Bernoulli **26**(1), 2020, 323–351
https://doi.org/10.3150/19-BEJ1127

Weak convergence of quantile and expectile processes under general assumptions

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We show weak convergence of quantile and expectile processes to Gaussian limit processes in the space of bounded functions endowed with an appropriate semimetric which is based on the concepts of epi- and hypo- convergence as introduced in A. Bücher, J. Segers and S. Volgushev (2014), '*When Uniform Weak Convergence Fails: Empirical Processes for Dependence Functions and Residuals via Epi- and Hypographs*', *Annals of Statistics* **42**. We impose assumptions for which it is known that weak convergence with respect to the supremum norm generally fails to hold. For quantiles, we consider stationary observations, where the marginal distribution function is assumed to be strictly increasing and continuous except for finitely many points and to admit strictly positive – possibly infinite – left- and right-sided derivatives. For expectiles, we focus on independent and identically distributed (i.i.d.) observations. Only a finite second moment and continuity at the boundary points but no further smoothness properties of the distribution function are required. We also show consistency of the bootstrap for this mode of convergence in the i.i.d. case for quantiles and expectiles.

Keywords: epi- and hypo convergence; expectile process; quantile process; weak convergence

1. Introduction

Quantiles are fundamental parameters of a probability distribution which have various applications in statistics and econometrics (Koenker [8]) as well as in finance (McNeil, Frey and Embrechts [9]). For distributions with finite first moments, expectiles are defined as minimizers of a weighted quadratic loss, similarly to quantiles which minimize a weighted absolute loss. Expectiles were introduced in Newey and Powell [11] and have found renewed interest as a coherent, elicitable class of risk measures (Bellini *et al.* [1], Ziegel [15]).

The asymptotic properties of sample quantiles and expectiles have been addressed in detail under suitable conditions. For quantiles, differentiability of the distribution function at the quantile with positive derivative implies asymptotic normality of the empirical quantile, and under a continuity assumption on the density one obtains weak convergence of the quantile process to a Gaussian limit process in the space of bounded functions with the supremum distance from the functional delta method (van der Vaart [13]). However, without the existence of a positive derivative of the distribution function at the quantile, the weak limit will be non-normal (Knight [7]), and thus process convergence to a Gaussian limit with respect to the supremum distance cannot hold true.

Similarly, for a distribution with finite second moment, the empirical expectile is asymptotically normally distributed if the distribution function is continuous at the expectile, but non-normally distributed otherwise (Holzmann and Klar [5]). For continuous distribution functions, process convergence of the empirical expectile process in the space of continuous functions also holds true, but for discontinuous distribution functions this can no longer be valid.

In this paper, we discuss convergence of expectile and quantile processes under more general conditions. We show that the expectile process of independent and identically distributed (i.i.d.) observations converges to a Gaussian limit in the semimetric space of bounded functions endowed with the hypi-semimetric as recently introduced in Bücher, Segers and Volgushev [2] under the assumption of a finite second moment only. Since the Gaussian limit process is discontinuous in general while the empirical expectile process is continuous, this convergence cannot hold with respect to the supremum distance. As we will see, the hypi-semimetric is appropriate in this situation. The discussion in Molchanov [10], p. 377, and in Bücher, Segers and Volgushev [2] relates the hypi-semimetric to the Skorohod M_2 metric for càdlàg functions, indicating that our results are true in this metric as well. Further, we consider quantile processes for general stationary and ergodic sequences. If a Donsker theorem for the associated empirical process of the stationary sequence is satisfied (Dehling, Durieu and Volny [4]), and if its marginal distribution function is strictly increasing, continuous except for finitely many points and if it admits strictly positive – possibly infinite – left- and right-sided derivatives, then the quantile process converges weakly to a Gaussian limit with respect to the hypi-semimetric. These limit theorems still imply weak convergence of important statistics such as Kolmogorov–Smirnov and Cramér–von Mises type statistics. We also show consistency of the n out of n bootstrap for i.i.d. observations for both expectile and quantile processes.

The paper is organized as follows. In Section 2, we briefly introduce weak convergence under the hypi-semimetric and derive the limit results mentioned above. Section 3 contains a short simulation study, which in particular illustrates the discontinuity of the limit process in the expectile case. Section 4 contains an outline of the proofs of the main results as well as details for the most relevant steps. In the supplement (Zwingmann and Holzmann [16]), we provide the remaining technical proofs, and also carry our results over to Skorohod M_2-convergence.

2. Weak convergence of quantile and expectile processes under the hypi-semimetric

2.1. Weak convergence under the hypi-semimetric

Let us briefly discuss the concept of hypi-convergence as introduced by Bücher, Segers and Volgushev [2]. Let $(\mathbb{T}, \mathrm{d})$ be a compact, separable metric space, and let $\ell^\infty(\mathbb{T})$ denote the space of all bounded functions $h : \mathbb{T} \to \mathbb{R}$. The lower- and upper-semicontinuous hulls of $h \in \ell^\infty(\mathbb{T})$ are defined by

$$h_\wedge(t) = \lim_{\varepsilon \searrow 0} \inf\{h(t') \mid \mathrm{d}(t, t') < \varepsilon\}, \qquad h_\vee(t) = \lim_{\varepsilon \searrow 0} \sup\{h(t') \mid \mathrm{d}(t, t') < \varepsilon\} \tag{2.1}$$

and satisfy $h_\wedge, h_\vee \in \ell^\infty(\mathbb{T})$ as well as $h_\wedge \le h \le h_\vee$. A sequence $h_n \in \ell^\infty(\mathbb{T})$ hypi-converges to a limit $h \in \ell^\infty(\mathbb{T})$, if it both epi-converges to $h_\wedge$, that is,

$$\begin{aligned} \text{for all } t, t_n \in \mathbb{T} \text{ with } t_n \to t: &\quad h_\wedge(t) \le \liminf_{n\to\infty} h_n(t_n) \\ \text{for all } t \in \mathbb{T} \text{ there exist } t_n \in \mathbb{T},\ t_n \to t: &\quad h_\wedge(t) = \lim_{n\to\infty} h_n(t_n), \end{aligned} \tag{2.2}$$

and hypo-converges to $h_\vee$, that is,

$$\begin{aligned} \text{for all } t, t_n \in \mathbb{T} \text{ with } t_n \to t: &\quad \limsup_{n\to\infty} h_n(t_n) \le h_\vee(t) \\ \text{for all } t \in \mathbb{T} \text{ there exist } t_n \in \mathbb{T},\ t_n \to t: &\quad \lim_{n\to\infty} h_n(t_n) = h_\vee(t). \end{aligned} \tag{2.3}$$

The limit function h is only determined in terms of its lower- and upper-semicontinuous hulls. Indeed, there is a semimetric, denoted by $\mathrm{d}_{\mathrm{hypi}}$, so that the convergence in (2.2) and (2.3) is equivalent to $\mathrm{d}_{\mathrm{hypi}}(h_n, h) \to 0$, see Bücher, Segers and Volgushev [2] for further details. To transfer the concept of weak convergence from metric to semimetric spaces, Bücher, Segers and Volgushev [2] consider the space $L^\infty(\mathbb{T})$ of equivalence classes $[h] = \{g \in \ell^\infty(\mathbb{T}) \mid \mathrm{d}_{\mathrm{hypi}}(h, g) = 0\}$. The convergence of a sequence of random elements (Y_n) in $g \in \ell^\infty(\mathbb{T})$ to a Borel-measurable Y is defined by weak convergence of $([Y_n])$ to $[Y]$ in the metric space $(L^\infty(\mathbb{T}), \mathrm{d}_{\mathrm{hypi}})$ in the sense of Hofmann-Jørgensen, see van der Vaart and Wellner [14] and Bücher, Segers and Volgushev [2]. Ordinary weak convergence of real-valued random variables will be denoted by $\xrightarrow{\mathcal{L}}$.

2.2. Convergence of the quantile process

We shall denote the α-quantile, $\alpha \in (0, 1)$, of a distribution function F by

$$F^{\mathrm{Inv}}(\alpha) = q_\alpha = \inf\{x \in \mathbb{R} : F(x) \ge \alpha\}.$$

The following Assumption A will imply semi-Hadamard differentiability w.r.t. the hypi-semimetric of the map which takes a function to its quasi-inverse. From the functional delta method in the version of Theorem B.7 in Bücher, Segers and Volgushev [2], a Donsker theorem for the empirical process then implies weak convergence of the quantile process.

In the following, we shall say that a function h is *càdlàg in a point* x of its domain if h is right continuous in x with existing left-sided limit; similarly for *làdcàg*. The space of functions which are càdlàg in every point of an interval $I \subset \mathbb{R}$ are denoted by $\mathcal{D}(I)$; $\mathcal{C}(I)$ denotes all continuous functions on I.

Assumption A. For given $0 < \alpha_l < \alpha_u < 1$ and $\varepsilon > 0$, the distribution function F is strictly increasing on $[q_{\alpha_l} - \varepsilon, q_{\alpha_u} + \varepsilon]$ and is continuous except at finitely many points $\{y_1, \dots, y_r\}$. In addition, F admits right- and left-sided derivatives – which may be infinite – at any point of $(q_{\alpha_l} - \varepsilon, q_{\alpha_u} + \varepsilon)$, that is

$$\partial^+(F)(q) = \lim_{h\to 0, h>0} \frac{F(q+h) - F(q)}{h} \quad \text{and} \quad \partial^-(F)(q) = \lim_{h\to 0, h>0} \frac{F(q) - F(q-h)}{h}$$

exist in $[0, \infty]$ for any $q \in (q_{\alpha_l} - \varepsilon, q_{\alpha_u} + \varepsilon)$. Further, both functions $q \mapsto \partial^+(F)(q)$ and $q \mapsto \partial^-(F)(q)$ are bounded from below by some constant $c > 0$, are càdlàg or làdcàg in every point except at $\{y_1, \dots, y_r\}$, have right- and left-sided limits in $\{y_1, \dots, y_r\}$ and are continuous in q_{α_l} and q_{α_u}.

Remark (Discussion of Assumption A). Assumption A is rather general, and contains the standard case of a continuously differentiable distribution function with strictly positive derivative (van der Vaart [13]). An example of an absolutely continuous distribution function F for which the derivative is unbounded can be obtained by gluing together in $\frac{1}{4}$ the function $x \mapsto -\sqrt{-x + \frac{1}{4}} + \frac{1}{2}$, $x \in [0, \frac{1}{4}]$ and a normal distribution with mean $\frac{1}{4}$ and variance 1. However, a strictly increasing, singular distribution function does not satisfy Assumption A.

For a compact subset $K \subset \mathbb{R}$, we denote by

$$\|\varphi\| = \|\varphi\|_K = \sup_{\tau \in K} \big|\varphi(\tau)\big|, \quad \varphi \in \ell^\infty(K),$$

the supremum norm on $\ell^\infty(K)$. For the subset $\mathcal{D}_0 \subseteq \mathcal{D}([q_{\alpha_l}, q_{\alpha_u}])$ of non-decreasing càdlàg functions consider the map

$$\Phi : \big(\mathcal{D}_0, \|\cdot\|\big) \longrightarrow \big(\ell^\infty\big([\alpha_l, \alpha_u]\big), \mathrm{d}_{\mathrm{hypi}}\big), \qquad \Phi(h) = h^{\mathrm{Inv}}. \tag{2.4}$$

We denote the set of discontinuities of a function $h \in \mathcal{D}([q_{\alpha_l}, q_{\alpha_u}])$ by $\mathrm{Dsc}(h)$. Further for $s \in \mathbb{R}$ we write s^{-1} instead of $\frac{1}{s}$. The following is the main technical result of this section.

Lemma 1. *Under Assumption A the map Φ is semi-Hadamard differentiable with respect to* $\mathrm{d}_{\mathrm{hypi}}$ *in F tangentially to*

$$\mathbb{W}_F = \big\{\varphi \in \mathcal{D}\big([q_{\alpha_l}, q_{\alpha_u}]\big) \mid \mathrm{Dsc}(\varphi) \subseteq \mathrm{Dsc}(F)\big\} \tag{2.5}$$

with semi-derivative given by

$$\dot{\Phi}(\varphi) = -\varphi \circ F^{\mathrm{Inv}} \cdot \big(\partial^-(F) \circ F^{\mathrm{Inv}}\big)^{-1}, \tag{2.6}$$

that is, for sequences $t_n > 0$, $t_n \to 0$ and $\varphi_n \in \mathcal{D}([q_{\alpha_l}, q_{\alpha_u}])$ which satisfy $\varphi_n \to \varphi \in \mathbb{W}_F$ with respect to $\|\cdot\|_{[q_{\alpha_l}, q_{\alpha_u}]}$ as well as $F + t_n\varphi_n \in \mathcal{D}_0$ we have that

$$\mathrm{d}_{\mathrm{hypi}}\big(t_n^{-1}\big(\Phi(F + t_n\varphi_n) - \Phi(F)\big), \dot{\Phi}(\varphi)\big) \to 0. \tag{2.7}$$

In (2.6) if $\partial^-(F)(F^{\mathrm{Inv}}(\alpha)) = \infty$ we set $(\partial^-(F)(F^{\mathrm{Inv}}(\alpha)))^{-1} = 0$. To formulate a limit theorem for the quantile process, consider a stationary and ergodic sequence $(Y_n)_{n \in \mathbb{N}}$ of real-valued random variables with marginal distribution function F. Given $n \in \mathbb{N}$ we let

$$F_n(x) = \frac{1}{n} \sum_{k=1}^{n} \mathbb{1}(Y_k \le x)$$

denote the empirical distribution function.

Assumption B. Assume that the ordinary empirical process $\sqrt{n}(F_n - F)$ of the sequence $(Y_n)_{n\in\mathbb{N}}$ converges weakly in $(\ell^\infty([q_{\alpha_l}, q_{\alpha_u}]), \|\cdot\|)$ to a process Z which concentrates on $\mathbb{W}_F$ in (2.5).

See, for example, Rio [12], Theorem 7.2, for conditions on the strong mixing coefficients which imply Assumption B.

From Assumption B, Lemma 1 and the functional delta method in the version of Theorem B.7 in Bücher, Segers and Volgushev [2], we obtain the following theorem.

Theorem 2. *Suppose that for given $0 < \alpha_l < \alpha_u < 1$, the stationary and ergodic sequence $(Y_n)_{n\in\mathbb{N}}$ of real-valued random variables satisfies Assumption B, and that its marginal distribution function F satisfies Assumption A. Then the weak convergence*

$$\sqrt{n}\big(F_n^{\mathrm{Inv}} - F^{\mathrm{Inv}}\big) \rightsquigarrow -\big(\partial^-(F)\circ F^{\mathrm{Inv}}\big)^{-1}\cdot Z\circ F^{\mathrm{Inv}}$$

in $(L^\infty([\alpha_l, \alpha_u]), \mathrm{d}_{\mathrm{hypi}})$ holds true.

From Propositions 2.3 and 2.4 in Bücher, Segers and Volgushev [2], hypi-convergence of the quantile process implies ordinary weak convergence of important statistics such as Kolmogorov–Smirnov or Cramér–von Mises type statistics.

Corollary 3. *Under the assumptions of Theorem 2, as $n \to \infty$ we have that*

$$\sqrt{n}\big\|F_n^{\mathrm{Inv}} - F^{\mathrm{Inv}}\big\|_{[\alpha_l,\alpha_u]} \xrightarrow{\mathcal{L}} \big\|\big(\partial^-(F)\circ F^{\mathrm{Inv}}\big)^{-1}\cdot Z\circ F^{\mathrm{Inv}}\big\|_{[\alpha_l,\alpha_u]}, \tag{2.8}$$

as well as

$$n^{p/2}\int_{\alpha_l}^{\alpha_u}\big|F_n^{\mathrm{Inv}}(\alpha) - F^{\mathrm{Inv}}(\alpha)\big|^p w(\alpha)\,\mathrm{d}\alpha \xrightarrow{\mathcal{L}} \int_{\alpha_l}^{\alpha_u}\big|\big(\partial^-(F)\circ F^{\mathrm{Inv}}\big)^{-1}(\alpha)\cdot Z\circ F^{\mathrm{Inv}}(\alpha)\big|^p w(\alpha)\,\mathrm{d}\alpha$$

for $p \geq 1$ and a bounded, non-negative weight function w on $[\alpha_l, \alpha_u]$.

Remark (Necessity of lower bound in Assumption A). We discuss the necessity of the lower bound on $\partial^\pm(F)(q)$ in Assumption A. Indeed, if for some quantile q_{α_0}, $\alpha_0 \in (\alpha_l, \alpha_u)$, the derivative f of F satisfies $f(q_{\alpha_0}) = 0$, then the discussion in Knight [7] shows that for a sequence $b_n/n \to 0$, $\sqrt{b_n}(\widehat{q}_{n,\alpha_0} - q_{\alpha_0})$ converges in distribution to a non-degenerate limit. However, the conclusion (2.8) of Theorem 2 in particular implies that $\sqrt{n}\|\widehat{q}_{n,\alpha} - q_\alpha\| = O_{\mathrm{P}}(1)$, which cannot simultaneously hold true.

Next we turn to a bootstrap version of Theorem 2. Here, we restrict ourselves to the i.i.d. situation for two reasons. First, we then do not need to get into variants of the bootstrap for dependent observations (Bühlmann [3]). More importantly, we were not able extend Lemma 1 to the uniform semi-Hadamard differentiability of Φ in (2.4) in the sense of van der Vaart and Wellner [14], p. 379, which requires (2.7) to hold if F is replaced by a sequence $F_n \to F$ w.r.t. $\mathrm{d}_{\mathrm{hypi}}$ and which seems to be required for the analysis of the bootstrap. Instead, the proof of Theorem 4 below uses a somewhat different approach, and uniform semi-Hadamard differentiability is proved for a different functional in Zwingmann and Holzmann [16], Lemma B.1.

Let $(Y_n)_{n\in\mathbb{N}}$ be an i.i.d. sequence of real-valued random variables with distribution function F. For $n \in \mathbb{N}$ let $Y_1^*, \dots, Y_n^*$ be a sample drawn from $Y_1, \dots, Y_n$ with replacement, that is, having distribution function F_n. Let F_n^* denote the empirical distribution function of $Y_1^*, \dots, Y_n^*$.

Theorem 4. *Let $(Y_n)_{n\in\mathbb{N}}$ be an i.i.d. sequence of real-valued random variables with distribution function F satisfying Assumption A. Then, the bootstrap quantile process $\sqrt{n}((F_n^*)^{\mathrm{Inv}} - F^{\mathrm{Inv}})(\alpha)$, $\alpha \in [\alpha_l, \alpha_u]$, converges weakly in $(L^\infty[\alpha_l, \alpha_u], \mathrm{d_{hypi}})$ to $(\partial^-(F) \circ F^{\mathrm{Inv}}(\alpha))^{-1} \cdot V_\alpha$, $\alpha \in [\alpha_l, \alpha_u]$ conditionally on $Y_1, Y_2, \dots$ in probability. Here (V_α) is a standard Brownian bridge process on* $[0, 1]$.

The proof of Theorem 4 is relegated to the technical supplement, Section B.

Remark (Failure of the bootstrap). The simple n out of n bootstrap does not apply for the empirical quantile at level τ if F is not differentiable at q_α (Knight [6]). Thus, Theorem 4 is somewhat surprising. Further insight is offered in Remark 7 below.

2.3. Convergence of the expectile process

For a random variable Y with distribution function F and $\mathbb{E}[|Y|] < \infty$, the τ-expectile $\mu_\tau = \mu_\tau(F)$, $\tau \in (0, 1)$, can be defined as the unique solution of $\mathbb{E}[I_\tau(x, Y)] = 0$, $x \in \mathbb{R}$, where

$$I_\tau(x, y) = \tau(y - x)\mathbb{1}(y \geq x) - (1 - \tau)(x - y)\mathbb{1}(y < x), \tag{2.9}$$

and $\mathbb{1}(\cdot)$ is the indicator function. Given a sequence of independent and identically distributed copies $Y_1, Y_2, \dots$ of Y and a natural number $n \in \mathbb{N}$, we let

$$\hat{\mu}_{\tau,n} = \mu_\tau(F_n), \qquad F_n(x) = \frac{1}{n}\sum_{k=1}^{n} \mathbb{1}(Y_k \leq x),$$

be the empirical τ-expectile and the empirical distribution function, respectively.

Theorem 5. *Suppose that $\mathbb{E}[Y^2] < \infty$. Given $0 < \tau_l < \tau_u < 1$ such that F is continuous in μ_{τ_l}, μ_{τ_u}, the standardized expectile process $\tau \mapsto \sqrt{n}(\hat{\mu}_{\tau,n} - \mu_\tau)$, $\tau \in [\tau_l, \tau_u]$, converges weakly in $(L^\infty[\tau_l, \tau_u], \mathrm{d_{hypi}})$ to the limit process $(\dot{\psi}_0^{\mathrm{Inv}}(Z)(\tau))_{\tau\in[\tau_l,\tau_u]}$. Here, $\dot{\psi}_0^{\mathrm{Inv}}(\varphi)(\tau) = \frac{\varphi(\tau)}{(\tau+(1-2\tau)F(\mu_\tau))}$, $\varphi \in \ell^\infty[\tau_l, \tau_u]$, and $(Z_\tau)_{\tau\in[\tau_l,\tau_u]}$ is a centred tight Gaussian process with continuous sample paths and covariance function $\mathrm{cov}(Z_\tau, Z_{\tau'}) = \mathbb{E}[I_\tau(\mu_\tau, Y)I_{\tau'}(\mu_{\tau'}, Y)]$ for $\tau, \tau' \in [\tau_l, \tau_u]$.*

Similar as above we have the following corollary using results from Bücher, Segers and Volgushev [2].

Corollary 6. *Let the assumptions of Theorem 5 be true, then we have as $n \to \infty$ that*

$$\sqrt{n}\|\hat{\mu}_{\cdot,n} - \mu_{\cdot}\|_{[\tau_l,\tau_u]} \xrightarrow{\mathcal{L}} \left\|\dot{\psi}_0^{\mathrm{Inv}}(Z)\right\|_{[\tau_l,\tau_u]}.$$

Further, for $p \geq 1$ and a bounded, non-negative weight function w on $[\tau_l, \tau_u]$,

$$n^{p/2} \int_{\tau_l}^{\tau_u} \left|(\hat{\mu}_{\tau,n} - \mu_\tau)\right|^p w(\tau)\,\mathrm{d}\tau \xrightarrow{\mathcal{L}} \int_{\tau_l}^{\tau_u} \left|\left(\dot{\psi}_0^{\mathrm{Inv}}(Z)\right)(\tau)\right|^p w(\tau)\,\mathrm{d}\tau.$$

Remark 7 (Point evaluation). Evaluation at a given point x is only a continuous operation under the hypi-semimetric if the limit function is continuous at x, see Proposition 2.2 in Bücher, Segers and Volgushev [2]. In particular, this does not apply to the expectile process if the distribution function F is discontinuous at μ_τ. Indeed, Theorem 7 in Holzmann and Klar [5] shows that the weak limit of the empirical expectile is not normal in this case.

Next we turn to the validity of the bootstrap. Given $n \in \mathbb{N}$ let $Y_1^*, \ldots, Y_n^*$ denote an i.i.d. sample drawn from $Y_1, \ldots, Y_n$ with replacement, that is, having distribution function F_n. Again, let F_n^* denote the empirical distribution function of $Y_1^*, \ldots, Y_n^*$, and let $\mu_{\tau,n}^* = \mu_\tau(F_n^*)$ denote the bootstrap expectile at level $\tau \in (0, 1)$.

Theorem 8. *Suppose that $\mathbb{E}[Y^2] < \infty$. Then, almost surely, conditionally on $Y_1, Y_2, \ldots$ the standardized bootstrap expectile process $\tau \mapsto \sqrt{n}(\mu_{\tau,n}^* - \hat{\mu}_{\tau,n})$, $\tau \in [\tau_l, \tau_u]$, converges weakly in $(L^\infty[\tau_l, \tau_u], \mathrm{d}_{\mathrm{hypi}})$ to $(\dot{\psi}_0^{\mathrm{Inv}}(Z)(\tau))_{\tau \in [\tau_l,\tau_u]}$, where the map $\dot{\psi}_0^{\mathrm{Inv}}$ and the process $(Z_\tau)_{\tau \in [\tau_l,\tau_u]}$ are as in Theorem 5.*

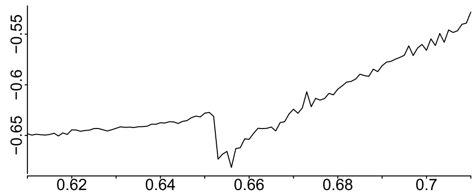
Remark (Finite second moments). Dropping the assumption of finite second moments in Theorems 5 and 8 leads to stable limit distributions for individual expectiles, see Holzmann and Klar [5]. Possibly this result could be generalized to process convergence, which is, however, beyond the scope of the present paper.

Remark (Convergence of the expectile process for dependent sequences). It would be of some interest to deduce the convergence of the expectile process from the weak convergence of the empirical process $\sqrt{n}(F_n - F)$ similarly as in Theorem 2. Semi-Hadamard differentiability of the map $\nu \mapsto \mu_{\cdot}(\nu)$ should suffice, however, our proof of Theorem 5 proceeds differently and hence we do not have an immediate extension for dependent sequences.

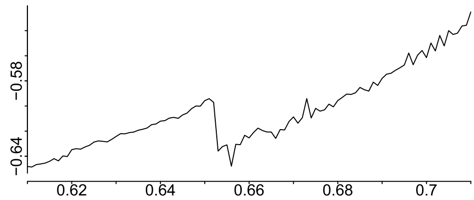
3. Numerical illustrations

In this section, we illustrate the asymptotic results for the expectile process in a short simulation. Let Y be a random variable with distribution function

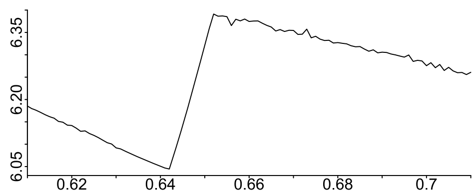
$$F(x) = \frac{9}{10} \int_{-\infty}^{x} \frac{1}{4\sqrt{2\pi}} \exp\left(-\frac{y^2}{32}\right) \mathrm{d}y + \frac{1}{10} \mathbb{1}(x \geq 1),$$



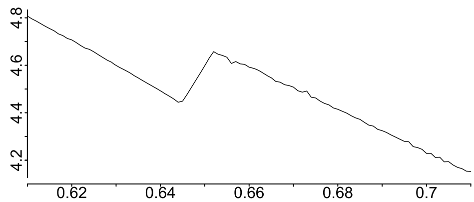
(a) Path evolving a downward jump.



(b) Path evolving a downward jump.



(c) Path evolving an upward jump.



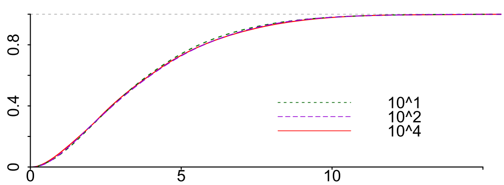
(d) Path evolving an upward jump.

Figure 1. The pictures show simulated paths of the empirical expectile process based on $n = 10^4$ observations of Y. If the path is negative (positive) around τ_0, a downward- (upward-) jump seems to evolve. This is plausible when considering the form of the hulls of $\dot{\psi}^{\mathrm{Inv}}$ in the limit process.

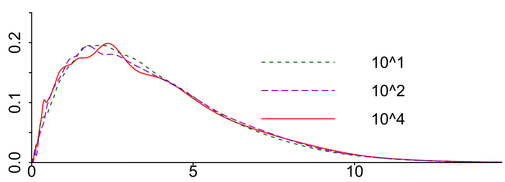
which is a mixture of a $\mathcal{N}(0, 16)$ random variable and a point mass in 1, so that $\mathbb{E}[Y] = \frac{1}{10}$ and $\mathbb{E}[Y^2] = 14.5$. We will concentrate on the weak convergence of the sup-norm of the empirical expectile process. Using equation (2.7) in Newey and Powell [11], we numerically find $\mu_{\tau_0} = 1$ for $\tau_0 \approx 0.6529449$, and investigate the expectile process on the interval $[0.6, 0.7]$.

Figure 1 contains four paths of the expectile process $\sqrt{n}(\hat{\mu}_{\tau,n} - \mu_\tau)$ for samples of size $n = 10^4$. All plotted paths seem to evolve a jump around τ_0.

Now we investigate the distribution of the supremum norm of the expectile process on the interval $[0.6, 0.7]$. To this end, we simulate $M = 10^4$ samples of sizes $n \in \{10, 10^2, 10^4\}$ and compute the expectile process and its supremum norm. Plots of the resulting empirical distribution functions and density estimates of this statistic are contained in Figure 2. The distribution of



(a) Estimated cumulative distribution function.



(b) Estimated density.

Figure 2. Figure (a) shows the cumulative distribution function of the supremum norm of $\sqrt{n}(\hat{\mu}_{\tau,n} - \mu_\tau)$, based on $M = 10^4$ samples of sizes $n \in \{10, 10^2, 10^4\}$, Figure (b) the corresponding density estimate.

Table 1. Empirical quantiles for $\sqrt{n}\|\widehat{\mu}_{\cdot,n} - \mu_{\cdot}\|$, based on 10^4 samples of sizes $n \in \{10^2, 10^3, 10^4\}$ and last averaged over $2 \cdot 10^2$ repetitions. The terms in brackets are the resulting standard deviations

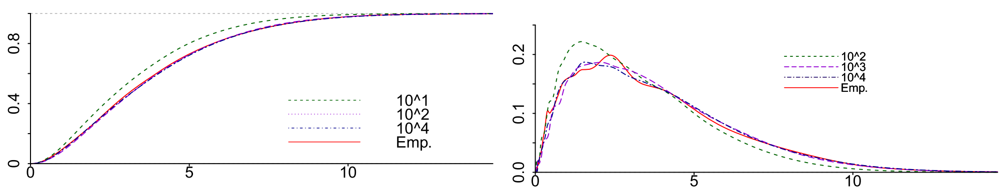
Size	Quantile								
n	1%	5%	10%	25%	50%	75%	90%	95%	99%
10^2	0.338 (0.015)	0.740 (0.017)	1.076 (0.017)	1.900 (0.023)	3.313 (0.033)	5.195 (0.038)	7.155 (0.047)	8.374 (0.069)	10.744 (0.135)
10^3	0.336 (0.015)	0.736 (0.018)	1.072 (0.020)	1.895 (0.023)	3.307 (0.030)	5.200 (0.043)	7.168 (0.054)	8.425 (0.072)	10.880 (0.14)
10^4	0.339 (0.015)	0.740 (0.017)	1.074 (0.019)	1.897 (0.023)	3.305 (0.030)	5.182 (0.038)	7.137 (0.061)	8.385 (0.074)	10.833 (0.149)

the supremum distance seems to converge quickly. In Table 1, we compute empirical quantiles over the M samples for the three sizes, which also stabilize already at $n = 10^2$.

Finally, to illustrate performance of the bootstrap, Figure 3 displays the distribution of $M = 10^4$ bootstrap samples of $\|\sqrt{n}(\mu^*_{\cdot,n} - \hat{\mu}_{\cdot,n})\|$ based on a single sample of size $n \in \{10^2, 10^3, 10^4\}$ from the n out of n bootstrap, together with the distribution of $\|\sqrt{n}(\hat{\mu}_{\cdot,n} - \mu_{\cdot})\|$. The bootstrap distribution for $n = 10^4$ is quite close to the empirical distribution. In Table 2, we also computed bootstrap quantiles, which are close to their empirical versions from Table 1.

4. Proofs

This section contains the proofs of the main results. Section 4.1 gives outlines of the proofs of Theorems 2, 5 and 8. Section 4.2 contains auxiliary results on hypi-convergence and lower- and upper-semicontinuous hulls. Section 4.3 contains details for the proof outlines from Section 4.1.



(a) Estimated bootstrap cumulative distribution function.

(b) Estimated bootstrap density.

Figure 3. Figure (a) shows the estimated cumulative bootstrap distribution function of $\|\sqrt{n}(\hat{\mu}^*_{\cdot,n} - \mu_{\cdot,n})\|$, figure (b) the estimated density thereof, obtained from $M = 10^4$ estimates of this statistic. The red line indicates the estimated empirical distribution and density function, respectively, taken from $\|\sqrt{n}(\mu_{\cdot,n} - \mu_{\cdot})\|$ for $n = 10^4$.

Table 2. Bootstrap quantiles for the supremum norm $\sqrt{n}\|\mu^*_{\cdot,n} - \widehat{\mu}_{\cdot,n}\|$, obtained from 10^4 estimates of this statistic, averaged over $2 \cdot 10^2$ repetitions. The bracketed numbers are the calculated standard deviations

Size	Quantile								
n	1%	5%	10%	25%	50%	75%	90%	95%	99%
10^2	0.339 (0.032)	0.734 (0.062)	1.065 (0.088)	1.885 (0.152)	3.300 (0.269)	5.190 (0.431)	7.155 (0.606)	8.406 (0.714)	10.842 (0.938)
10^3	0.344 (0.016)	0.741 (0.025)	1.074 (0.031)	1.901 (0.051)	3.321 (0.092)	5.215 (0.143)	7.195 (0.202)	8.465 (0.250)	10.938 (0.355)
10^4	0.345 (0.015)	0.742 (0.019)	1.077 (0.020)	1.904 (0.027)	3.324 (0.043)	5.217 (0.064)	7.200 (0.087)	8.469 (0.108)	10.955 (0.162)
Emp. 10^4	0.339	0.740	1.074	1.897	3.305	5.182	7.137	8.385	10.833

Some technicalities as well as the proof of Theorem 4 are deferred to the supplementary material, Zwingmann and Holzmann [16].

Notation

Let us recall some notation. For $s \in \mathbb{R}$ we will write s^{-1} instead of $\frac{1}{s}$. Given a function $g : \mathbb{R} \longrightarrow \mathbb{R}$, we will denote the pseudo inverse of g with g^{Inv}, that is $g^{\mathrm{Inv}}(y) = \inf\{x \in \mathbb{R} : g(x) \geq y\}$.

We write $\mathbb{E}_n[g(Y)] = \frac{1}{n}\sum_{k=1}^{n} g(Y_k)$, and use the abbreviation

$$\big\|\sqrt{n}(F_n - F)\big\|_{\mathcal{G}} = \sup_{g\in\mathcal{G}}\big|\sqrt{n}\big[\mathbb{E}_n\big[g(Y)\big] - \mathbb{E}\big[g(Y)\big]\big]\big|$$

for a class of measurable functions $\mathcal{G}$.

4.1. Outline of proofs of main results

In this section, we present an outline of the proofs of the main results. Additional details are provided in Section 4.3.

4.1.1. *Outline of the proofs of Lemma 1 and Theorem 2*

We give an outline of the proof of Lemma 1, which has Theorem 2 as an immediate consequence.

Step 1. Upper- and lower semicontinuous hulls of $\dot{\Phi}(\varphi)$ defined in (2.6).

Lemma 9. *Let Assumption A be true. Then it holds that*

$$\partial^-(F)(y-) = \partial^+(F)(y-) \quad \textit{and} \quad \partial^-(F)(y+) = \partial^+(F)(y+).$$

Lemma 10. *Let Assumption A hold for the distribution function F and let $\varphi \in \mathbb{W}_F$. Then $\varphi \circ F^{\mathrm{Inv}} \cdot (\partial^-(F) \circ F^{\mathrm{Inv}})^{-1}$ is càdlàg or làdcàg in every point $\alpha \in [\alpha_l, \alpha_u]$, and furthermore the following assertions are true.*

(i) *If F is continuous in q_α, then it holds that*

$$\big(\varphi \circ F^{\mathrm{Inv}} \cdot \big(\partial^-(F) \circ F^{\mathrm{Inv}}\big)^{-1}\big)_\wedge(\alpha) = \varphi(q_\alpha) \cdot \big(\big(\partial^-(F) \circ F^{\mathrm{Inv}}\big)^{-1}\big)_\wedge(\alpha) \cdot \mathbb{1}\big(\varphi(q_\alpha) > 0\big) + \varphi(q_\alpha) \cdot \big(\big(\partial^-(F) \circ F^{\mathrm{Inv}}\big)^{-1}\big)_\vee(\alpha) \cdot \mathbb{1}\big(\varphi(q_\alpha) < 0\big)$$

and

$$\big(\varphi \circ F^{\mathrm{Inv}} \cdot \big(\partial^-(F) \circ F^{\mathrm{Inv}}\big)^{-1}\big)_\vee(\alpha) = \varphi(q_\alpha) \cdot \big(\big(\partial^-(F) \circ F^{\mathrm{Inv}}\big)^{-1}\big)_\vee(\alpha) \cdot \mathbb{1}\big(\varphi(q_\alpha) > 0\big) + \varphi(q_\alpha) \cdot \big(\big(\partial^-(F) \circ F^{\mathrm{Inv}}\big)^{-1}\big)_\wedge(\alpha) \cdot \mathbb{1}\big(\varphi(q_\alpha) < 0\big)$$

with (*observe Lemma* 9)

$$\big(\big(\partial^-(F) \circ F^{\mathrm{Inv}}\big)^{-1}\big)_\wedge(\alpha) = \min\big\{\big(\partial^\pm(F)(q_\alpha-)\big)^{-1}, \big(\partial^\pm(F)(q_\alpha+)\big)^{-1}\big\} \quad \textit{and}$$
$$\big(\big(\partial^-(F) \circ F^{\mathrm{Inv}}\big)^{-1}\big)_\vee(\alpha) = \max\big\{\big(\partial^\pm(F)(q_\alpha-)\big)^{-1}, \big(\partial^\pm(F)(q_\alpha+)\big)^{-1}\big\}.$$

Now let $q_\alpha \in \mathrm{Dsc}(F)$ be a jump discontinuity of F.

(ii) *If $\alpha \in (F(q_\alpha-), F(q_\alpha))$, then*

$$\big(\varphi \circ F^{\mathrm{Inv}} \cdot \big(\partial^-(F) \circ F^{\mathrm{Inv}}\big)^{-1}\big)_\wedge(\alpha) = 0 = \big(\varphi \circ F^{\mathrm{Inv}} \cdot \big(\partial^-(F) \circ F^{\mathrm{Inv}}\big)^{-1}\big)_\vee(\alpha).$$

(iii) *If $\alpha = F(q_\alpha)$, then*

$$\big(\varphi \circ F^{\mathrm{Inv}} \cdot \big(\partial^-(F) \circ F^{\mathrm{Inv}}\big)^{-1}\big)_\wedge(\alpha) = \varphi(q_\alpha) \cdot \big(\big(\partial^-(F) \circ F^{\mathrm{Inv}}\big)^{-1}\big)_\vee(\alpha) \cdot \mathbb{1}\big(\varphi(q_\alpha) < 0\big),$$
$$\big(\varphi \circ F^{\mathrm{Inv}} \cdot \big(\partial^-(F) \circ F^{\mathrm{Inv}}\big)^{-1}\big)_\vee(\alpha) = \varphi(q_\alpha) \cdot \big(\big(\partial^-(F) \circ F^{\mathrm{Inv}}\big)^{-1}\big)_\vee(\alpha) \cdot \mathbb{1}\big(\varphi(q_\alpha) > 0\big),$$

where $((\partial^-(F) \circ F^{\mathrm{Inv}})^{-1})_\vee(\alpha) = \partial^\pm(F)(q_\alpha+)$.

(iv) *If $\alpha = F(q_\alpha-)$, then*

$$\begin{aligned}&\big(\varphi \circ F^{\mathrm{Inv}} \cdot \big(\partial^-(F) \circ F^{\mathrm{Inv}}\big)^{-1}\big)_\wedge(\alpha)\\ &\quad= \varphi(q_\alpha-) \cdot \big(\big(\partial^-(F) \circ F^{\mathrm{Inv}}\big)^{-1}\big)_\vee(\alpha) \cdot \mathbb{1}\big(\varphi(q_\alpha-) < 0\big),\\ &\big(\varphi \circ F^{\mathrm{Inv}} \cdot \big(\partial^-(F) \circ F^{\mathrm{Inv}}\big)^{-1}\big)_\vee(\alpha)\\ &\quad= \varphi(q_\alpha-) \cdot \big(\big(\partial^-(F) \circ F^{\mathrm{Inv}}\big)^{-1}\big)_\vee(\alpha) \cdot \mathbb{1}\big(\varphi(q_\alpha-) > 0\big),\end{aligned}$$

where $((\partial^-(F) \circ F^{\mathrm{Inv}})^{-1})_\vee(\alpha) = \partial^\pm(F)(q_\alpha-)$.

Step 2. Outline of the proof of the semi-Hadamard differentiability (2.7).

Let $t_n > 0$, $t_n \to 0$ and $\varphi_n \in \mathcal{D}([q_{\alpha_l}, q_{\alpha_u}])$ fulfilling $\varphi_n \to \varphi \in \mathbb{W}$ with respect to $\|\cdot\|$ as well as $F + t_n\varphi_n \in \mathcal{D}_0$. To show (2.7), we proceed as follows. First, observe that since $\dot{\Phi}(\varphi)$ is càdlàg or làdcàg in every point of $[\alpha_l, \alpha_u]$ by Lemma 10 (applied to $-\varphi$), the set $\mathcal{S} = [\alpha_l, \alpha_u] \setminus \mathrm{Dsc}(\dot{\Phi}(\varphi))$ is dense in $[\alpha_l, \alpha_u]$ and the restriction $\dot{\Phi}(\varphi)|_{\mathcal{S}}$ of $\dot{\Phi}(\varphi)$ to $\mathcal{S}$ is continuous. Note that $\alpha_l, \alpha_u \in \mathcal{S}$ by Assumption A. If we now form the lower- and upper- semicontinuous hulls in (2.1) restricted over the set $\mathcal{S}$, that is,

$$\big(\dot{\Phi}(\varphi)|_{\mathcal{S}}\big)_{\wedge}^{\mathcal{S}:[\alpha_l,\alpha_u]}(x) = \lim_{\varepsilon \searrow 0} \inf\big\{\dot{\Phi}(\varphi)\big(x'\big) \mid \mathrm{d}\big(x, x'\big) < \varepsilon,\ x' \in \mathcal{S}\big\},$$

and similarly for $(\dot{\Phi}(\varphi)|_{\mathcal{S}})_{\vee}^{\mathcal{S}:[\alpha_l,\alpha_u]}$, then since $\dot{\Phi}(\varphi)$ is càdlàg or làdcàg at every point we have that

$$\big(\dot{\Phi}(\varphi)|_{\mathcal{S}}\big)_{\wedge}^{\mathcal{S}:[\alpha_l,\alpha_u]} = \dot{\Phi}(\varphi)_{\wedge} \quad \text{and} \quad \big(\dot{\Phi}(\varphi)|_{\mathcal{S}}\big)_{\vee}^{\mathcal{S}:[\alpha_l,\alpha_u]} = \dot{\Phi}(\varphi)_{\vee}. \tag{4.1}$$

Hence from Bücher, Segers and Volgushev [2], Corollary A.7, is suffices to show that

$$\begin{aligned} &\liminf_n t_n^{-1}\big((F + t_n\varphi_n)^{\mathrm{Inv}}(\alpha_n) - F^{\mathrm{Inv}}(\alpha_n)\big) \geq \big(\dot{\Phi}(\varphi)\big)_{\wedge}(\alpha) \quad \text{and} \\ &\limsup_n t_n^{-1}\big((F + t_n\varphi_n)^{\mathrm{Inv}}(\alpha_n) - F^{\mathrm{Inv}}(\alpha_n)\big) \leq \big(\dot{\Phi}(\varphi)\big)_{\vee}(\alpha) \end{aligned} \tag{4.2}$$

in order to obtain (2.7). Thus in (2.2) and (2.3) we no longer need to exhibit the convergent subsequences.

We shall prove (4.2) by distinguishing the same cases as in Lemma 10. Full details are provided in Section 4.3.1 and in the technical supplement, Section A.

4.1.2. *Outline of proofs of Theorem 5 and 8*

Proof of Theorem 5 (Outline). We give an outline of the proof of Theorem 5. For a distribution function S with finite first moment let $I_\tau(x, S) = \int I_\tau(x, y)\,\mathrm{d}S(y)$ and $[\psi(\varphi, S)](\tau) = -I_\tau(\varphi(\tau), S)$, where $\tau \in [\tau_l, \tau_u]$ and $\varphi \in \ell^\infty[\tau_l, \tau_u]$. Set $\psi_0(\cdot) = \psi(\cdot, F)$ and $\psi_n(\cdot) = \psi(\cdot, \hat{F}_n)$.

Step 1. Weak convergence of $\sqrt{n}(\psi_0(\hat{\mu}_{\cdot,n}) - \psi_0(\mu_{\cdot}))$ to Z in $(\ell^\infty[\tau_l, \tau_u], \|\cdot\|)$.

This step uses standard results from empirical process theory based on bracketing properties of Lipschitz-continuous functions. The main issue in the proof of the lemma below is the Lipschitz-continuity of $\tau \mapsto \mu_\tau$, $\tau \in [\tau_l, \tau_u]$, for a general distribution function F.

Lemma 11. *In $(\ell^\infty[\tau_l, \tau_u], \|\cdot\|)$ we have the weak convergence*

$$\sqrt{n}\big(\psi_n(\mu_{\cdot}) - \psi_0(\mu_{\cdot})\big)(\tau) \to Z_\tau, \quad \tau \in [\tau_l, \tau_u]. \tag{4.3}$$

Further, given $\delta_n \searrow 0$ we have as $n \to \infty$ that

$$\begin{aligned} \sup_{\|\varphi\|_{[\tau_l,\tau_u]} \leq \delta_n} \sup_{\tau \in [\tau_l,\tau_u]} \sqrt{n}\big|&\psi_n(\mu_{\cdot} + \varphi)(\tau) - \psi_0(\mu_{\cdot} + \varphi)(\tau) \\ &- \big[\psi_n(\mu_{\cdot})(\tau) - \psi_0(\mu_{\cdot})(\tau)\big]\big| = o_{\mathrm{P}}(1). \end{aligned} \tag{4.4}$$

Since $\psi_0(\mu_\cdot) = \psi_n(\hat{\mu}_{\cdot,n}) = 0$, we can rewrite

$$\begin{aligned}\sqrt{n}\big(\psi_0(\widehat{\mu}_{\cdot,n}) - \psi_0(\mu_\cdot)\big) &= \sqrt{n}\big(\psi_0(\widehat{\mu}_{\cdot,n}) - \psi_n(\widehat{\mu}_{\cdot,n})\big) \\ &= \sqrt{n}\big[\psi_0(\mu_\cdot + \varphi_n) - \psi_n(\mu_\cdot + \varphi_n)\big] \qquad (4.5)\end{aligned}$$

for $\varphi_n(\cdot) = \widehat{\mu}_{\cdot,n} - \mu_\cdot$ and adding and subtracting $-\sqrt{n}(\psi_n(\mu_\cdot) - \psi_0(\mu_\cdot))$ yields

$$\begin{aligned}&\sqrt{n}\big(\psi_0\big(\mu_\cdot + \varphi_n(\cdot)\big) - \psi_n\big(\mu_\cdot + \varphi_n(\cdot)\big)\big) \\ &\quad = -\sqrt{n}\big(\psi_n(\mu_\cdot) - \psi_0(\mu_\cdot)\big) \\ &\qquad + \sqrt{n}\big[\psi_n(\mu_\cdot) - \psi_0(\mu_\cdot) - \big(\psi_n(\mu_\cdot + \varphi_n) - \psi_0(\mu_\cdot + \varphi_n)\big)\big]. \qquad (4.6)\end{aligned}$$

Due to the uniform consistency shown in Theorem 2, Holzmann and Klar [5], it holds that $\|\varphi_n\| = o_{\mathrm{P}}(1)$, such that the supremum (over $\tau \in [\tau_l, \tau_u]$) of the term in angle brackets above is smaller than (or equal to) the expression in (4.4). Using this together with (4.5) and (4.6) shows

$$\sqrt{n}\big(\psi_0(\widehat{\mu}_{\cdot,n}) - \psi_0(\mu_\cdot)\big) = -\sqrt{n}\big(\psi_n(\mu_\cdot) - \psi_0(\mu_\cdot)\big) + o_{\mathrm{P}}(1).$$

Then (4.3) and the fact that Z and $-Z$ have the same law conclude the proof of

$$\sqrt{n}\big(\psi_0(\widehat{\mu}_{\cdot,n}) - \psi_0(\mu_\cdot)\big) \rightsquigarrow Z \qquad (4.7)$$

in $(\ell^\infty([\tau_l, \tau_u]), \|\cdot\|)$, finishing Step 1.

Step 2. Invertibility of ψ_0 and semi-Hadamard differentiability of the inverse with respect to $\mathrm{d}_{\mathrm{hypi}}$.

The first part of Step 2 is observing the following lemma.

Lemma 12. *The map ψ_0 is invertible, and for the inverse map we have that $\psi_0^{\mathrm{Inv}}(\varphi) \in \ell^\infty[\tau_l, \tau_u]$ for any $\varphi \in \ell^\infty[\tau_l, \tau_u]$, so that $\psi_0 : \ell^\infty[\tau_l, \tau_u] \to \ell^\infty[\tau_l, \tau_u]$ is bijective.*

The next result then is the key technical ingredient in the proof of Theorem 5. The general definition of semi-Hadamard differentiability can be found in Definition B.6, Bücher, Segers and Volgushev [2].

Lemma 13. *The map ψ_0^{Inv} is semi-Hadamard differentiable with respect to the hypi-semimetric in $0 \in \mathcal{C}[\tau_l, \tau_u]$ tangentially to $\mathcal{C}[\tau_l, \tau_u]$ with semi-Hadamard derivative given by $\dot{\psi}_0^{\mathrm{Inv}}(\varphi)(\tau) = (\tau + (1-2\tau)F(\mu_\tau))^{-1}\varphi(\tau)$, $\varphi \in \ell^\infty[\tau_l, \tau_u]$, that is, we have*

$$t_n^{-1}\big(\psi_0^{\mathrm{Inv}}(t_n\varphi_n) - \psi_0^{\mathrm{Inv}}(0)\big) \to \dot{\psi}_0^{\mathrm{Inv}}(\varphi)$$

for any sequence $t_n \to 0$, $t_n > 0$ and $\varphi_n \in \ell^\infty[\tau_l, \tau_u]$ with $\varphi_n \to \varphi \in \mathcal{C}[\tau_l, \tau_u]$ with respect to $\mathrm{d}_{\mathrm{hypi}}$.

The proof of the lemma is based on an explicit representation of increments of ψ_0^{Inv}, and novel technical properties of convergence under the hypi-semimetric for products and quotients.

We observe here that until the former lemma the results did not depend on the hypi-semimetric. When wanting to obtain Theorem 5 directly for the M_2-topology, one can use the same steps, just replacing $\mathrm{d}_{\mathrm{hypi}}$ with M_2-convergence in the above lemma. We will give more details on that in the supplement, Section C.

Step 3. Conclusion with the generalized functional delta method.

From (4.7), Lemma 13 and the generalized functional delta method, Theorem B.7 in Bücher, Segers and Volgushev [2], we obtain

$$\sqrt{n}(\hat{\mu}_{\cdot,n} - \mu_\cdot) = \sqrt{n}\big(\psi_0^{\mathrm{Inv}}\big(\psi_0(\hat{\mu}_{\cdot,n})\big) - \psi_0^{\mathrm{Inv}}(0)\big) \to \dot{\psi}_0^{\mathrm{Inv}}(Z)$$

in $(L^\infty[\tau_l, \tau_u], \mathrm{d}_{\mathrm{hypi}})$.

Note that the generalized functional delta method is formulated for arbitrary (semi-)metric spaces, such that the conclusion would also work with respect to M_2-convergence, given Lemma 13 is proven with respect to the M_2-topology as well. □

Proof of Theorem 8 (Outline). The steps in the proof are similar to those of Theorem 5. In the analogous result to Lemma 11 and (4.7), we require the uniform consistency of $\mu^*_{\cdot,n}$ as in Holzmann and Klar [5], Theorem 1. The weak convergence statements require the changing classes central limit theorem, van der Vaart [13], Theorem 19.28. In the second step, we argue directly with the extended continuous mapping theorem, Theorem B.3 in Bücher, Segers and Volgushev [2]. □

4.2. Auxiliary results on hypi-convergence and hulls

A major technical issue in the above arguments is to determine hypi-convergence of sums, products and quotients of hypi-convergent functions. The proofs of Lemmas 14 and 15 are given below, while those of Lemmas 16 and 17 are provided in the supplement, Section E.

Lemma 14. *Let $\nu, \nu_n, \varphi_n \in \ell^\infty([l,u])$ and $\varphi \in \mathcal{C}[l,u]$.*

(i) *If $\mathrm{d}_{\mathrm{hypi}}(\varphi_n, \varphi) \to 0$ and $\mathrm{d}_{\mathrm{hypi}}(\nu_n, \nu) \to 0$ hold true, then $\nu_n\varphi_n$ hypi-converges to $\nu\varphi$. More precisely $\nu_n\varphi_n$ epi-converges to $(\nu\varphi)_\wedge$ and hypo-converges to $(\nu\varphi)_\vee$, where*

$$\begin{aligned}(\varphi\nu)_\wedge &= \varphi\big(\nu_\wedge \mathbb{1}(\varphi > 0) + \nu_\vee \mathbb{1}(\varphi < 0)\big),\\ (\varphi\nu)_\vee &= \varphi\big(\nu_\vee \mathbb{1}(\varphi > 0) + \nu_\wedge \mathbb{1}(\varphi < 0)\big).\end{aligned} \tag{4.8}$$

(ii) *If ν admits right- and left-sided limits in every $x \in [l,u]$, the functions $x \mapsto \nu(x-)$ and $x \mapsto \nu(x+)$ do the same. More precisely, the right-sided limit of both $\nu(x-)$ and $\nu(x+)$ is $\nu(x+)$, the left-sided limit for them is $\nu(x-)$.*

(iii) *Assume ν has left- and right-sided limits at every point in $[l,u]$. Then*

$$\nu_\wedge(x) = \min\big\{\nu(x-), \nu(x), \nu(x+)\big\}, \qquad \nu_\vee(x) = \max\big\{\nu(x-), \nu(x), \nu(x+)\big\}.$$

(iv) *If $\nu_n, \nu > 0$, the convergence $\mathrm{d}_{\mathrm{hypi}}(\frac{1}{\nu_n}, \frac{1}{\nu}) \to 0$ follows from $\mathrm{d}_{\mathrm{hypi}}(\nu_n, \nu) \to 0$.*

For the proof of Lemma 10 we need to determine the hulls of products of functions as follows.

Lemma 15. *Let $\varphi, h \in \mathcal{D}(I)$ and $t \in I$.*

(i) *If $(\varphi(t) - \varphi(t-))(h(t) - h(t-)) \geq 0$, then it holds that*

$$(\varphi h)_\wedge(t) = \min\big\{(\varphi_\wedge h_\wedge)(t), (\varphi_\vee h_\vee)(t)\big\}, \qquad (\varphi h)_\vee(t) = \max\big\{(\varphi_\wedge h_\wedge)(t), (\varphi_\vee h_\vee)(t)\big\}.$$

(ii) *If $h(t) = 0$, then it holds that*

$$(\varphi h)_\wedge(t) = \min\big\{\varphi(t-)h(t-), 0\big\} \quad \text{and} \quad (\varphi h)_\vee(t) = \max\big\{\varphi(t-)h(t-), 0\big\}.$$

(iii) *If $h(t-) = 0$, then it holds that*

$$(\varphi h)_\wedge(t) = \min\big\{\varphi(t)h(t), 0\big\} \quad \text{and} \quad (\varphi h)_\vee(t) = \max\big\{\varphi(t)h(t), 0\big\}.$$

The strength of the second assertion is that h and φ do not have to jump in the same direction as needed in the first statement. In general, if h does not jump in the same direction as φ, the equalities stated in (i) are not valid.

In addition, we shall require the following basic relations between lim sup and lim inf.

Lemma 16. *Let $(a_n)_n$, $(b_n)_n$ be bounded sequences. Then*

$$\liminf_{n\to\infty} a_n + \limsup_{n\to\infty} b_n \geq \liminf_{n\to\infty}(a_n + b_n) \geq \liminf_{n\to\infty} a_n + \liminf_{n\to\infty} b_n$$

and

$$\limsup_{n\to\infty} a_n + \liminf_{n\to\infty} b_n \leq \limsup_{n\to\infty}(a_n + b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n.$$

If $a_n > 0$, then

$$\liminf_n \frac{1}{a_n} = \frac{1}{\limsup_n a_n}.$$

Lemma 17. *Let $(b_n)_n$ be a bounded sequence and let $(a_n)_n$ be convergent with limit $a \in \mathbb{R}$. Then*

$$\liminf_n a_n b_n = \liminf_n a b_n, \qquad \limsup_n a_n b_n = \limsup_n a b_n.$$

Proof of Lemma 14. From the definition in (2.1), for a function $h \in \ell^\infty([l,u])$ the lower semi-continuous hull $h_\wedge$ at $x \in [l,u]$ is characterized by the following conditions

$$\begin{aligned} &\text{For any sequence } x_n \to x, \quad \text{we have } \liminf_{n\to\infty} h(x_n) \geq h_\wedge(x), \\ &\text{there is a sequence } x_n' \to x \quad \text{for which } \lim_{n\to\infty} h\big(x_n'\big) = h_\wedge(x), \end{aligned} \tag{4.9}$$

and similarly for $h_\vee$.

Ad (i): By continuity of φ we have $\varphi(x_n) \to \varphi(x)$ for any sequence $x_n \to x$. The statement (4.8) now follows immediately using (4.9) and Lemma 17 and noting that for $\varphi(x) < 0$,

$$\liminf_{n\to\infty} \varphi(x)\nu(x_n) = \varphi(x) \limsup_{n\to\infty} \nu(x_n), \qquad \limsup_{n\to\infty} \varphi(x)\nu(x_n) = \varphi(x) \liminf_{n\to\infty} \nu(x_n).$$

Further, by continuity of φ, the hypi-convergence of φ_n to φ actually implies the uniform convergence. Therefore, for any $x_n \to x$ we have that $\varphi_n(x_n) \to \varphi(x)$. Using the pointwise criteria (2.2) and (2.3) for hypi-convergence, Lemma 17 and (4.8) we obtain the asserted convergence $\varphi_n\nu_n \to \varphi\nu$ with respect to the hypi-semimetric.

Ad (ii): This is Lemma C.5, Bücher, Segers and Volgushev [2]; we give a proof here for convenience. We show $\lim_{x'\searrow x} \nu(x'-) = \nu(x+)$. First, we consider the expression $\liminf_{x'\searrow x} \nu(x'-)$. Observe that

$$\liminf_{x'\searrow x} \nu\big(x'-\big) = \lim_{\varepsilon\searrow 0} \inf_{x'\in(x,x+\varepsilon)} \lim_{x''\nearrow x'} \nu\big(x''\big) = \lim_{\varepsilon\searrow 0} \inf_{x'\in(x,x+\varepsilon)} \lim_{\delta\searrow 0} \inf_{x''\in(x'-\delta,x')} \nu\big(x''\big).$$

Therefore, choose any $\varepsilon > 0$ and $x' \in (x, x+\varepsilon)$. Then for some small $\delta > 0$ it holds that $(x'-\delta, x') \subset (x, x+\varepsilon)$ and thus

$$\lim_{\delta\searrow 0} \inf_{x''\in(x'-\delta,x')} \nu\big(x''\big) \geq \inf_{x''\in(x,x+\varepsilon)} \nu\big(x''\big)$$

is valid. Taking the infimum over $x' \in (x, x+\varepsilon)$ and then letting $\varepsilon \searrow 0$ yields

$$\lim_{\varepsilon\searrow 0} \inf_{x'\in(x,x+\varepsilon)} \lim_{\delta\searrow 0} \inf_{x''\in(x'-\delta,x')} \nu\big(x''\big) \geq \lim_{\varepsilon\searrow 0} \inf_{x''\in(x,x+\varepsilon)} \nu\big(x''\big) = \lim_{x''\searrow x} \nu\big(x''\big) = \nu(x+),$$

as ν has a right-sided limit in x. This means

$$\liminf_{x'\searrow x} \nu\big(x'-\big) \geq \nu(x+).$$

Similar we deduce that

$$\limsup_{x'\searrow x} \nu\big(x'-\big) \leq \nu(x+),$$

hence we obtain

$$\lim_{x'\searrow x} \nu\big(x'-\big) = \nu(x+)$$

as asserted. The remaining assertions are proven analogously.

Ad (iii): The proof of Lemma C.6, Bücher, Segers and Volgushev [2] show that for a function, which admits right- and left-sided limits, the supremum over a shrinking neighbourhood around a point x converges to the maximum of the three points $\nu(x-)$, $\nu(x)$ and $\nu(x+)$. The analogues statement holds for the infimum, which is the first part of (ii). From (ii) shown before, the maps $x \mapsto \nu(x-)$ and $x \mapsto \nu(x+)$ both have a right-sided limit equal to $\nu(x+)$ and a left-sided limit equal to $\nu(x-)$, hence this is also true for the functions $\nu_\vee(x) = \max\{\nu(x-), \nu(x), \nu(x+)\}$ and $\nu_\wedge(x) = \min\{\nu(x-), \nu(x)\nu(x+)\}$. From the above argument, we obtain

$$(\nu_\vee)_\wedge(x) = \min\Big\{\nu(x-), \nu(x+), \max\big\{\nu(x-), \nu(x), \nu(x+)\big\}\Big\} = \min\big\{\nu(x-), \nu(x+)\big\} \quad \text{and}$$
$$(\nu_\wedge)_\vee(x) = \max\Big\{\nu(x-), \nu(x+), \min\big\{\nu(x-), \nu(x), \nu(x+)\big\}\Big\} = \max\big\{\nu(x-), \nu(x+)\big\}.$$

If $\nu(x_0-) \le \nu(x_0) \le \nu(x_0+)$ or $\nu(x_0-) \ge \nu(x_0) \ge \nu(x_0+)$, we deduce

$$\nu_\vee(x_0) = \max\big\{\nu(x-), \nu(x+)\big\} = (\nu_\wedge)_\vee(x_0),$$
$$\nu_\wedge(x_0) = \min\big\{\nu(x-), \nu(x+)\big\} = (\nu_\vee)_\wedge(x_0).$$

Ad (iv): From Lemma 16 (last statement) and (4.9), we obtain $(\frac{1}{\nu})_\wedge = \frac{1}{\nu_\vee}$ and $(\frac{1}{\nu})_\vee = \frac{1}{\nu_\wedge}$. The hypi-convergence of $\frac{1}{\nu_n}$ to these hulls follows similarly from Lemma 16 (last statement) and the pointwise criteria (2.2) and (2.3) for hypi-convergence. □

Proof of Lemma 15. In (i) note that the assumption guarantees that φ and h jump in the same direction, such that

$$\varphi_\vee(t) = \varphi(t-) \quad \text{if and only if} \quad h_\vee(t) = h(t-)$$

and likewise for $\varphi_\wedge$ and $h_\wedge$. By Lemma 14, (iii), we thus know that

$$(\varphi h)_\vee(t) = \max\big\{\varphi(t-)h(t-), \varphi(t)h(t)\big\} = \max\big\{\varphi_\vee(t)h_\vee(t), \varphi_\wedge(t)h_\wedge(t)\big\}$$

and similar for $(\varphi h)_\wedge$.

For (ii) we use Lemma 14, (iii), to deduce

$$(\varphi h)_\wedge(t) = \min\big\{\varphi(t-)h(t-), \varphi(t)h(t)\big\} = \min\big\{\varphi(t-)h(t-), 0\big\}.$$

The second part of (ii) and the assertion in (iii) are proven analogous. □

4.3. Details for proof outlines from Section 4.1

4.3.1. *Details for the proofs of Lemma 1 and Theorem 2*

Proof of Lemma 9. Since the points in which F is differentiable form a dense set, we can choose a sequence $y_s \nearrow y$ such that $F'(y_s)$ exists. For this sequence it holds that

$$\partial^-(F)(y-) = \lim_{s\to\infty} \partial^-(F)(y_s) = \lim_{s\to\infty} \partial^+(F)(y_s) = \partial^+(F)(y-)$$

where all values are in the interval $[c, \infty]$. This shows the first equality, the second follows likewise. □

Proof of Lemma 10. We note that F^{Inv} is continuous as F is assumed to be strictly increasing. We work through the cases (i)–(iv), in each the arguments will imply that $\varphi \circ F^{\mathrm{Inv}} \cdot (\partial^-(F) \circ F^{\mathrm{Inv}})^{-1}$ is càdlàg or làdcàg in α.

Ad (i). As $\mathrm{Dsc}(\varphi) \subset \mathrm{Dsc}(F)$ and $\mathrm{Dsc}(F)$ is finite by Assumption A, $\varphi \circ F^{\mathrm{Inv}}$ is continuous in a neighborhood of α. Using (4.8) in Lemma 14, (i), it remains to determine the semicontinuous hulls of $(\partial^-(F) \circ F^{\mathrm{Inv}})^{-1}$. From Lemma 14, (iii), it follows that

$$
\begin{aligned}
&\big((\partial^-(F) \circ F^{\mathrm{Inv}})^{-1}\big)_\vee(\alpha) \\
&\quad = \max\big\{\big(\partial^-(F) \circ F^{\mathrm{Inv}}\big)^{-1}(\alpha-), \big(\partial^-(F) \circ F^{\mathrm{Inv}}\big)^{-1}(\alpha), \big(\partial^-(F) \circ F^{\mathrm{Inv}}\big)^{-1}(\alpha+)\big\} \\
&\quad = \max\big\{\big(\partial^-(F)(q_\alpha-)\big)^{-1}, \big(\partial^-(F)(q_\alpha+)\big)^{-1}\big\}
\end{aligned}
$$

where the second equality follows since $\partial^-(F)$ is càdlàg or làdcàg in q_α by Assumption A. Lemma 9 now implies the assertion, and the lower-semicontinuous hulls are determined likewise.

Ad (ii). Since $q_\alpha \in \mathrm{Dsc}(F)$ we have that $\partial^-(F)(q_\alpha) = \infty$. Note that $F^{\mathrm{Inv}}(\alpha') = q_\alpha$ for every $\alpha' \in (F(q_\alpha-), F(q_\alpha))$, hence $(\partial^-(F) \circ F^{\mathrm{Inv}})^{-1}(\alpha') = 0$ is valid from the convention $\frac{1}{\infty} = 0$. Therefore $(\partial^-(F) \circ F^{\mathrm{Inv}})^{-1}|_{(F(q_\alpha-), F(q_\alpha))} = 0$, and the conclusion follows since the hulls of a continuous function are equal to the function itself.

Ad (iii). Since F^{Inv} is continuous, increasing and $F^{\mathrm{Inv}}(\alpha') = q_\alpha$, $\alpha' \in (\alpha - \delta, \alpha]$ for some small $\delta > 0$, by right-continuity of φ it follows that $\varphi \circ F^{\mathrm{Inv}}$ is continuous for in a neighbourhood of α. Hence we may apply (4.8) in Lemma 14, (i), and it remains to compute the semicontinuous hulls of $(\partial^-(F) \circ F^{\mathrm{Inv}})^{-1}$. Using Lemma 14, (iii) we observe that

$$
\begin{aligned}
&\big((\partial^-(F) \circ F^{\mathrm{Inv}})^{-1}\big)_\vee(\alpha) \\
&\quad = \max\big\{\big(\partial^-(F) \circ F^{\mathrm{Inv}}\big)^{-1}(\alpha-), \big(\partial^-(F) \circ F^{\mathrm{Inv}}\big)^{-1}(\alpha), \big(\partial^-(F) \circ F^{\mathrm{Inv}}\big)^{-1}(\alpha+)\big\}. \qquad (4.10)
\end{aligned}
$$

In Assumption A, it is not assumed that $\partial^-(F)$ necessarily is càdlàg nor làdcàg in q_α, so that we have to argue differently compared to case (i). For a sequence $\alpha_n \nearrow \alpha$ and n big enough it holds that $F^{\mathrm{Inv}}(\alpha_n) = q_\alpha$, hence

$$
\big(\partial^-(F) \circ F^{\mathrm{Inv}}\big)^{-1}(\alpha-) = 0 = \big(\partial^-(F) \circ F^{\mathrm{Inv}}\big)^{-1}(\alpha),
$$

see case (ii). By non-negativity of $\partial^-(F)$ by Assumption A, we obtain that the maximum in (4.10) is

$$
\big((\partial^-(F) \circ F^{\mathrm{Inv}})^{-1}\big)_\vee(\alpha) = \big(\partial^-(F) \circ F^{\mathrm{Inv}}\big)^{-1}(\alpha+) = \partial^\pm(F)(q_\alpha+),
$$

where the last equality follows from Lemma 9. Similarly it follows that $((\partial^-(F) \circ F^{\mathrm{Inv}})^{-1})_\wedge(\alpha) = 0$.

Ad (iv). Note that $(\partial^-(F) \circ F^{\mathrm{Inv}})^{-1}(\alpha') = 0$ for $\alpha' \in [\alpha, \alpha + \delta)$, so that $(\partial^-(F) \circ F^{\mathrm{Inv}})^{-1}$ is càdlàg in α. The map $\varphi \circ F^{\mathrm{Inv}}$ is càdlàg since F^{Inv} is increasing and continuous and φ is càdlàg. Hence, we may apply Lemma 15, (ii). Using analogue arguments as in case (iii), we deduce

$((\partial^-(F)\circ F^{\mathrm{Inv}})^{-1})_\vee(\alpha)=\partial^\pm(F)(q_\alpha-)$ and $((\partial^-(F)\circ F^{\mathrm{Inv}})^{-1})_\wedge(\alpha)=0$, and the conclusion follows from Lemma 15, (ii). $\square$

Proof of Lemma 1. To show (4.2), observe that $\Delta_n=\|t_n\varphi_n\|$, we have $F-\Delta_n\le F+t_n\varphi_n\le F+\Delta_n$ and hence

$$\begin{aligned}F^{\mathrm{Inv}}(\alpha_n-\Delta_n)&=(F+\Delta_n)^{\mathrm{Inv}}(\alpha_n)\le(F+t_n\varphi_n)^{\mathrm{Inv}}(\alpha_n)\\&\le(F-\Delta_n)^{\mathrm{Inv}}(\alpha_n)=F^{\mathrm{Inv}}(\alpha_n+\Delta_n).\end{aligned}\tag{4.11}$$

Further, since $\Delta_n\to 0$ we obtain $\alpha_n+\Delta_n\to\alpha$ and therefore $F(\alpha_n\pm\Delta_n)\in(q_{\alpha_l}-\varepsilon,q_{\alpha_u}+\varepsilon)$ for n big enough by continuity of F in the boundary points q_{α_l} and q_{α_u}, where ε is given as in Assumption A. Without loss of generality, we assume that this inclusion holds for all n.

(i) If $\alpha\in(F(q_\alpha-),F(q_\alpha))$, we have $(\dot\Phi(\varphi))_\vee(\alpha)=(\dot\Phi(\varphi))_\wedge(\alpha)=0$ by Lemma 10. We shall show that actually

$$t_n^{-1}\big((F+t_n\varphi_n)^{\mathrm{Inv}}(\alpha_n)-F^{\mathrm{Inv}}(\alpha_n)\big)=0\tag{4.12}$$

for n large enough, which then in particular implies (4.2). Indeed, since $\alpha_n\to\alpha$ and $\alpha_n\pm\Delta_n\to\alpha$ it holds that $\alpha_n,\alpha_n\pm\Delta_n\in(F(q_\alpha-),F(q_\alpha))$ for n big enough, such that $F^{\mathrm{Inv}}(\alpha_n)=F^{\mathrm{Inv}}(\alpha_n\pm\Delta_n)=q_\alpha$. (4.11) then implies (4.12).

For the remaining cases, we start with some general observations. From the definition of the generalized inverse for non-decreasing functions, for $\varepsilon_n>0$ it holds that

$$\begin{aligned}&(F+t_n\varphi_n)\big((F+t_n\varphi_n)^{\mathrm{Inv}}(\alpha_n)-\varepsilon_n\big)\\&\quad\le\alpha_n\le(F+t_n\varphi_n)\big((F+t_n\varphi_n)^{\mathrm{Inv}}(\alpha_n)\big).\end{aligned}\tag{4.13}$$

The left inequality together with $\alpha_n\le F(F^{\mathrm{Inv}}(\alpha_n))$ implies

$$\begin{aligned}&F\big((F+t_n\varphi_n)^{\mathrm{Inv}}(\alpha_n)-\varepsilon_n\big)-F\big(F^{\mathrm{Inv}}(\alpha_n)\big)\\&\quad\le -t_n\varphi_n\big((F+t_n\varphi_n)^{\mathrm{Inv}}(\alpha_n)-\varepsilon_n\big).\end{aligned}$$

Expanding the left-hand side with $(F+t_n\varphi_n)^{\mathrm{Inv}}(\alpha_n)-\varepsilon_n-F^{\mathrm{Inv}}(\alpha_n)$ and reorganizing finally gives

$$\begin{aligned}&t_n^{-1}\big((F+t_n\varphi_n)^{\mathrm{Inv}}(\alpha_n)-F^{\mathrm{Inv}}(\alpha_n)\big)\\&\quad\le-\varphi_n\big((F+t_n\varphi_n)^{\mathrm{Inv}}(\alpha_n)-\varepsilon_n\big)\\&\qquad\times\left(\frac{F((F+t_n\varphi_n)^{\mathrm{Inv}}(\alpha_n)-\varepsilon_n)-F(F^{\mathrm{Inv}}(\alpha_n))}{(F+t_n\varphi_n)^{\mathrm{Inv}}(\alpha_n)-\varepsilon_n-F^{\mathrm{Inv}}(\alpha_n)}\right)^{-1}+\frac{\varepsilon_n}{t_n},\end{aligned}\tag{4.14}$$

where we note that the "big" fraction on the right-hand side of (4.14), by which we divided to get (4.14), is positive due to the strict monotonicity of F if we choose ε_n such that $(F+t_n\varphi_n)^{\mathrm{Inv}}(\alpha_n)-\varepsilon_n-F^{\mathrm{Inv}}(\alpha_n)\neq 0$. Similarly, since $\alpha_n>F(F^{\mathrm{Inv}}(\alpha_n)-\delta_n)$ for any $\delta_n>0$, from the right-hand side of (4.13), we obtain that

$$-t_n\varphi_n\big((F+t_n\varphi_n)^{\mathrm{Inv}}(\alpha_n)\big) \le F\big((F+t_n\varphi_n)^{\mathrm{Inv}}(\alpha_n)\big) - F\big(F^{\mathrm{Inv}}(\alpha_n)-\delta_n\big).$$

We expand the right-hand side with $(F+t_n\varphi_n)^{\mathrm{Inv}}(\alpha_n) - F^{\mathrm{Inv}}(\alpha_n)+\delta_n$, where we choose δ_n such that the latter term is non-zero, and reorganize the resulting inequality to obtain

$$\begin{aligned} & t_n^{-1}\big((F+t_n\varphi_n)^{\mathrm{Inv}}(\alpha_n) - F^{\mathrm{Inv}}(\alpha_n)\big) \\ & \quad \ge -\varphi_n\big((F+t_n\varphi_n)^{\mathrm{Inv}}(\alpha_n)\big) \\ & \quad \times \left(\frac{F((F+t_n\varphi_n)^{\mathrm{Inv}}(\alpha_n)) - F(F^{\mathrm{Inv}}(\alpha_n)-\delta_n)}{(F+t_n\varphi_n)^{\mathrm{Inv}}(\alpha_n) - F^{\mathrm{Inv}}(\alpha_n)+\delta_n}\right)^{-1} - \frac{\delta_n}{t_n}, \end{aligned} \tag{4.15}$$

where the "big" fraction is again positive by strict monotonicity of F.

In the following, we choose $\varepsilon_n, \delta_n = o(t_n)$, so that $t_n^{-1}\varepsilon_n$ in (4.14) and $t_n^{-1}\delta_n$ in (4.15) converge to zero. Further, by continuity of F^{Inv} and (4.11) the convergences

$$(F+t_n\varphi_n)^{\mathrm{Inv}}(\alpha_n) \to q_\alpha, \qquad F^{\mathrm{Inv}}(\alpha_n) \to q_\alpha \tag{4.16}$$

are valid. In particular, for large n, $(F+t_n\varphi_n)^{\mathrm{Inv}}(\alpha_n) - \varepsilon_n, F^{\mathrm{Inv}}(\alpha_n) - \delta_n \in (q_{\alpha_l} - \varepsilon, q_{\alpha_u} + \varepsilon)$, where ε is as in Assumption A.

Next, we aim to show (4.2) by using the bounds (4.14) and (4.15).

(ii) Let F be continuous at q_α. We start by bounding the "big" fractions in (4.14) and (4.15). By Assumption A, F is then continuous in a neighbourhood $(q_\alpha - \Delta, q_\alpha + \Delta)$ for some small $\Delta > 0$. From (4.16) and $\varepsilon_n, \delta_n \to 0$ it follows that the intervals

$$\begin{aligned} I_n &= \big[\min\big\{\big((F+t_n\varphi_n)^{\mathrm{Inv}}(\alpha_n) - \varepsilon_n\big), F^{\mathrm{Inv}}(\alpha_n)\big\}, \\ &\qquad \max\big\{\big((F+t_n\varphi_n)^{\mathrm{Inv}}(\alpha_n) - \varepsilon_n\big), F^{\mathrm{Inv}}(\alpha_n)\big\}\big], \\ J_n &= \big[\min\big\{(F+t_n\varphi_n)^{\mathrm{Inv}}(\alpha_n), \big(F^{\mathrm{Inv}}(\alpha_n) - \delta_n\big)\big\}, \\ &\qquad \max\big\{(F+t_n\varphi_n)^{\mathrm{Inv}}(\alpha_n), \big(F^{\mathrm{Inv}}(\alpha_n) - \delta_n\big)\big\}\big], \end{aligned} \tag{4.17}$$

are contained in $(q_\alpha - \Delta, q_\alpha + \Delta)$ for sufficiently large n. Hence, we can apply the extended mean value theorem for left- and right-sided derivatives to F over the intervals I_n and J_n to conclude that

$$\begin{aligned} & \min\big\{\big(\partial^-(F)(\xi_n)\big)^{-1}, \big(\partial^+(F)(\xi_n)\big)^{-1}\big\} \\ & \quad \le \left(\frac{F((F+t_n\varphi_n)^{\mathrm{Inv}}(\alpha_n) - \varepsilon_n) - F(F^{\mathrm{Inv}}(\alpha_n))}{(F+t_n\varphi_n)^{\mathrm{Inv}}(\alpha_n) - \varepsilon_n - F^{\mathrm{Inv}}(\alpha_n)}\right)^{-1} \\ & \quad \le \max\big\{\big(\partial^-(F)(\xi_n)\big)^{-1}, \big(\partial^+(F)(\xi_n)\big)^{-1}\big\} \end{aligned} \tag{4.18}$$

for some $\xi_n \in I_n$, and that

$$
\begin{aligned}
&\min\{(\partial^-(F)(\zeta_n))^{-1}, (\partial^+(F)(\zeta_n))^{-1}\} \\
&\quad \le \left(\frac{F((F+t_n\varphi_n)^{\mathrm{Inv}}(\alpha_n)) - F(F^{\mathrm{Inv}}(\alpha_n)-\delta_n)}{(F+t_n\varphi_n)^{\mathrm{Inv}}(\alpha_n) - F^{\mathrm{Inv}}(\alpha_n)+\delta_n}\right)^{-1} \\
&\quad \le \max\{(\partial^-(F)(\zeta_n))^{-1}, (\partial^+(F)(\zeta_n))^{-1}\}
\end{aligned}
\tag{4.19}
$$

for some $\zeta_n \in J_n$.

Further, to deal with the first factor on the right-hand sides of (4.14) and (4.15), note φ is also continuous in q_α since $\mathrm{Dsc}(\varphi) \subseteq \mathrm{Dsc}(F)$. The uniform convergence of φ_n to φ then implies that

$$
\varphi_n\big((F+t_n\varphi_n)^{\mathrm{Inv}}(\alpha_n)-\varepsilon_n\big), \varphi_n\big((F+t_n\varphi_n)^{\mathrm{Inv}}(\alpha_n)\big) \to \varphi(q_\alpha).
$$

Applying "lim sup" across (4.14), we obtain

$$
\begin{aligned}
&\limsup_n t_n^{-1}\big((F+t_n\varphi_n)^{\mathrm{Inv}}(\alpha_n) - F^{\mathrm{Inv}}(\alpha_n)\big) \\
&\quad \le \limsup_n \big(-\varphi(q_\alpha)\big)\left(\frac{F((F+t_n\varphi_n)^{\mathrm{Inv}}(\alpha_n)-\varepsilon_n) - F(F^{\mathrm{Inv}}(\alpha_n))}{(F+t_n\varphi_n)^{\mathrm{Inv}}(\alpha_n)-\varepsilon_n - F^{\mathrm{Inv}}(\alpha_n)}\right)^{-1} \qquad \text{Lemma 17} \\
&\quad \le \begin{cases} -\varphi(q_\alpha)\min\Big\{\liminf_n\big(\partial^-(F)(\xi_n)\big)^{-1}, \liminf_n\big(\partial^+(F)(\xi_n)\big)^{-1}\Big\} & \text{if } \varphi(q_\alpha)\ge 0, \\ -\varphi(q_\alpha)\max\Big\{\limsup_n\big(\partial^-(F)(\xi_n)\big)^{-1}, \limsup_n\big(\partial^+(F)(\xi_n)\big)^{-1}\Big\} & \text{if } \varphi(q_\alpha)\le 0; \end{cases}
\end{aligned}
\tag{4.20}
$$

where we use (4.18) for the second inequality. Since $\xi_n \to q_\alpha$, and since $\partial^-(F)$ and $\partial^+(F)$ are càdlàg or làdcàg in q_α by Assumption A, we further have

$$
\begin{aligned}
&\limsup_n\big(\partial^-(F)(\xi_n)\big)^{-1} \le \max\Big(\big(\partial^-(F)(q_\alpha+)\big)^{-1}, \big(\partial^-(F)(q_\alpha-)\big)^{-1}\Big), \\
&\limsup_n\big(\partial^+(F)(\xi_n)\big)^{-1} \le \max\Big(\big(\partial^+F\big)(q_\alpha+)\big)^{-1}, \big(\partial^+(F)(q_\alpha-)\big)^{-1}\Big),
\end{aligned}
$$

so that in case $\varphi(q_\alpha)\ge 0$ we obtain

$$
\begin{aligned}
&\limsup_n t_n^{-1}\big((F+t_n\varphi_n)^{\mathrm{Inv}}(\alpha_n) - F^{\mathrm{Inv}}(\alpha_n)\big) \\
&\quad \le -\varphi(q_\alpha)\max\big\{\big(\partial^\pm(F)(q_\alpha-)\big)^{-1}, \big(\partial^\pm(F)(q_\alpha+)\big)^{-1}\big\} \\
&\quad = \big(-\varphi\circ F^{\mathrm{Inv}}\cdot\big(\partial^-(F)\circ F^{\mathrm{Inv}}\big)^{-1}\big)_\vee(\alpha)
\end{aligned}
$$

by Lemma 10, (i) applied to $-\varphi$. The case $\varphi(q_\alpha)\le 0$ is dealt with similarly, together showing the second inequality in (4.2). The first follows similarly: Applying "lim inf" on (4.15) and using

(4.19) yields

$$
\begin{aligned}
&\liminf_n t_n^{-1}\big((F+t_n\varphi_n)^{\mathrm{Inv}}(\alpha_n)-F^{\mathrm{Inv}}(\alpha_n)\big)\\
&\geq \begin{cases} -\varphi(q_\alpha)\max\Big\{\limsup_n\big(\partial^-(F)(\zeta_n)\big)^{-1},\limsup_n\big(\partial^+(F)(\zeta_n)\big)^{-1}\Big\} & \text{if } \varphi(q_\alpha)\geq 0,\\ -\varphi(q_\alpha)\min\Big\{\liminf_n\big(\partial^-(F)(\zeta_n)\big)^{-1},\liminf_n\big(\partial^+(F)(\zeta_n)\big)^{-1}\Big\} & \text{if } \varphi(q_\alpha)\leq 0,\end{cases}\\
&\geq \Big(-\varphi\circ F^{\mathrm{Inv}}\cdot\big(\partial^-(F)\circ F^{\mathrm{Inv}}\big)^{-1}\Big)_{\wedge}(\alpha).
\end{aligned}
$$

This concludes the proof of case (ii). The remaining cases (iii) $\alpha = F(q_\alpha-)$ and (iv) $\alpha = F(q_\alpha)$ for $q_\alpha \in \mathrm{Dsc}(F)$ are dealt with in the supplementary material. □

4.3.2. *Details for the proof of Theorem 5*

Recall from Holzmann and Klar [5] the identity

$$I_\tau(x,F)=\tau\int_x^\infty\big(1-F(y)\big)\,\mathrm{d}y-(1-\tau)\int_{-\infty}^x F(y)\,\mathrm{d}y. \tag{4.21}$$

We start with some technical preliminaries.

Lemma 18. *We have that for $x_1, x_2 \in \mathbb{R}$,*

$$I_\tau(x_1,F)-I_\tau(x_2,F)=(x_2-x_1)\bigg[\tau+(1-2\tau)\int_0^1 F\big(x_2+s(x_1-x_2)\big)\,\mathrm{d}s\bigg]. \tag{4.22}$$

Lemma 19. *We have*

$$\min\{\tau_l,1-\tau_u\}\leq\tau+(1-2\tau)s\leq 3/2,\quad \tau\in[\tau_l,\tau_u],\ s\in[0,1]. \tag{4.23}$$

Next, we discuss Lipschitz-properties of relevant maps.

Lemma 20. *For any $x_1, x_2, y \in \mathbb{R}$ and $\tau \in [\tau_l, \tau_u]$,*

$$\big|I_\tau(x_1,y)-I_\tau(x_2,y)\big|\leq|x_2-x_1| \tag{4.24}$$

Further, for any $\tau, \tau' \in [\tau_l, \tau_u]$ and $x, y \in \mathbb{R}$,

$$\big|I_\tau(x,y)-I_{\tau'}(x,y)\big|\leq\big|\tau-\tau'\big|\big(|x|+|y|\big) \tag{4.25}$$

Finally, the map $\tau \mapsto \mu_\tau$, $\tau \in [\tau_l, \tau_u]$, is Lipschitz-continuous.

The proofs of Lemmas 18, 19 and 20 are given in Section D.

Details for Step 1.

Proof of Lemma 11. *Proof of (4.3).*

By Lemma 20, the function class

$$\mathcal{F} = \big\{ y \mapsto -I_\tau(\mu_\tau, y) \mid \tau \in [\tau_l, \tau_u] \big\}$$

is Lipschitz-continuous in the parameter τ for given y, and the Lipschitz constant (which depends on y) is square-integrable under F. Indeed, the triangle inequality first gives

$$\big| I_\tau(\mu_\tau, y) - I_{\tau'}(\mu_{\tau'}, y) \big| \le \big| I_\tau(\mu_\tau, y) - I_{\tau'}(\mu_\tau, y) \big| + \big| I_{\tau'}(\mu_\tau, y) - I_{\tau'}(\mu_{\tau'}, y) \big|.$$

Using (4.25) the first summand on the right fulfils

$$\big| I_\tau(\mu_\tau, y) - I_{\tau'}(\mu_\tau, y) \big| \le \big| \tau - \tau' \big| \big(|\mu_{\tau_l}| \vee |\mu_{\tau_u}| + |y| \big),$$

and the second is bounded by

$$\big| I_{\tau'}(\mu_\tau, y) - I_{\tau'}(\mu_{\tau'}, y) \big| \le |\mu_\tau - \mu_{\tau'}| \le \big| \tau - \tau' \big| \frac{|\mu_{\tau_u}| \vee |\mu_{\tau_l}| + \mathbb{E}[|Y|]}{a},$$

utilizing (4.24) and (D.3). Thus

$$\big| I_\tau(\mu_\tau, y) - I_{\tau'}(\mu_{\tau'}, y) \big| \le \big| \tau - \tau' \big| \big(C + |y| \big) \tag{4.26}$$

for some constant $C \ge 1$. By example 19.7 in combination with Theorem 19.5 in van der Vaart [13], $\mathcal{F}$ is a Donsker class, so that $\sqrt{n}(\psi_n(\mu_.) - \psi_0(\mu_.))$ converges to the process Z. The same reasoning as in Theorem 8, Holzmann and Klar [5], then shows continuity of the sample paths of Z with respect to the Euclidean distance on $[\tau_l, \tau_u]$.

Proof of (4.4).

Setting

$$\mathcal{F}_{\delta_n} = \big\{ y \mapsto I_\tau(\mu_\tau + x, y) - I_\tau(\mu_\tau, y) \mid |x| \le \delta_n, \tau \in [\tau_l, \tau_u] \big\}$$

we estimate that

$$\sup_{\|\varphi\|_{[\tau_l,\tau_u]} \le \delta_n} \sup_{\tau \in [\tau_l, \tau_u]} \sqrt{n} \big| \psi_n(\mu_. + \varphi)(\tau) - \psi_0(\mu_. + \varphi)(\tau) - \big[\psi_n(\mu_.)(\tau) - \psi_0(\mu_.)(\tau) \big] \big|$$

is smaller than $\|\sqrt{n}(F_n - F)\|_{\mathcal{F}_{\delta_n}}$. From the triangle inequality, for any $\tau, \tau' \in [\tau_l, \tau_u]$ and $x, x' \in [-\delta_1, \delta_1]$ we first obtain

$$\begin{aligned} &\big| I_\tau(\mu_\tau + x, y) - I_\tau(\mu_\tau, y) - \big(I_{\tau'}\big(\mu_{\tau'} + x', y\big) - I_{\tau'}(\mu_{\tau'}, y) \big) \big| \\ &\quad \le \big| I_\tau(\mu_\tau + x, y) - I_{\tau'}\big(\mu_{\tau'} + x', y\big) \big| + \big| I_\tau(\mu_\tau, y) - I_{\tau'}(\mu_{\tau'}, y) \big|, \end{aligned}$$

where the second term was discussed above and the first can be handled likewise to conclude

$$\big| I_\tau(\mu_\tau + x, y) - I_{\tau'}\big(\mu_{\tau'} + x', y\big) \big| \le \big(\big|\tau - \tau'\big| + \big|x - x'\big| \big) \big(C + \delta_1 + |y| \big) \tag{4.27}$$

with the same C as above. Hence,

$$\begin{aligned}&\big|I_\tau(\mu_\tau + x, y) - I_\tau(\mu_\tau, y) - \big(I_{\tau'}\big(\mu_{\tau'} + x', y\big) - I_{\tau'}(\mu_{\tau'}, y)\big)\big|\\&\quad\le m(y)\big(\big|\tau - \tau'\big| + \big|x - x'\big|\big)\end{aligned}$$

with Lipschitz-constant

$$m(y) = 2C + \delta_1 + 2|y|,$$

which is square-integrable by assumption on F. By example 19.7 in van der Vaart [13] the bracketing number $N_{[]}(\epsilon, \mathcal{F}_{\delta_1}, L_2(F))$ of $\mathcal{F}_{\delta_1}$ is of order ϵ^{-2}, so that for the bracketing integral

$$J_{[]}\big(\epsilon_n, \mathcal{F}_{\delta_n}, L_2(F)\big) \le J_{[]}\big(\epsilon_n, \mathcal{F}_{\delta_1}, L_2(F)\big) \to 0 \quad \text{as } \epsilon_n \to 0.$$

From (4.24), the class $\mathcal{F}_{\delta_n}$ has envelope δ_n, and hence using Corollary 19.35 in van der Vaart [13], we obtain

$$\mathbb{E}\big[\big\|\sqrt{n}(F_n - F)\big\|_{\mathcal{F}_{\delta_n}}\big] \le J_{[]}\big(\delta_n, \mathcal{F}_{\delta_n}, L_2(F)\big) \to 0. \tag{4.28}$$

An application of the Markov inequality ends the proof of (4.4). □

Details for Step 2.

Proof of Lemma 12. Given $\tau \in [\tau_l, \tau_u]$, by (4.22) and the lower bound in (4.23), the function $x \mapsto I_\tau(x, F)$ is strictly decreasing, and its image is all of $\mathbb{R}$. Hence, for any $z \in \mathbb{R}$ there is a unique x satisfying $I_\tau(x, F) = z$, which shows that ψ_0 is invertible.

Next for fixed $\varphi \in \ell^\infty[\tau_l, \tau_u]$ the preimage $((I_\tau(\cdot, F))^{\mathrm{Inv}}([-\|\varphi\|, \|\varphi\|])$ is by monotonicity an interval $[L_\tau, U_\tau]$, $|L_\tau|, |U_\tau| < \infty$. By (4.21),

$$I_\tau(x, F) = \tau\left\{\int_x^\infty \big(1 - F(y)\big)\,\mathrm{d}y + \int_{-\infty}^x F(y)\,\mathrm{d}y\right\} - \int_{-\infty}^x F(y)\,\mathrm{d}y,$$

thus the map $\tau \mapsto I_\tau(x, F)$ is increasing, showing $L_{\tau'} \le L_\tau$ and $U_{\tau'} \le U_\tau$ for $\tau \ge \tau'$. Hence, the solution of $z = I_\tau(x, F)$ for $z \in [-\|\varphi\|, \|\varphi\|]$ lies in $[L_{\tau_l}, U_{\tau_u}]$, which means that $\psi_0^{\mathrm{Inv}}(\varphi)$ is bounded. □

Before we turn to the proof of Lemma 13, we require the following technical assertions about ψ_0^{Inv}.

Lemma 21. *Given $t > 0$ and $\nu \in \ell^\infty[\tau_l, \tau_u]$, we have that*

$$\begin{aligned}&t^{-1}\big(\psi_0^{\mathrm{Inv}}(t\nu) - \psi_0^{\mathrm{Inv}}(0)\big)(\tau)\\&\quad= \nu(\tau)\left\{\tau + (1 - 2\tau)\int_0^1 F\big(\mu_\tau + s\big(\psi_0^{\mathrm{Inv}}(t\nu)(\tau) - \mu_\tau\big)\big)\,\mathrm{d}s\right\}^{-1}.\end{aligned} \tag{4.29}$$

In particular, if $\nu_n \in \ell^\infty[\tau_l, \tau_u]$ *with* $\|\nu_n\| \to 0$, *then* $\|\psi_0^{\mathrm{Inv}}(\nu_n)(\cdot) - \mu_\cdot\| \to 0$, *so that for any* $\tau \in [\tau_l, \tau_u]$ *and* $\tau_n \to \tau$,

$$\psi_0^{\mathrm{Inv}}(\nu_n)(\tau_n) - \mu_{\tau_n} \to 0. \tag{4.30}$$

The proof of Lemma 21 is given in Section D.

We now introduce the following notation for the sequel. Given $\varphi \in \ell^\infty[\tau_l, \tau_u]$ let $c_\varphi \in \ell^\infty[\tau_l, \tau_u]$ be defined by

$$c_\varphi(\tau) = \tau + (1 - 2\tau) \int_0^1 F\big(\mu_\tau + s\varphi(\tau)\big)\,\mathrm{d}s. \tag{4.31}$$

Proof of Lemma 13. Let $t_n \to 0$, $t_n > 0$, $(\varphi_n)_n \subset \ell^\infty[\tau_l, \tau_u]$ with $\varphi_n \to \varphi \in \mathcal{C}[\tau_l, \tau_u]$ with respect to $\mathrm{d}_{\mathrm{hypi}}$ and thus uniformly by Proposition 2.1 in Bücher, Segers and Volgushev [2]. From (4.29), and using the notation (4.31) we can write

$$t_n^{-1}\big(\psi_0^{\mathrm{Inv}}(t_n\varphi_n) - \psi_0^{\mathrm{Inv}}(0)\big) = \varphi_n / c_{\kappa_n}, \qquad \kappa_n(\tau) = \psi_0^{\mathrm{Inv}}(t_n\varphi_n)(\tau) - \mu_\tau$$

and we need to show that

$$\varphi_n / c_{\kappa_n} \to \dot{\psi}_0^{\mathrm{Inv}}(\varphi) = \varphi / c_0 \tag{4.32}$$

with respect to $\mathrm{d}_{\mathrm{hypi}}$, where $c_0(\tau) = \tau + (1 - 2\tau)F(\mu_\tau)$.

Now, since $\varphi_n \to \varphi$ uniformly and φ is continuous, to obtain (4.32) if suffices by Lemma 14, (i) and (iv), to show that $c_{\kappa_n} \to c_0$ under $\mathrm{d}_{\mathrm{hypi}}$. To this end, by Lemma A.4, Bücher, Segers and Volgushev [2] and Lemma 14, (iv), it suffices to show that under $\mathrm{d}_{\mathrm{hypi}}$

$$h_n(\tau) = \int_0^1 F\big(\mu_\tau + s\big(\psi_0^{\mathrm{Inv}}(t_n\varphi_n)(\tau) - \mu_\tau\big)\big)\,\mathrm{d}s \to h(\tau) := F(\mu_\tau), \tag{4.33}$$

for which we shall use Corollary A.7 in Bücher, Segers and Volgushev [2]. Let

$$\mathbb{T} = [\tau_l, \tau_u], \qquad \mathcal{S} = \mathbb{T} \setminus \big\{\tau \in [\tau_l, \tau_u] \mid F \text{ is not continuous in } \mu_\tau\big\},$$

so that $\mathcal{S}$ is dense in $\mathbb{T}$ and $h|_{\mathcal{S}}$ is continuous. Observe that $\tau_l, \mu_{\tau_u} \in \mathcal{S}$ by assumption. Using the notation from Bücher, Segers and Volgushev [2], Appendix A.2, for $F(\mu_\tau-) = \lim_{\tau' \nearrow \tau} F(\mu_{\tau'})$ we have that

$$(h|_{\mathcal{S}})_\wedge^{\mathcal{S}:\mathbb{T}} = h_\wedge = F(\mu_\cdot-) \quad \text{and} \quad (h|_{\mathcal{S}})_\vee^{\mathcal{S}:\mathbb{T}} = h_\vee = h, \tag{4.34}$$

where the first equalities follow from the discussion in Bücher, Segers and Volgushev [2], Appendix A.2, and the second equalities from Lemma 14, (iii) below. If we show that

(i) for all $\tau \in [\tau_l, \tau_u]$ with $\tau_n \to \tau$ it holds that $\liminf_n h_n(\tau_n) \geq F(\mu_\tau-)$ and
(ii) for all $\tau \in [\tau_l, \tau_u]$ with $\tau_n \to \tau$ it holds that $\limsup_n h_n(\tau_n) \leq F(\mu_\tau)$,

Corollary A.7 in Bücher, Segers and Volgushev [2] implies (4.33), which concludes the proof of the convergence in (4.32).

To this end, concerning (i), we compute that

$$F(\mu_\tau-) \le \int_0^1 \liminf_n F\big(\mu_{\tau_n} + s\big(\psi_0^{\mathrm{Inv}}(t_n\varphi_n)(\tau_n) - \mu_{\tau_n}\big)\big)\,\mathrm{d}s$$
$$\le \liminf_n \int_0^1 F\big(\mu_{\tau_n} + s\big(\psi_0^{\mathrm{Inv}}(t_n\varphi_n)(\tau_n) - \mu_{\tau_n}\big)\big)\,\mathrm{d}s = \liminf_n h_n(\tau_n),$$

where the first inequality follows from (4.30) and the fact that $F(\mu_\tau-) \le F(\mu_\tau)$, and the second inequality follows from Fatou's lemma. For (ii), we argue analogously

$$F(\mu_\tau) \ge \int_0^1 \limsup_n F\big(\mu_{\tau_n} + s\big(\psi_0^{\mathrm{Inv}}(t_n\varphi_n)(\tau_n) - \mu_{\tau_n}\big)\big)\,\mathrm{d}s$$
$$\ge \limsup_n \int_0^1 F\big(\mu_{\tau_n} + s\big(\psi_0^{\mathrm{Inv}}(t_n\varphi_n)(\tau_n) - \mu_{\tau_n}\big)\big)\,\mathrm{d}s = \limsup_n h_n(\tau_n).$$

This concludes the proof of the lemma. □

4.3.3. *Details for the proof of Theorem 8*

We let $\psi_n^*(\varphi)(\tau) = -I_\tau(\varphi(\tau), F_n^*)$, $\varphi \in \ell^\infty[\tau_l, \tau_u]$, and denote by P_n^* the conditional law of $Y_1^*, \dots, Y_n^*$ given $Y_1, \dots, Y_n$, and by $\mathbb{E}_n^*$ expectation under this conditional law.

Lemma 22. *We have, almost surely, conditionally on $Y_1, Y_2, \dots,$ the following statements.*

(i) *If $\mathbb{E}[|Y|] < \infty$, then*

$$\sup_{\tau\in[\tau_l,\tau_u]} \big|\mu_{\tau,n}^* - \hat\mu_{\tau,n}\big| = o_{\mathrm{P}_n^*}(1). \tag{4.35}$$

Now assume $\mathbb{E}[Y^2] < \infty$.

(ii) *Weakly in $(\ell^\infty[\tau_l, \tau_u], \|\cdot\|)$ it holds that*

$$\sqrt{n}\big(\psi_n^*(\hat\mu_{\cdot,n}) - \psi_n(\hat\mu_{\cdot,n})\big) \to Z \tag{4.36}$$

with Z as in Theorem 5.

(iii) *For every sequence $\delta_n \to 0$ it holds that*

$$\begin{aligned}\sup_{\|\varphi\|\le\delta_n} \sup_{\tau\in[\tau_l,\tau_u]} \sqrt{n}\big|&\psi_n^*(\hat\mu_{\cdot,n}+\varphi)(\tau) - \psi_n(\hat\mu_{\cdot,n}+\varphi)(\tau)\\ &- \big[\psi_n^*(\hat\mu_{\cdot,n})(\tau) - \psi_n(\hat\mu_{\cdot,n}+)(\tau)\big]\big| = o_{\mathrm{P}_n^*}(1).\end{aligned} \tag{4.37}$$

(iv) *Weakly in $(\ell^\infty[\tau_l, \tau_u], \|\cdot\|)$ we have that*

$$\sqrt{n}\big(\psi_n\big(\mu_{\cdot,n}^*\big) - \psi_n(\hat\mu_{\cdot,n})\big) \to Z. \tag{4.38}$$

The proof of Lemma 22 is contained in Section D.

Lemma 23. *The map ψ_n is invertible. Further, if $t_n \to 0$, $\varphi \in \mathcal{C}[\tau_l, \tau_u]$ and $\varphi_n \to \varphi$ with respect to $\mathrm{d}_{\mathrm{hypi}}$ and hence uniformly, we have that almost surely, conditionally on $Y_1, Y_2, \dots$,*

$$t_n^{-1}\big(\psi_n^{\mathrm{Inv}}(t_n\varphi_n) - \psi_n^{\mathrm{Inv}}(0)\big) \to \dot{\psi}_0^{\mathrm{Inv}}(\varphi) \tag{4.39}$$

with respect to the hypi-semimetric.

Proof. The first part follows from Lemma 12 with F in ψ_0 replaced by F_n in ψ_n as no specific assumptions on F were used in that lemma. For (4.39), with the same calculations as for Lemma 21 we obtain the representation

$$\begin{aligned}
&t_n^{-1}\big(\psi_n^{\mathrm{Inv}}(t_n\varphi_n) - \psi_n^{\mathrm{Inv}}(0)\big)(\tau)\\
&\quad= \varphi_n(\tau)\bigg\{\tau + (1-2\tau)\int_0^1 F_n\big(\hat{\mu}_{\tau,n} + s\big(\psi_n^{\mathrm{Inv}}(t_n\varphi_n)(\tau) - \hat{\mu}_{\tau,n}\big)\big)\,\mathrm{d}s\bigg\}^{-1},
\end{aligned}$$

and we have to prove hypi-convergence to $\dot{\psi}_0^{\mathrm{Inv}}(\varphi)$. By the same reductions as in the proof of Theorem 13, it suffices to prove the hypi-convergence of

$$h_n(\tau) = \int_0^1 F_n\big(\hat{\mu}_{\tau,n} + s\big(\psi_n^{\mathrm{Inv}}(t_n\varphi_n)(\tau) - \hat{\mu}_{\tau,n}\big)\big)\,\mathrm{d}s$$

to $h(\tau)$ for almost every sequence $Y_1, Y_2, \dots$. To this end, observe that for any $s \in [0,1]$ the sequence $\hat{\mu}_{\tau,n} + s(\psi_n^{\mathrm{Inv}}(t_n\varphi_n)(\tau) - \hat{\mu}_{\tau,n})$ converges to μ_τ almost surely by the same arguments as in Lemma 21. Since μ_τ is continuous in τ, for any sequence $\tau_n \to \tau$ the almost sure convergence $\hat{\mu}_{\tau_n,n} + s(\psi_n^{\mathrm{Inv}}(t_n\varphi_n)(\tau_n) - \hat{\mu}_{\tau_n,n}) \to \mu_\tau$ holds. By adding and subtracting $F_n(\hat{\mu}_{\tau,n} + s(\psi_n^{\mathrm{Inv}}(t_n\varphi_n)(\tau) - \hat{\mu}_{\tau,n}))$ and using Lemma 16, we now can estimate

$$\begin{aligned}
F(\mu_\tau-) &\le \int_0^1 \liminf_n\big(F_n\big(\hat{\mu}_{\tau,n} + s\big(\psi_n^{\mathrm{Inv}}(t_n\varphi_n)(\tau) - \hat{\mu}_{\tau,n}\big)\big)\big)\\
&\qquad + \limsup_n\big(F\big(\hat{\mu}_{\tau,n} + s\big(\psi_n^{\mathrm{Inv}}(t_n\varphi_n)(\tau) - \hat{\mu}_{\tau,n}\big)\big)\\
&\qquad - F_n\big(\hat{\mu}_{\tau,n} + s\big(\psi_n^{\mathrm{Inv}}(t_n\varphi_n)(\tau) - \hat{\mu}_{\tau,n}\big)\big)\big)\,\mathrm{d}s\\
&\le \int_0^1 \liminf_n F_n\big(\hat{\mu}_{\tau,n} + s\big(\psi_n^{\mathrm{Inv}}(t_n\varphi_n)(\tau) - \hat{\mu}_{\tau,n}\big)\big)\,\mathrm{d}s + \limsup_n \|F_n - F\|_{\mathbb{R}}\\
&\le \liminf_n \int_0^1 F_n\big(\hat{\mu}_{\tau,n} + s\big(\psi_n^{\mathrm{Inv}}(t_n\varphi_n)(\tau) - \hat{\mu}_{\tau,n}\big)\big)\,\mathrm{d}s = \liminf_n h_n(\tau_n)
\end{aligned}$$

almost surely, where the 'lim sup' vanishes due to the Glivenko–Cantelli-theorem for the empirical distribution function. Similarly, we almost surely have

$$\begin{aligned} F(\mu_\tau) &\geq \int_0^1 \limsup_n F_n\big(\hat{\mu}_{\tau,n} + s\big(\psi_n^{\mathrm{Inv}}(t_n\varphi_n)(\tau) - \hat{\mu}_{\tau,n}\big)\big)\,\mathrm{d}s + \liminf_n \|F_n - F\|_{\mathbb{R}} \\ &\geq \limsup_n \int_0^1 F_n\big(\hat{\mu}_{\tau,n} + s\big(\psi_n^{\mathrm{Inv}}(t_n\varphi_n)(\tau) - \hat{\mu}_{\tau,n}\big)\big)\,\mathrm{d}s = \limsup_n h_n(\tau_n). \end{aligned}$$

The proof is concluded as that of Theorem 13 by using Corollary A.7, Bücher, Segers and Volgushev [2]. □

Proof of Theorem 8. Set $t_n = 1/\sqrt{n}$ and define the function $g_n(\varphi) = t_n^{-1}(\psi_n^{\mathrm{Inv}}(t_n\varphi) - \psi_n^{\mathrm{Inv}}(0))$. Then from (4.39) the hypi-convergence $g_n(\varphi_n) \to \dot{\psi}_0^{\mathrm{Inv}}(\varphi)$ holds almost surely, whenever $\varphi \in \mathcal{C}[\tau_l, \tau_u]$ and $\varphi_n \to \varphi$ with respect to $\mathrm{d}_{\mathrm{hypi}}$. In addition $\sqrt{n}(\psi_n(\mu^*_{\cdot,n}) - \psi_n(\hat{\mu}_{\cdot,n})) \to Z$ conditional in distribution with respect to the sup-norm, almost surely, by (4.36), where Z is continuous almost surely. Hence the convergence is also valid with respect to $\mathrm{d}_{\mathrm{hypi}}$, such that

$$\sqrt{n}\big(\mu^*_{\cdot,n} - \hat{\mu}_{\cdot,n}\big) = g_n\big(\sqrt{n}\big(\psi_n\big(\mu^*_{\cdot,n}\big) - \psi_n(\hat{\mu}_{\cdot,n})\big)\big) \to \dot{\psi}_0^{\mathrm{Inv}}(Z)$$

holds conditionally in distribution, almost surely, by using the extended continuous mapping theorem, Theorem B.3, in Bücher, Segers and Volgushev [2]. □

Acknowledgements

The authors are grateful to an associate editor and a referee for detailed and helpful comments. Tobias Zwingmann acknowledges financial support from of the Cusanuswerk for providing a dissertation scholarship.

Supplementary Material

Supplement to "Weak convergence of quantile and expectile processes under general assumptions" (DOI: 10.3150/19-BEJ1127SUPP; .pdf). The supplement contains the remaining technical details.

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Received August 2017 and revised April 2019