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Long-time heat kernel estimates and upper rate functions of Brownian motion type for symmetric jump processes

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Let X be a symmetric jump process on $\mathbb{R}^d$ such that the corresponding jumping kernel $J(x, y)$ satisfies

$$J(x,y) \le \frac{c}{|x-y|^{d+2}\log^{1+\varepsilon}(e+|x-y|)}$$

for all $x, y \in \mathbb{R}^d$ with $|x-y| \ge 1$ and some constants $c, \varepsilon > 0$. Under additional mild assumptions on $J(x, y)$ for $|x-y| < 1$, we show that $C\sqrt{r \log\log r}$ with some constant $C > 0$ is an upper rate function of the process X, which enjoys the same form as that for Brownian motions. The approach is based on heat kernel estimates of large time for the process X. As a by-product, we also obtain two-sided heat kernel estimates of large time for symmetric jump processes whose jumping kernels are comparable to

$$\frac{1}{|x-y|^{d+2+\varepsilon}}$$

for all $x, y \in \mathbb{R}^d$ with $|x-y| \ge 1$ and some constant $\varepsilon > 0$.

Keywords: Dirichlet form; heat kernel; symmetric jump process; upper rate function

1. Introduction and main results

In this paper, we are concerned with upper rate functions, which are a quantitative expression of conservativeness, for a class of symmetric jump processes on $\mathbb{R}^d$. In particular, we investigate conditions on jumping kernels such that the corresponding upper rate functions are of the iterated logarithm type.

It is well known that by Kolmogorov's test (see, e.g., [16], 4.12), the function $R(t) = \sqrt{ct\log\log t}$ with constant $c > 0$ is an upper rate function for the standard Brownian motion on $\mathbb{R}^d$ if and only if $c > 2$. This fact immediately implies Khintchine's law of the iterated logarithm. Similar results of this type are true even for a large class of Lévy processes. For example, earlier Gnedenko [15] (see also [20], Proposition 48.9) showed that if a Lévy process $X = (\{X_t\}_{t\ge 0}, \mathbb{P})$

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on $\mathbb{R}$ satisfies $\mathbb{E}X_1 = 0$ and $\mathbb{E}X_1^2 < \infty$, then

$$\limsup_{t\to\infty} \frac{|X_t|}{\sqrt{2t\log\log t}} = \left(\mathbb{E}X_1^2\right)^{1/2}, \quad \text{a.s.}$$

Sirao [22] also obtained analogous results in terms of integral tests on the distribution function of X. We note that such results as [15,22] do not hold in general for Lévy processes with the infinite second moment, for instance, symmetric α-stable processes with $\alpha \in (0,2)$ (see [17] or [19], Theorem 2.1).

The purpose of this paper is to establish upper rate functions of the form $\sqrt{t\log\log t}$ for a class of non-Lévy symmetric jump processes generated by regular Dirichlet forms on $L^2(\mathbb{R}^d; \mathrm{d}x)$, which we introduce later. Let $J(x,y)$ be a non-negative measurable function on $\mathbb{R}^d \times \mathbb{R}^d$, and set

$$\mathcal{D} = \left\{ f \in L^2\left(\mathbb{R}^d; \mathrm{d}x\right) \,\middle|\, \iint_{x\neq y} \left(f(y) - f(x)\right)^2 J(x,y)\,\mathrm{d}x\,\mathrm{d}y < \infty \right\},$$

$$\mathcal{E}(f,f) = \iint_{x\neq y} \left(f(y) - f(x)\right)^2 J(x,y)\,\mathrm{d}x\,\mathrm{d}y, \qquad f \in \mathcal{D}.$$

Throughout this paper, we always impose the following.

Assumption 1.1. The function $J(x,y)$ satisfies

(i) $J(x,y) = J(y,x)$ for all $x \neq y$;
(ii) there exist constants $0 < \kappa_1 \le \kappa_2 < \infty$ and $0 < \alpha_1 \le \alpha_2 < 2$ such that for all $x, y \in \mathbb{R}^d$ with $0 < |x-y| < 1$,

$$\frac{\kappa_1}{|x-y|^{d+\alpha_1}} \le J(x,y) \le \frac{\kappa_2}{|x-y|^{d+\alpha_2}}; \tag{1.1}$$

(iii)

$$\sup_{x\in\mathbb{R}^d} \int_{\{|x-y|\ge 1\}} J(x,y)\,\mathrm{d}y < \infty. \tag{1.2}$$

Under (1.1) and (1.2), it is obvious that

$$\sup_{x\in\mathbb{R}^d} \int \left(1 \wedge |x-y|^2\right) J(x,y)\,\mathrm{d}y < \infty. \tag{1.3}$$

Denote by $C_c^{\mathrm{lip}}(\mathbb{R}^d)$ the set of Lipschitz continuous functions on $\mathbb{R}^d$ with compact support. Then, due to (1.3), we have $C_c^{\mathrm{lip}}(\mathbb{R}^d) \subset \mathcal{D}$. Let $\mathcal{F}$ be the closure of $C_c^{\mathrm{lip}}(\mathbb{R}^d)$ with respect to the norm $\|f\|_{\mathcal{E}_1} := \sqrt{\mathcal{E}(f,f) + \|f\|_2^2}$ on $\mathcal{D}$. Then it is easy to check that the bilinear form $(\mathcal{E}, \mathcal{F})$ is a symmetric regular Dirichlet form on $L^2(\mathbb{R}^d; \mathrm{d}x)$, see, for example, [14], Example 1.2.4. The function $J(x,y)$ is called the jumping kernel corresponding to $(\mathcal{E},\mathcal{F})$. Associated with the regular Dirichlet form $(\mathcal{E},\mathcal{F})$ is a symmetric Hunt process $X=(\{X_t\}_{t\ge 0},\{\mathbb{P}^x\}_{x\in\mathbb{R}^d\setminus\mathcal{N}})$ with state space $\mathbb{R}^d\setminus\mathcal{N}$, where $\mathcal{N}\subset\mathbb{R}^d$ is a properly exceptional set for $(\mathcal{E},\mathcal{F})$.

The main result is as follows.

Theorem 1.2. *Let $X=(\{X_t\}_{t\ge 0},\{\mathbb{P}^x\}_{x\in\mathbb{R}^d\setminus\mathcal{N}})$ be the symmetric Hunt process generated by the regular Dirichlet form $(\mathcal{E},\mathcal{F})$ as above. Let $J(x,y)$ be the jumping kernel corresponding to $(\mathcal{E},\mathcal{F})$ such that Assumption 1.1 holds. Then, we have the following two statements.*

(1) *If there exist positive constants c and ε such that for any $x,y\in\mathbb{R}^d$ with $|x-y|\ge 1$,*

$$J(x,y)\le \frac{c}{|x-y|^{d+2}\log^{1+\varepsilon}(e+|x-y|)}, \tag{1.4}$$

then there exists a constant $C_0>0$ such that for all $x\in\mathbb{R}^d\setminus\mathcal{N}$,

$$\mathbb{P}^x\big(|X_t-x|\le C_0\sqrt{t\log\log t}\ \textit{for all sufficiently large } t\big)=1. \tag{1.5}$$

(2) *If there exists a positive constant c such that for any $x,y\in\mathbb{R}^d$ with $|x-y|\ge 1$,*

$$J(x,y)\le \frac{c}{|x-y|^{d+2}},$$

and

$$\sup_{x\in\mathbb{R}^d}\int_{\mathbb{R}^d}|x-y|^2 J(x,y)\,\mathrm{d}y<\infty, \tag{1.6}$$

then there exists a constant $c_0>0$ such that for all $x\in\mathbb{R}^d\setminus\mathcal{N}$,

$$\mathbb{P}^x\big(|X_t-x|\le c_0\sqrt{t\log\log t}\ \textit{for all sufficiently large } t\big)=0.$$

The condition (1.6) implies that the jumping kernel of X has the finite second moment. It is clear that (1.6) holds true when (1.1) and (1.4) are satisfied. (1.5) indicates that the function $C_0\sqrt{t\log\log t}$ is the so-called upper rate function of the process X, which describes the forefront of the process X. As we mentioned before, $\sqrt{(2+\varepsilon)t\log\log t}$ with $\varepsilon>0$ is an upper rate function for the standard Brownian motion on $\mathbb{R}^d$. Therefore, Theorem 1.2 shows that if the jumping kernel of X satisfies the condition as in Theorem 1.2(1), then X enjoys upper rate functions of the Brownian motion type. Moreover, according to Theorem 1.2(2), these rate functions are sharp up to constant. For instance, Theorem 1.2(1) is valid, if the jumping kernel $J(x,y)$ satisfies Assumption 1.1(i), (ii) and that for any $x,y\in\mathbb{R}^d$ with $|x-y|\ge 1$, $J(x,y)=0$ (finite range) or $J(x,y)\asymp 1/|x-y|^{d+2+\varepsilon}$ with any $\varepsilon>0$ (polynomial decay).

Here it should be noted that the arguments of [15,22] heavily depend on the independent increments property for Lévy processes (see [19], Sections 2 and 3, for more details), while in the present setting such characterization is not available. To overcome this difficulty, we prove Theorem 1.2 by using heat kernel estimates. The idea of obtaining rate functions via heat kernel estimates has appeared in the literatures before, see [21] and the references therein. There are a few differences and difficulties in the present paper.

(1) In [21], Theorem 3.1, we used two-sided heat kernel estimates to derive the explicit probability estimates for exit times, which are crucial to obtain upper rate functions for the process. However, for symmetric jump processes of variable order (see (1.1)), it seems impossible to present two-sided heat kernel estimates, see [2] for details. Instead of this approach, here we turn to consider heat kernel estimates only for large time, which is enough to yield the rate function of the process.

(2) There are a lot of works on heat kernel estimates for symmetric jump processes on $\mathbb{R}^d$ generated by non-local symmetric Dirichlet forms, see [2,3,6,7,9,13] and the references therein. Among them, Chen, Kim and Kumagai [7] established two-sided heat kernel estimates for symmetric jump processes such that their jumping kernels decay (sub/super)exponentially in a explicit way, see [7], (1.6) and (1.7). On the other hand, we can obtain nice upper bounds of heat kernel estimates for processes whose jumping kernels decay polynomially and involve the logarithmic factor (Theorem 3.2). Moreover, we can establish two-sided heat kernel estimates of large time for symmetric jump processes whose jumping kernels are comparable to $|x-y|^{-(d+2+\varepsilon)}$ for all $x, y \in \mathbb{R}^d$ with $|x-y| \geq 1$ and some constant $\varepsilon > 0$ (Corollary 3.11).

By analogy with Brownian motions, one may guess that in order to prove Theorem 1.2, it suffices to get Gaussian type upper bound estimates for the heat kernel. However, as far as we have discussed in this paper, such upper bounds are only true for some interval of large time, not for all large time. This is quite different from the Brownian motion case, and so we need further considerations on the heat kernel bounds (Theorem 3.2 and the proof of Theorem 1.2 in the last section).

Bass and Kumagai [4] proved the convergence to symmetric diffusion processes of continuous time random walks on $\mathbb{Z}^d$ with unbounded range. In particular, they assumed the uniform finite second moment condition on conductances similar to (1.6) on jumping kernels, see [4], (A3) in page 2043. For the proof of the convergence result, they obtained sharp on-diagonal heat kernel estimates, Hölder regularity of parabolic functions and Harnack inequalities. Our result can be regarded as an another approach to get the diffusivity of symmetric jump processes with jumping kernels having the finite second moment.

Remark 1.3. Recently, it is proved in [1], Theorem 5.2, that for a class of symmetric jump processes on $\mathbb{R}^d$, if their jumping kernels have the matching upper and lower bounds, then Khintchine's law of the iterated logarithm holds if and only if the jumping kernels have the finite second moment. Their approach is also based on heat kernel estimates.

The remainder of this paper is arranged as follows. In the next section, we recall some known results for heat kernel of the process X, and then present related assumptions used in our paper. Section 3 is devoted to establish upper bounds and lower bounds of heat kernel for large time. In particular, Theorems 3.2 and 3.6 are interesting on their own. Then the proof of Theorem 1.2 will be presented in the last section.

For any two positive measurable functions f and g, $f \asymp g$ means that there is a constant $c > 1$ such that $c^{-1} f \leq g \leq cf$.

2. Known results and assumptions

Recall that $X = (\{X_t\}_{t\geq 0}, \{\mathbb{P}^x\}_{x\in\mathbb{R}^d\setminus\mathcal{N}})$ is the Hunt process associated with $(\mathcal{E},\mathcal{F})$, which can start from any point in $\mathbb{R}^d\setminus\mathcal{N}$. Let $P(t,x,\mathrm{d}y)$ be the transition probability of X. The transition semigroup $\{P_t, t\geq 0\}$ of X is defined for $x\in\mathbb{R}^d\setminus\mathcal{N}$ by

$$P_t f(x) = \mathbb{E}^x\big(f(X_t)\big) = \int_{\mathbb{R}^d} f(y)P(t,x,\mathrm{d}y), \quad f\geq 0, t\geq 0.$$

The following result has been proved in [2], Theorem 1.2, and [7], Proposition 3.1.

Theorem 2.1 ([2], Theorem 1.2, and [7], Proposition 3.1)). *Under Assumption 1.1, there are a properly exceptional set $\mathcal{N}\subset\mathbb{R}^d$, a non-negative symmetric kernel $p(t,x,y)$ defined on $(0,\infty)\times(\mathbb{R}^d\setminus\mathcal{N})\times(\mathbb{R}^d\setminus\mathcal{N})$ such that $P(t,x,\mathrm{d}y) = p(t,x,y)\,\mathrm{d}y$, and*

$$p(t,x,y)\leq c_0\big(t^{-d/\alpha_1}\vee t^{-d/2}\big), \quad t>0, x,y\in\mathbb{R}^d\setminus\mathcal{N}$$

holds with some constant $c_0>0$. Moreover, there is an $\mathcal{E}$-nest $\{F_k : k\geq 1\}$ of compact subsets of $\mathbb{R}^d$ so that

$$\mathcal{N} = \mathbb{R}^d\setminus\bigcup_{k=1}^{\infty} F_k$$

and that for each fixed $t>0$ and $y\in\mathbb{R}^d\setminus\mathcal{N}$, the map $x\mapsto p(t,x,y)$ is continuous on each F_k.

To obtain upper bounds of off-diagonal estimates for $p(t,x,y)$, we will use the following Davies' method, see [5]. Note that, the so-called *carré du champ* associated with $(\mathcal{E},\mathcal{F})$ is given by

$$\Gamma(f,g)(x) = \int_{\mathbb{R}^d}\big(f(y)-f(x)\big)\big(g(y)-g(x)\big)J(x,y)\,\mathrm{d}y, \quad f,g\in\mathcal{F}.$$

We can extend $\Gamma(f,f)$ to any non-negative measurable function f, whenever it is pointwise well defined.

The following proposition immediately follows from Theorem 2.1 and [5], Corollary 3.28.

Proposition 2.2. *Suppose that Assumption 1.1 holds. Then, there exists a constant $c_0>0$ such that for any $x,y\in\mathbb{R}^d\setminus\mathcal{N}$ and $t>0$,*

$$p(t,x,y)\leq c_0\big(t^{-d/\alpha_1}\vee t^{-d/2}\big)\exp\big(E(2t,x,y)\big),$$

where

$$E(t,x,y) := -\sup\big\{\big|\psi(x)-\psi(y)\big| - t\Lambda(\psi) : \psi\in C_c^{\mathrm{lip}}\big(\mathbb{R}^d\big) \text{ with } \Lambda(\psi)<\infty\big\}$$

and

$$\Lambda(\psi) := \big\|e^{-2\psi}\Gamma\big(e^{\psi},e^{\psi}\big)\big\|_\infty.$$

In the next section, we will consider the following two assumptions on the jumping kernel $J(x, y)$ for $x, y \in \mathbb{R}^d$ with $|x - y| \geq 1$.

(A) There are a constant $c > 0$ and an increasing function $\phi : [1, \infty) \to (1, \infty]$ such that for all $x, y \in \mathbb{R}^d$ with $|x - y| \geq 1$,

$$J(x, y) \leq \frac{c}{|x - y|^{d+2}\phi(|x - y|)} \tag{2.1}$$

and

$$\int_1^\infty \frac{\mathrm{d}r}{r\phi(r)} < \infty. \tag{2.2}$$

Moreover, the function

$$\Phi(s) := \left(\int_s^\infty \frac{\mathrm{d}r}{r\phi(r)}\right)^{-1}, \quad s \geq 1$$

satisfies

- the function $s \mapsto \log \Phi(s)/s$ is decreasing on $[1, \infty)$;
- there is a constant $\gamma > 0$ such that

$$\sup_{s \geq 1} \frac{\Phi(s)}{\phi^\gamma(s)} < \infty. \tag{2.3}$$

(B) There is a constant $c > 0$ such that for all $x, y \in \mathbb{R}^d$ with $|x - y| \geq 1$,

$$J(x, y) \leq \frac{c}{|x - y|^{d+2}}. \tag{2.4}$$

It also holds that

$$\sup_{x \in \mathbb{R}^d} \int_{\{|x-y| \geq 1\}} |x - y|^2 J(x, y)\,\mathrm{d}y < \infty. \tag{2.5}$$

Because ϕ is increasing on $[1, \infty)$, (2.1) is stronger than (2.4). Since (2.2) implies (2.5), **(A)** is stronger than **(B)**. For instance,, $\phi(r) = (1 + r)^\theta$, $\phi(r) = \log^{1+\theta}(e + r)$ and $\phi(r) = \log(e + r)\log^{1+\theta}\log(e^e + r)$ for any $\theta > 0$ satisfy the conditions in **(A)**. On the other hand, under (1.1) and (2.5),

$$\sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} |x - y|^2 J(x, y)\,\mathrm{d}y < \infty.$$

In particular, there is a constant $c_1 > 0$ such that for any $K > 0$,

$$\sup_{x \in \mathbb{R}^d} \int_{\{|x-y| > K\}} J(x, y)\,\mathrm{d}y \leq \frac{c_1}{K^2}. \tag{2.6}$$

3. Heat kernel estimates

Throughout this section, we always suppose that Assumption 1.1 holds. We will derive upper and lower bound estimates of the heat kernel for large time, respectively.

3.1. Heat kernel upper bound

Proposition 3.1. *Under Assumption* **(B)**, *there exist positive constants t_0 and c such that for all $t \ge t_0$ and $x, y \in \mathbb{R}^d \setminus \mathcal{N}$,*

$$p(t,x,y) \le \begin{cases} \dfrac{c}{t^{d/2}}, & t \ge |x-y|^2, \\ \dfrac{ct}{|x-y|^{d+2}}, & t \le |x-y|^2. \end{cases}$$

Proof. We mainly follow the proof of [3], Theorem 1.4, but here we suppose that the time parameter t is large. By Theorem 2.1, there are constants $t_0, c_0 > 0$ such that for all $x, y \in \mathbb{R}^d \setminus \mathcal{N}$ and $t \ge t_0$,

$$p(t,x,y) \le c_0 t^{-d/2}.$$

Thus, we only need to verify the off-diagonal estimate for $p(t,x,y)$.

We first introduce truncated Dirichlet forms associated with $(\mathcal{E}, \mathcal{F})$. For $0 < K < \infty$, define

$$\mathcal{E}^{(K)}(u,v) = \iint_{\{0<|x-y|<K\}} \big(u(x)-u(y)\big)\big(v(x)-v(y)\big) J(x,y)\,\mathrm{d}x\,\mathrm{d}y, \quad u, v \in \mathcal{F}.$$

Then by (2.6),

$$\begin{aligned} \iint_{\{|x-y|\ge K\}} \big(u(x)-u(y)\big)^2 J(x,y)\,\mathrm{d}x\,\mathrm{d}y &\le 4\int_{\mathbb{R}^d} u(x)^2 \left(\int_{\{|x-y|\ge K\}} J(x,y)\,\mathrm{d}y\right)\mathrm{d}x \\ &\le \frac{c_1}{K^2}\|u\|_2^2, \end{aligned}$$

which yields that

$$\begin{aligned} \mathcal{E}(u,u) &= \mathcal{E}^{(K)}(u,u) + \iint_{\{|x-y|\ge K\}} \big(u(x)-u(y)\big)^2 J(x,y)\,\mathrm{d}x\,\mathrm{d}y \\ &\le \mathcal{E}^{(K)}(u,u) + \frac{c_1}{K^2}\|u\|_2^2. \end{aligned} \tag{3.1}$$

In particular, $(\mathcal{E}^{(K)}, \mathcal{F})$ is a regular Dirichlet form on $L^2(\mathbb{R}^d; \mathrm{d}x)$.

Let $P^{(K)}(t,x,\mathrm{d}y)$ be the transition probability associated with $(\mathcal{E}^{(K)}, \mathcal{F})$. Then, by (3.1) and the proof of [2], Theorem 1.2, (or [7], Proposition 3.1), there exist positive constants c_2, c_3 and t_1 such that for all $t \ge t_1$ and $x, y \in \mathbb{R}^d \setminus \mathcal{N}$,

$$P^{(K)}(t,x,\mathrm{d}y) = p^{(K)}(t,x,y)\,\mathrm{d}y$$

and

$$p^{(K)}(t,x,y) \le c_2 t^{-d/2} \exp\left(\frac{c_3 t}{K^2}\right). \tag{3.2}$$

Next, we will obtain the off-diagonal estimate for $p^{(K)}(t,x,y)$, by applying Proposition 2.2 to $(\mathcal{E}^{(K)}, \mathcal{F})$. For fixed points $x_0, y_0 \in \mathbb{R}^d$, let $R = |x_0 - y_0|$ and $K = R/\theta$ for some $\theta > 0$, which will be determined later. For $\lambda > 0$, we define the function $\psi \in C_c^{\mathrm{lip}}(\mathbb{R}^d)$ by

$$\psi(x) = \left[\lambda\big(R - |x - y_0|\big)\right] \vee 0.$$

Then, by the inequality $(e^r - 1)^2 \le r^2 e^{2|r|}$ for $r \in \mathbb{R}$ and the fact that $|\psi(x) - \psi(y)| \le \lambda|x-y|$ for all $x, y \in \mathbb{R}^d$, we get

$$\begin{aligned}
\Gamma_K(\psi)(x) &:= e^{-2\psi(x)} \Gamma^{(K)}\big(e^{\psi}, e^{\psi}\big)(x) \\
&= \int_{\{0<|x-y|<K\}} \big(e^{\psi(y)-\psi(x)} - 1\big)^2 J(x,y)\,\mathrm{d}y \\
&\le \int_{\{0<|x-y|<K\}} \big(\psi(x) - \psi(y)\big)^2 e^{2|\psi(x)-\psi(y)|} J(x,y)\,\mathrm{d}y \\
&\le e^{2\lambda K} \lambda^2 \int_{\{0<|x-y|<K\}} |x-y|^2 J(x,y)\,\mathrm{d}y \\
&\le c_4 \lambda^2 e^{2\lambda K} \le c_5 \frac{e^{3\lambda K}}{K^2},
\end{aligned} \tag{3.3}$$

where in the third inequality we used (2.5) and the last inequality follows from the fact that $r^2 \le 2e^r$ for all $r \ge 0$. Hence,

$$\Lambda_K(\psi) := \big\|\Gamma_K(\psi)\big\|_\infty \le c_5 \frac{e^{3\lambda K}}{K^2},$$

which implies that

$$E^{(K)}(t, x_0, y_0) \le -\big|\psi(x_0) - \psi(y_0)\big| + \Lambda(\psi) t \le c_5 \frac{e^{3\lambda K}}{K^2} t - \lambda R. \tag{3.4}$$

In what follows, we assume that $t < K^2$. In (3.4), if we take

$$\lambda = \frac{1}{3K} \log\left(\frac{K^2}{t}\right),$$

then

$$E^{(K)}(t, x_0, y_0) \le -\frac{R}{3K} \log\left(\frac{K^2}{t}\right) + \frac{c_5}{K^2} \frac{K^2}{t} t = c_5 - \frac{\theta}{3} \log\left(\frac{K^2}{t}\right)$$

so that by (3.2) and Proposition 2.2,

$$\begin{aligned}p^{(K)}(t,x_0,y_0) &\le c_6 t^{-d/2}\exp\left(\frac{c_3 t}{K^2}+E^{(K)}(2t,x_0,y_0)\right)\\ &\le c_6 t^{-d/2}\exp\left(c_3+c_5-\frac{\theta}{3}\log\left(\frac{K^2}{2t}\right)\right)\\ &= c_7 t^{-d/2}\left(\frac{2t}{K^2}\right)^{\theta/3}.\end{aligned}$$

Hence by letting $\theta = 3(d+2)/2$, we have

$$p^{(K)}(t,x_0,y_0) \le c_7 t^{-d/2}\left(\frac{2t}{K^2}\right)^{(d+2)/2} = \frac{c_8 t}{K^{d+2}} = \frac{c_8\theta^{d+2}t}{|x_0-y_0|^{d+2}}. \tag{3.5}$$

We finally obtain the off-diagonal upper bound of $p(t,x,y)$. In fact, by Meyer's construction (see, e.g., [3], Lemma 3.1(c), or [2], Lemma 3.7(b)), (3.5) and (2.4),

$$p(t,x_0,y_0) \le p^{(K)}(t,x_0,y_0) + t\sup_{|x-y|\ge K} J(x,y) \le \frac{c_9 t}{|x_0-y_0|^{d+2}}. \tag{3.6}$$

Therefore, the proof is complete. □

Theorem 3.2. *Suppose that Assumption* **(A)** *holds. Then, for any* $\kappa \ge 1$, *there exist positive constants* $\theta_0 \in (0,1)$, $t_0 \ge 1$ *and* c_i $(i=1,2)$ *such that for all* $t \ge t_0$,

$$p(t,x,y) \le \begin{cases} \dfrac{c_1}{t^{d/2}}, & t \ge |x-y|^2,\\ \dfrac{c_1}{t^{d/2}}\exp\left(-\dfrac{c_2|x-y|^2}{t}\right), & \dfrac{\theta_0|x-y|^2}{\log\Phi(|x-y|)} \le t \le |x-y|^2,\\ U\big(t,|x-y|,\phi,\Phi,\kappa\big), & t \le \dfrac{\theta_0|x-y|^2}{\log\Phi(|x-y|)},\end{cases}$$

where

$$U\big(t,|x-y|,\phi,\Phi,\kappa\big) := \frac{c_1}{t^{d/2}\Phi(|x-y|/\kappa)^{\kappa/8}} \wedge \frac{c_1 t}{|x-y|^{d+2}} + \frac{c_1 t}{|x-y|^{d+2}\phi(|x-y|/\kappa)}.$$

Proof. We use the same notations as in those of Proposition 3.1. By Theorem 2.1, we only need to consider off-diagonal estimates, that is, the case that $t \le |x-y|^2$. We split the proof into two parts. Even though the proof below is based on the Davies method, the argument is much more delicate than that of Proposition 3.1.

Let $K \ge 1$. For fixed points $x_0, y_0 \in \mathbb{R}^d$ with $|x_0-y_0| \ge 1$, let $R = |x_0-y_0|$. For $\lambda > 0$, define the function $\psi \in C_c^{\mathrm{lip}}(\mathbb{R}^d)$ by

$$\psi(x) = \big[\lambda\big(R-|x-y_0|\big)\big] \vee 0.$$

Then by the same argument as in (3.3), and by Assumption 1.1(ii) and Assumption **(A)**,

$$\begin{aligned}\Gamma_K(\psi)(x) &= \int_{\{0<|x-y|<K\}} \big(e^{\psi(y)-\psi(x)}-1\big)^2 J(x,y)\,\mathrm{d}y \\ &\le \lambda^2 \int_{\{0<|x-y|<K\}} |x-y|^2 e^{2\lambda|x-y|} J(x,y)\,\mathrm{d}y \\ &= \lambda^2 \int_{\{0<|x-y|<1\}} |x-y|^2 e^{2\lambda|x-y|} J(x,y)\,\mathrm{d}y \\ &\quad + \lambda^2 \int_{\{1\le|x-y|<K\}} |x-y|^2 e^{2\lambda|x-y|} J(x,y)\,\mathrm{d}y \\ &\le \lambda^2 e^{2\lambda} \sup_{x\in\mathbb{R}^d} \int_{\{0<|x-y|<1\}} |x-y|^2 J(x,y)\,\mathrm{d}y \\ &\quad + c_1\lambda^2 \int_{\{1\le|x-y|<K\}} \frac{e^{2\lambda|x-y|}}{|x-y|^d\phi(|x-y|)}\,\mathrm{d}y \\ &=: \text{(I)} + \text{(II)}. \end{aligned} \tag{3.7}$$

(1) We first derive the desired Gaussian upper bound. For any $\theta > 0$, let η be a positive constant such that $\eta/\theta < 1/4$. Assume that $K = R$ and $t \ge \theta K^2/\log\Phi(K)$. We set $\lambda = \eta K/t$. Since $K \ge 1$ and the function $s \mapsto \log\Phi(s)/s$ is decreasing on $[1,\infty)$ by Assumption **(A)**,

$$e^{2\lambda} = e^{2\eta K/t} \le \exp\left(2\eta \frac{\log\Phi(K)}{\theta K}\right) \le e^{2\eta\log\Phi(1)/\theta} = \Phi(1)^{2\eta/\theta},$$

and so

$$\text{(I)} \le c_2\Phi(1)^{2\eta/\theta}\lambda^2 \le c_2\big(1+\Phi(1)\big)^{2\eta/\theta}\lambda^2 \le c_2\big(1+\Phi(1)\big)^{1/2}\lambda^2 =: c_3\lambda^2.$$

If $1 \le r \le K$, then, also due to the decreasing property of the function $s \mapsto \log\Phi(s)/s$,

$$e^{2\lambda r} = e^{2\eta K r/t} \le \exp\left(2\eta r \frac{\log\Phi(K)}{\theta K}\right) \le \exp\left(2\eta r\frac{\log\Phi(r)}{\theta r}\right) = \Phi(r)^{2\eta/\theta},$$

which implies that

$$\begin{aligned}\text{(II)} &\le c_1\lambda^2 \int_{\{|x-y|\ge 1\}} \frac{\Phi(|x-y|)^{2\eta/\theta}}{|x-y|^d\phi(|x-y|)}\,\mathrm{d}y = c_4\lambda^2\int_1^\infty \frac{\Phi(r)^{2\eta/\theta}}{r\phi(r)}\,\mathrm{d}r \\ &= c_4\lambda^2 \int_1^\infty \frac{1}{r\phi(r)}\left(\int_r^\infty \frac{1}{s\phi(s)}\,\mathrm{d}s\right)^{-2\eta/\theta}\mathrm{d}r = \frac{c_4\lambda^2}{1-(2\eta/\theta)}\left(\int_1^\infty \frac{1}{s\phi(s)}\,\mathrm{d}s\right)^{1-(2\eta/\theta)} \\ &\le 2c_4\lambda^2\left(1+\int_1^\infty\frac{1}{s\phi(s)}\,\mathrm{d}s\right) =: c_5\lambda^2.\end{aligned}$$

Hence by (3.7),

$$\Lambda_K(\psi) = \left\|\Gamma_K(\psi)\right\|_\infty \le (c_3 + c_5)\lambda^2 =: C_*\lambda^2.$$

In particular, we have

$$E^{(K)}(t, x_0, y_0) \le \Lambda_K(\psi)t - \left|\psi(x_0) - \psi(y_0)\right| \le C_*\lambda^2 t - \lambda R = -\eta(1 - \eta C_*)\frac{K^2}{t}.$$

This along with Proposition 2.2 yields that there is a constant $c_6 > 0$ such that for all $t \ge \theta K^2/\log\phi(K)$,

$$p^{(K)}(t, x_0, y_0) \le c_6 t^{-d/2}\exp\left\{\frac{c_0 t}{K^2} - \frac{\eta(1-\eta C_*)}{2}\frac{K^2}{t}\right\}. \tag{3.8}$$

We note that the constants c_6 and C_* above are independent of η and θ.

In what follows, we assume that

$$\frac{\theta K^2}{\log\Phi(K)} \le t \le K^2.$$

Since $t/K^2 \le 1$, we have by (3.8),

$$p^{(K)}(t, x_0, y_0) \le c_7 t^{-d/2}\exp\left\{-\frac{\eta(1-\eta C_*)}{2}\frac{K^2}{t}\right\}.$$

Then by the first inequality in (3.6) and (2.1),

$$\begin{aligned} p(t, x_0, y_0) &\le p^{(K)}(t, x_0, y_0) + t\sup_{|x-y|\ge K} J(x, y) \\ &\le c_7 t^{-d/2}\exp\left\{-\frac{\eta(1-\eta C_*)}{2}\frac{K^2}{t}\right\} + \frac{c_8 t}{K^{d+2}\phi(K)}. \end{aligned} \tag{3.9}$$

Let η_* be a positive constant such that

$$\frac{\eta_*(1-\eta_* C_*)}{2\theta} \in \left(0, 1 \wedge \frac{1}{\gamma}\right),$$

where γ is the constant in Assumption **(A)**. Then by (2.3), there is a constant $c_9 > 0$ such that

$$\begin{aligned} \exp\left\{-\frac{\eta_*(1-\eta_* C_*)}{2}\frac{K^2}{t}\right\} &\ge \exp\left\{-\frac{\eta_*(1-\eta_* C_*)}{2}\frac{\log\Phi(K)}{\theta}\right\} \\ &= \frac{1}{\Phi(K)^{\eta_*(1-\eta_* C_*)/(2\theta)}} \ge \frac{c_9}{\phi(K)}. \end{aligned}$$

By noting that

$$\frac{1}{t^{d/2}} = \frac{t}{t^{(d+2)/2}} \ge t\left(\frac{1}{K^2}\right)^{(d+2)/2} = \frac{t}{K^{d+2}},$$

we get

$$\frac{t}{K^{d+2}\phi(K)} \le c_9^{-1} t^{-d/2} \exp\left\{-\frac{\eta_*(1-\eta_* C_*)}{2}\frac{K^2}{t}\right\}.$$

Hence if we take $\eta = \eta_*$ in (3.9), then

$$\begin{aligned} p(t, x_0, y_0) \ &\le \ c_7 t^{-d/2} \exp\left\{-\frac{\eta_*(1-\eta_* C_*)}{2}\frac{K^2}{t}\right\} \\ &\quad + c_{10} t^{-d/2} \exp\left\{-\frac{\eta_*(1-\eta_* C_*)}{2}\frac{K^2}{t}\right\} \\ &=: c_* t^{-d/2} \exp\left\{-\frac{\eta_*(1-\eta_* C_*)}{2}\frac{|x_0-y_0|^2}{t}\right\}. \end{aligned}$$

Namely, for each fixed $\theta > 0$, we get the desired Gaussian bound for any $t > 0$ and $x, y \in \mathbb{R}^d$ such that

$$\frac{\theta|x-y|^2}{\log \Phi(|x-y|)} \le t \le |x-y|^2.$$

(2) Let $\kappa \ge 1$. Here we let $K = R/\kappa$. Since we can choose t_0 in the statement large enough, we may and do assume that $|x_0 - y_0|$ is large enough such that $|x_0 - y_0| \ge \kappa$, and so $K \ge 1$. Below we assume that

$$t \le \frac{\theta_0 R^2}{\log \Phi(R)}$$

for some $\theta_0 > 0$ small enough, which will be determined later.

Let

$$\lambda = \frac{\log \Phi(K)}{4K}.$$

Since the function $s \mapsto \log \Phi(s)/s$ on $[1, \infty)$ is decreasing by Assumption **(A)**,

$$e^{2\lambda r} = \exp\left(r\frac{\log \Phi(K)}{2K}\right) \le \exp\left(r\frac{\log \Phi(r)}{2r}\right) = \Phi(r)^{1/2}, \quad 1 \le r \le K.$$

Hence by (3.7),

$$\Lambda_K(\psi) \le c_0 \lambda^2,$$

where $c_0 > 0$ is independent of θ_0, κ and λ. In particular, by choosing $\theta_0 \in (0, 1)$ so small that $c_0 \kappa \theta_0 \le 2$, we have

$$\begin{aligned} E^{(K)}(t, x_0, y_0) &\le \Lambda_K(\psi)t - \left|\psi(x_0) - \psi(y_0)\right| \\ &\le c_0 \lambda^2 t - \lambda R \end{aligned}$$

$$
\begin{aligned}
&\leq \frac{c_0}{16}\left(\frac{\log \Phi(K)}{K}\right)^2 \frac{\theta_0 R^2}{\log \Phi(R)} - \frac{\log \Phi(K)}{4K} R \\
&= \frac{\kappa}{4} \log \Phi(K)\left(-1 + \frac{c_0 \kappa \theta_0}{4} \frac{\log \Phi(K)}{\log \Phi(\kappa K)}\right) \\
&\leq -\frac{\kappa}{8} \log \Phi(K),
\end{aligned}
$$

where we used $\kappa \geq 1$ and the increasing property of the function $\Phi(r)$ in the last inequality. We then have by Proposition 2.2,

$$
p^{(K)}(t, x_0, y_0) \leq c_1 t^{-d/2} \frac{1}{\Phi(K)^{\kappa/8}},
$$

which yields that by the same way as in (3.9),

$$
p(t, x_0, y_0) \leq c_1 t^{-d/2} \frac{1}{\Phi(|x_0 - y_0|/\kappa)^{\kappa/8}} + \frac{c_2 t}{|x_0 - y_0|^{d+2} \phi(|x_0 - y_0|/\kappa)}.
$$

Noting that Assumption **(B)** is weaker than Assumption **(A)**, we know from Proposition 3.1 that for any $x_0, y_0 \in \mathbb{R}^d \setminus \mathcal{N}$ and $t \geq t_0$ with $t \leq |x_0 - y_0|^2$,

$$
p(t, x_0, y_0) \leq \frac{c_3 t}{|x_0 - y_0|^{d+2}}.
$$

Since ϕ is an increasing function on $[1, \infty)$ and $|x_0 - y_0| \geq \kappa$, we have $\phi(|x_0 - y_0|/\kappa) \geq \phi(1)$ so that

$$
\frac{t}{|x_0 - y_0|^{d+2} \phi(|x_0 - y_0|/\kappa)} \leq \frac{t}{\phi(1)|x_0 - y_0|^{d+2}}.
$$

Therefore, we finally obtain

$$
p(t, x_0, y_0) \leq \frac{c_4}{t^{d/2} \Phi(|x - y|/\kappa)^{\kappa/8}} \wedge \frac{c_4 t}{|x - y|^{d+2}} + \frac{c_4 t}{|x - y|^{d+2} \phi(|x - y|/\kappa)}.
$$

Combining the conclusions in (1) and (2) above, we get the desired assertion. □

Remark 3.3. (i) According to Theorem 3.2, we can obtain [7], Theorem 3.3, when $\phi(r) = \exp(cr^\beta)$ for some constants $c > 0$ and $\beta \in (0, 1]$. By [7], (1.14) in Theorem 1.2, we know that upper bound estimates in Theorem 3.2 are sharp up to constants in this case.

(ii) By part (1) of the argument for Theorem 3.2, we indeed prove that for any $\theta > 0$, there are constants $c_i = c_i(\theta) > 0$ $(i = 1, 2)$ such that for all $t \geq t_0$ and $x, y \in \mathbb{R}^d$ with

$$
\frac{\theta |x - y|^2}{\log \Phi(|x - y|)} \leq t \leq |x - y|^2,
$$

it holds that

$$
p(t, x, y) \leq \frac{c_1}{t^{d/2}} \exp\left(-\frac{c_2 |x - y|^2}{t}\right).
$$

As a consequence of Theorem 3.2, we have the following statement about upper bound estimates of the heat kernel for a new class of symmetric jump processes.

Corollary 3.4. *Assume that there are positive constants ε, c_0 such that for all $x, y \in \mathbb{R}^d$ with $|x-y| \ge 1$,*

$$J(x,y) \le \frac{c_0}{|x-y|^{d+2+\varepsilon}}.$$

Then, there exist positive constants $t_0 \ge 1$, $\theta_0 > 0$ and c_i $(i=1,2)$ such that for all $t \ge t_0$,

$$p(t,x,y) \le \begin{cases} \dfrac{c_1}{t^{d/2}}, & t \ge |x-y|^2, \\ \dfrac{c_1}{t^{d/2}} \exp\Big(-\dfrac{c_2|x-y|^2}{t}\Big), & \dfrac{\theta_0|x-y|^2}{\log(1+|x-y|)} \le t \le |x-y|^2, \\ \dfrac{c_1 t}{|x-y|^{d+2+\varepsilon}}, & t \le \dfrac{\theta_0|x-y|^2}{\log(1+|x-y|)}. \end{cases}$$

Proof. By adjusting the constant c in Assumption **(A)** properly, we can take $\phi(r) = (1+r)^\varepsilon$ and so $c_1(1+r)^\varepsilon \le \Phi(r) \le c_2(1+r)^\varepsilon$ for all $r \ge 1$. Hence by Theorem 3.2, there exists $t_0 \ge 1$ such that for any $t \ge t_0$, we have the desired assertion if $t \ge |x-y|^2$ or if $\frac{\theta_0|x-y|^2}{\log(1+|x-y|)} \le t \le |x-y|^2$. Next, we assume that $t_0 \le t \le \frac{\theta_0|x-y|^2}{\log(1+|x-y|)}$. Since there exists $c_5 > 0$ such that $|x-y| \ge c_5$, by taking $\kappa \ge 1$ so large enough that $\varepsilon\kappa/8 \ge d+2+\varepsilon$ in Theorem 3.2, we find that

$$\begin{aligned} U\big(t, |x-y|, \phi, \Phi, \kappa\big) &\le \frac{c_3}{t^{d/2}|x-y|^{d+2+\varepsilon}} \wedge \frac{c_3 t}{|x-y|^{d+2}} + \frac{c_3 t}{|x-y|^{d+2+\varepsilon}} \\ &\le \frac{c_4}{t^{d/2}|x-y|^{d+2+\varepsilon}} + \frac{c_3 t}{|x-y|^{d+2+\varepsilon}} \\ &\le \frac{c_5 t}{|x-y|^{d+2+\varepsilon}}. \end{aligned}$$

At the last inequality, we again used the fact that $t \ge t_0 \ge 1$. Combining all conclusions above, we prove the desired assertion. □

To study rate functions of the process X corresponding to the test function $\phi(r) = \log^{1+\varepsilon} r$, we also need the following.

Proposition 3.5. *Suppose that Assumption* **(A)** *is satisfied. Then for any $\delta \in (0,1)$, there exist positive constants t_0, θ_0 and c_1, c_2 such that*

$$p(t,x,y) \le \frac{c_1 t}{|x-y|^{d+2}\log^{(d+2)\delta/2}\log(e+\Phi(c_2|x-y|))}$$

for all $t \ge t_0$ and $x, y \in \mathbb{R}^d \setminus \mathcal{N}$ with

$$t_0 \le t \le \frac{\theta_0|x-y|^2}{\log\Phi(|x-y|)}.$$

Proof. For fixed points $x_0, y_0 \in \mathbb{R}^d$ and $\theta > 0$, we let $R = |x_0 - y_0|$ and $K = R/\theta$. Since t_0 can be large enough, we may and do assume that R is large enough. We use the approach of Proposition 3.1 and start from the estimate (3.4). Taking

$$\lambda = \frac{1}{3K} \log\left(\frac{K^2 \log^\delta \log \Phi(K)}{t}\right),$$

we have

$$E^{(K)}(t, x_0, y_0) \le -\frac{\theta}{3} \log\left(\frac{K^2 \log^\delta \log \Phi(K)}{t}\right) + c_* \log^\delta \log \Phi(K),$$

where c_* is the constant c_5 in (3.4). If

$$t \le \frac{c_0 K^2}{\log \Phi(K)}$$

for some $c_0 > 0$, then for $K \ge 1$ large enough,

$$\frac{\theta}{6} \log\left(\frac{K^2 \log^\delta \log \Phi(K)}{t}\right) \ge \frac{\theta}{6} \log\left(\frac{\log \Phi(K) \log^\delta \log \Phi(K)}{c_0}\right) \ge c_* \log^\delta \log \Phi(K),$$

due to the fact that $\delta \in (0, 1)$. Hence, for $K \ge 1$ large enough, we have

$$E^{(K)}(t, x_0, y_0) \le -\frac{\theta}{6} \log\left(\frac{K^2 \log^\delta \log \Phi(K)}{t}\right),$$

which along with Proposition 2.2 yields that

$$\begin{aligned} p^{(K)}(t, x_0, y_0) &\le c_1 t^{-d/2} \exp\left(-\frac{\theta}{6} \log\left(\frac{K^2 \log^\delta \log \Phi(K)}{2t}\right)\right) \\ &= c_1 t^{-d/2} \left(\frac{2t}{K^2 \log^\delta \log \Phi(K)}\right)^{\theta/6}. \end{aligned}$$

Setting $\theta = 3(d+2)$, we get

$$p^{(K)}(t, x_0, y_0) \le c_2 \frac{t}{K^{d+2} \log^{(d+2)\delta/2} \log \Phi(K)}.$$

This along with the first inequality in (3.6), (2.1) and (2.3) in Assumption **(A)** and the fact that $|x_0 - y_0| = \theta K$ gives us that

$$\begin{aligned} p(t, x_0, y_0) &\le p^{(K)}(t, x_0, y_0) + t \sup_{|x-y| \ge K} J(x, y) \\ &\le \frac{c_3 t}{|x_0 - y_0|^{d+2} \log^{(d+2)\delta/2} \log \Phi(c_4 |x_0 - y_0|)} \end{aligned}$$

$$
\begin{aligned}
&+\frac{c_5 t}{|x_0-y_0|^{d+2}\Phi^{1/\gamma}(c_4|x_0-y_0|)}\\
&\le \frac{c_6 t}{|x_0-y_0|^{d+2}\log^{(d+2)\delta/2}\log\Phi(c_4|x_0-y_0|)}.
\end{aligned}
$$

The proof is complete. □

3.2. Heat kernel lower bound

In this subsection, we establish the following lower bound estimates for the heat kernel.

Theorem 3.6. *Under Assumption* (***B***), *there exist positive constants* t_0 *and* c_i $(i=1,2,3)$ *such that for all* $t \ge t_0$ *and* $x, y \in \mathbb{R}^d \setminus \mathcal{N}$,

$$
p(t,x,y) \ge \begin{cases} c_1 t^{-d/2}, & |x-y|^2 \le t,\\ c_1 t^{-d/2}\exp\Big(-\dfrac{c_2|x-y|^2}{t}\Big), & c_3|x-y| \le t \le |x-y|^2. \end{cases}
$$

We first explain the main idea of the proof of Theorem 3.6. Following the approach of [2], we introduce a class of modifications for the jumping kernel $J(x,y)$. Let κ_2 be the constant in (1.1). For $\delta \in (0,1)$, define

$$
J^{(\delta)}(x,y) := J(x,y)\mathbf{1}_{\{|x-y|\ge\delta\}} + \frac{\kappa_2}{|x-y|^{d+\alpha_2}}\mathbf{1}_{\{0<|x-y|<\delta\}} \tag{3.10}
$$

and

$$
\mathcal{D}^\delta := \Big\{u \in L^2\big(\mathbb{R}^d;\mathrm{d}x\big) \;\Big|\; \iint_{x\neq y}\big(u(x)-u(y)\big)^2 J^{(\delta)}(x,y)\,\mathrm{d}x\,\mathrm{d}y < \infty\Big\}.
$$

Then by Assumption 1.1, we have for any $\delta \in (0,1)$

$$
\begin{aligned}
\iint_{\{|x-y|\ge\delta\}}\big(u(x)-u(y)\big)^2 J(x,y)\,\mathrm{d}x\,\mathrm{d}y &\le 4\int u(x)^2\Big(\int_{\{|x-y|\ge\delta\}} J(x,y)\,\mathrm{d}y\Big)\,\mathrm{d}x\\
&\le c_1(\delta)\int u(x)^2\,\mathrm{d}x
\end{aligned}
$$

and so

$$
\begin{aligned}
&\iint_{x\neq y}\big(u(x)-u(y)\big)^2 J^{(\delta)}(x,y)\,\mathrm{d}x\,\mathrm{d}y + \|u\|^2_{L^2(\mathbb{R}^d;\mathrm{d}x)}\\
&\quad\asymp \iint_{x\neq y}\frac{(u(x)-u(y))^2}{|x-y|^{d+\alpha_2}}\,\mathrm{d}x\,\mathrm{d}y + \|u\|^2_{L^2(\mathbb{R}^d;\mathrm{d}x)}.
\end{aligned} \tag{3.11}
$$

Therefore, for all $\delta \in (0,1)$,

$$\mathcal{D}^\delta = \left\{ u \in L^2(\mathbb{R}^d; \mathrm{d}x) \,\Big|\, \iint_{x \neq y} \frac{(u(x)-u(y))^2}{|x-y|^{d+\alpha_2}} \,\mathrm{d}x\,\mathrm{d}y < \infty \right\};$$

that is, $\mathcal{D}^\delta$ is independent of $\delta \in (0,1)$.

Let $(\mathcal{E}^\delta, \mathcal{D}^\delta)$ be a bilinear form on $L^2(\mathbb{R}^d; \mathrm{d}x)$ given by

$$\mathcal{E}^\delta(u,v) = \iint_{\mathbb{R}^d \times \mathbb{R}^d} \big(u(x)-u(y)\big)\big(v(x)-v(y)\big) J^{(\delta)}(x,y)\,\mathrm{d}x\,\mathrm{d}y, \quad u, v \in \mathcal{D}^\delta,$$

and let $\mathcal{F}^\delta$ be the closure of $C_c^{\mathrm{lip}}(\mathbb{R}^d)$ with respect to the norm $\|f\|_{\mathcal{E}_1^\delta} := \sqrt{\mathcal{E}^\delta(f,f) + \|f\|_2^2}$ in $\mathcal{D}^\delta$. Then, $(\mathcal{E}^\delta, \mathcal{F}^\delta)$ is a regular Dirichlet form on $L^2(\mathbb{R}^d; \mathrm{d}x)$. Moreover, according to (3.11) and the argument of [2], Lemma 2.5, we have $\mathcal{F}^\delta = \mathcal{D}^\delta$.

Associated with the regular Dirichlet form $(\mathcal{E}^\delta, \mathcal{F}^\delta)$ is a symmetric Hunt process $Y^\delta = (\{Y_t^\delta\}_{t \ge 0}, \{\mathbb{P}^x\}_{x \in \mathbb{R}^d \setminus \mathcal{N}})$ with state space $\mathbb{R}^d \setminus \mathcal{N}_\delta$, where $\mathcal{N}_\delta \subset \mathbb{R}^d$ is a properly exceptional set for $(\mathcal{E}^\delta, \mathcal{F}^\delta)$. By [18], Main result, the process Y^δ is conservative. We also see from Theorem 2.1 that there exists a non-negative kernel $q^\delta(t,x,y)$ on $(0,\infty) \times (\mathbb{R}^d \setminus \mathcal{N}_\delta) \times (\mathbb{R}^d \setminus \mathcal{N}_\delta)$ such that for any non-negative function f on $\mathbb{R}^d$,

$$\mathbb{E}^x f\big(Y_t^\delta\big) = \int_{\mathbb{R}^d} q^\delta(t,x,y) f(y)\,\mathrm{d}y, \quad t > 0 \text{ and } x \in \mathbb{R}^d \setminus \mathcal{N}_\delta$$

and there is a constant $c_2 > 0$ such that

$$q^\delta(t,x,y) \le c_2\big(t^{-d/2} \vee t^{-d/\alpha_1}\big), \quad t > 0 \text{ and } x, y \in \mathbb{R}^d \setminus \mathcal{N}_\delta. \tag{3.12}$$

Moreover, there exists an $\mathcal{E}^\delta$-nest $\{F_k^\delta\}_{k \ge 1}$ of compact sets such that

$$\mathcal{N}_\delta = \mathbb{R}^d \setminus \bigcup_{k=1}^\infty F_k^\delta$$

and for each fixed $t > 0$ and $y \in \mathbb{R}^d \setminus \mathcal{N}_\delta$, the map $x \mapsto q^\delta(t,x,y)$ is continuous on each F_k^δ. Here we should note that the constant c_2 in (3.12) can be chosen to be independent of $\delta \in (0,1)$. Indeed, by the definition of $J^{(\delta)}(x,y)$,

$$J^{(\delta)}(x,y) \ge \frac{\kappa_1}{|x-y|^{d+\alpha_1}} \mathbf{1}_{\{|x-y|<1\}} + J(x,y)\mathbf{1}_{\{|x-y| \ge 1\}} =: J_l(x,y)$$

for any $\delta \in (0,1)$ and $x, y \in \mathbb{R}^d$. Then by following the argument of [2], Theorem 1.2, and [7], Proposition 3.1, we see that c_2 can be determined by $J_l(x,y)$, which is independent of δ.

Actually, under Assumption **(B)**, we can also get the following near-diagonal lower bound of $q^\delta(t,x,y)$, which is the key to Theorem 3.6.

Proposition 3.7. *Under Assumption* **(B)**, *there exist constants* $t_0 > 0$ *and* $c_0 = c_0(t_0) > 0$, *which are independent of* $\delta \in (0, 1)$, *such that for any* $t \geq t_0$ *and* $x, y \in \mathbb{R}^d \setminus \mathcal{N}_\delta$ *with* $|x - y|^2 \leq t$,

$$q^\delta(t, x, y) \geq c_0 t^{-d/2}.$$

We will prove Proposition 3.7 later, and present the proof of Theorem 3.6 first.

Proof of Theorem 3.6. (1) We first claim that there exist an $\mathcal{E}$-properly exceptional set $\mathcal{N}$ and constants $t_0, c_0 > 0$ such that for any $t \geq t_0$ and $x, y \in \mathbb{R}^d \setminus \mathcal{N}$ with $|x - y|^2 \leq t$,

$$p(t, x, y) \geq c_0 t^{-d/2}.$$

Indeed, let $\{\delta_n\}_{n=1}^\infty$ be a decreasing sequence in $(0, 1)$ such that $\delta_n \to 0$ as $n \to \infty$. Then, by [2], page 1969, Theorem 2.3, $(\mathcal{E}^{\delta_n}, \mathcal{F}^{\delta_n})$ converges to $(\mathcal{E}, \mathcal{F})$ in the sense of Mosco as $n \to \infty$. Since $J^{(\delta)}(x, y) \geq J(x, y)$ by definition, we have $\mathcal{F}^\delta \subset \mathcal{F}$ and

$$\mathcal{E}^\delta(u, u) \geq \mathcal{E}(u, u) \quad \text{for any } u \in \mathcal{F}^\delta.$$

Therefore, any $\mathcal{E}^\delta$-exceptional set can be regarded as an $\mathcal{E}$-exceptional set. Namely, we can choose an $\mathcal{E}$-exceptional set $\mathcal{N}$ so that $\bigcup_{n=1}^\infty \mathcal{N}_{\delta_n} \subset \mathcal{N}$. On account of this, the desired assertion follows from Proposition 3.7 and [2], pages 1990–1991, Proof of Theorem 1.3.

(2) Next, we prove Theorem 3.6 by following the argument of [6], Theorem 3.6. Note that if $t \geq t_0$ and $|x - y|^2 \leq t$, then our assertion follows from (1). In what follows, we assume that $\sqrt{t_0}|x - y| \leq t \leq |x - y|^2$.

Let l be the maximum of positive integers such that

$$\frac{t}{l} \leq \left(\frac{|x - y|}{l}\right)^2.$$

Since

$$\frac{|x - y|^2}{t} - 1 \leq l \leq \frac{|x - y|^2}{t}, \tag{3.13}$$

we have

$$\frac{1}{2}\left(\frac{|x - y|}{l}\right)^2 \leq \frac{t}{l} \leq \left(\frac{|x - y|}{l}\right)^2 \tag{3.14}$$

and

$$\frac{t}{l} \geq \frac{t^2}{|x - y|^2} \geq t_0. \tag{3.15}$$

Let $\{x_i\}_{0 \leq i \leq 6l}$ be a sequence on the line segment joining $x_0 = x$ and $x_{6l} = y$ such that

$$|x_k - x_{k-1}| = \frac{|x - y|}{6l} \quad \text{for any } k = 1, \ldots, 6l. \tag{3.16}$$

Take a sequence $\{y_i\}_{0\le i\le 6l}$ such that $y_0 = x$, $y_{6l} = y$ and $y_k \in B(x_k, (6l)^{-1}|x-y|)$ for all $1 \le k \le 6l-1$. Then, (3.16) and (3.14) imply that for any $1 \le k \le 6l$,

$$|y_k - y_{k-1}| \le |y_k - x_k| + |x_k - x_{k-1}| + |x_{k-1} - y_{k-1}| \le 3 \cdot \frac{|x-y|}{6l} = \frac{|x-y|}{2l} \le \sqrt{\frac{t}{l}}.$$

Hence by (3.15) and (1), there exists a constant $C = C(t_0) \in (0, 1)$ such that

$$p\left(\frac{t}{l}, y_{k-1}, y_k\right) \ge C\left(\frac{t}{l}\right)^{-d/2}, \quad 1 \le k \le 6l.$$

This, together with the Chapman–Kolmogorov equation implies that

$$\begin{aligned} &p(t, x, y) \\ &\quad = \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} p(t/l, x, y_1) \cdots p(t/l, y_{6l-1}, y)\,\mathrm{d}y_1 \cdots \mathrm{d}y_{6l-1} \\ &\quad \ge \int_{B(x_1,(6l)^{-1}|x-y|)} \cdots \int_{B(x_{6l-1},(6l)^{-1}|x-y|)} p(t/l, x, y_1) \cdots p(t/l, y_{6l-1}, y)\,\mathrm{d}y_1 \cdots \mathrm{d}y_{6l-1} \\ &\quad \ge C\left(\frac{t}{l}\right)^{-d/2} \prod_{k=1}^{6l-1} \left\{ C\left(\frac{t}{l}\right)^{-d/2} \left|B\left(x_k, (6l)^{-1}|x-y|\right)\right| \right\} \\ &\quad \ge c_1 \left(\frac{t}{l}\right)^{-d/2} C^{6l}, \end{aligned}$$

where in the second inequality $|\cdot|$ denotes the d-dimensional Lebesgue measure, and the last inequality follows from (3.14). Note that, by (3.13), we have

$$C^{6l} \ge e^{-c_2 l} \ge \exp\left(-c_2 \frac{|x-y|^2}{t}\right),$$

which, along with the estimate above, yields the desired assertion. □

The remainder of this subsection is devoted to the proof of Proposition 3.7. For this, we need Lemmas 3.9 and 3.10 below. These two lemmas are concerned with a class of scaled processes for the subprocess of Y^δ on a ball.

We begin with some results which are due to [2,6,9,13]. Let $B(x,r)$ be an open ball with radius $r > 0$ centered at $x \in \mathbb{R}^d$, and $B_r = B(0, r)$. Denote by Y^{δ,B_r} the subprocess of Y^δ on B_r. Let $q^{\delta,B_r}(t, x, y)$ and $(\mathcal{E}^{\delta,B_r}, \mathcal{F}^{\delta,B_r})$ be the heat kernel (also called Dirichlet heat kernel in the literature) and the regular Dirichlet form associated with Y^{δ,B_r}, respectively.

For a fixed $r > 0$, define

$$Y_t^{\delta,(r)} := r^{-1} Y^\delta_{r^2 t}.$$

Then $Y^{\delta,(r)} = (\{Y_t^{\delta,(r)}\}_{t\ge 0}, \{\mathbb{P}^x\}_{x\in\mathbb{R}^d\setminus\mathcal{N}_\delta})$ is a symmetric Hunt process on $\mathbb{R}^d \setminus \mathcal{N}_\delta$ such that the associated Dirichlet form $(\mathcal{E}^{\delta,(r)}, \mathcal{F}^{\delta,(r)})$ on $L^2(\mathbb{R}^d; \mathrm{d}x)$ is given by

$$\mathcal{E}^{\delta,(r)}(u,v) = \iint_{\mathbb{R}^d\times\mathbb{R}^d} \big(u(x)-u(y)\big)\big(v(x)-v(y)\big) r^{d+2} J^{(\delta)}(rx,ry)\,\mathrm{d}x\,\mathrm{d}y$$

and

$$\mathcal{F}^{\delta,(r)} = \left\{ u \in L^2\big(\mathbb{R}^d;\mathrm{d}x\big) \,\Big|\, \iint_{\mathbb{R}^d\times\mathbb{R}^d} \frac{(u(x)-u(y))^2}{|x-y|^{d+\alpha_2}}\,\mathrm{d}x\,\mathrm{d}y < \infty \right\}.$$

Moreover, the associated heat kernel $q_r^\delta(t,x,y)$ satisfies

$$q_r^\delta(t,x,y) = r^d q^\delta\big(r^2 t, rx, ry\big). \tag{3.17}$$

Let $Y^{\delta,(r),B_1}$ be the subprocess of $Y^{\delta,(r)}$ on B_1. Then the associated Dirichlet heat kernel $q_r^{\delta,B_1}(t,x,y)$ is given by

$$q_r^{\delta,B_1}(t,x,y) = r^d q^{\delta,B_r}\big(r^2 t, rx, ry\big), \quad t>0 \text{ and } x,y \in B_1 \setminus \mathcal{N}_\delta.$$

We denote by $(\mathcal{E}^{\delta,(r),B_1}, \mathcal{F}^{\delta,(r),B_1})$ the associated regular Dirichlet form on $L^2(B_1;\mathrm{d}x)$.

In the following, let

$$\Phi(x) = C_\Phi\big(1-|x|^2\big)^{\frac{12}{2-\alpha_2}} \mathbf{1}_{B_1}(x), \quad x \in \mathbb{R}^d$$

for some constant $C_\Phi > 0$ so that $\int_{B_1} \Phi(x)\,\mathrm{d}x = 1$. For each fixed $x_1 \in B_1 \setminus \mathcal{N}$, $r \ge 1$ and $\varepsilon \in (0,1)$, define

$$u_r(t,x) := q_r^{\delta,B_1}(t,x,x_1), \qquad u_r^\varepsilon(t,x) := u_r(t,x) + \varepsilon$$

and

$$H_\varepsilon(t) := \int_{B_1} \Phi(y) \log u_r^\varepsilon(t,y)\,\mathrm{d}y.$$

Proposition 3.8. *Under Assumption* **(B)**, *the next two assertions hold.*

(i) *For each $t>0$, the function $\Phi(\cdot)/u_r^\varepsilon(t,\cdot)$ belongs to $\mathcal{F}^{\delta,(r),B_1}$.*
(ii) *The function $H_\varepsilon(t)$ is differentiable on $(0,\infty)$ and for each $t>0$,*

$$H'_\varepsilon(t) = -\mathcal{E}^{\delta,(r),B_1}\left(u_r(t,\cdot), \frac{\Phi(\cdot)}{u_r^\varepsilon(t,\cdot)}\right). \tag{3.18}$$

Proof. (i) For any $x,y \in B_1$,

$$\frac{\Phi(x)}{u_r^\varepsilon(t,x)} \le \frac{1}{\varepsilon}\Phi(x)$$

and

$$\begin{aligned}\left|\frac{\Phi(x)}{u_r^\varepsilon(t,x)}-\frac{\Phi(y)}{u_r^\varepsilon(t,y)}\right| &\le \frac{1}{u_r^\varepsilon(t,x)}\left|\Phi(x)-\Phi(y)\right|+\Phi(y)\left|\frac{1}{u_r^\varepsilon(t,x)}-\frac{1}{u_r^\varepsilon(t,y)}\right| \\ &= \frac{1}{u_r^\varepsilon(t,x)}\left|\Phi(x)-\Phi(y)\right|+\frac{\Phi(y)}{u_r^\varepsilon(t,x)u_r^\varepsilon(t,y)}\left|u_r^\varepsilon(t,x)-u_r^\varepsilon(t,y)\right| \\ &\le \frac{1}{\varepsilon}\left|\Phi(x)-\Phi(y)\right|+\frac{C_\Phi}{\varepsilon^2}\left|u_r(t,x)-u_r(t,y)\right|.\end{aligned}$$

Then our assertion follows by the strong version of the normal contraction property (e.g., see the proof of [14], Theorem 1.4.2(ii)).

(ii) By (i), the right hand side of (3.18) is finite for any $t>0$. Then our assertion follows by the same way as in [2], Lemmas 4.1 and 4.7, and [13], Proposition 3.7. □

Lemma 3.9. *Under Assumption* **(B)**, *there exist positive constants c_1 and c_2 such that for any $\varepsilon\in(0,1)$, $\delta\in(0,1)$, $x_1\in B_1\setminus\mathcal{N}_\delta$, $t>0$ and $r\ge 1$,*

$$H_\varepsilon'(t)\ge -c_1+c_2\int_{B_1}\left(\log u_r^\varepsilon(t,y)-H_\varepsilon(t)\right)^2\Phi(y)\,\mathrm{d}y. \tag{3.19}$$

Proof. We mainly follow the argument of [2], Lemma 4.7. By Proposition 3.8(ii),

$$\begin{aligned} H_\varepsilon'(t) &= -\mathcal{E}^{\delta,(r),B_1}\left(u_r(t,\cdot),\frac{\Phi(\cdot)}{u_r^\varepsilon(t,\cdot)}\right) \\ &= -\iint_{B_1\times B_1}\left(u_r^\varepsilon(t,y)-u_r^\varepsilon(t,x)\right)\frac{u_r^\varepsilon(t,x)\Phi(y)-u_r^\varepsilon(t,y)\Phi(x)}{u_r^\varepsilon(t,x)u_r^\varepsilon(t,y)} \\ &\quad\times r^{d+2}J^{(\delta)}(rx,ry)\,\mathrm{d}x\,\mathrm{d}y \\ &\quad-2\int_{B_1}\Phi(x)\left(r^{d+2}\int_{B_1^c}J^{(\delta)}(rx,ry)\,\mathrm{d}y\right)\frac{u_r(t,x)}{u_r^\varepsilon(t,x)}\,\mathrm{d}x. \end{aligned}\tag{3.20}$$

Let $a=u_r^\varepsilon(t,y)/u_r^\varepsilon(t,x)$ and $b=\Phi(y)/\Phi(x)$. Since $s+1/s-2\ge(\log s)^2$ for any $s>0$, we have

$$\begin{aligned} &\left(u_r^\varepsilon(t,y)-u_r^\varepsilon(t,x)\right)\frac{u_r^\varepsilon(t,x)\Phi(y)-u_r^\varepsilon(t,y)\Phi(x)}{u_r^\varepsilon(t,x)u_r^\varepsilon(t,y)} \\ &\quad=\Phi(x)\left(1-a+b-\frac{b}{a}\right) \\ &\quad=\Phi(x)\left[(1-\sqrt{b})^2-\sqrt{b}\left(\frac{a}{\sqrt{b}}+\frac{\sqrt{b}}{a}-2\right)\right] \end{aligned}$$

$$
\begin{aligned}
&\le \Phi(x)\left[(1-\sqrt{b})^2-\sqrt{b}\left(\log\frac{a}{\sqrt{b}}\right)^2\right]\\
&=\left(\sqrt{\Phi(x)}-\sqrt{\Phi(y)}\right)^2-\sqrt{\Phi(x)\Phi(y)}\left[\log\left(\frac{u_r^{\varepsilon}(t,y)}{\sqrt{\Phi(y)}}\right)-\log\left(\frac{u_r^{\varepsilon}(t,x)}{\sqrt{\Phi(x)}}\right)\right]^2.
\end{aligned}
$$

Using this inequality with $0\le u_r(t,x)/u_r^{\varepsilon}(t,x)\le 1$, we obtain by (3.20),

$$
\begin{aligned}
H_{\varepsilon}'(t) \ge &-\iint_{B_1\times B_1}\left(\sqrt{\Phi(x)}-\sqrt{\Phi(y)}\right)^2 r^{d+2}J^{(\delta)}(rx,ry)\,\mathrm{d}x\,\mathrm{d}y\\
&+\iint_{B_1\times B_1}\sqrt{\Phi(x)\Phi(y)}\left[\log\left(\frac{u_r^{\varepsilon}(t,y)}{\sqrt{\Phi(y)}}\right)-\log\left(\frac{u_r^{\varepsilon}(t,x)}{\sqrt{\Phi(x)}}\right)\right]^2 r^{d+2}J^{(\delta)}(rx,ry)\,\mathrm{d}x\,\mathrm{d}y\\
&-2\int_{B_1}\Phi(x)\left(r^{d+2}\int_{B_1^c}J^{(\delta)}(rx,ry)\,\mathrm{d}y\right)\mathrm{d}x\\
=&-\iint_{\mathbb{R}^d\times\mathbb{R}^d}\left(\sqrt{\Phi(x)}-\sqrt{\Phi(y)}\right)^2 r^{d+2}J^{(\delta)}(rx,ry)\,\mathrm{d}x\,\mathrm{d}y\\
&+\iint_{B_1\times B_1}\sqrt{\Phi(x)\Phi(y)}\left[\log\left(\frac{u_r^{\varepsilon}(t,y)}{\sqrt{\Phi(y)}}\right)-\log\left(\frac{u_r^{\varepsilon}(t,x)}{\sqrt{\Phi(x)}}\right)\right]^2 r^{d+2}J^{(\delta)}(rx,ry)\,\mathrm{d}x\,\mathrm{d}y\\
=:&-(\mathrm{I})+(\mathrm{II}).
\end{aligned}
$$

To give a lower bound of the last expression above, we first show that there exists a constant $C_1>0$, which is independent of $\delta\in(0,1)$ and $\varepsilon\in(0,1)$, such that

$$
(\mathrm{I})\le C_1\left(\int_{\mathbb{R}^d}\left|\nabla\sqrt{\Phi(x)}\right|^2\mathrm{d}x+\int_{B_1}\Phi(x)\,\mathrm{d}x\right). \tag{3.21}
$$

To do so, we write

$$
\begin{aligned}
(\mathrm{I}) =& \iint_{\{0<|x-y|<1/r\}}\left(\sqrt{\Phi(x)}-\sqrt{\Phi(y)}\right)^2 r^{d+2}J^{(\delta)}(rx,ry)\,\mathrm{d}x\,\mathrm{d}y\\
&+\iint_{\{1/r\le|x-y|<1\}}\left(\sqrt{\Phi(x)}-\sqrt{\Phi(y)}\right)^2 r^{d+2}J^{(\delta)}(rx,ry)\,\mathrm{d}x\,\mathrm{d}y\\
&+\iint_{\{|x-y|\ge1\}}\left(\sqrt{\Phi(x)}-\sqrt{\Phi(y)}\right)^2 r^{d+2}J^{(\delta)}(rx,ry)\,\mathrm{d}x\,\mathrm{d}y\\
=:&\ (\mathrm{I})_1+(\mathrm{I})_2+(\mathrm{I})_3.
\end{aligned}
$$

By Assumption 1.1(ii) and [6], (3.9), there exists a positive constant c_1, which is independent of $\delta\in(0,1)$ and $r\ge1$, such that

$$
(\mathrm{I})_1\le\kappa_1 r^{d+2}\iint_{\{0<|x-y|<1/r\}}\frac{(\sqrt{\Phi(x)}-\sqrt{\Phi(y)})^2}{|rx-ry|^{d+\alpha_2}}\,\mathrm{d}x\,\mathrm{d}y\le c_1\int_{\mathbb{R}^d}\left|\nabla\sqrt{\Phi}(x)\right|^2\mathrm{d}x.
$$

Since $6/(2-\alpha_2) > 1$, the function $\sqrt{\Phi(x)} = \sqrt{C_\Phi}(1-|x|^2)^{\frac{6}{2-\alpha_2}}\mathbf{1}_{B_1}(x)$ is Lipschitz continuous; that is, there exists a positive constant c_Φ such that

$$\left|\sqrt{\Phi(x)} - \sqrt{\Phi(y)}\right| \le c_\Phi |x-y| \quad \text{for any } x, y \in \mathbb{R}^d.$$

We note that for any $\delta \in (0,1)$, $J^{(\delta)}(rx, ry) = J(rx, ry)$ for $x, y \in \mathbb{R}^d$ and $r > 1$ with $|rx - ry| \ge 1$. Therefore, there exist positive constants c_{2i} $(i = 1,2,3)$, which are independent of $r \ge 1$ and $\delta \in (0,1)$, such that

$$\begin{aligned}
(\mathrm{I})_2 &\le c_{21} r^{d+2} \iint_{\{1/r \le |x-y| < 1\}} \left(\sqrt{\Phi(x)} - \sqrt{\Phi(y)}\right)^2 J(rx, ry)\,\mathrm{d}x\,\mathrm{d}y \\
&\le c_{22} r^{d+2} \int_{B_2} \left(\int_{\{1/r \le |x-y| < 1\}} |x-y|^2 J(rx, ry)\,\mathrm{d}y\right)\mathrm{d}x \\
&\le \frac{c_{22}}{r^d} \int_{B_{2r}} \left(\int_{\{|x-y| \ge 1\}} |x-y|^2 J(x, y)\,\mathrm{d}y\right)\mathrm{d}x \\
&\le c_{23} = c_{23} \int_{B_1} \Phi(x)\,\mathrm{d}x,
\end{aligned}$$

where we used Assumption **(B)** in the last inequality. We also have

$$\begin{aligned}
(\mathrm{I})_3 &\le c_{31} r^{d+2} \int_{B_1} \left(\int_{\{|x-y| \ge 1\}} J(rx, ry)\,\mathrm{d}y\right)\mathrm{d}x \\
&= \frac{c_{31} r^2}{r^d} \int_{B_r} \left(\int_{\{|x-y| \ge r\}} J(x, y)\,\mathrm{d}y\right)\mathrm{d}x \\
&\le c_{32} = c_{32} \int_{B_1} \Phi(x)\,\mathrm{d}x
\end{aligned}$$

for some positive constants c_{3i} $(i = 1,2)$, which are independent of $r \ge 1$ and $\delta \in (0,1)$. We thus arrive at (3.21).

We next show that there exist positive constants c and c', which are independent of $\varepsilon \in (0,1)$, $\delta \in (0,1)$, $x_1 \in B_1 \setminus \mathcal{N}_\delta$, $t > 0$ and $r \ge 1$, such that

$$(\mathrm{II}) \ge -c + c' \int_{B_1} \left(\log u_r^\varepsilon(t,x) - H_\varepsilon(t)\right)^2 \Phi(x)\,\mathrm{d}x. \tag{3.22}$$

To do so, we first prove that

$$\int_{B_1} \left[\log\left(\frac{u_r^\varepsilon(t,x)}{\sqrt{\Phi(x)}}\right)\right]^2 \mathrm{d}x < \infty. \tag{3.23}$$

Since (3.12) implies that

$$u_r(t,x) = q_r^{\delta, B_1}(t, x, x_1) = r^d q^{\delta, B_r}\left(r^2 t, rx, rx_1\right) \le c'' r^d \left[\left(r^2 t\right)^{-d/2} \vee \left(r^2 t\right)^{-d/\alpha_1}\right],$$

we have

$$\varepsilon \le u_r^\varepsilon(t,x) = u_r(t,x) + \varepsilon \le c'' r^d \big[\big(r^2 t\big)^{-d/2} \vee \big(r^2 t\big)^{-d/\alpha_1} \big] + \varepsilon$$

so that

$$0 \le \big(\log u_r^\varepsilon(t,x)\big)^2 \le \big[|\log \varepsilon| \vee \big| \log\big(c'' r^d \big(\big(r^2 t\big)^{-d/2} \vee \big(r^2 t\big)^{-d/\alpha_1} \big) + \varepsilon \big) \big| \big]^2.$$

Hence,

$$\int_{B_1} \big(\log u_r^\varepsilon(t,x)\big)^2 \,\mathrm{d}x < \infty.$$

Noting that

$$\begin{aligned}\left[\log\left(\frac{u_r^\varepsilon(t,x)}{\sqrt{\Phi(x)}} \right) \right]^2 &= \big(\log u_r^\varepsilon(t,x) - \log \sqrt{\Phi(x)} \big)^2 \\ &\le 2 \big(\log u_r^\varepsilon(t,x)\big)^2 + 2 \big(\log \sqrt{\Phi(x)} \big)^2\end{aligned}$$

and

$$\int_{B_1} \big(\log \sqrt{\Phi(x)} \big)^2 \,\mathrm{d}x < \infty,$$

we get (3.23).

We next give a lower bound of (II). By (1.1) and (3.10), we have for all $r \ge 1$ and $x, y \in \mathbb{R}^d$,

$$r^{d+2} J^{(\delta)}(rx, ry) \ge r^{d+2} \frac{\kappa_1}{|rx - ry|^{d+\alpha_1}} \mathbf{1}_{\{|x-y|<1/r\}} = r^{2-\alpha_1} \frac{\kappa_1}{|x-y|^{d+\alpha_1}} \mathbf{1}_{\{|x-y|<1/r\}}.$$

Then by (3.23) and the weighted Poincaré inequality ([12], Corollary 6, see also the argument in [7], Theorem 4.1, and [6], Proposition 3.2), we obtain

$$\begin{aligned}(\mathrm{II}) &\ge r^{2-\alpha_1} \iint_{B_1 \times B_1} \sqrt{\Phi(x)\Phi(y)} \left(\log\left(\frac{u_r^\varepsilon(t,y)}{\sqrt{\Phi(y)}} \right) - \log\left(\frac{u_r^\varepsilon(t,x)}{\sqrt{\Phi(x)}} \right) \right)^2 \\ &\quad \times \frac{\kappa_1}{|x-y|^{d+\alpha_1}} \mathbf{1}_{\{|x-y|<1/r\}} \,\mathrm{d}x \,\mathrm{d}y \\ &\ge c_4 \int_{B_1} \left[\log\left(\frac{u_r^\varepsilon(t,x)}{\sqrt{\Phi(x)}} \right) - \left(\int_{B_1} \log\left(\frac{u_r^\varepsilon(t,y)}{\sqrt{\Phi(y)}} \right) \Phi(y) \,\mathrm{d}y \right) \right]^2 \Phi(x) \,\mathrm{d}x \\ &= c_4 \int_{B_1} \left[\log\left(\frac{u_r^\varepsilon(t,x)}{\sqrt{\Phi(x)}} \right) - \left(H_\varepsilon(t) - \frac{1}{2} \int_{B_1} \Phi(y) \log \Phi(y) \,\mathrm{d}y \right) \right]^2 \Phi(x) \,\mathrm{d}x \qquad (3.24)\end{aligned}$$

for some positive constant $c_4 = c_4(\kappa_1, d, \alpha_1, \Phi)$, which is independent of $\delta \in (0,1)$, $x_1 \in B_1 \setminus \mathcal{N}_\delta$, $t > 0$, $r \ge 1$ and $\varepsilon \in (0,1)$. Moreover, since

$$\big(\log u_r^\varepsilon(t,x) - H_\varepsilon(t)\big)^2 \le 2\left[\log\left(\frac{u_r^\varepsilon(t,x)}{\sqrt{\Phi(x)}}\right) - \left(H_\varepsilon(t) - \frac{1}{2}\int_{B_1}\Phi(y)\log\Phi(y)\,\mathrm{d}y\right)\right]^2 + 2\left(\frac{1}{2}\log\Phi(x) - \frac{1}{2}\int_{B_1}\Phi(y)\log\Phi(y)\,\mathrm{d}y\right)^2,$$

the last expression in (3.24) is greater than

$$\frac{c_4}{2}\int_{B_1}\big(\log u_r^\varepsilon(t,x) - H_\varepsilon(t)\big)^2\Phi(x)\,\mathrm{d}x - c_5$$

for

$$c_5 = \frac{c_4}{4}\int_{B_1}\left(\log\Phi(x) - \int_{B_1}\Phi(y)\log\Phi(y)\,\mathrm{d}y\right)^2\Phi(x)\,\mathrm{d}x,$$

whence (3.22) follows.

Combining (3.21) with (3.22), we have (3.19). The proof is complete. □

Lemma 3.10. *Under Assumption* **(B)**, *there exist constants $t_0 \in (0,1)$ small enough and $c_* = c_*(t_0) \ge 1$ such that the following assertions hold.*

(i) *For all $\delta \in (0,1)$, $r \ge c_*$, $t \in [t_0/8, 2t_0]$ and $x \in \mathbb{R}^d \setminus \mathcal{N}_\delta$,*

$$\mathbb{P}^x\left(\big|Y_t^{\delta,(r)} - Y_0^{\delta,(r)}\big| > \frac{1}{4}\right) \le \frac{1}{12}.$$

(ii) *For all $\delta \in (0,1)$, $r \ge c_*$, $t \in [t_0/8, t_0]$ and $x_1 \in B_{1/2} \setminus \mathcal{N}_\delta$,*

$$\int_{B(x_1,1/4)} u_r(t,x)\,\mathrm{d}x \ge \frac{3}{4}.$$

Proof. (i) By (3.17) and the change of variables, we have for all $t > 0$ and $x \in \mathbb{R}^d \setminus \mathcal{N}_\delta$,

$$\begin{aligned}
\mathbb{P}^x\left(\big|Y_t^{\delta,(r)} - Y_0^{\delta,(r)}\big| > \frac{1}{4}\right) &= \int_{\{|y-x|\ge 1/4\}} q_r^\delta(t,x,y)\,\mathrm{d}y \\
&= r^d\int_{\{|y-x|\ge 1/4\}} q^\delta\big(r^2t, rx, ry\big)\,\mathrm{d}y \\
&= \int_{\{|y-rx|\ge r/4\}} q^\delta\big(r^2t, rx, y\big)\,\mathrm{d}y \\
&= \int_{\{|y-rx|\ge r/4,\,|y-rx|^2\ge r^2t\}} q^\delta\big(r^2t, rx, y\big)\,\mathrm{d}y
\end{aligned}$$

$$
\begin{aligned}
&\quad + \int_{\{|y-rx|\ge r/4, r^2t>|y-rx|^2\}} q^\delta\big(r^2t, rx, y\big)\,\mathrm{d}y\\
&=: (\mathrm{I}) + (\mathrm{II}).
\end{aligned}
$$

Since the jumping kernel $J^{(\delta)}(x,y)$ fulfills Assumption **(B)**, we see by Proposition 3.1 that there are constants c_i $(i=1,2)>0$ and $t_1>0$ (both are independent of $\delta\in(0,1)$) such that for all $r^2t\ge t_1$ and $x\in\mathbb{R}^d\setminus\mathcal{N}_\delta$,

$$
(\mathrm{I}) \le \int_{\{|y-rx|\ge r/4, |y-rx|^2\ge r^2t\}} \frac{c_1 r^2 t}{|y-rx|^{d+2}}\,\mathrm{d}y \le c_1 r^2 t \int_{\{|y-rx|\ge r/4\}} \frac{\mathrm{d}y}{|y-rx|^{d+2}} = c_2 t.
$$

On the other hand, if $t\le 1/16$, then $r^2t\le r^2/16$, and so $(\mathrm{II})=0$. Therefore, if we choose $t_2>0$ small enough such that

$$
t_2 \le \frac{1}{32} \quad \text{and} \quad c_2 t_2 \le \frac{1}{24},
$$

then for any $r\ge\sqrt{8t_1/t_2}$ and $t\in[t_2/8, 2t_2]$,

$$
\mathbb{P}^x\left(\big|Y_t^{\delta,(r)} - Y_0^{\delta,(r)}\big| > \frac14\right) \le \frac{1}{12}.
$$

The desired assertion follows by taking $t_0=t_2$ and $c_*=1\vee\sqrt{8t_1/t_2}$.

(ii) For an open subset D of $\mathbb{R}^d$, let $\tau_D^{Y^{\delta,(r)}}$ be the exit time of $Y^{\delta,(r)}$ from D. Since $q_r^{\delta,B_1}(t,x,x_1)=q_r^{\delta,B_1}(t,x_1,x)$,

$$
\begin{aligned}
\int_{B(x_1,1/4)} u_r(t,x)\,\mathrm{d}x &= \int_{B(x_1,1/4)} q_r^{\delta,B_1}(t,x,x_1)\,\mathrm{d}x\\
&= \int_{B(x_1,1/4)} q_r^{\delta,B_1}(t,x_1,x)\,\mathrm{d}x\\
&= \mathbb{P}^{x_1}\big(\big|Y_t^{\delta,(r),B_1} - x_1\big| < 1/4\big)\\
&= \mathbb{P}^{x_1}\big(\big|Y_t^{\delta,(r)} - x_1\big| < 1/4, t<\tau_{B_1}^{Y^{\delta,(r)}}\big).
\end{aligned} \tag{3.25}
$$

Noting that

$$
\begin{aligned}
1 &= \mathbb{P}^{x_1}\big(\big|Y_t^{\delta,(r)} - x_1\big| < 1/4, t<\tau_{B_1}^{Y^{\delta,(r)}}\big) + \mathbb{P}^{x_1}\big(\big|Y_t^{\delta,(r)} - x_1\big| < 1/4, \tau_{B_1}^{Y^{\delta,(r)}}\le t\big)\\
&\quad + \mathbb{P}^{x_1}\big(\big|Y_t^{\delta,(r)} - x_1\big| \ge 1/4\big)\\
&\le \mathbb{P}^{x_1}\big(\big|Y_t^{\delta,(r)} - x_1\big| < 1/4, t<\tau_{B_1}^{Y^{\delta,(r)}}\big) + \mathbb{P}^{x_1}\big(\tau_{B_1}^{Y^{\delta,(r)}}\le t\big) + \mathbb{P}^{x_1}\big(\big|Y_t^{\delta,(r)} - x_1\big| \ge 1/4\big),
\end{aligned}
$$

we get by (3.25),

$$
\int_{B(x_1,1/4)} u_r(t,x)\,\mathrm{d}x \ge 1 - \mathbb{P}^{x_1}\big(\tau_{B_1}^{Y^{\delta,(r)}}\le t\big) - \mathbb{P}^{x_1}\big(\big|Y_t^{\delta,(r)} - x_1\big| \ge 1/4\big). \tag{3.26}
$$

Let $X = (\{X_t\}_{t\ge 0}, \{\mathbb{P}^x\}_{x\in\mathbb{R}^d})$ be the strong Markov process on $\mathbb{R}^d$ and τ_D the exit time of X from D. Then by the same way as in [3], (2.18), the strong Markov property implies that for any $x \in \mathbb{R}^d$, $t > 0$ and $r > 0$,

$$\begin{aligned}\mathbb{P}^x(\tau_{B(x,r)} \le t) &\le \mathbb{P}^x\big(\tau_{B(x,r)} \le t, |X_{2t} - x| \le r/2\big) + \mathbb{P}^x\big(|X_{2t} - x| \ge r/2\big)\\ &\le \mathbb{P}^x\big(\tau_{B(x,r)} \le t, |X_{2t} - X_{\tau_{B(x,r)}}| \ge r/2\big) + \mathbb{P}^x\big(|X_{2t} - x| \ge r/2\big)\\ &\le \sup_{s\le t, |z-x|\ge r} \mathbb{P}^z\big(|X_{2t-s} - z| \ge r/2\big) + \mathbb{P}^x\big(|X_{2t} - x| \ge r/2\big)\\ &\le 2 \sup_{s\in[t,2t], z\in\mathbb{R}^d} \mathbb{P}^z\big(|X_s - z| \ge r/2\big). \end{aligned}\tag{3.27}$$

Applying it to $\{Y_t^{\delta,(r)}\}_{t\ge 0}$, we see that for any $x_1 \in B_{1/2} \setminus \mathcal{N}_\delta$,

$$\mathbb{P}^{x_1}\big(\tau_{B_1}^{Y^{\delta,(r)}} \le t\big) \le \mathbb{P}^{x_1}\big(\tau_{B(x_1,1/2)}^{Y^{\delta,(r)}} \le t\big) \le 2 \sup_{s\in[t,2t], z\in\mathbb{R}^d} \mathbb{P}^z\big(\big|Y_s^{\delta,(r)} - z\big| \ge 1/4\big).$$

Then by (i), we obtain for any $x_1 \in B_{1/2} \setminus \mathcal{N}_\delta$ and $t \in [t_0/8, t_0]$,

$$\begin{aligned}&\mathbb{P}^{x_1}\big(\tau_{B_1}^{Y^{\delta,(r)}} \le t\big) + \mathbb{P}^{x_1}\big(\big|Y_t^{\delta,(r)} - x_1\big| \ge 1/4\big)\\ &\le 2 \sup_{s\in[t,2t], z\in\mathbb{R}^d} \mathbb{P}^z\big(\big|Y_s^{\delta,(r)} - z\big| \ge 1/4\big) + \mathbb{P}^{x_1}\big(\big|Y_t^{\delta,(r)} - x_1\big| \ge 1/4\big)\\ &\le 2\cdot\frac{1}{12} + \frac{1}{12} = \frac{1}{4}.\end{aligned}$$

Hence the proof is complete by (3.26). $\square$

Now, we are in position to give the proof of Proposition 3.7.

Proof of Proposition 3.7. Let $t_0 \in (0, 1)$ and $c_* \ge 1$ be the same constants as in Lemma 3.10. We first prove that there exists a positive constant $c = c(t_0)$ such that for all $\delta \in (0, 1)$, $r \ge c_*$, $x_1 \in B_{1/2} \setminus \mathcal{N}_\delta$ and $t_1 \in [t_0/4, t_0]$,

$$\int_{B_1} \Phi(y) \log q_r^{\delta,B_1}(t_1, y, x_1)\,\mathrm{d}y \ge -c.$$

Our approach here is similar to that of [11], Lemmas 3.3.1–3.3.3, and [13], Proof of Theorem 2.5. Fix $\varepsilon \in (0, 1)$, $\delta \in (0, 1)$, $x_1 \in B_{1/2} \setminus \mathcal{N}_\delta$, $r \ge c_*$ and $t \in [t_0/8, t_0]$. Let K be a constant such that $|B(x_1, 1/4))|e^{-K} = 1/4$, and define

$$D_t^\varepsilon := \big\{x \in B(x_1, 1/4) \mid u_r^\varepsilon(t, x) \ge e^{-K}\big\}.$$

Then

$$\int_{B(x_1,1/4)\setminus D_t^\varepsilon} u_r(t, x)\,\mathrm{d}x \le \int_{B(x_1,1/4)\setminus D_t^\varepsilon} u_r^\varepsilon(t, x)\,\mathrm{d}x \le e^{-K}\big|B(x_1, 1/4)\big| = \frac{1}{4}.$$

Since $r \geq 1$ and $t \leq 1$ by assumption, we get from (3.12) that

$$\begin{aligned} u_r(t,x) &= r^d q^{\delta,B_r}\big(r^2 t, rx, rx_1\big) \leq r^d q^\delta\big(r^2 t, rx, rx_1\big) \\ &\leq c_1 r^d \big(\big(r^2 t\big)^{-d/2} \vee \big(r^2 t\big)^{-d/\alpha_1}\big) \leq c_1 t^{-d/\alpha_1}, \end{aligned} \tag{3.28}$$

where c_1 is a positive constant independently of $\delta \in (0,1)$, $r \geq 1$ and $x, x_1 \in B_{1/2} \setminus \mathcal{N}_\delta$. Then

$$\int_{D_t^\varepsilon} u_r(t,x)\,\mathrm{d}x \leq \frac{c_1}{t^{d/\alpha_1}}\big|D_t^\varepsilon\big|.$$

Combining all the estimates above with Lemma 3.10(ii), we have

$$\frac{3}{4} \leq \int_{B(x_1,1/4)} u_r(t,x)\,\mathrm{d}x = \int_{D_t^\varepsilon} u_r(t,x)\,\mathrm{d}x + \int_{B(x_1,1/4)\setminus D_t^\varepsilon} u_r(t,x)\,\mathrm{d}x \leq \frac{c_1}{t^{d/\alpha_1}}\big|D_t^\varepsilon\big| + \frac{1}{4};$$

that is,

$$\big|D_t^\varepsilon\big| \geq \frac{t^{d/\alpha}}{2c_1} \geq \frac{1}{2c_1}\left(\frac{t_0}{8}\right)^{d/\alpha} \quad \text{for all } t \in [t_0/8, t_0].$$

Furthermore, by following the argument in [6], pages 851–852, and using Lemma 3.9, there exists a positive constant $c_2 = c_2(t_0)$, which is independent of $\varepsilon \in (0,1)$, $\delta \in (0,1)$, $r \geq c_*$ and $x_1 \in B_{1/2} \setminus \mathcal{N}_\delta$, such that for any $t_1 \in [t_0/4, t_0]$,

$$H_\varepsilon(t_1) = \int_{B_1} \Phi(y) \log u_r^\varepsilon(t_1, y)\,\mathrm{d}y \geq -c_2. \tag{3.29}$$

Note that if $0 < \varepsilon < 1 \wedge (2c_1/t_0^{d/\alpha_1})$, then by (3.28),

$$\frac{\varepsilon t_1^{d/\alpha_1}}{2c_1} \leq \frac{t_1^{d/\alpha_1}}{2c_1} u_r^\varepsilon(t_1, y) = \frac{t_1^{d/\alpha_1}}{2c_1}\big(u_r(t_1,y) + \varepsilon\big) \leq \frac{1}{2} + \frac{t_0^{d/\alpha_1}\varepsilon}{2c_1} \leq 1.$$

Therefore, by the monotone convergence theorem,

$$\int_{B_1} \Phi(y) \log\left(\frac{t_1^{d/\alpha_1}}{2c_1} u_r^\varepsilon(t_1,y)\right)\mathrm{d}y \to \int_{B_1} \Phi(y) \log\left(\frac{t_1^{d/\alpha_1}}{2c_1} u_r(t_1,y)\right)\mathrm{d}y \quad (\varepsilon \downarrow 0).$$

Then by letting $\varepsilon \downarrow 0$ in (3.29), we get

$$\int_{B_1} \Phi(y) \log q_r^{\delta,B_1}(t_1, y, x_1)\,\mathrm{d}y = \int_{B_1} \Phi(y) \log u_r(t_1,y)\,\mathrm{d}y \geq -c_2,$$

which is the desired inequality.

We next discuss the lower bound of $q^\delta(t,x,y)$. By Jensen's inequality, there exists a positive constant $c_3=c_3(t_0,\Phi)$ such that for all $\delta\in(0,1)$, $r\ge c_*$, $t_1\in[t_0/4,t_0]$ and $x_0,x_1\in B_{1/2}\setminus\mathcal{N}_\delta$,

$$\begin{aligned}\log q_r^{\delta,B_1}(2t_1,x_0,x_1)&=\log\left(\int_{B_1} q_r^{\delta,B_1}(t_1,x_0,y)q_r^{\delta,B_1}(t_1,y,x_1)\,\mathrm{d}y\right)\\&\ge\log\left(\int_{B_1} q_r^{\delta,B_1}(t_1,x_0,y)q_r^{\delta,B_1}(t_1,y,x_1)\Phi(y)\,\mathrm{d}y\right)-\log\|\Phi\|_\infty\\&\ge\int_{B_1}\log\big(q_r^{\delta,B_1}(t_1,x_0,y)q_r^{\delta,B_1}(t_1,y,x_1)\big)\Phi(y)\,\mathrm{d}y-\log\|\Phi\|_\infty\\&=\int_{B_1}\Phi(y)\log q_r^{\delta,B_1}(t_1,x_0,y)\,\mathrm{d}y+\int_{B_1}\Phi(y)q_r^{\delta,B_1}(t_1,y,x_1)\,\mathrm{d}y\\&\quad-\log\|\Phi\|_\infty\\&\ge-c_3;\end{aligned}$$

that is,

$$q_r^{\delta,B_1}(t,x_0,x_1)\ge e^{-c_3}\quad\text{for all } t\in[t_0/2,2t_0]. \tag{3.30}$$

As we see from the proof of Lemma 3.10, the positive constant t_0 can be arbitrary small. In what follows, without loss of generality, we may and can assume that $0<t_0<1/4$. Then for any $t\in[1/2,2]$, there exists a positive integer $k_t\ge1$ such that $t-k_tt_0/2\in[t_0/2,2t_0]$. In fact,

$$0<\frac{1}{t_0}-4\le\frac{t-2t_0}{t_0/2}\le k_t\le\frac{t-t_0/2}{t_0/2}\le\frac{4}{t_0}-1 \tag{3.31}$$

and

$$\frac{t-t_0/2}{t_0/2}-\frac{t-2t_0}{t_0/2}=3.$$

By the semigroup property and (3.30), we have for any $t\in[1/2,2]$ and $x_0,x_1\in B_{1/2}\setminus\mathcal{N}_\delta$,

$$\begin{aligned}r^dq^{\delta,B_r}\big(r^2t,rx_0,rx_1\big)&=q_r^{\delta,B_1}(t,x_0,x_1)\\&=\int_{B_1}q_r^{\delta,B_1}(t-t_0/2,x_0,z_1)q_r^{\delta,B_1}(t_0/2,z_1,x_1)\,\mathrm{d}z_1\\&\ge\int_{B_{1/2}}q_r^{\delta,B_1}(t-t_0/2,x_0,z_1)q_r^{\delta,B_1}(t_0/2,z_1,x_1)\,\mathrm{d}z_1\\&\ge e^{-c_3}\int_{B_{1/2}}q_r^{\delta,B_1}(t-t_0/2,x_0,z_1)\,\mathrm{d}z_1.\end{aligned}$$

By the same way, the last term above is equal to

$$
\begin{aligned}
& e^{-c_3} \int_{B_{1/2}} \left(\int_{B_1} q_r^{\delta, B_1}(t - 2 \cdot t_0/2, x_0, z_2) q_r^{\delta, B_1}(t_0/2, z_2, z_1) \, \mathrm{d}z_2 \right) \mathrm{d}z_1 \\
& \quad \geq e^{-2c_3} \int_{B_{1/2}} \left(\int_{B_{1/2}} q_r^{\delta, B_1}(t - 2 \cdot t_0/2, x_0, z_2) \, \mathrm{d}z_2 \right) \mathrm{d}z_1.
\end{aligned}
$$

By repeating this procedure and using (3.31), there exists a positive constant $c_4 = c_4(t_0, \Phi)$ such that for all $\delta \in (0, 1)$, $r \geq c_*$, $t \in [1/2, 2]$ and $x_0, x_1 \in B_{1/2} \setminus \mathcal{N}_\delta$,

$$
\begin{aligned}
r^d q^{\delta, B_r}\big(r^2 t, r x_0, r x_1\big) & \geq e^{-k_t c_3} \int_{B_{1/2}} \cdots \int_{B_{1/2}} q_r^{\delta, B_1}(t - k_t t_0/2, x_0, z_{k_t}) \, \mathrm{d}z_{k_t} \cdots \mathrm{d}z_1 \\
& \geq e^{-(k_t+1)c_3} |B_{1/2}|^{k_t} \geq c_4,
\end{aligned}
\tag{3.32}
$$

where c_4 is independent of t.

By taking $t = 1$ in (3.32), we find that for all $\delta \in (0, 1)$, $r \geq c_*$ and $x_0, x_1 \in B_{1/2} \setminus \mathcal{N}_\delta$,

$$
q^{\delta, B_r}\big(r^2, r x_0, r x_1\big) \geq \frac{c_4}{r^d}.
$$

Letting $r = \sqrt{t}$ in the estimate above, we have for any for $t \geq c_*^2$ and $x_0, x_1 \in B_{1/2} \setminus \mathcal{N}_\delta$,

$$
q^{\delta, B_{\sqrt{t}}}(t, \sqrt{t} x_0, \sqrt{t} x_1) \geq \frac{c_4}{t^{d/2}};
$$

that is,

$$
q^{\delta, B_{\sqrt{t}}}(t, x_0, x_1) \geq \frac{c_4}{t^{d/2}}, \quad x_0, x_1 \in B_{\sqrt{t}/2} \setminus \mathcal{N}_\delta.
$$

By the space-uniformity of $\mathbb{R}^d$, we can replace the center of any ball by $z_0 \in \mathbb{R}^d$ in the argument above. Hence for any $t \geq c_*^2$, $z_0 \in \mathbb{R}^d$ and $x, y \in B(z_0, \sqrt{t}/2) \setminus \mathcal{N}_\delta$,

$$
q^\delta(t, x, y) \geq q^{\delta, B(z_0, \sqrt{t})}(t, x, y) \geq \frac{c_4}{t^{d/2}}.
$$

Note that for any $x, y \in \mathbb{R}^d$ with $|x - y|^2 \leq t$, there exists a point $z_0 \in \mathbb{R}^d$ such that $x, y \in B(z_0, \sqrt{t}/2)$. Therefore, our assertion is valid for $t \geq c_*^2$. □

At the end of this section, we present two-sided heat kernel estimates for jump processes, upper bounds of which have been established in Corollary 3.4.

Corollary 3.11. *Assume that there is a constant $\varepsilon > 0$ such that for all $x, y \in \mathbb{R}^d$ with $|x - y| \geq 1$,*

$$
J(x, y) \asymp \frac{1}{|x - y|^{d+2+\varepsilon}}.
$$

Then, there exist positive constants $t_0 \ge 1$, $\theta_0 > 0$ and c_0 such that for all $t \ge t_0$,

$$p(t,x,y) \asymp \begin{cases} \dfrac{1}{t^{d/2}}, & t \ge |x-y|^2, \\ \dfrac{1}{t^{d/2}} \exp\Big(-\dfrac{c_0|x-y|^2}{t}\Big), & \dfrac{\theta_0|x-y|^2}{\log(1+|x-y|)} \le t \le |x-y|^2, \\ \dfrac{1}{|x-y|^{d+2+\varepsilon}}, & t \le \dfrac{\theta_0|x-y|^2}{\log(1+|x-y|)}. \end{cases}$$

Here we note that the constants c_0 and θ_0 in the formula above should be different for upper and lower bounds.

Proof. The upper bound estimates have been proved in Corollary 3.4, so we need verify lower bounds. According to Theorem 3.6, we have got the first two cases, that is, $t \ge |x-y|^2$ and $\frac{\theta_0|x-y|^2}{\log(1+|x-y|)} \le t \le |x-y|^2$. Then, the proof is complete, if we prove that there exist constants $t_0 \ge 1$ and $c_1, c_2 > 0$ such that for all $t_0 \le t \le c_1|x-y|^2$,

$$p(t,x,y) \ge \frac{c_2}{|x-y|^{d+2+\varepsilon}}. \tag{3.33}$$

(1) First, we claim that there are positive constants c_0 and t_0 such that for all $t \ge t_0$ and $x \in \mathbb{R}^d \setminus \mathcal{N}$,

$$\mathbb{P}^x(\tau_{B(x,c_0\sqrt{t})} \le t) \le 1/2. \tag{3.34}$$

Indeed, we recall (3.27): for any $x \in \mathbb{R}^d \setminus \mathcal{N}$ and $t, r > 0$,

$$\mathbb{P}^x(\tau_{B(x,r)} \le t) \le 2 \sup_{s \le t, z \in \mathbb{R}^d} \mathbb{P}^z\big(|X_{2t-s} - z| \ge r/2\big). \tag{3.35}$$

Now, according to upper bound estimates for $p(t,x,y)$ in Corollary 3.4, there is a constant $t_0 > 0$ such that for all $t \ge t_0$, $r^2 \ge t$ and $x \in \mathbb{R}^d \setminus \mathcal{N}$,

$$\begin{aligned} \mathbb{P}^x\big(|X_t - x| \ge r\big) &\le c_1\bigg(\int_{\{|y-x|\ge r\}} t^{-d/2} \exp\big(-c_2|x-y|^2/t\big)\,\mathrm{d}y + \int_{\{|y-x|\ge r\}} \frac{t}{|x-y|^{d+2+\varepsilon}}\,\mathrm{d}y\bigg) \\ &\le c_3\bigg(\int_{r^2/t}^\infty e^{-c_2 s} s^{d/2-1}\,\mathrm{d}s + \int_r^\infty \frac{t}{s^{3+\varepsilon}}\,\mathrm{d}s\bigg) \\ &\le c_4\bigg(e^{-c_5 r^2/t} + \frac{t}{r^{2+\varepsilon}}\bigg). \end{aligned}$$

In particular, taking $r \ge c_6 t^{1/2}$ for some c_6 large enough, we find that

$$\mathbb{P}^x\big(|X_t - x| \ge r\big) \le 1/4.$$

This along with (3.35) yields (3.34).

(2) Next, we will use the approach of [10], Section 4.4. Fix $t \ge t_0$ and $x, y \in \mathbb{R}^d \setminus \mathcal{N}$ with $|x-y| \ge 4c_0 t^{1/2}$, where c_0 is the constant in (3.34). It follows from the Chapman–Kolmogorov equation and Theorem 3.6 that

$$\begin{aligned} p(2t, x, y) &= \int_{\mathbb{R}^d} p(t, x, z) p(t, z, y)\, \mathrm{d}z \\ &\ge \Big(\inf_{|z-y| \le 2c_0 t^{1/2}} p(t, z, y) \Big) \int_{\{|y-z| \le 2c_0 t^{1/2}\}} p(t, x, z)\, \mathrm{d}z \\ &\ge c_1 t^{-d/2} \mathbb{P}^x \big(X_t \in B\big(y, 2c_0 t^{1/2}\big)\big). \end{aligned}$$

For any $x \in \mathbb{R}^d$ and $r > 0$, define

$$\sigma_{B(x,r)} = \inf\big\{t > 0 : X_t \in B(x, r)\big\}.$$

By the strong Markov property,

$$\begin{aligned} &\mathbb{P}^x \big(X_t \in B\big(y, 2c_0 t^{1/2}\big)\big) \\ &\quad\ge \mathbb{P}^x \Big(\sigma_{B(y, c_0 t^{1/2})} \le t/2; \sup_{s \in [\sigma_{B(y,c_0 t^{1/2})}, t]} |X_s - X_{\sigma_{B(y,c_0 t^{1/2})}}| \le c_0 t^{1/2} \Big) \\ &\quad\ge \mathbb{P}^x (\sigma_{B(y,c_0 t^{1/2})} \le t/2) \inf_{z \in B(y, c_0 t^{1/2})} \mathbb{P}^z (\tau_{B(z, c_0 t^{1/2})} > t) \\ &\quad\ge \frac{1}{2} \mathbb{P}^x (\sigma_{B(y,c_0 t^{1/2})} \le t/2), \end{aligned}$$

where we used (3.34) in the last inequality. Furthermore, by the Lévy system formula (see [3], page 151, and [8], Appendix A) and the fact that $|x-y| \ge 4c_0 t^{1/2}$,

$$\begin{aligned} \mathbb{P}^x (\sigma_{B(y,c_0 t^{1/2})} \le t/2) &\ge \mathbb{P}^x \big(X_{(t/2) \wedge \tau_{B(x, c_0 t^{1/2})}} \in B\big(y, c_0 t^{1/2}\big)\big) \\ &\ge c_2 \mathbb{E}^x \bigg(\int_0^{(t/2) \wedge \tau_{B(x,c_0 t^{1/2})}} \int_{B(y, c_0 t^{1/2})} \frac{\mathrm{d}z}{|X_s - z|^{d+2+\varepsilon}}\, \mathrm{d}s \bigg) \\ &\ge c_3 t^{d/2+1} \mathbb{P}^x (\tau_{B(x, c_0 t^{1/2})} \ge t/2) \frac{1}{|x-y|^{d+2+\varepsilon}} \\ &\ge c_4 t^{d/2+1} \frac{1}{|x-y|^{d+2+\varepsilon}}, \end{aligned}$$

where in the third inequality we used the facts that $|x-y| \ge 4c_0 t^{1/2}$, and for all $s \in (0, (t/2) \wedge \tau_{B(x, c_0 t^{1/2})})$ and $z \in B(y, c_0 t^{1/2})$,

$$|X_s - z| \le |X_s - x| + |x - y| + |y - z| \le 2c_0 t^{1/2} + |x - y| \le 2|x - y|;$$

and the last inequality follows from (3.34). Combining all the inequalities above, we find that $t \ge t_0$ and $x, y \in \mathbb{R}^d \setminus \mathcal{N}$ with $|x-y| \ge 4c_0 t^{1/2}$,

$$p(2t, x, y) \ge \frac{c_4 t}{|x-y|^{d+2+\varepsilon}},$$

which proves (3.33). □

4. Proof of Theorem 1.2

Proof of Theorem 1.2. Throughout this proof, we set $\psi(r) = \sqrt{r \log\log r}$. Recall that $\tau_{B(x,r)} = \inf\{t > 0 : X_t \notin B(x,r)\}$ for any $x \in \mathbb{R}^d$ and $r > 0$.

(1) In this case, $\phi(s) = \log^{1+\varepsilon}(e+s)$ and so $c_*^{-1} \log^{\varepsilon}(e+s) \le \Phi(s) \le c_* \log^{\varepsilon}(e+s)$ for some constant $c_* \ge 1$. We follow the proof of [21], Theorem 3.1(1), first. Setting $t_k = 2^k$, we have for any $c > 0$, $k \ge 2$ and $x \in \mathbb{R}^d \setminus \mathcal{N}$,

$$\begin{aligned}
&\mathbb{P}^x\big(|X_s - x| \ge c\psi(s) \text{ for some } s \in [t_{k-1}, t_k]\big) \\
&\quad \le \mathbb{P}^x\Big(\sup_{s\in[t_{k-1},t_k]} |X_s - x| \ge c\psi(t_{k-1})\Big) \le \mathbb{P}^x\big(\tau_{B(x,c\psi(t_{k-1}))} \le t_k\big) \\
&\quad \le 2 \sup_{s\le t_k, z\in\mathbb{R}^d} \mathbb{P}^z\big(|X_{t_{k+1}-s} - z| \ge c\psi(t_{k-1})/2\big),
\end{aligned} \tag{4.1}$$

where in the last inequality we used (3.27).

For any $\kappa \ge 1$, let θ_0 be the constant in Theorem 3.2. In the following, let $C := C(\kappa) > 0$ which is chosen later. We first take $\theta_0^* > C$ large enough such that, if $r \ge \theta_0^*\psi(t)$, then $t \le \frac{\theta_0 r^2}{\log\Phi(r)}$; if $r \le \theta_0^*\psi(t)$, then $t \ge \frac{\theta_0' r^2}{\log\Phi(r)}$ for some constant $\theta_0' \in (0,1)$. Below, we fix this κ and θ_0^*, and let $\delta > 0$ be arbitrarily first. For any $x \in \mathbb{R}^d \setminus \mathcal{N}$ and $t > 1$ large enough, according to Theorem 3.2, Remark 3.3(ii) and Proposition 3.5 (with $\delta = 1/2$),

$$\begin{aligned}
&\mathbb{P}^x\big(|X_t - x| \ge C\psi(t)\big) \\
&\quad = \int_{\{|y-x|\ge C\psi(t)\}} p(t,x,y)\,dy \\
&\quad \le \frac{c_1}{t^{d/2}} \int_{\{C\psi(t)\le|y-x|\le\theta_0^*\psi(t)\}} \exp\Big(-\frac{c_2|x-y|^2}{t}\Big)\,dy \\
&\qquad + c_3 \int_{\{\theta_0^*\psi(t)\le|y-x|\le c_4\sqrt{t\log^{1+\delta} t}\}} \Big(t^{-d/2}\frac{1}{\log^{\kappa\varepsilon/8}|x-y|} + \frac{t}{|x-y|^{d+2}\log^{1+\varepsilon}|x-y|}\Big)\,dy \\
&\qquad + c_5 \int_{\{|y-x|\ge c_4\sqrt{t\log^{1+\delta} t}\}} \frac{t}{|x-y|^{d+2}\log^{(d+2)/4}\log\log(1+|x-y|)}\,dy \\
&\quad =: I_1 + I_2 + I_3,
\end{aligned}$$

where the constants c_i $(i = 1, \dots, 5)$ may depend on κ and δ. First, it holds that

$$\begin{aligned} I_2 &\le c_{21}\left[\big(t\log^{1+\delta} t\big)^{d/2}\big(t^{-d/2}\log^{-\kappa\varepsilon/8} t\big) + \int_{\theta_0^*\psi(t)}^{\infty} \frac{t}{r^3\log^{1+\varepsilon} r}\,\mathrm{d}r\right] \\ &\le c_{22}\left[\log^{-((\kappa\varepsilon/8)-((1+\delta)d/2))} t + \frac{1}{\log^{1+\varepsilon} t}\right]. \end{aligned}$$

Taking $\kappa \ge 1$ large enough such that $\kappa\varepsilon/8 \ge (1+\delta)d/2 + 1 + \varepsilon$, we find that

$$I_2 \le \frac{c_{23}}{\log^{1+\varepsilon} t}.$$

Second, we fix κ as above. We find that

$$\begin{aligned} I_1 &\le \frac{c_{11}}{t^{d/2}} \int_{\{|y-x|\ge C\psi(t)\}} \exp\left(-\frac{c_2|x-y|^2}{t}\right)\mathrm{d}y \\ &\le c_{12}\int_{C^2\log\log t}^{\infty} \exp(-c_2 s)s^{d/2-1}\,\mathrm{d}s \le c_{13}(\log t)^{-C^2c_2/2}, \end{aligned}$$

where c_2 depends on κ above. Choosing $C > 1$ large enough such that $C^2c_2/2 \ge 1+\varepsilon$, we get that

$$I_1 \le \frac{c_{14}}{\log^{1+\varepsilon} t}.$$

Third, it is easy to see that

$$I_3 \le \frac{c_{31}}{\log^{1+\delta} t}.$$

In particular, letting $\delta = \varepsilon$,

$$I_3 \le \frac{c_{32}}{\log^{1+\varepsilon} t}.$$

Below, we fix C chosen above. By all the estimates above, we obtain that there is a constant $C_1 > 0$ such that for any $x \in \mathbb{R}^d \setminus \mathcal{N}$ and $t > 1$ large enough,

$$\mathbb{P}^x\big(|X_t - x| \ge C\psi(t)\big) \le \frac{C_1}{\log^{1+\varepsilon} t}. \tag{4.2}$$

According to (4.1) and (4.2), we know that there are constants $C_0, C_2 > 0$ such that for all $k \ge 2$ and $x \in \mathbb{R}^d \setminus \mathcal{N}$,

$$\mathbb{P}^x\big(|X_s - x| \ge C_0\psi(s) \text{ for some } s \in [t_{k-1}, t_k]\big) \le \frac{C_2}{k^{1+\varepsilon}}.$$

This together with the Borel–Cantelli lemma proves the first desired assertion.

(2) For any $c > 0$ and $k \geq 1$, set $t_k = 2^k$ and

$$B_k = \big\{|X_{t_{k+1}} - X_{t_k}| \geq c\psi(t_{k-1})\big\}.$$

Denote by $(F_t)_{t\geq 0}$ the natural filtration of the process X. Then, for every $x \in \mathbb{R}^d \setminus \mathcal{N}$ and $k \geq 1$, by the Markov property and Theorem 3.6,

$$\begin{aligned}
\mathbb{P}^x(B_k|F_{t_k}) \geq & \min_{z\in\mathbb{R}^d\setminus\mathcal{N}} \mathbb{P}^z\big(|X_{t_k} - z| \geq c\psi(t_{k-1})\big) \\
\geq & \min_{z\in\mathbb{R}^d\setminus\mathcal{N}} \int_{\{c\psi(t_{k-1})\leq|y-z|\leq t_k\}} p(t_k, z, y)\,\mathrm{d}y \\
\geq & c_1 t_k^{-d/2} \min_{z\in\mathbb{R}^d\setminus\mathcal{N}} \int_{\{c\psi(t_{k-1})\leq|y-z|\leq t_k\}} \exp\Big(-\frac{c_2|z-y|^2}{t_k}\Big)\,\mathrm{d}y \\
\geq & c_3 \int_{c^2\log\log(t_{k-1})/2}^{t_k} e^{-c_2 s} s^{d/2-1}\,\mathrm{d}s \\
\geq & c_4 k^{-c^2 c_2}.
\end{aligned}$$

Choosing $c > 0$ small enough such that $c^2 c_2 \in (0, 1]$, we have

$$\sum_{k=1}^{\infty} \mathbb{P}^x(B_k|F_{t_k}) = \infty.$$

Then by the second Borel–Cantelli lemma,

$$\mathbb{P}^x(\limsup B_k) = 1.$$

This yields the desired assertion, see e.g. the proof of [21], Theorem 3.1(2). □

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