Section 2.4, only has the existence of a strong solution of some SDE associated with the truncation of the coupling operator.

Let us return to the framework of Section 3.1. We will assume that the pure jump Lévy process Z in the SDE (1) is rotationally symmetric and that its Lévy measure is of the form $\nu(dz) := q(|z|)\,dz$ for some nonnegative measurable function $q(r)$. Let $q_0(r) \le q(r)$, that is, $q_0(|z|)\,dz$ is also a rotationally symmetric Lévy measure.

For any $x, y, z \in \mathbb{R}^d$, let $R_{x,y}(z)$ be the reflection defined in (9). We consider the following jump system on $\mathbb{R}^{2d}$:

$$(x, y) \longmapsto \begin{cases} \big(x+z, y+z+(x-y)\big), & q_0\big(|z|\big) \wedge q_0\big(|x-y+z|\big)\,dz; \\ \big(x+z, y+R_{x,y}(z)\big), & \big[q_0\big(|z|\big) - q_0\big(|z|\big) \wedge q_0\big(|x-y+z|\big)\big]\,dz; \\ (x+z, y+z), & \big[q\big(|z|\big) - q_0\big(|z|\big)\big]\,dz. \end{cases} \tag{17}$$

If we choose in (8) $n = 2$, $\Psi_1(z) = z + (x-y)$, $\Psi_2(z) = R_{x,y}(z)$, $\nu_1(dz) = q_0(|z|) \wedge q_0(|x-y+z|)\,dz$ and $\nu_2(dz) = [q_0(|z|) - q_0(|z|) \wedge q_0(|x-y+z|)]\,dz$, then (17) can be derived from (8). Observing that $R_{x,y}(x-y) = y-x$ $(x \neq y)$ and $R_{x,y}(z_1+z_2) = R_{x,y}(z_1) + R_{x,y}(z_2)$ for any $z_1, z_2 \in \mathbb{R}^d$ we see

$$q_0\big(\big|\Psi_1^{-1}(z)\big|\big) \wedge q_0\big(\big|x-y+\Psi_1^{-1}(z)\big|\big) = q_0\big(|y-x+z|\big) \wedge q_0\big(|z|\big)$$

and

$$\begin{aligned} &q_0\big(\big|R_{x,y}^{-1}(z)\big|\big) - q_0\big(\big|R_{x,y}^{-1}(z)\big|\big) \wedge q_0\big(\big|x-y+R_{x,y}^{-1}(z)\big|\big) \\ &\quad = q_0\big(|z|\big) - q_0\big(|z|\big) \wedge q_0\big(\big|R_{x,y}^{-1}(y-x+z)\big|\big) \\ &\quad = q_0\big(|z|\big) - q_0\big(|z|\big) \wedge q_0\big(|y-x+z|\big). \end{aligned}$$

This shows that (7) is satisfied.

We are now going to construct the coupling process for the coupling (17). We continue to use the notation introduced in Section 3.2. Denote by $\breve{N}(dt, dz, du)$ and $\breve{N}_0(dt, dz, du)$ the extended Poisson random measure on $\mathbb{R}_+ \times \mathbb{R}^d \times [0,1]$ whose compensators are given by $dt\,q(|z|)\,dz\,du$ and $dt\,q_0(|z|)\,dz\,du$, respectively. In order to keep notation simple, we set $r(z, X_{t-} - Y_{t-}) := (q_0(|z|) \wedge q_0(X_{t-} - Y_{t-} + z))/q_0(|z|)$. Consider the following SDE:

$$\begin{cases} dX_t = b(X_t)\,dt + \displaystyle\int_{\mathbb{R}^d \times [0,1]} z\breve{N}(dt, dz, du), \\ dY_t = b(Y_t)\,dt + \displaystyle\int_{\mathbb{R}^d \times [0,1]} z\big(\breve{N}(dt, dz, du) - \breve{N}_0(dt, dz, du)\big) \\ \qquad + \displaystyle\int_{\mathbb{R}^d \times [0,1]} R_{X_{t-}, Y_{t-}}(z)\mathbb{1}_{(r(z, X_{t-}-Y_{t-}), 1]}(u)\breve{N}_0(dt, dz, du) \\ \qquad + \displaystyle\int_{\mathbb{R}^d \times [0,1]} (X_{t-} - Y_{t-} + z)\mathbb{1}_{(r(z, X_{t-}-Y_{t-}), 1]}(u)\breve{N}_0(dt, dz, du) \\ \qquad - \displaystyle\int_{\mathbb{R}^d \times [0,1]} (X_{t-} - Y_{t-} + z)\big[\mathbb{1}_{(0,1)}\big(\big|z + (X_{t-} - Y_{t-})\big|\big) - \mathbb{1}_{(0,1)}\big(|z|\big)\big] \\ \qquad \times \mathbb{1}_{(r(z, X_{t-}-Y_{t-}), 1]}(u)\,dt\,q_0\big(|z|\big)\,dz\,du. \end{cases} \tag{18}$$

Proposition 3.1. *If b is Lipschitz continuous on $\mathbb{R}^d$, then the SDE (18) has a unique strong solution up to the coupling time τ. The generator of this solution is determined by the coupling (17).*

Proof. As in (16), we see that $\int q_0(|z|) \wedge q_0(|z+x-y|)\,dz < \infty$ for $x \neq y$. We can rewrite the SDE for Y_t in the following way

$$
\begin{aligned}
dY_t = {} & b(Y_t)\,dt + \int_{\mathbb{R}^d\times[0,1]} z\big(\breve{N}(dt,dz,du) - \breve{N}_0(dt,dz,du)\big) \\
& + \int_{\mathbb{R}^d\times[0,1]} R_{X_{t-},Y_{t-}}(z)\breve{N}_0(dt,dz,du) \\
& - \int_{\mathbb{R}^d\times[0,1]} R_{X_{t-},Y_{t-}}(z)\mathbb{1}_{(r(z,X_{t-}-Y_{t-}),1]}(u)\breve{N}_0(dt,dz,du) \\
& + \int_{\mathbb{R}^d\times[0,1]} (X_{t-}-Y_{t-}+z)\mathbb{1}_{(r(z,X_{t-}-Y_{t-}),1]}(u)\breve{N}_0(dt,dz,du) \\
& - \int_{\mathbb{R}^d\times[0,1]} (X_{t-}-Y_{t-}+z)\big[\mathbb{1}_{(0,1)}\big(\big|z+(X_{t-}-Y_{t-})\big|\big) - \mathbb{1}_{(0,1)}\big(|z|\big)\big] \\
& \times \mathbb{1}_{(r(z,X_{t-}-Y_{t-}),1]}(u)\,dt q_0\big(|z|\big)\,dz\,du \\
= {} & b(Y_t)\,dt + \int_{\mathbb{R}^d\times[0,1]} z\big(\breve{N}(dt,dz,du) - \breve{N}_0(dt,dz,du)\big) \\
& + \int_{\mathbb{R}^d\times[0,1]} R_{X_{t-},Y_{t-}}(z)\breve{N}_0(dt,dz,du) \\
& - \int_{\mathbb{R}^d\times[0,1]} R_{X_{t-},Y_{t-}}(z)\mathbb{1}_{(r(z,X_{t-}-Y_{t-}),1]}(u)N_0(dt,dz,du) \\
& + \int_{\mathbb{R}^d\times[0,1]} (X_{t-}-Y_{t-}+z)\mathbb{1}_{(r(z,X_{t-}-Y_{t-}),1]}(u)N_0(dt,dz,du) \\
& + \int_{\{|z|\le 1\}\times[0,1]} R_{X_{t-},Y_{t-}}(z)\mathbb{1}_{(r(z,X_{t-}-Y_{t-}),1]}(u)\,dt q_0\big(|z|\big)\,dz\,du \\
& - \int_{\{|z|\le 1\}\times[0,1]} (X_{t-}-Y_{t-}+z)\mathbb{1}_{(r(z,X_{t-}-Y_{t-}),1]}(u)\,dt q_0\big(|z|\big)\,dz\,du \\
& - \int_{\mathbb{R}^d\times[0,1]} (X_{t-}-Y_{t-}+z)\big[\mathbb{1}_{(0,1)}\big(\big|z+(X_{t-}-Y_{t-})\big|\big) - \mathbb{1}_{(0,1)}\big(|z|\big)\big] \\
& \times \mathbb{1}_{(r(z,X_{t-}-Y_{t-}),1]}(u)\,dt q_0\big(|z|\big)\,dz\,du.
\end{aligned}
$$

We will now rearrange the last three terms involving $dt q_0(|z|)\,dz\,du$:

$$
\begin{aligned}
& \int_{\{|z|<1\}\times[0,1]} R_{X_{t-},Y_{t-}}(z)\mathbb{1}_{(r(z,X_{t-}-Y_{t-}),1]}(u)\,dt q_0\big(|z|\big)\,dz\,du \\
& \quad = \int_{\{|z|<1\}\times[0,1]} z\mathbb{1}_{[0,r(z,Y_{t-}-X_{t-})]}(u)\,dt q_0\big(|z|\big)\,dz\,du,
\end{aligned}
$$

which follows from $R_{x,y}(x-y) = y-x$ $(x \neq y)$ and $R_{x,y}(z_1+z_2) = R_{x,y}(z_1) + R_{x,y}(z_2)$ for $z_1, z_2 \in \mathbb{R}^d$. On the other hand,

$$\begin{aligned}
&\int_{\mathbb{R}^d\times[0,1]} (X_{t-}-Y_{t-}+z)\big[\mathbb{1}_{(0,1)}\big(\big|z+(X_{t-}-Y_{t-})\big|\big) - \mathbb{1}_{(0,1)}\big(|z|\big)\big] \\
&\qquad \times \mathbb{1}_{(r(z,X_{t-}-Y_{t-}),1]}(u)\,dt q_0\big(|z|\big)\,dz\,du \\
&\quad = \int_{\{|z+(X_{t-}-Y_{t-})|<1\}\times[0,1]} (X_{t-}-Y_{t-}+z)\mathbb{1}_{(r(z,X_{t-}-Y_{t-}),1]}(u)\,dt q_0\big(|z|\big)\,dz\,du \\
&\qquad - \int_{\{|z|<1\}\times[0,1]} (X_{t-}-Y_{t-}+z)\mathbb{1}_{(r(z,X_{t-}-Y_{t-}),1]}(u)\,dt q_0\big(|z|\big)\,dz\,du \\
&\quad = \int_{\{|z|<1\}\times[0,1]} z\mathbb{1}_{[0,r(z,Y_{t-}-X_{t-})]}(u)\,dt q_0\big(|z|\big)\,dz\,du \\
&\qquad - \int_{\{|z|<1\}\times[0,1]} (X_{t-}-Y_{t-}+z)\mathbb{1}_{(r(z,X_{t-}-Y_{t-}),1]}(u)\,dt q_0\big(|z|\big)\,dz\,du.
\end{aligned}$$

This means that the equation for Y_t becomes simpler:

$$\begin{aligned}
dY_t = b(Y_t)\,dt &+ \int_{\mathbb{R}^d\times[0,1]} z\big(\breve{N}(dt,dz,du) - \breve{N}_0(dt,dz,du)\big) \\
&+ \int_{\mathbb{R}^d\times[0,1]} R_{X_{t-},Y_{t-}}(z)\breve{N}_0(dt,dz,du) \\
&- \int_{\mathbb{R}^d\times[0,1]} R_{X_{t-},Y_{t-}}(z)\mathbb{1}_{(r(z,X_{t-}-Y_{t-}),1]}(u)N_0(dt,dz,du) \\
&+ \int_{\mathbb{R}^d\times[0,1]} (X_{t-}-Y_{t-}+z)\mathbb{1}_{(r(z,X_{t-}-Y_{t-}),1]}(u)N_0(dt,dz,du).
\end{aligned}$$

For fixed $z \in \mathbb{R}^d$, the function $(x,y) \mapsto R_{x,y}(z)$ is locally Lipschitz continuous on $\{(x,y) \in \mathbb{R}^{2d} : x \neq y\}$. The remaining two terms driven by

$$\mathbb{1}_{(r(z,X_{t-}-Y_{t-}),1]}(u)N_0(dt,dz,du)$$

may be regarded as stochastic integrals with respect to a finite Poisson measure; this is again due to the fact that $\int q_0(|z|) \wedge q_0(z+x-y)\,dz < \infty$ for any $x \neq y$. Using the standard interlacing technique, we see that the SDE (18) has a unique strong solution up to the coupling time τ, see [20], Chapter IV.9. $\square$

4. Regularity properties and successful coupling of SDEs with additive Lévy noise revisited

4.1. Regularity properties of the semigroup

We will now apply our coupling technique, to study regularity properties of the semigroup associated with the SDE (1). Let $(X_t)_{t\geq 0}$ be the unique strong solution to (1) and denote by $(P_t)_{t\geq 0}$ its transition semigroup. We define

$$B(r) := \frac{1}{r} \sup_{|x-y|=r} \langle b(x) - b(y), x - y \rangle, \quad r > 0.$$

Theorem 4.1. *Let Z be a pure jump Lévy process on $\mathbb{R}^d$ with Lévy measure ν.*

(a) *Let Z be rotationally symmetric and define for $r > 0$*

$$\psi(r) = \int_{\{|z|\leq r\}} |z|^2 \nu(dz) \quad \textit{and} \quad \Phi(r) = \int_0^r \int_u^1 \frac{1}{\psi(s/4)}\, ds\, du.$$

If

$$\text{(i)} \quad \int_0^1 \frac{s}{\psi(s)}\, ds < \infty \quad \textit{and} \quad \text{(ii)} \quad \limsup_{r\to 0} \left(B(r) \int_r^1 \frac{1}{\psi(s/4)}\, ds \right) < \frac{2}{d}, \tag{19}$$

then there exists a constant $c > 0$ such that for any $f \in B_b(\mathbb{R}^d)$, $x \in \mathbb{R}^d$ and $t > 0$,

$$\limsup_{y\to x} \frac{|P_t f(x) - P_t f(y)|}{\Phi(|x-y|)} \leq c\left(1 \vee \frac{1}{t}\right).$$

(b) *For an arbitrary pure jump Lévy process Z and $r > 0$ define*

$$\psi(r) = r^2 \inf_{|x|\leq r} \big(\nu \wedge (\delta_x * \nu)\big)\big(\mathbb{R}^d\big) \quad \textit{and} \quad \Phi(r) = \int_0^r \int_u^1 \frac{1}{\psi(s/2)}\, ds\, du. \tag{20}$$

If

$$\text{(i)} \quad \int_0^1 \frac{s}{\psi(s)}\, ds < \infty \quad \textit{and} \quad \text{(ii)} \quad \limsup_{r\to 0} \left(B(r) \int_r^1 \frac{1}{\psi(s/2)}\, ds \right) < \frac{1}{2}, \tag{21}$$

then there exists a constant $c > 0$ such that for any $f \in B_b(\mathbb{R}^d)$, $x \in \mathbb{R}^d$ and $t > 0$,

$$\limsup_{y\to x} \frac{|P_t f(x) - P_t f(y)|}{\Phi(|x-y|)} \leq c\left(1 \vee \frac{1}{t}\right).$$

Remark 4.2.

(a) Recently, the authors of [3] introduced the *local coupling property* for Markov processes and proved that it is equivalent to the following condition

$$\lim_{y\to x} \big\| p_t(x,\cdot) - p_t(y,\cdot) \big\|_{\mathrm{Var}} = 0 \quad \text{for all } x \in \mathbb{R}^d \text{ and } t > 0, \tag{22}$$

where $p_t(x,\cdot)$ is the transition probability of the Markov process $X := (X_t)_{t\ge 0}$. Let $(P_t)_{t\ge 0}$ be the transition semigroup associated of the process X. Since

$$\bigl\| p_t(x,\cdot) - p_t(y,\cdot)\bigr\|_{\mathrm{Var}} = \sup_{\|f\|_\infty \le 1} \bigl|P_t f(x) - P_t f(y)\bigr|,$$

(22) implies that $(P_t)_{t\ge 0}$ is a strong Feller semigroup, that is, for any $t > 0$, P_t maps the set of bounded measurable functions into the set of bounded continuous functions. On the other hand, using [33], Chapter 1, Propositions 5.8 and 5.12, we can deduce (22) from the strong Feller property. Therefore, the local coupling property and the strong Feller property coincide for Markov semigroups.

(b) Since we have $\Phi(0) = 0$, $\Phi'(r) > 0$ and $\Phi''(r) < 0$ on $(0,\infty)$, $(x, y) \mapsto \Phi(|x-y|)$ is a distance function in $\mathbb{R}^d$. Therefore, Theorem 4.1 guarantees the regularity of the semigroup associated with the SDE (1) which in turn implies the strong Feller property.

(c) If $b \equiv 0$, that is, if $X = Z$ is a pure jump Lévy process, the conditions (19.ii) and (20.ii) are trivially satisfied. Theorem 4.1(a) seems to be new even for Lévy processes. If $X = Z$ is a rotationally symmetric Lévy process, Theorem 4.1(a) also extends [2], Theorem 2.2, where only the one-dimensional case is discussed.

(d) According to [27], Example 1.2, Theorem 4.1(b) holds for any Lévy measure ν satisfying

$$\nu(dz) \ge \mathbb{1}_{(0,1]}(z_1)\frac{c}{|z|^{d+\alpha}}\,dz$$

for some $\alpha \in (0,2)$ and $c > 0$. If we take, for example, $\nu(dz) = \mathbb{1}_{(0,1]}(z_1)\frac{c}{|z|^{d+\alpha}}\,dz$, then ν is a Lévy measure with zero symmetric part, and such settings are not covered by [3], Theorem 7, which treats only the one-dimensional case.

In the proof of Theorem 4.1(b), we will apply the coupling operator $\widetilde{L}$ from Section 3.1 with $\eta = \frac{1}{2}$. We begin with the following simple estimate.

Lemma 4.3. *Let $\widetilde{L}$ be the coupling operator given by the jump system (10) with $\eta = \frac{1}{2}$. Pick $f \in C[0,2] \cap C^2(0,2]$ such that $f(0) = 0$, $f' \ge 0$, $f'' \le 0$ and f'' is increasing on $(0,2]$. For any $x, y \in \mathbb{R}^d$ with $0 < |x-y| \le 1$,*

$$\widetilde{L} f\bigl(|x-y|\bigr) \le f'\bigl(|x-y|\bigr)\frac{\langle b(x)-b(y), x-y\rangle}{|x-y|} + \frac{2}{d} f''\bigl(2|x-y|\bigr)\int_{\{|z|\le |x-y|/2\}} |z|^2 \nu(dz).$$

Proof. Let $f \in C[0,2] \cap C^2(0,2]$ with $f(0) = 0$ and $x, y \in \mathbb{R}^d$ with $0 < |x-y| \le 1$.

$$\begin{aligned}\widetilde{L} f\bigl(|x-y|\bigr) &= f'\bigl(|x-y|\bigr)\frac{\langle b(x)-b(y), x-y\rangle}{|x-y|}\\ &\quad + \int_{\{|z|\le |x-y|/2\}}\Bigl[f\bigl(\bigl|(x-y) + \bigl(z - R_{x,y}(z)\bigr)\bigr|\bigr) - f\bigl(|x-y|\bigr)\end{aligned}$$

$$
\begin{aligned}
&- f'\big(|x-y|\big)\frac{\langle x-y, z\rangle}{|x-y|}\mathbb{1}_{(0,1)}\big(|z|\big)\\
&+ f'\big(|x-y|\big)\frac{\langle x-y, R_{x,y}(z)\rangle}{|x-y|}\mathbb{1}_{(0,1)}\big(|z|\big)\Big]\nu(dz),
\end{aligned}
$$

where we use $|R_{x,y}(z)| = |z|$. Observe that $\widetilde{L}f(|x-y|) = \widetilde{L}f(|y-x|)$ and $R_{xy}(z) = R_{yx}(z)$. This allows us to symmetrize the above expression and we get

$$
\begin{aligned}
\widetilde{L}f\big(|x-y|\big) &= f'\big(|x-y|\big)\frac{\langle b(x)-b(y), x-y\rangle}{|x-y|}\\
&\quad + \frac{1}{2}\int_{\{|z|\le|x-y|/2\}}\big[f\big(\big|(x-y)+\big(R_{x,y}(z)-z\big)\big|\big)\\
&\quad + f\big(\big|(x-y)+\big(z-R_{x,y}(z)\big)\big|\big) - 2f\big(|x-y|\big)\big]\nu(dz)\\
&= f'\big(|x-y|\big)\frac{\langle b(x)-b(y), x-y\rangle}{|x-y|}\\
&\quad + \frac{1}{2}\int_{\{|z|\le|x-y|/2\}}\bigg[f\bigg(|x-y|\bigg(1+\frac{2\langle x-y, z\rangle}{|x-y|^2}\bigg)\bigg)\\
&\quad + f\bigg(|x-y|\bigg(1-\frac{2\langle x-y, z\rangle}{|x-y|^2}\bigg)\bigg) - 2f\big(|x-y|\big)\bigg]\nu(dz);
\end{aligned}
$$

to see the last equality, use that $|z| \le |x-y|/2$.

We assume now, in addition, that $f' \ge 0$, $f'' \le 0$ and f'' is increasing. For any $\delta \in [0, r]$,

$$
f(r+\delta) + f(r-\delta) - 2f(r) = \int_r^{r+\delta}\int_{s-\delta}^{s} f''(u)\,du\,ds \le f''(r+\delta)\delta^2. \tag{23}
$$

Using again the fact that ν is rotationally symmetric, we get

$$
\begin{aligned}
&\widetilde{L}f\big(|x-y|\big)\\
&\le f'\big(|x-y|\big)\frac{\langle b(x)-b(y), x-y\rangle}{|x-y|} + 2f''\big(2|x-y|\big)\int_{\{|z|\le|x-y|/2\}}\frac{\langle x-y, z\rangle^2}{|x-y|^2}\nu(dz)\\
&= f'\big(|x-y|\big)\frac{\langle b(x)-b(y), x-y\rangle}{|x-y|} + 2f''\big(2|x-y|\big)\int_{\{|z|\le|x-y|/2\}}|z_1|^2\nu(dz)\\
&= f'\big(|x-y|\big)\frac{\langle b(x)-b(y), x-y\rangle}{|x-y|} + \frac{2}{d}f''\big(2|x-y|\big)\int_{\{|z|\le|x-y|/2\}}|z|^2\nu(dz).
\end{aligned}
$$

□

Proof of Theorem 4.1(a). Fubini's theorem shows

$$
\Phi(r) = \int_0^r\int_u^1\frac{ds\,du}{\psi(s/4)} = \int_0^1\frac{s\wedge r}{\psi(s/4)}\,ds, \quad r>0.
$$

Since $r \mapsto \psi(r)$ is increasing, Φ is well defined under (19.i); moreover, $\Phi'(r) = \int_r^1 \frac{ds}{\psi(s/4)} > 0$, $\Phi''(r) = -\frac{1}{\psi(r/4)} < 0$, and Φ'' is increasing since $r \mapsto \psi(r)$ is increasing. According to Lemma 4.3 and (19.ii), there exist constants $\epsilon_0 \in (0,1]$ and $c_0 > 0$ such that for $x, y \in \mathbb{R}^d$ with $0 < |x-y| \le \epsilon_0$,

$$\widetilde{L}\Phi\big(|x-y|\big) \le -c_0. \tag{24}$$

Let $(X_t, Y_t)_{t\ge 0}$ be the coupling process constructed at the end of Section 3.1. Denote by $\widetilde{\mathbb{P}}^{(x,y)}$ and $\widetilde{\mathbb{E}}^{(x,y)}$ the probability law and the expectation of $(X_t, Y_t)_{t\ge 0}$ such that $(X_0, Y_0) = (x, y)$, respectively. For $\epsilon_0 \in (0,1]$ as above and any $n \ge 1$ we set

$$\sigma_{\epsilon_0} := \inf\big\{t \ge 0 : |X_t - Y_t| > \epsilon_0\big\},$$
$$\tau_n := \inf\big\{t \ge 0 : |X_t - Y_t| \le 1/n\big\}.$$

It is clear that $\lim_{n\to\infty} \tau_n = \tau$, where τ is the coupling time.

Let $x, y \in \mathbb{R}^d$ with $0 < |x-y| < \epsilon_0$ and choose n so large that $|x-y| > 1/n$. Because of the monotonicity of Φ, Dynkin's formula and (24), we get for all $t > 0$,

$$\begin{aligned}
0 &\le \Phi(\epsilon_0)\mathbb{P}^{(x,y)}(\sigma_{\epsilon_0} < \tau_n \wedge t) \\
&\le \widetilde{\mathbb{E}}^{(x,y)} \Phi\big(|X_{t\wedge\tau_n\wedge\sigma_{\epsilon_0}} - Y_{t\wedge\tau_n\wedge\sigma_{\epsilon_0}}|\big) \\
&= \Phi\big(|x-y|\big) + \widetilde{\mathbb{E}}^{(x,y)}\bigg(\int_0^{t\wedge\tau_n\wedge\sigma_{\epsilon_0}} \widetilde{L}\Phi\big(|X_s - Y_s|\big)\,ds\bigg) \\
&\le \Phi\big(|x-y|\big) - c_0\widetilde{\mathbb{E}}^{(x,y)}(t\wedge\tau_n\wedge\sigma_{\epsilon_0}).
\end{aligned}$$

Rearranging this inequality and letting $t \to \infty$ yields

$$c_0\widetilde{\mathbb{E}}^{(x,y)}(\tau_n \wedge \sigma_{\epsilon_0}) + \Phi(\epsilon_0)\widetilde{\mathbb{P}}^{(x,y)}(\sigma_{\epsilon_0} < \tau_n) \le \Phi\big(|x-y|\big).$$

Therefore, we can use Markov's inequality and get

$$\begin{aligned}
\widetilde{\mathbb{P}}^{(x,y)}(\tau_n > t) &\le \widetilde{\mathbb{P}}^{(x,y)}(\tau_n \wedge \sigma_{\epsilon_0} > t) + \widetilde{\mathbb{P}}^{(x,y)}(\sigma_{\epsilon_0} < \tau_n) \\
&\le \frac{\widetilde{\mathbb{E}}^{(x,y)}(\tau_n \wedge \sigma_{\epsilon_0})}{t} + \frac{\Phi(|x-y|)}{\Phi(\epsilon_0)} \le \Phi\big(|x-y|\big)\bigg[\frac{1}{tc_0} + \frac{1}{\Phi(\epsilon_0)}\bigg].
\end{aligned}$$

Letting $n \to \infty$, we find that

$$\widetilde{\mathbb{P}}^{(x,y)}(\tau > t) \le \Phi\big(|x-y|\big)\bigg[\frac{1}{tc_0} + \frac{1}{\Phi(\epsilon_0)}\bigg].$$

Finally, we have for any $f \in B_b(\mathbb{R}^d)$, $x \in \mathbb{R}^d$ and $t > 0$,

$$\limsup_{y\to x} \frac{|P_t f(x) - P_t f(y)|}{\Phi(|x-y|)} = \limsup_{y\to x} \frac{|\widetilde{\mathbb{E}}^{(x,y)}[f(X_t) - f(Y_t)]|}{\Phi(|x-y|)}$$

$$
\begin{aligned}
&= \limsup_{y\to x} \frac{|\widetilde{\mathbb{E}}^{(x,y)}[(f(X_t)-f(Y_t))\mathbb{1}_{\{\tau>t\}}]|}{\Phi(|x-y|)} \\
&\le 2\|f\|_\infty \limsup_{y\to x} \frac{\widetilde{\mathbb{P}}^{(x,y)}(\tau>t)}{\Phi(|x-y|)} \\
&\le 2\left[\frac{1}{tc_0}+\frac{1}{\Phi(\epsilon_0)}\right] \le c_1\left(1\vee\frac{1}{t}\right). \qquad \square
\end{aligned}
$$

In order to prove Theorem 4.1(b), we use the refined basic coupling from Section 3.2.

Lemma 4.4. *Let $\kappa>0$ be the constant from* (13) *and denote by $\widetilde{L}$ the coupling operator given by the jump system* (13). *Let $f\in C[0,2\kappa]\cap C^2(0,2\kappa]$ such that $f(0)=0$, $f'\ge 0$, $f''\le 0$ and f'' is increasing on $(0,2\kappa]$. For all $x,y\in\mathbb{R}^d$ with $0<|x-y|\le\kappa$,*

$$
\widetilde{L}f\big(|x-y|\big) \le f'\big(|x-y|\big)\frac{\langle b(x)-b(y),x-y\rangle}{|x-y|} + \frac{1}{2}\mu_{(x-y)}\big(\mathbb{R}^d\big)|x-y|^2 f''\big(2|x-y|\big),
$$

where $\mu_x(dz)=[\nu\wedge(\delta_x*\nu)](dz)$ for all $x\in\mathbb{R}^d$.

Proof. If $x,y\in\mathbb{R}^d$ satisfy $0<|x-y|\le\kappa$, then we have $(x-y)_\kappa=(x-y)$. Using the jump system (13), we get for any $x,y\in\mathbb{R}^d$ with $0<|x-y|\le\kappa$

$$
\begin{aligned}
\widetilde{L}f\big(|x-y|\big) &= f'\big(|x-y|\big)\frac{\langle b(x)-b(y),x-y\rangle}{|x-y|} \\
&\quad + \left[\frac{1}{2}\mu_{(y-x)}\big(\mathbb{R}^d\big)\big(f\big(2|x-y|\big)-f\big(|x-y|\big)\big) - \frac{1}{2}\mu_{(y-x)}\big(\mathbb{R}^d\big)f\big(|x-y|\big)\right] \\
&= f'\big(|x-y|\big)\frac{\langle b(x)-b(y),x-y\rangle}{|x-y|} \\
&\quad + \frac{1}{2}\mu_{(x-y)}\big(\mathbb{R}^d\big)\big[f\big(2|x-y|\big)-2f\big(|x-y|\big)\big],
\end{aligned}
$$

where we use the identity $\mu_{(x-y)}(\mathbb{R}^d)=\mu_{(y-x)}(\mathbb{R}^d)$. This, together with the assumptions on f and (23), yields the assertion. $\square$

Theorem 4.1(b) can now be proved along the same lines as Theorem 4.1(a): all we have to do is to use Lemma 4.4 instead of Lemma 4.3. Since the arguments are similar, we leave the details to the reader.

4.2. Successful coupling

Let $X:=(X_t)_{t\ge0}$ be the unique strong solution to (1). A coupling $(X_t,Y_t)_{t\ge0}$ of the process X is called *successful*, if the coupling time τ of $(X_t,Y_t)_{t\ge0}$ defined by (12) is almost surely finite.

If the process X has a successful coupling, then we say that the process X has the *coupling property*.

Theorem 4.5. *Let Z be a pure jump Lévy process on $\mathbb{R}^d$ with Lévy measure ν such that*

$$\int_0^1 \frac{s}{\psi(s)}\,ds < \infty, \tag{25}$$

where ψ is defined in Theorem 4.1. If the drift $b(x)$ satisfies

$$\langle b(x) - b(y), x - y\rangle \le 0, \quad x, y \in \mathbb{R}^d, \tag{26}$$

then the process X has the coupling property.

Proof. For simplicity, we only consider rotationally symmetric Z. Let $\widetilde{L}$ be the coupling operator given by the jump system (10) with $\eta = \frac{1}{2}$. Fix $r_0 > 0$, and $f \in C[0, 2r_0] \cap C^2(0, 2r_0]$ such that $f(0) = 0$, $f' \ge 0$, $f'' \le 0$ and f'' is increasing on $(0, 2r_0]$. Following the argument of Lemma 4.3 and using (26), we get for any $x, y \in \mathbb{R}^d$ with $0 < |x - y| \le r_0$,

$$\widetilde{L} f\big(|x - y|\big) \le \frac{2}{d} f''\big(2|x - y|\big) \int_{\{|z| \le |x-y|/2\}} |z|^2 \nu(dz). \tag{27}$$

As in the proof of Theorem 4.1(a), let $(X_t, Y_t)_{t\ge0}$ be the coupling process constructed at the end of Section 3.1. Denote by $\widetilde{\mathbb{P}}^{(x,y)}$ and $\widetilde{\mathbb{E}}^{(x,y)}$ the probability law and the expectation of $(X_t, Y_t)_{t\ge0}$ such that $(X_0, Y_0) = (x, y)$. For $n \ge 1$, we set

$$\sigma_n := \inf\big\{t \ge 0 : |X_t - Y_t| > n\big\}, \qquad \tau_n := \inf\big\{t \ge 0 : |X_t - Y_t| \le 1/n\big\}.$$

It is clear that $\lim_{n\to\infty} \tau_n = \tau$ where τ is the coupling time.

Fix $x, y \in \mathbb{R}^d$ and pick n and m so large that $1/n \le |x - y| \le m$. Set

$$\Phi_m(r) = \int_0^r \int_u^{2m} \frac{ds\,du}{\psi(s/4)}, \quad r > 0.$$

Following the proof of Theorem 4.1(a) and applying (27) to the function $f(r) = \Phi_m(r)$, we get

$$\widetilde{\mathbb{E}}^{(x,y)}(\tau_n \wedge \sigma_m) \le \frac{\Phi_m(|x - y|)}{c_0},$$

where $c_0 > 0$ does not depend on x, y, n and m. On the other hand, applying (27) to the function $f(r) = r$ and following again the proof of Theorem 4.1(a), we obtain

$$\widetilde{\mathbb{P}}^{(x,y)}(\sigma_m < \tau_n) \le \frac{|x - y|}{m}.$$

Therefore, for any $x, y \in \mathbb{R}^d$ and $t > 0$,

$$\begin{aligned}\widetilde{\mathbb{P}}^{(x,y)}(\tau_n > t) &\le \widetilde{\mathbb{P}}^{(x,y)}(\tau_n \wedge \sigma_m > t) + \widetilde{\mathbb{P}}^{(x,y)}(\sigma_m < \tau_n) \\ &\le \frac{\widetilde{\mathbb{E}}^{(x,y)}(\tau_n \wedge \sigma_m)}{t} + \frac{|x-y|}{m} \\ &\le \frac{\Phi_m(|x-y|)}{tc_0} + \frac{|x-y|}{m}.\end{aligned}$$

Letting $n \to \infty$, then $t \to \infty$ and finally $m \to \infty$, we see

$$\widetilde{\mathbb{P}}^{(x,y)}(\tau = \infty) = 0.$$

This finishes the proof. □

Remark 4.6.

(a) Condition (26) is necessary for the coupling property for Ornstein–Uhlenbeck processes driven by symmetric α-stable processes, see [37], Remark 1.3(2).

(b) For a pure jump Lévy process, the condition (26) is trivially satisfied. Note that Theorem 4.5 is new even for Lévy processes. In particular, Theorem 4.5 and the coupling used in its proof indicate that both coupling by reflection for rotationally symmetric Lévy noises with the parameter $\eta = \frac{1}{2}$ in Section 3.1 and the refined basic coupling for general Lévy noises in Section 3.2 are successful, whenever (25) is satisfied. For symmetric α-stable processes we have $\psi(r) = c_\alpha r^{2-\alpha}$, that is, (25) holds; this is also true for the process from Remark 4.2(d).

5. Comparison of coupling operators & coupling operators for SDEs with multiplicative Lévy noise

5.1. Comparison of coupling operators for Lévy processes mentioned in Section 3

The construction of (17) is motivated by optimal coupling of Gaussian distributions, see [17], Theorem 1.4, or [19], Section 3, on the one hand, and by reflection coupling for Brownian motion, see [25], Section 2, or [19], Section 2, on the other hand. Folklore wisdom from the theory of optimal transport tells us that we should use most of the common mass of two probability distributions if we want to obtain a coupling with nice properties. In this sense, the first row in (17) is a natural choice, see [28], Section 2.1, for further details, and this is also the idea behind basic coupling (14). The problem is, how we should use the remaining mass.

If the Lévy measure is rotationally symmetric, we use reflection of the remaining mass, cf. the second row of (17). This approach is essentially due to [28], Section 2.2, where $q_0(|z|) = q(|z|)\mathbb{1}_{\{|z|<m\}}$ for some large $m \gg 1$. For Lévy processes which are subordinate to a Brownian motion, [5] shows that this type of coupling with $m = \infty$ is indeed a Markovian maximal coupling. For further discussions in this direction, we refer our readers to [5], Section 5.

In a general setting, one can try to use independent coupling with the remaining mass; this approach often has poor properties. Intuitively, a much better solution should be to couple the remaining mass synchronously, but it turns out that such a construction does not produce a coupling. In the preliminary construction of the refined basic coupling (14), the two marginal processes jump to the same place only with *half* of the maximal probability (see the first row in (14)), while with the other half we perform a transformation which doubles the distance between the two marginal processes (see the second row in (14)). With the remaining probability we let the marginal processes move synchronously, see the third row in (14). With a view towards the refined basic coupling (14), it seems sometimes to be better not to have the marginals jump to the same place with the *maximal possible* probability, but to use some of the mass for coupling the marginals in a more convenient way.

In what follows, let f be a non-decreasing and concave function on $[0,\infty)$ such that $f(0)=0$. We will compare $\widetilde{L}f$ with the three coupling operators introduced in Section 3. There are several good reasons to compare the corresponding coupling operators $\widetilde{L}f$: (i) the estimates for $\widetilde{L}f$ in Lemmas 4.3 and 4.4 play a crucial role in the study of the regularity and coupling properties in Theorem 4.1 and Theorem 4.5, respectively. Roughly speaking, the smaller $\widetilde{L}f$ is, the better become results like Theorem 4.1 and 4.5. (ii) recall from [9], Definition 2.3, that a coupling operator $\bar{L}_f$ is said to be f*-optimal* (f is a non-decreasing and concave function on $[0,\infty)$ such that $f(0)=0$), if

$$\bar{L}_f f\big(|x_1-x_2|\big)=\inf_{\widetilde{L}}\widetilde{L}f\big(|x_1-x_2|\big),\quad x_1,x_2\in\mathbb{R}^d,$$

where the infimum ranges over all coupling operators $\widetilde{L}$. This means, in particular, that the infimum is attained for the coupling operator $\bar{L}_f$. The f-optimal Markov operator has been efficiently used to consider the spectral gap or the exponential L^2-convergence of symmetric Markov processes, see [10], Chapter 2.4. Hence, it is natural to study optimal (in the sense of Chen) couplings for Lévy noise. In contrast to the diffusion or the birth-and-death process cases – see [9], Theorem 3.2 and Section 5 – there seems to be no general structure formula for coupling operators relating to Lévy noise. Therefore, we concentrate on the three couplings presented in Section 3.

Let $Z=(Z_t)_{t\ge 0}$ be a rotationally symmetric pure jump Lévy process whose Lévy measure is of the form $\nu(dz)=q(|z|)\,dz$ for some measurable function $q(r)\ge 0$, $r>0$. We consider the following two cases.

Case 1 (Jumps of infinite range). Denote by $\widetilde{L}_r$ the "coupling-by-reflection" operator with $\eta=\infty$, cf. Section 3.1, and by $\widetilde{L}_{r,b}$ the "combined reflection-and-basic" coupling operator constructed in Section 3.3 with $q_0(|z|)=q(|z|)$. For any $f\in C[0,\infty)\cap C^2(0,\infty)$ with $f(0)=0$ and $f'\ge 0$, and any $x,y\in\mathbb{R}^d$ with $x\ne y$, we have

$$\begin{aligned}
&\widetilde{L}_r f\big(|x-y|\big)\\
&\quad=\int_{\mathbb{R}^d}\bigg[f\big(\big|x-y+z-R_{x,y}(z)\big|\big)-f\big(|x-y|\big)-f'\big(|x-y|\big)\frac{\langle x-y,z\rangle}{|x-y|}\mathbb{1}_{(0,1)}\big(|z|\big)\\
&\qquad+f'\big(|x-y|\big)\frac{\langle x-y,R_{x,y}(z)\rangle}{|x-y|}\mathbb{1}_{(0,1)}\big(|z|\big)\bigg]q\big(|z|\big)\,dz.
\end{aligned}$$

Since $\widetilde{L}_r f(|x-y|) = \widetilde{L}_r f(|y-x|)$ and $R_{x,y}(z) = R_{y,x}(z)$, we can symmetrize the above expression, and get

$$
\begin{aligned}
&\widetilde{L}_r f\big(|x-y|\big) \\
&\quad = \frac{1}{2}\int_{\mathbb{R}^d} \big[f\big(\big|x-y+z-R_{x,y}(z)\big|\big) + f\big(\big|x-y+R_{x,y}(z)-z\big|\big) - 2f\big(|x-y|\big)\big] q\big(|z|\big)\,dz.
\end{aligned}
$$

For the other coupling operator, we get

$$
\begin{aligned}
&\widetilde{L}_{r,b} f\big(|x-y|\big) \\
&\quad = \int_{\mathbb{R}^d} \bigg[f\big(\big|x+z-y-z-(x-y)\big|\big) - f\big(|x-y|\big) - f'\big(|x-y|\big)\frac{\langle x-y, z\rangle}{|x-y|}\mathbb{1}_{(0,1)}\big(|z|\big) \\
&\qquad + f'\big(|x-y|\big)\frac{\langle x-y, z+(x-y)\rangle}{|x-y|}\mathbb{1}_{(0,1)}\big(\big|z+(x-y)\big|\big)\bigg] q\big(|z|\big)\wedge q\big(|z+x-y|\big)\,dz \\
&\qquad + \int_{\mathbb{R}^d} \bigg[f\big(\big|x+z-y-R_{x,y}(z)\big|\big) - f\big(|x-y|\big) - f'\big(|x-y|\big)\frac{\langle x-y, z\rangle}{|x-y|}\mathbb{1}_{(0,1)}\big(|z|\big) \\
&\qquad + f'\big(|x-y|\big)\frac{\langle x-y, R_{x,y}(z)\rangle}{|x-y|}\mathbb{1}_{(0,1)}\big(|z|\big)\bigg] \big[q\big(|z|\big) - q\big(|z|\big)\wedge q\big(|z+x-y|\big)\big]\,dz \\
&\quad = -\int_{\mathbb{R}^d} f\big(\big|x+z-y-R_{x,y}(z)\big|\big) q\big(|z|\big)\wedge q\big(|z+x-y|\big)\,dz \\
&\qquad + \int_{\mathbb{R}^d} \bigg[f\big(\big|x+z-y-R_{x,y}(z)\big|\big) - f\big(|x-y|\big) - f'\big(|x-y|\big)\frac{\langle x-y, z\rangle}{|x-y|}\mathbb{1}_{(0,1)}\big(|z|\big) \\
&\qquad + f'\big(|x-y|\big)\frac{\langle x-y, R_{x,y}(z)\rangle}{|x-y|}\mathbb{1}_{(0,1)}\big(|z|\big)\bigg] q\big(|z|\big)\,dz.
\end{aligned}
$$

In the last equality, we use that $q(z)\wedge q(z+x-y)\,dz$, $x \neq y$, is a finite measure on $\mathbb{R}^d$ as well as the following identity which one easily checks using (in the last line) the change of variables $z \rightsquigarrow R_{x,y}(z)$ and $R_{x,y}(x-y) = R_{x,y}(y-x)$:

$$
\begin{aligned}
&\int_{\mathbb{R}^d} \frac{\langle x-y, z+(x-y)\rangle}{|x-y|}\mathbb{1}_{(0,1)}\big(\big|z+(x-y)\big|\big) q\big(|z|\big)\wedge q\big(|z+x-y|\big)\,dz \\
&\quad = \int_{\mathbb{R}^d} \frac{\langle x-y, z\rangle}{|x-y|}\mathbb{1}_{(0,1)}\big(|z|\big) q(z-x+y)\wedge q\big(|z|\big)\,dz \\
&\quad = \int_{\mathbb{R}^d} \frac{\langle x-y, R_{x,y}(z)\rangle}{|x-y|}\mathbb{1}_{(0,1)}\big(|z|\big) q\big(|z|\big)\wedge q\big(|z+x-y|\big)\,dz.
\end{aligned}
$$

Using symmetry as above, we can swap the roles of x and y in the second term of the right-hand side for $\widetilde{L}_{r,b} f(|x-y|)$, and get

$$\begin{aligned}
&\widetilde{L}_{r,b} f\big(|x-y|\big) \\
&\quad = -\int_{\mathbb{R}^d} f\big(\big|x+z-y-R_{x,y}(z)\big|\big) q\big(|z|\big) \wedge q\big(|z+x-y|\big)\,dz \\
&\qquad + \frac{1}{2}\int_{\mathbb{R}^d} \big[f\big(\big|x-y+z-R_{x,y}(z)\big|\big) + f\big(\big|x-y+R_{x,y}(z)-z\big|\big) - 2f\big(|x-y|\big)\big] q\big(|z|\big)\,dz.
\end{aligned}$$

Comparing the formulae for $\widetilde{L}_r f(|x-y|)$ and $\widetilde{L}_{r,b} f(|x-y|)$ we see that

$$\widetilde{L}_{r,b} f\big(|x-y|\big) \le \widetilde{L}_r f\big(|x-y|\big), \quad x,y\in\mathbb{R}^d. \tag{28}$$

Case 2 (Jumps of finite range). Denote by $\widetilde{L}_r$ the "coupling-by-reflection" operator with $\eta = \frac{1}{2}$, cf. Section 3.1, and by $\widetilde{L}_b$ the "refined-basic-coupling" operator constructed in the same way as (13) with $\nu_1(dz) = \nu_2(dz) = \frac{1}{2}\mathbb{1}_{\{|z|\le|x-y|/2\}} q(|z|)\,dz$ in Section 3.2. For any $f \in C[0,\infty) \cap C^2(0,\infty)$ with $f(0) = 0$ and $f'' \le 0$, we get with the symmetrization argument for $\widetilde{L}_r$ from Case 1 that

$$\begin{aligned}
\widetilde{L}_r f\big(|x-y|\big) = \frac{1}{2}\int_{\{|z|\le|x-y|/2\}} \bigg[f\bigg(|x-y| + \frac{2\langle x-y,z\rangle}{|x-y|}\bigg) \\
+ f\bigg(|x-y| - \frac{2\langle x-y,z\rangle}{|x-y|}\bigg) - 2f\big(|x-y|\big)\bigg] q\big(|z|\big)\,dz
\end{aligned}$$

and

$$\begin{aligned}
\widetilde{L}_b f\big(|x-y|\big) = \frac{1}{2}\big[f\big(|x-y| - |x-y|\wedge\kappa\big) + f\big(|x-y| + |x-y|\wedge\kappa\big) - 2f\big(|x-y|\big)\big] \\
\times \int_{\mathbb{R}^d} \big[q\big(|z|\big)\mathbb{1}_{\{|z|\le|x-y|/2\}}\big] \wedge \big[q\big(|z+x-y|\big)\mathbb{1}_{\{|z+x-y|\le|x-y|/2\}}\big]\,dz.
\end{aligned}$$

By the mean value theorem and $f'' \le 0$, it is easy to see that

$$\widetilde{L}_r f\big(|x-y|\big) \le 0.$$

On the other hand, we have for all $x,y,z\in\mathbb{R}^d$

$$\begin{aligned}
0 &\le \big[q\big(|z|\big)\mathbb{1}_{\{|z|\le|x-y|/2\}}\big] \wedge \big[q\big(|z+x-y|\big)\mathbb{1}_{\{|z+x-y|\le|x-y|/2\}}\big] \\
&\le \big[q\big(|z|\big)\mathbb{1}_{\{|z|\le|x-y|/2\}}\big] \wedge \big[q\big(|z+x-y|\big)\mathbb{1}_{\{|z|\ge|x-y|/2\}}(z)\big] = 0 \quad \text{a.e.}
\end{aligned} \tag{29}$$

This shows that

$$\widetilde{L}_b f\big(|x-y|\big) = 0, \quad x,y\in\mathbb{R}^d.$$

Let $\widetilde{L}_{r,b}$ denote the "combined reflection-and-basic" coupling operator constructed in Section 3.3 with $q_0(|z|) = \mathbb{1}_{\{|z|\le|x-y|/2\}}q(|z|)$. Using (29) it is easy to see that

$$
\begin{aligned}
&\widetilde{L}_{r,b}f\big(|x-y|\big)\\
&= \int_{\mathbb{R}^d}\bigg[f\big(\big|x+z-y-z-(x-y)\big|\big) - f\big(|x-y|\big) - f'\big(|x-y|\big)\frac{\langle x-y,z\rangle}{|x-y|}\mathbb{1}_{(0,1)}\big(|z|\big)\\
&\quad + f'\big(|x-y|\big)\frac{\langle x-y,z+x-y\rangle}{|x-y|}\mathbb{1}_{(0,1)}\big(\big|z+(x-y)\big|\big)\bigg]\\
&\quad \times \big[q\big(|z|\big)\mathbb{1}_{\{|z|\le|x-y|/2\}}\big]\wedge\big[q\big(|z+x-y|\big)\mathbb{1}_{\{|z+x-y|\le|x-y|/2\}}\big]\,dz\\
&\quad + \int_{\{|z|\le|x-y|/2\}}\bigg[f\bigg(|x-y|+\frac{2\langle x-y,z\rangle}{|x-y|}\bigg) - f\big(|x-y|\big)\\
&\quad - f'\big(|x-y|\big)\frac{\langle x-y,z\rangle}{|x-y|}\mathbb{1}_{(0,1)}\big(|z|\big) + f'\big(|x-y|\big)\frac{\langle x-y,R_{x,y}(z)\rangle}{|x-y|}\mathbb{1}_{(0,1)}\big(|z|\big)\bigg]\\
&\quad \times \Big(q\big(|z|\big) - \big[q\big(|z|\big)\mathbb{1}_{\{|z|\le|x-y|/2\}}\big]\wedge\big[q\big(|z+x-y|\big)\mathbb{1}_{\{|z+x-y|\le|x-y|/2\}}\big]\Big)\,dz\\
&= \int_{\{|z|\le|x-y|/2\}}\bigg[f\bigg(|x-y|+\frac{2\langle x-y,z\rangle}{|x-y|}\bigg) - f\big(|x-y|\big)\\
&\quad - f'\big(|x-y|\big)\frac{\langle x-y,z\rangle}{|x-y|}\mathbb{1}_{(0,1)}\big(|z|\big) + f'\big(|x-y|\big)\frac{\langle x-y,R_{x,y}(z)\rangle}{|x-y|}\mathbb{1}_{(0,1)}\big(|z|\big)\bigg]q\big(|z|\big)\,dz.
\end{aligned}
$$

The symmetrization argument used in Case 1 allows us to change the roles of x and y, and we get

$$
\begin{aligned}
\widetilde{L}_{r,b}f\big(|x-y|\big) &= \frac{1}{2}\int_{\{|z|\le|x-y|/2\}}\bigg[f\bigg(|x-y|+\frac{2\langle x-y,z\rangle}{|x-y|}\bigg)\\
&\quad + f\bigg(|x-y|-\frac{2\langle x-y,z\rangle}{|x-y|}\bigg) - 2f\big(|x-y|\big)\bigg]q\big(|z|\big)\,dz\\
&= \widetilde{L}_r f\big(|x-y|\big).
\end{aligned}
$$

If $f \in C[0,\infty)\cap C^2(0,\infty)$ with $f(0)=0$ and $f''\le 0$, these calculations show that

$$
\widetilde{L}_{r,b}f\big(|x-y|\big) = \widetilde{L}_r f\big(|x-y|\big) \le \widetilde{L}_b f\big(|x-y|\big) = 0 \quad \text{for all } x,y\in\mathbb{R}^d.
$$

Remark 5.1. Lévy processes which are subordinate to a Brownian motions are particular examples of rotationally symmetric Lévy processes. Thus, the conclusion of Case 1 shows that the coupling defined by (17) is, for subordinated Brownian motions and from a coupling operator point-of-view, *optimal* among the three couplings mentioned in Section 3. In particular, according to the proof of Theorem 4.5, Remark 4.6 and (28), the coupling defined by (17) is successful for a large class of subordinated Brownian motions.

On the other hand, one essential point in the proof of Case 2 uses the fact that – if the Lévy has finite-range jumps – the jumping density disappears, $q(|z|) \wedge q(|z+x-y|) = 0$, for $x, y \in \mathbb{R}^d$ which are sufficiently far apart, i.e. $|x-y| \gg 1$. In this case, the second row of (17) disappears, and (17) essentially becomes (10). This illustrates the advantage of the refined basic coupling (13): it applies both to finite-range jumps and non-necessarily rotationally symmetric Lévy processes.

5.2. Coupling operators for SDEs with multiplicative Lévy noise

It is possible to extend the coupling idea explained in the previous sections to SDEs with *multiplicative* Lévy noise

$$dX_t = b(X_t)\,dt + \sigma(X_{t-})\,dZ_t, \quad X_0 = x \in \mathbb{R}^d, \tag{30}$$

where $b : \mathbb{R}^d \to \mathbb{R}^d$ is measurable, $\sigma : \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^d$ is continuous, and $Z = (Z_t)_{t \ge 0}$ is a pure jump Lévy process on $\mathbb{R}^d$ with Lévy measure ν. Since the drift term b is always coupled synchronously, we only need to consider how to couple multiplicative Lévy noise. The multiplicative term $\sigma(x)$ affects the jumps in a way that the jump height $\Delta Z_t = Z_t - Z_{t-}$ is not simply added to X_{t-} (as in the additive noise case) but it is first transformed by the matrix $\sigma(X_{t-})$ and then added. This means that in our coupling scheme (8) we have to

$$\text{replace } \Psi_i(z) \text{ by } \sigma(y)\Psi_i(z).$$

More precisely, for any $1 \le i < n+1 \le \infty$, let $\Psi_i : \mathbb{R}^d \to \mathbb{R}^d$ be a bijective continuous map and ν_i a nonnegative measure on $(\mathbb{R}^d, \mathscr{B}(\mathbb{R}^d))$ such that $\sum_{i=1}^n \nu_i \le \nu$. Now we change the general formula (8) for the basic coupling with additive noise to

$$(x, y) \longmapsto \begin{cases} \big(x + \sigma(x)z,\ y + \sigma(y)\Psi_i(z)\big), & \mu_{\nu_i,\Psi_i}(dz) \text{ for } 1 \le i < n+1; \\[2ex] \big(x + \sigma(x)z,\ y + \sigma(y)z\big), & \left(\nu - \displaystyle\sum_{i=1}^n \mu_{\nu_i,\Psi_i}\right)(dz). \end{cases} \tag{31}$$

As in the proof of Proposition 2.2, we can verify that (31) determines a coupling operator for the infinitesimal generator of the SDE (30) if (7) is satisfied. It is natural that in the presence of multiplicative Lévy noise, the maps $\Psi_i(z)$ should depend on the coefficient $\sigma(x)$. In view of the results from Sections 3.1 and 3.2, let us discuss the following examples.

Example 5.2 (Coupling by reflection for multiplicative Lévy noise). Assume that Z is a pure jump rotationally symmetric Lévy process in $\mathbb{R}^d$ with Lévy measure ν. Let $n = 2$, $\nu_1(dz) = \nu_2(dz) = \frac{1}{2}\mathbb{1}_{\{|z| \le \eta|x-y|\}}\nu(dz)$ for some $\eta \in (0, \infty]$,

$$\Psi_1(z) = \sigma(y)^{-1}\sigma(x)R_{x,y}(z) \quad \text{and} \quad \Psi_2(z) = \Psi_1^{-1}(z) = R_{x,y}\big(\sigma(x)^{-1}\sigma(y)z\big);$$

where $R_{x,y}(z)$ is the reflection operator defined in (9). It is easy to see from the rotational invariance of the Lévy measure ν and the properties of $R_{x,y}(z)$, that setting $\sigma(x) = \mathrm{id}_d$ reduces (31) to (10).

Example 5.3 (Refined basic coupling for multiplicative Lévy noise). Let Z be an arbitrary pure jump Lévy process in $\mathbb{R}^d$, $n = 2$ and $\nu_1 = \nu_2 = \nu/2$. For any $\kappa > 0$ and $x, y \in \mathbb{R}^d$ with $x \neq y$, let

$$\Psi_1(z) = \Psi_{\kappa,x,y}(z) := \sigma(y)^{-1}\big(\sigma(x)z + (x-y)_\kappa\big)$$

and

$$\Psi_2(z) = \Psi_1^{-1}(z) = \sigma(x)^{-1}\big(\sigma(y)z - (x-y)_\kappa\big).$$

Again, if $\sigma(x) = \mathrm{id}_d$, (31) becomes (13). This coupling was first introduced in [22] when studying the regularity of semigroups and the ergodicity of the solution to the SDE (30).

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