

The eigenstructure of the sample covariance matrices of high-dimensional stochastic volatility models with heavy tails

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We consider a p -dimensional time series where the dimension p increases with the sample size n . The resulting data matrix \mathbf{X} follows a stochastic volatility model: each entry consists of a positive random volatility term multiplied by an independent noise term. The volatility multipliers introduce dependence in each row and across the rows. We study the asymptotic behavior of the eigenvalues and eigenvectors of the sample covariance matrix $\mathbf{X}\mathbf{X}'$ under a regular variation assumption on the noise. In particular, we prove Poisson convergence for the point process of the centered and normalized eigenvalues and derive limit theory for functionals acting on them, such as the trace. We prove related results for stochastic volatility models with additional linear dependence structure and for stochastic volatility models where the time-varying volatility terms are extinguished with high probability when n increases. We provide explicit approximations of the eigenvectors which are of a strikingly simple structure. The main tools for proving these results are large deviation theorems for heavy-tailed time series, advocating a unified approach to the study of the eigenstructure of heavy-tailed random matrices.

Keywords: cluster Poisson limit; convergence; dependent entries; Fréchet distribution; infinite variance stable limit; large deviations; largest eigenvalues; point process; regular variation; sample autocovariance matrix; trace

1. The stochastic volatility model

Stochastic volatility models are popular in econometrics [5], mathematical finance [1,19,20] where they are used for option and derivative securities pricing, insurance mathematics [12, 26], time series [14,29], dependence modeling [11] and many other applied research areas. In a classical Black–Scholes framework, the volatility is assumed constant. Empirical studies, however, have shown that many observed features of implied volatility surfaces, such as the so-called volatility smile, can only be explained by assuming a stochastic or even non-stationary volatility sequence over time; see, for example, the discussion in [37,38]. Therefore a wide variety of stochastic volatility models has been proposed and well studied over the last few years. Stochastic volatility models are heavily used within the fields of financial economics and mathematical finance to capture the impact of time-varying volatility on financial markets and decision making. Time-varying volatility is endemic in financial markets. This was observed early on, for example, by Mandelbrot [28], Fama [18], and Black and Scholes [8].

The aforementioned literature on stochastic volatility models deals with univariate or low-dimensional multivariate time series. Here we focus on a high-dimensional stochastic volatility time series whose dimension may grow with the sample size. To be precise, we study a p -dimensional stochastic volatility time series, and assuming that p is large, we analyze the dependence structure of n observations from this time series via spectral properties of the sample covariance matrix. We discuss two cases: a stochastic volatility field with dependence and whose marginal distribution does not change over time, and an iid stochastic volatility field with time-varying marginal distribution, both under the assumption of observations coming from a distribution with infinite fourth moment. This is quite a typical situation for financial and actuarial time series; see for example the Danish fire insurance data considered in [34], Example 4.2, emerging market stock returns [23,27] and exchange rates data [22]. For such time series it is also common to study so-called tail risk measures to describe the impact of extreme scenarios [9,25].

In the first part of this paper, we consider the $p \times n$ -dimensional data matrix

$$\mathbf{X} = \mathbf{X}_n = (X_{it})_{i=1,\dots,p;t=1,\dots,n},$$

where (X_{it}) has the structure of a *stochastic volatility model*, i.e.,

$$X_{it} = \sigma_{it} Z_{it}, \quad i, t = 1, 2, \dots, \tag{1.1}$$

and (σ_{it}) is a strictly stationary random field of non-negative random variables independent of the i.i.d. random field (Z_{it}) . In Section 3, we introduce additional dependence in the stochastic volatility model. In what follows, X, σ, Z , denote generic elements of these fields. Stochastic volatility models are common in financial time series analysis; see, for example, [1]. The present model is an extension allowing for dependence through time and across the rows of the data matrix. It is convenient to think of (1.1) as a model where each row stands for a time series of log-returns of a speculative price series from a large portfolio, for example, a stock index such as the Standard & Poor’s 500 where each of the 500 rows of \mathbf{X} could represent the log-returns of the stock price of a particular US-based company in a given period of time.

We will study the *eigenstructure*, that is eigenvalues and eigenvectors, of the $p \times p$ *sample covariance matrix* $\mathbf{S} = \mathbf{X}\mathbf{X}'$ with entries

$$S_{ij} = \sum_{t=1}^n X_{it} X_{jt}, \quad i, j = 1, \dots, p,$$

under the assumption that the dimension $p = p_n$ converges to infinity together with the sample size n . In what follows, we drop the double index for the diagonal entries S_{ii} and simply write S_i . In the model (1.1) the dependence across the rows and through time is described by the structure of the volatility field (σ_{it}) . We will assume that the noise variable Z is *heavy-tailed* in the sense that it satisfies the regular variation condition

$$\mathbb{P}(Z > x) \sim q_+ \frac{L(x)}{x^\alpha} \quad \text{and} \quad \mathbb{P}(Z < -x) \sim q_- \frac{L(x)}{x^\alpha}, \quad x \rightarrow \infty, \tag{1.2}$$

for some $\alpha \in (0, 4)$, constants $q_+, q_- \geq 0$ such that $q_+ + q_- = 1$, and a slowly varying function L . We assume $\mathbb{E}[Z] = 0$ whenever $\mathbb{E}[|Z|] < \infty$ and also that the non-negative σ has a much lighter tail than Z in the sense that all moments of σ are finite.

The considered random field (X_{it}) is flexible as regards second order dependence. If $\alpha > 1$, we have $\text{cov}(X_{it}, X_{js}) = 0$ for $(i, t) \neq (j, s)$. On the other hand, $\text{cov}(|X_{it}|^r, |X_{js}|^r)$, $r > 0$, may decay arbitrarily slowly to zero when $|i - j|$ or $|t - s|$ goes to infinity, provided these covariances exist. Arbitrary decay rates can be achieved, for example, by assuming that $(\log \sigma_{it})$ is a stationary Gaussian field with a suitable covariance structure. As a matter of fact, a large part of the literature on stochastic volatility time series models deals with the case when the log-volatility is stationary Gaussian; see [1] for surveys on the topic stochastic volatility.

Thanks to regular variation and the i.i.d.-ness of the noise (Z_{it}) , the extremal dependence structure of (X_{it}) is characterized by the fact that the finite-dimensional distributions of (X_{it}) are multivariate regularly varying with index α and have asymptotically independent marginals; we refer to [10,34,35] for introductions to multivariate regular variation. Indeed, applications of Breiman’s lemma (Lemma B.5.1 in [10]) imply that

$$\mathbb{P}(\pm\sigma_{it}Z_{it} > x) \sim \mathbb{E}[\sigma^\alpha]\mathbb{P}(\pm Z > x), \quad x \rightarrow \infty. \tag{1.3}$$

Thus, the marginal distributions are regularly varying with index α . Moreover, for $(i, t) \neq (j, s)$, by another Breiman argument,

$$\begin{aligned} \frac{\mathbb{P}(\sigma_{it}|Z_{it}| > x, \sigma_{js}|Z_{js}| > x)}{\mathbb{P}(|Z| > x)} &= \frac{\mathbb{P}(\min(\sigma_{it}|Z_{it}|, \sigma_{js}|Z_{js}|) > x)}{\mathbb{P}(|Z| > x)} \\ &\leq \frac{\mathbb{P}(\max(\sigma_{it}, \sigma_{js}) \min(|Z_{it}|, |Z_{js}|) > x)}{\mathbb{P}(|Z| > x)} \\ &\sim \mathbb{E}[\max(\sigma_{it}, \sigma_{js})^\alpha]\mathbb{P}(|Z| > x) \rightarrow 0, \quad x \rightarrow \infty. \end{aligned}$$

This means that X_{it} and X_{js} are asymptotically independent in the sense of extreme value theory. Writing

$$\mathbf{X}^{(d)} = (X_{it})_{i,t=1,\dots,d}, \quad \mathbf{Z}^{(d)} = (Z_{it})_{i,t=1,\dots,d}, \quad d \geq 1,$$

the previous calculations on the marginals combined with standard arguments from regular variation calculus (see [10,34,35]) ensure that

$$\frac{\mathbb{P}(x^{-1}\mathbf{Z}^{(d)} \in \cdot)}{\mathbb{P}(|Z| > x)} \xrightarrow{v} \nu_\alpha(\cdot), \quad \frac{\mathbb{P}(x^{-1}\mathbf{X}^{(d)} \in \cdot)}{\mathbb{P}(|Z| > x)} \xrightarrow{v} \mathbb{E}[\sigma^\alpha]\nu_\alpha(\cdot), \quad x \rightarrow \infty.$$

Here \xrightarrow{v} denotes vague convergence in $\overline{\mathbb{R}}^{d \times d} \setminus \{\mathbf{0}\}$, $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty, -\infty\}$, the limiting measure ν_α is concentrated on the axes, and its restriction to any of the axes has Lebesgue density given by

$$\alpha|x|^{-\alpha-1}[q_+\mathbf{1}(x > 0) + q_-\mathbf{1}(x < 0)].$$

The fact that ν_α is concentrated on the axes is another way of defining asymptotic independence of the components of \mathbf{X} .

Since we are interested in the sample covariance matrix \mathbf{S} in the heavy-tailed case we observe that its diagonal entries $S_i = \sum_{t=1}^n X_{it}^2$ and off-diagonal entries $S_{ij} = \sum_{t=1}^n X_{it}X_{jt}$ for $i \neq j$ have rather distinct tails. A first indication is the fact that, on one hand, by a Breiman argument,

$$P(X^2 > x) \sim \mathbb{E}[\sigma^\alpha] \mathbb{P}(Z^2 > x) \sim \mathbb{E}[\sigma^\alpha] x^{-\alpha/2} L(\sqrt{x}), \tag{1.4}$$

while, on the other hand, by a result in Embrechts and Goldie [16], for independent copies Z_1, Z_2 of Z ,

$$\mathbb{P}(Z_1 Z_2 > x) \sim \tilde{q}_+ \frac{\ell(x)}{x^\alpha} \quad \text{and} \quad \mathbb{P}(Z_1 Z_2 < -x) \sim \tilde{q}_- \frac{\ell(x)}{x^\alpha}, \tag{1.5}$$

where ℓ is a slowly varying function, $\tilde{q}_+, \tilde{q}_- \geq 0$ and $\tilde{q}_+ + \tilde{q}_- = 1$. Hence by Breiman’s lemma, for $(i, t) \neq (j, s)$,

$$\mathbb{P}(\pm X_{it} X_{js} > x) \sim \mathbb{E}[(\sigma_{it} \sigma_{js})^\alpha] \mathbb{P}(\pm Z_1 Z_2 > x), \quad x \rightarrow \infty. \tag{1.6}$$

We assume $\alpha \in (0, 4)$. In this case, (1.4) and (1.6) imply that the diagonal entries (S_i) of \mathbf{S} dominate all off-diagonal elements S_{ij} in the sense that the asymptotic behavior of the eigenvalues of \mathbf{S} is completely determined by the diagonal $\text{diag}(\mathbf{S})$ of \mathbf{S} . This phenomenon is described in Theorem 2.1. It is well known in the i.i.d. case when $p = p_n \rightarrow \infty$ (see [13,15,21]). Pioneering work for the largest eigenvalue of \mathbf{S} under a more restrictive growth condition on p and $\alpha \in (0, 2)$ is due to Soshnikov [39,40] and Auffinger et al. [2]. For constant p the same property was observed for the stochastic volatility model (1.1) in Janßen et al. [24].

The diagonal elements S_i are the eigenvalues of the matrix $\text{diag}(\mathbf{S})$. They approximate the eigenvalues of the sample covariance matrix \mathbf{S} ; see (2.1). Given this approximation, large deviation results from Mikosch and Wintenberger [30,31] for the partial sums S_i are used to derive the convergence of the point processes of the centered and normalized eigenvalues of \mathbf{S} towards an inhomogeneous Poisson process; see Theorem 2.3. A similar point process convergence in the i.i.d. case under the assumption that p and n are proportional was proved in [39,40] for $\alpha \in (0, 2)$ and later extended in [2] to $\alpha \in [2, 4)$. In their proofs, the authors used truncation techniques and a challenging combinatorial approach.

Based on Theorem 2.3, the convergence of the point process of the eigenvalues in the case $\alpha \in (0, 4)$ allows one to derive limit theory for the largest eigenvalues of \mathbf{S} and functionals acting on them. In particular, the centered and normalized largest eigenvalue of \mathbf{S} converges to a Fréchet distributed random variable with parameter $\alpha/2$. In [21], this was shown for an i.i.d. random field (X_{it}) .

In Section 3, we introduce additional dependence in the stochastic volatility model. We consider the $p \times p$ matrix $\mathbf{Y} = \mathbf{A}^{1/2} \mathbf{X}$ where $\mathbf{A} = \mathbf{A}_n$ are deterministic positive definite $p \times p$ matrices with uniformly bounded spectra. In Theorem 3.4, it is essentially shown that the eigenvalues of $\mathbf{Y}\mathbf{Y}'$ are approximated by those of the matrix $\text{diag}(\mathbf{S}) \text{diag}(\mathbf{A})$ and Theorem 3.7 yields an approximation for the eigenvectors of $\mathbf{Y}\mathbf{Y}'$.

In Section 4, we consider another modification of the stochastic volatility model (1.1). We assume that the distribution of σ is a function of n and write $\sigma^{(n)}$ for a generic random variable from the i.i.d. random field $(\sigma_{it}^{(n)})$, $n \geq 1$. The possible values of $\sigma^{(n)}$, $n \geq 1$, are $0 = s_0 < s_1 \cdots < s_m$ for some $m \geq 1$ and we assume that $q_0^{(n)} = \mathbb{P}(\sigma^{(n)} = 0) \rightarrow 1$ and that the limits

$\lim_{n \rightarrow \infty} n\mathbb{P}(\sigma^{(n)} = s_j) > 0$, $j = 1, \dots, m$, exist (finite or infinite). This means that there is a large probability of extinction of the i.i.d. entries $X_{it}^{(n)} = \sigma_{it}^{(n)} Z_{it}$ of the data matrix \mathbf{X}_n when n is large. This model was introduced in [3] for $m = 1$, $1 - q_0^{(n)} = n^{-v}$ for some $v \in (0, 1]$ and $p/n \rightarrow \gamma \in (0, \infty)$. In Theorem 4.3, we again show that the eigenvalues of \mathbf{S} are asymptotically given by $\text{diag}(\mathbf{S})$. The main difference to Theorem 2.1 is that the normalization needed for the eigenvalues of \mathbf{S} is of significantly smaller magnitude depending on the speed at which $q_0^{(n)}$ approaches 1. The method of proof of our results is different from those in [3] and works for more general growth rates of p ; we again use large deviation techniques and exploit the approximation of the eigenvalues of \mathbf{S} by those of $\text{diag}(\mathbf{S})$. We also derive the point process convergence of the eigenvalues of \mathbf{S} , find approximations for the eigenvectors and we derive results for $\mathbf{Y} = \mathbf{A}^{1/2}\mathbf{X}$ where \mathbf{A} is a deterministic positive definite matrix.

In Sections 5–8, we provide the proofs of the aforementioned results.

Some basic notation

Eigenvalues and eigenvectors

For any $p \times p$ positive semidefinite matrix \mathbf{C} , we denote its ordered eigenvalues by

$$\lambda_1(\mathbf{C}) \geq \dots \geq \lambda_p(\mathbf{C}).$$

If, for $k \leq p$, the multiplicity of $\lambda_k(\mathbf{C})$ is 1, then there exists a unique unit eigenvector $\mathbf{v}_k(\mathbf{C})$ associated with $\lambda_k(\mathbf{C})$, i.e. $\|\mathbf{v}_k(\mathbf{C})\|_{\ell_2} = 1$ (Euclidean norm) and

$$\mathbf{C}\mathbf{v}_k(\mathbf{C}) = \lambda_k(\mathbf{C})\mathbf{v}_k(\mathbf{C}),$$

such that the first non-zero coordinate of $\mathbf{v}_k(\mathbf{C})$ is positive. We will use the latter orientation convention throughout this paper for eigenvectors.

Spectral norm and diagonal matrix

For any $p \times p$ matrix \mathbf{C} , the spectral norm $\|\mathbf{C}\|_2$ is $\sqrt{\lambda_1(\mathbf{C}\mathbf{C}^\prime)}$. Moreover, $\text{diag}(\mathbf{C})$ denotes the diagonal matrix which has the same diagonal as \mathbf{C} . Sometimes we will simply refer to $\text{diag}(\mathbf{C})$ as the diagonal of \mathbf{C} .

Normalization

Typically, we use a sequence (a_k) satisfying $k\mathbb{P}(|Z| > a_k) \rightarrow 1$ as $k \rightarrow \infty$ for the normalization of eigenvalues.

2. Convergence results for the stochastic volatility model

We start with a fundamental approximation of the sample covariance matrix \mathbf{S} in spectral norm.

Theorem 2.1. Consider the stochastic volatility model (1.1). We assume the following conditions:

- (1) A growth condition for the integer sequence $p = p_n \rightarrow \infty$:

$$p = p_n = n^\beta \ell(n), \quad n \geq 1, \tag{C_p(\beta)}$$

where ℓ is a slowly varying function and $\beta \in (0, 1]$.

- (2) The regular variation condition (1.2) on Z for some $\alpha \in (0, 2) \cup (2, 4)$ and $\mathbb{E}[Z] = 0$ if $\mathbb{E}[|Z|] < \infty$.
- (3) Finiteness of all moments $\mathbb{E}[\sigma^r]$ for $r > 0$.

Then

$$a_{np}^{-2} \|\mathbf{S} - \text{diag}(\mathbf{S})\|_2 \xrightarrow{\mathbb{P}} 0, \quad n \rightarrow \infty.$$

This theorem provides a first indication that the spectral properties of \mathbf{S} might be similar to those of $\text{diag}(\mathbf{S})$ which has a simple structure. The normalizing sequence is of the form $a_{np}^2 = (np)^{2/\alpha} \ell_1(np)$ for some slowly varying function ℓ_1 . Note that the provided approximation of \mathbf{S} does not hold for $\alpha > 4$ when the fourth moment of X is finite. In fact, one obtains completely different types of limit results for the eigenstructure of \mathbf{S} ; see [13,21] and the monograph [4] for a detailed overview and more references. The approximation of the sample covariance matrix by its diagonal is featured in the heavy-tailed case only.

The proof of Theorem 2.1 is provided in Section 5.

Remark 2.2. Assume $\beta > 1$ in $(C_p(\beta))$. If we keep the remaining assumptions of Theorem 2.1, the same proof as for the latter result yields

$$a_{np}^{-2} \|\mathbf{X}'\mathbf{X} - \text{diag}(\mathbf{X}'\mathbf{X})\|_2 \xrightarrow{\mathbb{P}} 0, \quad n \rightarrow \infty.$$

On the other hand, the non-zero eigenvalues of $\mathbf{S} = \mathbf{X}\mathbf{X}'$ and $\mathbf{X}'\mathbf{X}$ are the same. This observation is useful when determining the asymptotic behavior of the eigenvalues of \mathbf{S} in the case $\beta > 1$.

In view of Weyl’s inequality (see [6]), we may conclude from Theorem 2.1 that

$$a_{np}^{-2} \max_{i=1,\dots,p} |\lambda_i(\mathbf{S}) - \lambda_i(\text{diag}(\mathbf{S}))| \leq a_{np}^{-2} \|\mathbf{S} - \text{diag}(\mathbf{S})\|_2 \xrightarrow{\mathbb{P}} 0, \quad n \rightarrow \infty. \tag{2.1}$$

Using (2.1), it is possible to study the asymptotic behavior of the point process of the scaled eigenvalues $(a_{np}^{-2} \lambda_i(\mathbf{S}))_{i=1,\dots,p}$, as formulated in the next theorem.

Theorem 2.3. Assume the conditions of Theorem 2.1. In addition, we assume the following conditions.

- (4) (σ_{it}) is a strictly stationary ergodic field and the sequence (σ_{1t}^2) is strongly mixing with rate function $\alpha_j \leq c j^{-a}$ for some constants $c > 0$ and $a > 1$.
- (5) $\sigma^2 \leq M$ a.s. for some constant $M > 0$.

Then we have the following weak convergence result for the point processes with state space $\mathbb{R} \setminus \{0\}$:

$$N_n = \sum_{i=1}^p \varepsilon_{a_{np}^{-2}(\lambda_i(\mathbf{S}) - c_n)} \xrightarrow{d} N, \quad n \rightarrow \infty,$$

where N is a Poisson process on $\mathbb{R} \setminus \{0\}$ with mean measure $\mu_\alpha(x, \infty) = \mathbb{E}[\sigma^\alpha]x^{-\alpha/2}$ and $\mu_\alpha(-\infty, -x) = 0$ for $x > 0$. Furthermore, ε_x denotes the Dirac measure in the point x and

$$c_n = \begin{cases} 0, & \text{if } \alpha \in (0, 2), \\ n\mathbb{E}[X^2], & \text{if } \alpha \in (2, 4). \end{cases} \tag{2.2}$$

The proof will be given in Section 6. We notice that this result is the same as for the iid field $((\mathbb{E}[\sigma^\alpha])^{1/\alpha} Z_{it})$; see [21], Theorem 3.10 and Lemma 3.8. This means that dependence within the light-tailed σ -field influences the limiting point process only through a multiplicative factor.

Remark 2.4. In view of Remark 2.2, an analogous result holds if $\beta > 1$ in $(C_p(\beta))$.

The limiting process in Theorem 2.3 has representation

$$N = \sum_{i=1}^{\infty} \varepsilon_{(\Gamma_i / \mathbb{E}[\sigma^\alpha])^{-2/\alpha}}, \tag{2.3}$$

where $\Gamma_i = E_1 + \dots + E_i$ for i.i.d. standard exponential random variable $s(E_i)$. From this result, it follows that

$$a_{np}^{-2}(\mathbb{E}[\sigma^\alpha])^{2/\alpha}(\lambda_1(\mathbf{S}) - c_n, \dots, \lambda_k(\mathbf{S}) - c_n) \xrightarrow{d} (\Gamma_1^{-2/\alpha}, \dots, \Gamma_k^{-2/\alpha}) \tag{2.4}$$

for fixed $k \geq 1$. In particular,

$$a_{np}^{-2}(\mathbb{E}[\sigma^\alpha])^{2/\alpha}(\lambda_1(\mathbf{S}) - c_n) \xrightarrow{d} \Gamma_1^{-2/\alpha},$$

and the limiting variable has a Fréchet distribution with parameter $\alpha/2$. Now one can apply the folklore from extreme value theory to derive limit theory for continuous functionals of $(\lambda_1(\mathbf{S}), \dots, \lambda_k(\mathbf{S}))$. Moreover, a continuous mapping argument also shows that

$$a_{np}^{-2} \sum_{i=1}^p (\lambda_i(\mathbf{S}) - c_n) = a_{np}^{-2}(\text{trace}(\mathbf{S}) - pc_n)$$

converges in distribution to a totally skewed to the right $\alpha/2$ -stable limit; see [13,15,21].

3. Introducing more dependence in the stochastic volatility model

In this section, we will extend our stochastic volatility model by including some additional dependence between the entries of \mathbf{X} .

To this end, let $\mathbf{A} = \mathbf{A}_n$ be a sequence of deterministic, positive definite $p \times p$ matrices with bounded spectrum, that is $(\|\mathbf{A}_n\|_2)$ is uniformly bounded. If the entries of \mathbf{X} are independent with mean 0 and variance 1, then the columns of

$$\mathbf{Y} = \mathbf{A}^{1/2}\mathbf{X} \tag{3.1}$$

have covariance matrix \mathbf{A} . Here $\mathbf{A}^{1/2}$ is the symmetric, positive definite square root of \mathbf{A} .

Remark 3.1. The positive definite \mathbf{A} can be diagonalized: $\mathbf{A} = \mathbf{O}\mathbf{T}\mathbf{O}'$ where \mathbf{O} is an orthogonal matrix and \mathbf{T} is diagonal and positive definite. By assumption, $\mathbf{T}^{1/2}$ exists and we get $\mathbf{A}^{1/2} = \mathbf{O}\mathbf{T}^{1/2}\mathbf{O}'$.

The transformation (3.1) is very important in multivariate statistics since it creates a sample with dependence structure \mathbf{A} from an i.i.d. sample and vice versa.

Now assume that \mathbf{X} follows the stochastic volatility model (1.1). While the dependence among the (X_{it}) is only due to the dependence among the light-tailed (σ_{it}) , the dependence of the heavy-tailed components in the entries of $\mathbf{Y} = (Y_{it})$ is determined by \mathbf{A} . Our main goal in this section is to approximate the eigenvalues and eigenvectors of

$$\mathbf{Y}\mathbf{Y}' = \mathbf{A}^{1/2}\mathbf{X}\mathbf{X}'\mathbf{A}^{1/2} = \mathbf{A}^{1/2}\mathbf{S}\mathbf{A}^{1/2}.$$

As regards eigenvalues, we note that the spectra of $\mathbf{A}^{1/2}\mathbf{S}\mathbf{A}^{1/2}$ and $\mathbf{S}\mathbf{A}$ coincide. Matrices, such as $\mathbf{S}\mathbf{A}$, which are a product of a sample covariance matrix and the inverse of another covariance matrix are called *multivariate F-matrices* [4]. The limiting spectral distribution of *F*-matrices was studied among others in [42]. *F*-matrices also play an important role in MANOVA. Wachter [41] analyzed the generalized eigenvalue problem

$$\det(\mathbf{S} - \lambda\mathbf{A}^{-1}) = 0, \tag{3.2}$$

where \mathbf{A} can be stochastic but is independent of \mathbf{X} . Since \mathbf{A} is positive definite its inverse can be interpreted as a covariance matrix. Solutions λ of (3.2) are eigenvalues of $\mathbf{A}^{1/2}\mathbf{S}\mathbf{A}^{1/2}$, see [4,32].

The entries of the matrix \mathbf{Y} possess a quite general dependence structure. Nevertheless the approximation of the eigenvalues of the associated sample covariance matrix $\mathbf{Y}\mathbf{Y}'$ is straightforward.

Theorem 3.2. *We consider the matrix $\mathbf{Y} = \mathbf{A}^{1/2}\mathbf{X}$, where \mathbf{X} follows the stochastic volatility model (1.1). We assume the following conditions:*

- *The growth condition $(C_p(\beta))$ with $\beta \in (0, 1]$.*
- *The regular variation condition (1.2) on Z for some $\alpha \in (0, 2) \cup (2, 4)$ and $\mathbb{E}[Z] = 0$ if $\mathbb{E}[|Z|] < \infty$.*

- Finiteness of all moments $\mathbb{E}[\sigma^r]$ for $r > 0$.
- $\mathbf{A} = \mathbf{A}_n$ constitutes a sequence of deterministic, positive definite $p \times p$ matrices with uniformly bounded spectra.

Then

$$a_{np}^{-2} \max_{i=1, \dots, p} |\lambda_i(\mathbf{A}^{1/2} \mathbf{S} \mathbf{A}^{1/2}) - \lambda_i(\text{diag}(\mathbf{S}) \mathbf{A})| \xrightarrow{\mathbb{P}} 0.$$

Proof. By Weyl’s inequality (see [6]), Theorem 2.1 and the uniform boundedness of $(\|\mathbf{A}\|_2)$, we have

$$\begin{aligned} & a_{np}^{-2} \max_{i=1, \dots, p} |\lambda_i(\mathbf{A}^{1/2} \mathbf{S} \mathbf{A}^{1/2}) - \lambda_i(\text{diag}(\mathbf{S}) \mathbf{A})| \\ & \leq a_{np}^{-2} \|\mathbf{S} \mathbf{A} - \text{diag}(\mathbf{S}) \mathbf{A}\|_2 \leq a_{np}^{-2} \|\mathbf{S} - \text{diag}(\mathbf{S})\|_2 \|\mathbf{A}\|_2 \xrightarrow{\mathbb{P}} 0, \quad n \rightarrow \infty. \quad \square \end{aligned}$$

In applications involving high-dimensional data sets, it is common to only allow for dependence between certain key variables, which corresponds to many entries of \mathbf{A} being zero. Therefore, we introduce a sparseness condition on \mathbf{A} under which we can derive asymptotic spectral properties of $\text{diag}(\mathbf{S}) \mathbf{A}$.

We say that $\mathbf{A} = (A_{ij}) \in \mathbb{R}^{p \times p}$ is a *band matrix* with *bandwidth* m if $A_{ij} = 0$ whenever $|i - j| > m$. If $A_{1\bullet}, \dots, A_{p\bullet} \in \mathbb{R}^{1 \times p}$ denote the rows of \mathbf{A} , we have

$$\text{diag}(\mathbf{S}) \mathbf{A} = (S_1 A'_{1\bullet}, \dots, S_p A'_{p\bullet})'.$$

For $1 \leq k \leq p$, there are $\binom{p}{k}$ ways to choose k of the p rows of \mathbf{A} . Each choice is uniquely described by an element of the set

$$\Pi_{k,p} = \{\mathbf{a} = (a_1, \dots, a_k) \in \{1, \dots, p\}^k : a_1 < \dots < a_k\},$$

where the coordinates of \mathbf{a} contain the indices of the selected $\mathbf{A}_{i\bullet}$. For $\mathbf{a} \in \Pi_{k,p}$ define

$$J_{k,p}(\mathbf{a}, \mathbf{A}) = \begin{cases} 1, & \text{if } \sum_{i=1}^k \sum_{j=1; |j-a_i|>k}^p |A_{a_i,j}| > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Remark 3.3. In other words, $J_{k,p}(\mathbf{a}, \mathbf{A})$ is 0 if, after inspection of the rows $A_{a_1\bullet}, \dots, A_{a_k\bullet}$ and no further information about \mathbf{A} , it is still possible that \mathbf{A} is a band matrix with bandwidth k . In fact, \mathbf{A} is a band matrix with bandwidth k if and only if $J_{k,p}(\mathbf{a}, \mathbf{A}) = 0$ for all $\mathbf{a} \in \Pi_{k,p}$. Also note that $A_{ii} > 0$ for all i since \mathbf{A} is symmetric and positive definite.

For $\tilde{\mathbf{a}} \in \Pi_{k,p}$ chosen uniformly at random, the probability $\mathbb{P}(J_{k,p}(\tilde{\mathbf{a}}, \mathbf{A}) = 1)$ is given by

$$P_n(\mathbf{A}, k) := \binom{p}{k}^{-1} \sum_{\mathbf{a} \in \Pi_{k,p}} J_{k,p}(\mathbf{a}, \mathbf{A}).$$

The following condition holds if the matrices (\mathbf{A}_n) are “nearly banded”.

Condition (NB): For the sequence of matrices (\mathbf{A}_n)

$$\text{there exists a sequence } k = k_p \rightarrow \infty, k_p^3 = o(p) \text{ such that } \lim_{n \rightarrow \infty} P_n(\mathbf{A}, k) = 0. \quad (NB)$$

By construction, a sequence of band matrices (\mathbf{A}) with bandwidths (k) such that $k_p^3 = o(p)$ satisfies condition (NB) since $P_n(\mathbf{A}, k) = 0$ for all n . Roughly speaking, $P_n(\mathbf{A}, k)$ is small if only a small number of rows relative to the dimension p violate the band matrix structure. In particular, a change of finitely many rows does not influence the validity of condition (NB).

Under condition (NB), we can simplify $\lambda_i(\text{diag}(\mathbf{S})\mathbf{A})$ which appeared as approximation of the eigenvalues of $\mathbf{Y}\mathbf{Y}'$ in Theorem 3.2. We have the following result.

Theorem 3.4 (Eigenvalues of $\mathbf{Y}\mathbf{Y}'$). Consider the setting and the conditions of Theorem 3.2. In addition, we assume the following:

- (\mathbf{A}_n) satisfies condition (NB).
- The rows of $(\sigma_{it})_{i,t \geq 1}$ are i.i.d., strictly stationary ergodic sequences. Moreover, they are strongly mixing with rate function $\alpha_j \leq c j^{-a}$ for some constants $c > 0$ and $a > 1$.

1. If $\alpha \in (0, 2)$, then

$$a_{np}^{-2} \max_{i=1, \dots, p} \left| \lambda_i(\mathbf{A}^{1/2} \mathbf{S} \mathbf{A}^{1/2}) - \lambda_i(\text{diag}(\mathbf{S}) \text{diag}(\mathbf{A})) \right| \xrightarrow{\mathbb{P}} 0. \quad (3.3)$$

2. If $\alpha \in (2, 4)$, then

$$a_{np}^{-2} \max_{i=1, \dots, p} \left| \lambda_i(\mathbf{A}^{1/2} (\mathbf{S} - c_n \mathbf{I}) \mathbf{A}^{1/2}) - \lambda_i(\text{diag}(\mathbf{S} - c_n \mathbf{I}) \text{diag}(\mathbf{A})) \right| \xrightarrow{\mathbb{P}} 0,$$

with centering c_n defined in (2.2).

While $\mathbf{Y}\mathbf{Y}' = \mathbf{A}^{1/2} \mathbf{S} \mathbf{A}^{1/2}$ is a product of large matrices with complicated eigenstructure, the eigenvalues of $\text{diag}(\mathbf{S}) \text{diag}(\mathbf{A})$ are very easy to find.

Remark 3.5. In the case $\alpha \in (2, 4)$ we note that $\mathbf{A}^{1/2} (\mathbf{S} - c_n \mathbf{I}) \mathbf{A}^{1/2} = \mathbf{Y}\mathbf{Y}' - \mathbb{E}[\mathbf{Y}\mathbf{Y}']$. If $\mathbf{A} = \text{diag}(\mathbf{A})$ then the centering is not needed and (3.3) also holds for $\alpha \in (2, 4)$; compare with [21], Theorem 3.11.

Proof. We start with the case $\alpha \in (0, 2)$. Let (k) be the integer sequence from condition (NB). Since $k \rightarrow \infty$, we have $a_{np}^{-2} \lambda_k(\text{diag}(\mathbf{S})) \xrightarrow{\mathbb{P}} 0$ which implies

$$a_{np}^{-2} \lambda_{k+1}(\text{diag}(\mathbf{S})\mathbf{A}) \leq a_{np}^{-2} \lambda_{k+1}(\text{diag}(\mathbf{S})) \|\mathbf{A}\|_2 \xrightarrow{\mathbb{P}} 0.$$

Therefore, it is sufficient to prove

$$a_{np}^{-2} \max_{i=1, \dots, k} \left| \lambda_i(\text{diag}_k(\mathbf{S})\mathbf{A}) - \lambda_i(\text{diag}_k(\mathbf{S}) \text{diag}(\mathbf{A})) \right| \xrightarrow{\mathbb{P}} 0, \quad (3.4)$$

where $\text{diag}_k(\mathbf{S})$ is created from $\text{diag}(\mathbf{S})$ by only keeping its k largest entries and setting the others to 0.

Define the random indices L_1, \dots, L_p via

$$S_{L_1} = \lambda_1(\text{diag}(\mathbf{S})) > \dots > S_{L_p} = \lambda_p(\text{diag}(\mathbf{S})) \quad \text{a.s.} \tag{3.5}$$

In other words, S_{L_i} is the i th order statistic of S_1, \dots, S_p . We have

$$\text{diag}_k(\mathbf{S})\mathbf{A} = (0_p, \dots, 0_p, S_{\pi_1}A'_{\pi_1 \bullet}, 0_p, \dots, 0_p, S_{\pi_2}A'_{\pi_2 \bullet}, \dots, S_{\pi_k}A'_{\pi_k \bullet}, 0_p, \dots, 0_p)',$$

where $\pi_1 < \dots < \pi_k$ are the order statistics of L_1, \dots, L_k and 0_p is the p -dimensional zero vector. Since the S_i 's are i.i.d., L_1, \dots, L_k have a uniform distribution on the set of distinct k -tuples from $(1, \dots, p)$. Therefore, the k -tuple $\pi = (\pi_1, \dots, \pi_k)$ is uniformly distributed on $\Pi_{k,p}$.

Define the set $B_n = \{J_{k,p}(\pi, \mathbf{A}) = 0\}$. From condition **(NB)** and the fact that π is uniformly distributed on $\Pi_{k,p}$, we see that $\mathbb{P}(B_n) \rightarrow 1$. On B_n , we have for $1 \leq i \leq k$,

$$S_{\pi_i}A_{\pi_i \bullet} = (0, \dots, 0, S_{\pi_i}A_{\pi_i, \pi_i - k}, S_{\pi_i}A_{\pi_i, \pi_i - k + 1}, \dots, S_{\pi_i}A_{\pi_i, \pi_i + k}, 0, \dots, 0).$$

Consider the set

$$C_n = \{|L_i - L_j| > 2k, i, j = 1, \dots, k, i \neq j\}. \tag{3.6}$$

Since L_1, \dots, L_k are uniformly distributed on the set of distinct k -tuples from $(1, \dots, p)$ we have

$$\lim_{n \rightarrow \infty} \mathbb{P}(C_n^c) \leq \lim_{n \rightarrow \infty} k(k-1) \frac{2pk(p-2) \dots (p-k+1)}{p(p-1) \dots (p-k+1)} \leq \lim_{n \rightarrow \infty} \frac{2k^3}{p-1} = 0,$$

where condition **(NB)** was used for the last equality.

On $B_n \cap C_n$, the matrix $\text{diag}_k(\mathbf{S})\mathbf{A}$ is block diagonal with $(2k+1) \times (2k+1)$ blocks $\mathbf{Q}_i, i \leq k$. The matrix \mathbf{Q}_i is zero everywhere except for its $(k+1)$ st row which is

$$(S_{\pi_i}A_{\pi_i, \pi_i - k}, S_{\pi_i}A_{\pi_i, \pi_i - k + 1}, \dots, S_{\pi_i}A_{\pi_i, \pi_i + k}), \quad i \leq k.$$

The $(k+1, k+1)$ entry of \mathbf{Q}_i is at position (π_i, π_i) of $\text{diag}_k(\mathbf{S})\mathbf{A}$. Therefore, the only non-zero eigenvalue of \mathbf{Q}_i is $S_{\pi_i}A_{\pi_i, \pi_i}$. We conclude that on $B_n \cap C_n$

$$\lambda_i(\text{diag}_k(\mathbf{S})\mathbf{A}) = \lambda_i(\text{diag}_k(\mathbf{S}) \text{diag}(\mathbf{A})), \quad 1 \leq i \leq k. \tag{3.7}$$

This finishes the proof of (3.4).

In the case $\alpha \in (2, 4)$, we replace \mathbf{S}, S_i by $\mathbf{S} - c_n\mathbf{I}, S_i - c_n$, respectively, and use the same proof as for $\alpha \in (0, 2)$. □

Define $\tilde{L}_i, i = 1, \dots, p$ via

$$(S_{\tilde{L}_i} - c_n)A_{\tilde{L}_i, \tilde{L}_i} = \lambda_i(\text{diag}(\mathbf{S} - c_n\mathbf{I}) \text{diag}(\mathbf{A})).$$

The random variable \tilde{L}_i encodes the location of the i th largest value of the entries of $\text{diag}(\mathbf{S} - c_n\mathbf{I}) \text{diag}(\mathbf{A})$.

Remark 3.6. As a by-product of the proof of Theorem 3.4 we get that, with probability tending to 1, $\{\tilde{L}_1, \dots, \tilde{L}_k\} = \{L_1, \dots, L_k\}$ for any fixed $k \geq 1$.

Next we approximate the eigenvectors of $\mathbf{Y}\mathbf{Y}'$. To this end, let $\mathbf{e}_j = (0, \dots, 0, 1, 0, \dots, 0)'$, $j = 1, \dots, p$, denote the canonical basis vectors of \mathbb{R}^p . We define $\text{sign}(\mathbf{A}^{1/2}\mathbf{e}_{\tilde{L}_j})$ as the sign of the first non-zero coordinate of the vector $\mathbf{A}^{1/2}\mathbf{e}_{\tilde{L}_j}$.

From the point process convergence in Theorem 2.3, one can deduce that the largest eigenvalues of \mathbf{S} are separated. Indeed they converge in distribution to the $(\Gamma_i^{-2/\alpha})$ in the representation of the limiting point process N ; see (2.3) and (2.4). Combining this with Theorem 3.4, the aforementioned separation property is inherited by the eigenvalues of $\mathbf{Y}\mathbf{Y}'$ which simplifies the identification of associated eigenvectors. It turns out that the unit eigenvectors of $\mathbf{Y}\mathbf{Y}'$ are approximated by the properly normalized $(\mathbf{A}^{1/2}\mathbf{e}_j)$ as shown in the next theorem.

Theorem 3.7 (Eigenvectors of $\mathbf{Y}\mathbf{Y}'$). Consider the setting and the conditions of Theorem 3.4. In addition, we assume $\sigma^2 \leq M$ a.s. for some constant $M > 0$.

1. If $\alpha \in (0, 2)$, then

$$\|\mathbf{v}_j(\mathbf{A}^{1/2}\mathbf{S}\mathbf{A}^{1/2}) - c_{\mathbf{A},j}\mathbf{A}^{1/2}\mathbf{e}_{\tilde{L}_j}\|_{\ell_2} \xrightarrow{\mathbb{P}} 0, \quad n \rightarrow \infty, j \geq 1, \quad (3.8)$$

with the normalization and orientation constants

$$c_{\mathbf{A},j} = \|\mathbf{A}^{1/2}\mathbf{e}_{\tilde{L}_j}\|_{\ell_2}^{-1} \text{sign}(\mathbf{A}^{1/2}\mathbf{e}_{\tilde{L}_j}).$$

2. If $\alpha \in (2, 4)$, then

$$\|\mathbf{v}_j(\mathbf{A}^{1/2}(\mathbf{S} - c_n\mathbf{I})\mathbf{A}^{1/2}) - c_{\mathbf{A},j}\mathbf{A}^{1/2}\mathbf{e}_{\tilde{L}_j}\|_{\ell_2} \xrightarrow{\mathbb{P}} 0, \quad n \rightarrow \infty, j \geq 1.$$

Proof. We focus on the case $\alpha \in (0, 2)$. Recall that $\mathbf{A}^{1/2}\mathbf{S}\mathbf{A}^{1/2}$ and $\mathbf{S}\mathbf{A}$ have the same eigenvalues. For any eigenvalue λ of $\mathbf{S}\mathbf{A}$ with associated eigenvector \mathbf{v} , that is, $\mathbf{S}\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$, we have

$$\mathbf{A}^{1/2}\mathbf{S}\mathbf{A}^{1/2}(\mathbf{A}^{1/2}\mathbf{v}) = \lambda(\mathbf{A}^{1/2}\mathbf{v}).$$

In words, \mathbf{v} is an eigenvector of $\mathbf{S}\mathbf{A}$ if and only if $\mathbf{A}^{1/2}\mathbf{v}$ is an eigenvector of $\mathbf{A}^{1/2}\mathbf{S}\mathbf{A}^{1/2}$; and both eigenvectors are associated with the same eigenvalue. For the proof of (3.8), it is therefore enough to show

$$\|\mathbf{v}_j(\mathbf{S}\mathbf{A}) - \mathbf{e}_{\tilde{L}_j}\|_{\ell_2} \xrightarrow{\mathbb{P}} 0, \quad n \rightarrow \infty, j \geq 1. \quad (3.9)$$

Fix $j \geq 1$ and let (k) be the integer sequence from condition (NB). We will follow the lines of the proof of Theorem 3.11 in [21].

By Theorem 2.1 and the observation $a_{np}^{-2}\|\text{diag}(\mathbf{S}) - \text{diag}_k(\mathbf{S})\|_2 \xrightarrow{\mathbb{P}} 0$, we see that

$$a_{np}^{-2} \max_{i=1, \dots, p} \|\mathbf{S}\mathbf{A}\mathbf{e}_i - \text{diag}_k(\mathbf{S})\mathbf{A}\mathbf{e}_i\|_{\ell_2} \leq a_{np}^{-2} \|\mathbf{S}\mathbf{A} - \text{diag}_k(\mathbf{S})\mathbf{A}\|_2 \xrightarrow{\mathbb{P}} 0, \quad n \rightarrow \infty, \quad (3.10)$$

and consequently

$$\varepsilon^{(n)} := a_{np}^{-2} \|\mathbf{SA} \mathbf{e}_{\tilde{L}_j} - S_{\tilde{L}_j} A_{\tilde{L}_j, \tilde{L}_j} \mathbf{e}_{\tilde{L}_j}\|_{\ell_2} \xrightarrow{\mathbb{P}} 0. \tag{3.11}$$

Before we can apply Proposition A.7 in [21] we need to show that, with probability converging to 1, there are no other eigenvalues in a suitably small interval around $\lambda_j(\mathbf{SA})$.

Let $\xi > 1$. We define the set

$$\Omega_n = \Omega_n(j, \xi) = \{\lambda_j(\mathbf{SA}) - \lambda_i(\mathbf{SA}) > \xi \varepsilon^{(n)} : i \neq j = 1, \dots, p\}.$$

Using (3.11) and Theorem 2.3, we obtain

$$\lim_{n \rightarrow \infty} \mathbb{P}(\Omega_n^c) = \lim_{n \rightarrow \infty} \mathbb{P}(a_{np}^{-2} \min\{\lambda_{j-1}(\mathbf{SA}) - \lambda_j(\mathbf{SA}), \lambda_j(\mathbf{SA}) - \lambda_{j+1}(\mathbf{SA})\} \leq \xi \varepsilon^{(n)}) = 0.$$

From the proof of Theorem 3.4 recall the definitions of the sets B_n and C_n . By Proposition A.7 in [21], the unit eigenvector $\mathbf{v}_j(\mathbf{SA})$ and the projection $\text{Proj}_{\mathbf{e}_{\tilde{L}_j}}(\mathbf{v}_j(\mathbf{SA}))$ of the vector $\mathbf{v}_j(\mathbf{SA})$ onto the linear space generated by $\mathbf{e}_{\tilde{L}_j}$ satisfy for fixed $\delta > 0$:

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \mathbb{P}(\|\mathbf{v}_j(\mathbf{SA}) - \text{Proj}_{\mathbf{e}_{\tilde{L}_j}}(\mathbf{v}_j(\mathbf{SA}))\|_{\ell_2} > \delta) \\ & \leq \limsup_{n \rightarrow \infty} \mathbb{P}(\{\|\mathbf{v}_j(\mathbf{SA}) - \text{Proj}_{\mathbf{e}_{\tilde{L}_j}}(\mathbf{v}_j(\mathbf{SA}))\|_{\ell_2} > \delta\} \cap \Omega_n \cap B_n \cap C_n) \\ & \quad + \limsup_{n \rightarrow \infty} \mathbb{P}((\Omega_n \cap B_n \cap C_n)^c) \\ & \leq \limsup_{n \rightarrow \infty} \mathbb{P}(\{2\varepsilon^{(n)} / (\xi \varepsilon^{(n)} - \varepsilon^{(n)}) > \delta\} \cap \Omega_n \cap B_n \cap C_n) \\ & \leq \limsup_{n \rightarrow \infty} \mathbb{P}(\{2/(\xi - 1) > \delta\}) = \mathbf{1}_{\{2/(\xi-1) > \delta\}}. \end{aligned}$$

The right-hand side is zero for sufficiently large ξ . Since both $\mathbf{v}_j(\mathbf{SA})$ and $\mathbf{e}_{\tilde{L}_j}$ are unit vectors and $\|\text{Proj}_{\mathbf{e}_{\tilde{L}_j}}(\mathbf{v}_j(\mathbf{SA}))\|_{\ell_2} \leq 1$, this means that $\|\mathbf{v}_j(\mathbf{SA}) - \mathbf{e}_{\tilde{L}_j}\|_{\ell_2} \xrightarrow{\mathbb{P}} 0$. This finishes the proof of (3.9).

For $\alpha \in (2, 4)$, the proof is identical after replacing \mathbf{S}, S_i by $\mathbf{S} - c_n \mathbf{I}, S_i - c_n$, respectively. \square

4. A stochastic volatility model with thinning

In this section, we consider a modification of the stochastic volatility model $X_{it} = \sigma_{it} Z_{it}$ introduced in (1.1). We keep the i.i.d. structure of the random field (Z_{it}) , the regular variation condition (1.2) on Z and the independence of (σ_{it}) and (Z_{it}) but we allow that σ_{it} varies with n :

$$X_{it}^{(n)} = \sigma_{it}^{(n)} Z_{it}, \quad n = 1, 2, \dots \tag{4.1}$$

Here $(\sigma_{it}^{(n)})_{i,t \in \mathbb{N}}$ is a field of i.i.d. non-negative random variables with a generic element $\sigma^{(n)}$ whose distribution may change with n . To be precise, we assume the following condition.

Assumption (A σ). For given $0 = s_0 < s_1 < \dots < s_m < \infty$ and $m \geq 1$,

$$\mathbb{P}(\sigma^{(n)} = s_i) = q_i^{(n)}, \quad i = 0, \dots, m, n = 1, 2, \dots, \tag{A\sigma}$$

$\lim_{n \rightarrow \infty} q_0^{(n)} = 1$ and the limits $\lim_{n \rightarrow \infty} nq_i^{(n)} > 0, i = 1, \dots, m$, exist.

Remark 4.1. The restriction to positive $s_i, i = 1, \dots, m$, is for notational convenience only. Also the assumption $\lim_{n \rightarrow \infty} q_0^{(n)} = 1$ which implies $\sigma^{(n)} \xrightarrow{\mathbb{P}} 0$ is for simplicity of presentation only. It implies that the matrix \mathbf{X} is sparse. If $\mathbb{E}[(\sigma^{(n)})^\alpha]$ had a positive limit w , the asymptotic spectral behavior of $\mathbf{S} = \mathbf{X}\mathbf{X}'$ constructed from $\mathbf{X} = (\sigma_{it}^{(n)} Z_{it})$ and $\mathbf{X} = (w^{1/\alpha} Z_{it})$, respectively, would be the same and one could work with the normalizing sequence a_{np}^2 . However, if $\mathbb{E}[(\sigma^{(n)})^\alpha] \rightarrow 0$, one needs to take this decay into account and adjust the normalizing sequence to obtain non-trivial asymptotic results.

We will assume the condition $(C_p(\beta))$ for some $\beta \in (0, 1]$ and use a normalizing sequence (b_n) such that

$$np\mathbb{E}[(\sigma^{(n)})^\alpha]\mathbb{P}(|Z| > b_n) \rightarrow 1, \quad n \rightarrow \infty.$$

Since $q_0^{(n)} \rightarrow 1$ we have $\mathbb{E}[(\sigma^{(n)})^\alpha] \rightarrow 0$. The additional condition $\lim_{n \rightarrow \infty} nq_i^{(n)} > 0$ means that the expected number of non-zero σ 's in a row of \mathbf{X} is positive. It ensures that $\lim_{n \rightarrow \infty} np\mathbb{E}[(\sigma^{(n)})^\alpha] = \infty$, hence $b_n \rightarrow \infty$. An alternative way of defining (b_n) would be

$$b_n = a_{[np\mathbb{E}[(\sigma^{(n)})^\alpha]]}. \tag{4.2}$$

Remark 4.2. We observe that for any $v > 0$,

$$\min_{i=1, \dots, m} s_i^v (1 - q_0^{(n)}) \leq \mathbb{E}[(\sigma^{(n)})^v] \leq \max_{i=1, \dots, m} s_i^v (1 - q_0^{(n)}),$$

hence all moments $\mathbb{E}[(\sigma^{(n)})^v]$ are of the same order as $1 - q_0^{(n)}$.

For fixed n , relations (1.3) and (1.6) remain valid but we will need results for these tails when $x = x_n \rightarrow \infty$ as $n \rightarrow \infty$. By the uniform convergence theorem for regularly varying functions, we have (see (1.2), (1.3) and (1.5) for the definitions of q_\pm and \tilde{q}_\pm)

$$\frac{\mathbb{P}(\pm\sigma^{(n)}Z > x_n)}{\mathbb{P}(|Z| > x_n)} \sim q_\pm \mathbb{E}[(\sigma^{(n)})^\alpha], \tag{4.3}$$

$$\frac{\mathbb{P}(\pm\sigma_1^{(n)}\sigma_2^{(n)}Z_1Z_2 > x_n)}{\mathbb{P}(|Z_1Z_2| > x_n)} \sim \tilde{q}_\pm \mathbb{E}[(\sigma_1^{(n)}\sigma_2^{(n)})^\alpha]. \tag{4.4}$$

The following result asserts that in the thinned stochastic volatility model (4.1) the sample covariance matrix is approximated by its diagonal under the new normalization b_n . It is an analog of Theorem 2.1.

Theorem 4.3. *Consider the stochastic volatility model (4.1). We assume the following conditions:*

- *The regular variation condition (1.2) for some $\alpha \in (0, 2) \cup (2, 4)$ and $\mathbb{E}[Z] = 0$ if $\mathbb{E}[|Z|] < \infty$.*
- *The growth condition $(C_p(\beta))$ for $p = p_n \rightarrow \infty$ for some $\beta \in (0, 1]$.*
- *Condition $(A\sigma)$ on the distribution of $\sigma^{(n)}$.*

Then

$$b_n^{-2} \|\mathbf{S} - \text{diag}(\mathbf{S})\|_2 \xrightarrow{\mathbb{P}} 0, \quad n \rightarrow \infty. \tag{4.5}$$

Theorem 2.1 and Theorem 4.3 show that neither the dependence structure in the σ -field nor a time-dependent distribution of σ change the core structure of \mathbf{S} , which is solely determined by the dependence in the heavy-tailed Z -field. Linear dependence among the Z_{it} 's, for instance, was studied in [13]. The resulting approximation of \mathbf{S} in this case is block diagonal.

The proof of Theorem 4.3 is given in Section 7.

By an application of Weyl's inequality, we may conclude from (4.5) that

$$b_n^{-2} \max_{i=1, \dots, p} |\lambda_i(\mathbf{S}) - \lambda_i(\text{diag}(\mathbf{S}))| \leq b_n^{-2} \|\mathbf{S} - \text{diag}(\mathbf{S})\|_2 \xrightarrow{\mathbb{P}} 0, \quad n \rightarrow \infty. \tag{4.6}$$

Using (4.6) and a continuous mapping argument, we can derive the limit of the point processes of the eigenvalues of the sample covariance matrix \mathbf{S} .

Theorem 4.4. *Assume the conditions of Theorem 4.3 and, in addition to $(A\sigma)$, for those $j \in \{1, \dots, m\}$ for which $\lim_{n \rightarrow \infty} nq_j^{(n)} = \infty$,*

$$pe^{-cnq_j^{(n)}} \rightarrow 0, \quad n \rightarrow \infty, \text{ for each } c > 0. \tag{4.7}$$

Then we have the following weak convergence of the point processes with state space $\mathbb{R} \setminus \{0\}$:

$$N_n = \sum_{i=1}^p \varepsilon_{b_n^{-2}(\lambda_i(\mathbf{S}) - c_n)} \xrightarrow{d} N, \quad n \rightarrow \infty.$$

Here N is a Poisson process on $\mathbb{R} \setminus \{0\}$ with mean measure $\mu_\alpha(x, \infty) = x^{-\alpha/2}$ and $\mu_\alpha(-\infty, -x) = 0$ for $x > 0$, and

$$c_n = \begin{cases} 0, & \text{if } \alpha \in (0, 2), \\ n\mathbb{E}[(X^{(n)})^2], & \text{if } \alpha \in (2, 4). \end{cases}$$

The proof is given in Section 8. This theorem generalizes the results in Auffinger and Tang [3] who considered the case $p/n \rightarrow \gamma \in (0, \infty)$, $m = 1$ and $1 - q_0^{(n)} = n^{-v}$ for some $v \in [0, 1]$. Condition (4.7) ensures that $nq_j^{(n)} \rightarrow \infty$ sufficiently fast. For example, if $p = n^\beta$ for some $\beta \in (0, 1]$ and $q_n^{(j)} \geq n^{-v}$ for some $v \in (0, 1)$ then for any fixed $c > 0$,

$$pe^{-cnq_j^{(n)}} \leq n^\beta e^{-cn^{1-v}} \rightarrow 0.$$

Theorem 4.4 shows that the limiting point processes of the thinned stochastic volatility model and the original one (see Theorem 2.3) are the same. Typically, thinning decreases the magnitude of the eigenvalues $\lambda_i(\mathbf{S} - c_n)$ which is accounted for by a smaller normalization b_n compared with a_{np}^2 used in Theorem 2.3. Indeed, from (4.2) one sees that $b_n a_{np}^{-2} \rightarrow 0$.

Next, we study the matrix $\mathbf{Y} = \mathbf{A}^{1/2}\mathbf{X}$ and the corresponding sample covariance matrix $\mathbf{Y}\mathbf{Y}'$ under thinning.

Theorem 4.5. *We consider the matrix $\mathbf{Y} = \mathbf{A}^{1/2}\mathbf{X}$, where \mathbf{X} follows the model (4.1). We assume the following conditions:*

- *The regular variation condition (1.2) for some $\alpha \in (0, 2) \cup (2, 4)$ and $\mathbb{E}[Z] = 0$ if $\mathbb{E}[|Z|] < \infty$.*
- *The growth condition $C_p(\beta)$ for $p = p_n \rightarrow \infty$ for some $\beta \in (0, 1]$.*
- *Condition (A σ) on the distribution of $\sigma^{(n)}$.*
- *$\mathbf{A} = \mathbf{A}_n$ constitutes a sequence of deterministic, positive definite $p \times p$ matrices with uniformly bounded spectra.*

Then

$$b_n^{-2} \max_{i=1, \dots, p} |\lambda_i(\mathbf{A}^{1/2}\mathbf{S}\mathbf{A}^{1/2}) - \lambda_i(\text{diag}(\mathbf{S})\mathbf{A})| \xrightarrow{\mathbb{P}} 0.$$

The proof of this result is identical to the proof of Theorem 3.2, using Theorem 4.3 instead of Theorem 2.1.

Moreover the same arguments that proved Theorems 3.4 and 3.7, using Theorems 4.3 and 4.5 instead of Theorems 2.1 and 3.2, respectively, show the following result.

Theorem 4.6 (Eigenvalues and eigenvectors of $\mathbf{Y}\mathbf{Y}'$). *Consider the setting and the conditions of Theorem 4.5. In addition, we assume that (\mathbf{A}_n) satisfies condition (NB).*

1. *If $\alpha \in (0, 2)$, we have for the eigenvalues of $\mathbf{Y}\mathbf{Y}'$,*

$$b_n^{-2} \max_{i=1, \dots, p} |\lambda_i(\mathbf{A}^{1/2}\mathbf{S}\mathbf{A}^{1/2}) - \lambda_i(\text{diag}(\mathbf{S}) \text{diag}(\mathbf{A}))| \xrightarrow{\mathbb{P}} 0,$$

and for the eigenvectors of $\mathbf{Y}\mathbf{Y}'$,

$$\|\mathbf{v}_j(\mathbf{A}^{1/2}\mathbf{S}\mathbf{A}^{1/2}) - c_{\mathbf{A},j}\mathbf{A}^{1/2}\mathbf{e}_{\tilde{L}_j}\|_{\ell_2} \xrightarrow{\mathbb{P}} 0, \quad n \rightarrow \infty, j \geq 1,$$

with the normalization and orientation constants

$$c_{\mathbf{A},j} = \|\mathbf{A}^{1/2} \mathbf{e}_{\tilde{L}_j}\|_{\ell_2}^{-1} \text{sign}(\mathbf{A}^{1/2} \mathbf{e}_{\tilde{L}_j}).$$

2. If $\alpha \in (2, 4)$, the eigenvalues of $\mathbf{Y}\mathbf{Y}' - \mathbb{E}[\mathbf{Y}\mathbf{Y}']$ satisfy

$$b_n^{-2} \max_{i=1, \dots, p} |\lambda_i(\mathbf{A}^{1/2}(\mathbf{S} - c_n \mathbf{I})\mathbf{A}^{1/2}) - \lambda_i(\text{diag}(\mathbf{S} - c_n \mathbf{I}) \text{diag}(\mathbf{A}))| \xrightarrow{\mathbb{P}} 0,$$

and for the eigenvectors of $\mathbf{Y}\mathbf{Y}' - \mathbb{E}[\mathbf{Y}\mathbf{Y}']$ we have

$$\|\mathbf{v}_j(\mathbf{A}^{1/2}(\mathbf{S} - c_n \mathbf{I})\mathbf{A}^{1/2}) - c_{\mathbf{A},j} \mathbf{A}^{1/2} \mathbf{e}_{\tilde{L}_j}\|_{\ell_2} \xrightarrow{\mathbb{P}} 0, \quad n \rightarrow \infty, j \geq 1.$$

In view of Remark 2.2, one can easily extend the results in this section to the case $\beta > 1$ in $(C_p(\beta))$.

5. Proof of Theorem 2.1

The proof is similar to the one of Theorem 3.5 in [21]: one has to replace $a_{np}^{-2} Z_{it}$ by $a_{np}^{-2} Z_{it} \sigma_{it}$ and solve a few additional technical difficulties stemming from the dependence in the σ -field. By assumption $\mathbb{E}[Z] = \mathbb{E}[X] = 0$ whenever these expectations are finite. Since the Frobenius norm $\|\cdot\|_F$ is an upper bound of the spectral norm we have

$$\begin{aligned} a_{np}^{-4} \|\mathbf{S} - \text{diag}(\mathbf{S})\|_2^2 &\leq a_{np}^{-4} \|\mathbf{S} - \text{diag}(\mathbf{S})\|_F^2 \\ &= a_{np}^{-4} \sum_{i,j=1; i \neq j}^p \sum_{t=1}^n X_{it}^2 X_{jt}^2 + a_{np}^{-4} \sum_{i,j=1; i \neq j}^p \sum_{t_1, t_2=1; t_1 \neq t_2}^n X_{i,t_1} X_{j,t_1} X_{i,t_2} X_{j,t_2} \\ &= a_{np}^{-4} \sum_{i,j=1; i \neq j}^p \sum_{t=1}^n X_{it}^2 X_{jt}^2 [\mathbf{1}(|Z_{it} Z_{jt}| > a_{np}^2) + \mathbf{1}(|Z_{it} Z_{jt}| \leq a_{np}^2)] + I_2^{(n)} \\ &= I_{11}^{(n)} + I_{12}^{(n)} + I_2^{(n)}. \end{aligned}$$

Thus it suffices to show that each of the expressions on the right-hand side converges to zero in probability. We have by Markov’s inequality for any $\epsilon > 0$ and sufficiently small $\delta \in (0, 1)$,

$$\mathbb{P}(I_{11}^{(n)} > \epsilon) \leq \sum_{i,j=1, i \neq j}^p n \mathbb{P}(|Z_1 Z_2| > a_{np}^2) \leq c \frac{np^2}{a_{np}^{2\alpha(1-\delta)}} \rightarrow 0.$$

Here we also used (1.5).

The case $\alpha \in (0, 2)$. An application of Markov's inequality, finiteness of all moments of σ and Karamata's theorem for $\alpha < 2$ show that for $\epsilon > 0$

$$\begin{aligned} \mathbb{P}(I_{12}^{(n)} > \epsilon) &\leq c \frac{n}{a_{np}^4} \sum_{i,j=1, i \neq j}^p \mathbb{E}[|Z_1 Z_2|^2 \mathbf{1}(|Z_1 Z_2| \leq a_{np}^2)] \\ &\leq cnp^2 \mathbb{P}(|Z_1 Z_2| > a_{np}^2) \rightarrow 0, \quad n \rightarrow \infty. \end{aligned}$$

The probability $\mathbb{P}(I_2^{(n)} > \epsilon)$ can be handled in a similar way by applying a Karamata argument.

The case $\alpha \in (2, 4)$. Before we proceed we provide an auxiliary result. Consider the following decomposition

$$[\mathbf{S} - \text{diag}(\mathbf{S})]^2 = \mathbf{D} + \mathbf{F} + \mathbf{R},$$

where

$$\mathbf{D} = (D_{ij})_{i,j=1,\dots,p} = \text{diag}([\mathbf{S} - \text{diag}(\mathbf{S})]^2).$$

The $p \times p$ matrix \mathbf{F} has a zero-diagonal and

$$F_{ij} = \sum_{u=1; u \neq i, j}^p \sum_{t=1}^n X_{it} X_{jt} X_{ut}^2, \quad 1 \leq i \neq j \leq p.$$

The $p \times p$ matrix \mathbf{R} has a zero-diagonal and

$$R_{ij} = \sum_{u=1; u \neq i, j}^p \sum_{t_1=1}^n \sum_{t_2=1; t_2 \neq t_1}^n X_{i,t_1} X_{j,t_2} X_{u,t_1} X_{u,t_2}, \quad 1 \leq i \neq j \leq p.$$

The following is the analog of Lemma 4.1 in [21].

Lemma 5.1. *Assume the conditions of Theorem 2.1 and $\alpha \in (2, 4)$. Then $a_{np}^{-4}(\|\mathbf{D}\|_2 + \|\mathbf{F}\|_2 + \|\mathbf{R}\|_2) \xrightarrow{\mathbb{P}} 0$.*

In view of this lemma, we have

$$a_{np}^{-4} \|\mathbf{S} - \text{diag}(\mathbf{S})\|_2^2 = a_{np}^{-4} \|\mathbf{S} - \text{diag}(\mathbf{S})\|_2^2 = a_{np}^{-4} \|\mathbf{D} + \mathbf{F} + \mathbf{R}\|_2 \xrightarrow{\mathbb{P}} 0.$$

This finishes the proof of Theorem 2.1. Our final goal is to prove Lemma 5.1.

Proof of the D-part. We have for $i = 1, \dots, p$,

$$\begin{aligned} D_{ii} &= \sum_{u=1}^p \sum_{t=1}^n X_{it}^2 X_{ut}^2 \mathbf{1}(i \neq u) + \sum_{u=1}^p \sum_{t_1=1}^n \sum_{t_2=1}^n X_{i,t_1} X_{u,t_1} X_{u,t_2} X_{i,t_2} \mathbf{1}(i \neq u) \mathbf{1}(t_1 \neq t_2) \\ &= M_{ii} + N_{ii}. \end{aligned}$$

We write \mathbf{M} and \mathbf{N} for diagonal matrices constructed from (M_{ii}) and (N_{ii}) such that $\mathbf{D} = \mathbf{M} + \mathbf{N}$. First bounding $\|\mathbf{N}\|_2$ by the Frobenius norm and then applying Markov's inequality and using the fact that the Z 's are centered, one can prove that $a_{np}^{-4}\|\mathbf{N}\|_2 \xrightarrow{\mathbb{P}} 0$. Writing $A_{i,u} = \{|\sum_{t=1}^n X_{it}^2 X_{ut}^2| > a_{np}^2\}$, we have for $i = 1, \dots, p$,

$$M_{ii} = \sum_{u=1, u \neq i}^p \sum_{t=1}^n X_{it}^2 X_{ut}^2 [\mathbf{1}_{A_{i,u}} + \mathbf{1}_{A_{i,u}^c}] = M_{ii}^{(1)} + M_{ii}^{(2)}.$$

On one hand, $\|\mathbf{M}^{(2)}\|_2 \leq pa_{np}^2$. Hence $a_{np}^{-4}\|\mathbf{M}^{(2)}\|_2 \xrightarrow{\mathbb{P}} 0$. On the other hand, we obtain with Markov's inequality for $\epsilon > 0$ and $r > 0$,

$$\begin{aligned} & \mathbb{P}(\|\mathbf{M}^{(1)}\|_2 > \epsilon a_{np}^4) \\ &= \mathbb{P}\left(\max_{i=1, \dots, p} |M_{ii}^{(1)}| > \epsilon a_{np}^4\right) \\ &\leq \mathbb{P}\left(\max_{i=1, \dots, p} \left| \sum_{u=1, u \neq i}^p \sum_{t=1}^n \sigma_{it}^2 \sigma_{ut}^2 \mathbf{1}_{\left(\max_{1 \leq j \leq p, 1 \leq s \leq n} \sigma_{js} > (np)^{1/(4r)}\right)} Z_{it}^2 Z_{ut}^2 \mathbf{1}_{A_{i,u}} \right| > \epsilon a_{np}^4\right) \\ &\quad + \mathbb{P}\left(\max_{i=1, \dots, p} \left| \sum_{u=1, u \neq i}^p \sum_{t=1}^n \sigma_{it}^2 \sigma_{ut}^2 \mathbf{1}_{\left(\max_{1 \leq j \leq p, 1 \leq s \leq n} \sigma_{js} \leq (np)^{1/(4r)}\right)} Z_{it}^2 Z_{ut}^2 \mathbf{1}_{A_{i,u}} \right| > \epsilon a_{np}^4\right) \\ &\leq np \mathbb{P}(|\sigma| > (np)^{1/(4r)}) \\ &\quad + \mathbb{P}\left(\max_{i=1, \dots, p} \left| \sum_{u=1, u \neq i}^p \sum_{t=1}^n Z_{it}^2 Z_{ut}^2 \mathbf{1}_{\left(\sum_{t=1}^n Z_{it}^2 Z_{ut}^2 > a_{np}^4 / (np)^{1/r}\right)} \right| > \epsilon a_{np}^4 / (np)^{1/r}\right) \\ &= J_1 + J_2. \end{aligned}$$

Since $\mathbb{E}[\sigma^{4r}] < \infty$ we have $J_1 \rightarrow 0$. We also have for large n , sufficiently large $r > 0$, by the von Bahr and Esséen inequality (see Petrov [33], 2.6.20 on page 82) for $q < \alpha/2$ close to $\alpha/2$,

$$\begin{aligned} J_2 &\leq p^2 \mathbb{P}\left(\sum_{t=1}^n Z_{1t}^2 Z_{2t}^2 > a_{np}^4 / (np)^{1/r}\right) \sim p^2 \mathbb{P}\left(\sum_{t=1}^n (Z_{1t}^2 Z_{2t}^2 - (\mathbb{E}[Z^2])^2) > a_{np}^4 / (np)^{1/r}\right) \\ &\leq cp^2 \frac{(np)^{q/r}}{a_{np}^{4q}} \mathbb{E}\left[\left|\sum_{t=1}^n (Z_{1t}^2 Z_{2t}^2 - (\mathbb{E}[Z^2])^2)\right|^q\right] \leq c \frac{p^2 n (np)^{q/r}}{a_{np}^{4q}} \rightarrow 0, \quad n \rightarrow \infty. \quad \square \end{aligned}$$

Proof of the F- and R-parts. The key observation is that $X = Z\sigma$ is regularly varying with index α . Choose \tilde{a}_n such that $\mathbb{P}(X > \tilde{a}_n) \sim n^{-1}$. The sequences a_n and \tilde{a}_n only differ by a slowly varying function which is negligible for the techniques in [21]. These techniques also work under the dependence stemming from the σ -field. Therefore, the proofs of the **F**- and **R**-parts are identical to [21]. □

6. Proof of Theorem 2.3

In view of (2.1) a continuous mapping argument shows that the points $(\lambda_i(\mathbf{S}) - c_n)/a_{np}^2$ in N_n may be replaced by the points $(S_i - c_n)/a_{np}^2$. We denote the resulting point process by

$$\tilde{N}_n = \sum_{i=1}^p \varepsilon_{a_{np}^{-2}(S_i - c_n)}.$$

We intend to use Kallenberg's theorem for proving $\tilde{N}_n \xrightarrow{d} N$; see Resnick [35], Proposition 3.22. For this reason, we have to show the following relations as $n \rightarrow \infty$,

$$\mathbb{E}[\tilde{N}_n(x, \infty)] \rightarrow \mathbb{E}[N(x, \infty)] = \mathbb{E}[\sigma^\alpha] x^{-\alpha/2}, \quad x > 0, \quad (6.1)$$

$$\mathbb{E}[\tilde{N}_n(-\infty, -x)] \rightarrow \mathbb{E}[N(-\infty, -x)] = 0, \quad x > 0, \quad (6.2)$$

$$\mathbb{P}(\tilde{N}_n(e_i, d_i] = 0, i = 1, \dots, m) \rightarrow \mathbb{P}(N(e_i, d_i] = 0, i = 1, \dots, m), \quad (6.3)$$

where $0 < e_1 < d_1 < \dots < e_m < d_m < \infty$, $m \geq 1$, are any positive numbers. We observe that for $S = S_1$,

$$\mathbb{E}[\tilde{N}_n(x, \infty)] = p\mathbb{P}(S > a_{np}^2 x + c_n), \quad (6.4)$$

$$\mathbb{E}[\tilde{N}_n(-\infty, -x)] = p\mathbb{P}(S < -a_{np}^2 x + c_n). \quad (6.5)$$

Then (6.1) and (6.2) will be a consequence of the following large deviation result which is a straightforward application of Theorem 4.2 in Mikosch and Wintenberger [30].

Lemma 6.1. *Assume the conditions of Theorem 2.3. Write $\gamma_n = n^{2/\alpha+\epsilon}$ for any $\epsilon > 0$.*

1. *If $\alpha \in (0, 2)$ we have*

$$\sup_{y \geq \gamma_n} \left| \frac{\mathbb{P}(S > y)}{n\mathbb{P}(X^2 > y)} - 1 \right| \rightarrow 0. \quad (6.6)$$

2. *If $\alpha \in (2, 4)$ we also assume that $(\sigma_t) = (\sigma_{it})$ is strongly mixing with rate (α_j) such that $\alpha_j \leq c j^{-a}$ for some $a > 1$, $c > 0$. Then we have*

$$\begin{aligned} \sup_{y \geq \gamma_n} \left| \frac{\mathbb{P}(\sum_{t=1}^n \sigma_{it}^2 (Z_{it}^2 - \mathbb{E}[Z^2]) > y)}{n\mathbb{P}(X^2 > y)} - 1 \right| &\rightarrow 0, \\ \sup_{y \geq \gamma_n} \frac{\mathbb{P}(\sum_{t=1}^n \sigma_{it}^2 (Z_{it}^2 - \mathbb{E}[Z^2]) \leq -y)}{n\mathbb{P}(X^2 > y)} &\rightarrow 0. \end{aligned}$$

Then (6.1) and (6.2) follow for $\alpha \in (0, 2)$ in view of (6.4), (6.5) and by choosing $y = a_{np}^2 x$ in (6.6). Indeed, in view of Breiman's lemma,

$$p\mathbb{P}(S > a_{np}^2 x) \sim np\mathbb{P}(X^2 > a_{np}^2 x) \sim \mathbb{E}[\sigma^\alpha] np\mathbb{P}(|Z| > a_{np}\sqrt{x}) \rightarrow \mathbb{E}[\sigma^\alpha] x^{-\alpha/2}. \quad (6.7)$$

The case $\alpha \in (2, 4)$ follows in the same way but we also have to show that the right-hand side in

$$p \frac{\mathbb{P}(a_{np}^{-2} |\sum_{t=1}^n (\sigma_t^2 - \mathbb{E}[\sigma^2])| > x)}{n \mathbb{P}(X^2 > a_{np}^2)} \sim cp^2 \mathbb{P}\left(a_{np}^{-2} \left| \sum_{t=1}^n (\sigma_t^2 - \mathbb{E}[\sigma^2]) \right| > x\right)$$

converges to zero. By Markov’s inequality, the right-hand expression is bounded by

$$cx^{-4} \underbrace{\frac{(np)^2}{a_{np}^8}}_{\rightarrow 0} \mathbb{E} \left[\left(n^{-1/2} \sum_{t=1}^n (\sigma_t^2 - \mathbb{E}[\sigma^2]) \right)^4 \right].$$

In view of the growth rate of (α_j) and the fact that $\sigma^2 \leq M$ a.s., Theorem 2.5 in [36] shows that the moments on the right-hand side converges to a constant, hence (6.7) converges to zero for $\alpha \in (2, 4)$.

Write \mathcal{F}_σ for the σ -algebra generated by (σ_{it}) . In what follows, we use the notation $\mathbb{P}_\sigma(\cdot) := \mathbb{P}(\cdot | \mathcal{F}_\sigma)$ and $\mathbb{E}_\sigma[\cdot] := \mathbb{E}[\cdot | \mathcal{F}_\sigma]$ for conditional probabilities and expectations with respect to \mathcal{F}_σ . By independence between (σ_{it}) and (Z_{it}) we have

$$\mathbb{P}(\tilde{N}_n(e_i, d_i) = 0, i = 1, \dots, m) = \mathbb{E} \left[\prod_{i=1}^m \mathbb{P}_\sigma(\tilde{N}_n(e_i, d_i) = 0) \right].$$

We intend to show that $\tilde{N}_n(e_i, d_i) \xrightarrow{d} \text{Pois}(\mu_\alpha(e_i, d_i))$ given \mathcal{F}_σ . Then (6.3) follows. By Poisson’s limit theorem (see Billingsley [7], Theorem 23.2), the latter limit holds if

$$\mathbb{E}_\sigma[\tilde{N}_n(e_i, d_i)] \rightarrow \mu_\alpha(e_i, d_i).$$

Lemma 6.2. *Assume the conditions of Theorem 2.3. For $\alpha \in (0, 2) \cup (2, 4)$ and $x > 0$, we have*

$$\mathbb{E}_\sigma[\tilde{N}_n(x, \infty)] = \sum_{i=1}^p \mathbb{P}_\sigma((S_i - c_n)/a_{np}^2 > x) \rightarrow \mu_\alpha(x, \infty), \tag{6.8}$$

$$\mathbb{E}_\sigma[\tilde{N}_n(-\infty, -x)] = \sum_{i=1}^p \mathbb{P}_\sigma((S_i - c_n)/a_{np}^2 < -x) \rightarrow 0. \tag{6.9}$$

Proof. We only show (6.8), the relation (6.9) can be proved in a similar way. We start with the case $\alpha \in (0, 2)$ and briefly comment on the case $\alpha \in (2, 4)$ at the end of this proof. We will show that

$$\sup_{i=1, \dots, p} \left| \frac{\mathbb{P}_\sigma(S_i/a_{np}^2 > x)}{\sum_{t=1}^n \sigma_{it}^\alpha \mathbb{P}(Z^2 > a_{np}^2 x)} - 1 \right| \rightarrow 0, \quad n \rightarrow \infty. \tag{6.10}$$

Then by definition of (a_{np}^2) and the ergodic theorem for (σ_{it}) ,

$$\begin{aligned} \sum_{i=1}^p \mathbb{P}_\sigma(S_i/a_{np}^2 > x) &\sim np\mathbb{P}(Z^2 > a_{np}^2 x) \left(\frac{1}{np} \sum_{i=1}^p \sum_{t=1}^n \sigma_{it}^\alpha \right) \\ &\rightarrow \mathbb{E}[\sigma^\alpha] x^{-\alpha/2} = \mu_\alpha(x, \infty). \end{aligned}$$

For ease of presentation, in the proof of (6.10) we assume that $x = 1$. Let $i \in \{1, \dots, p\}$. For small $\epsilon > 0$ we have

$$\begin{aligned} &\mathbb{P}_\sigma(S_i/a_{np}^2 > 1) \\ &\leq \sum_{t=1}^n \mathbb{P}_\sigma(\sigma_{it}^2 Z_{it}^2 > a_{np}^2 (1 - \epsilon)) + \mathbb{P}_\sigma\left(S_i - \max_{s=1, \dots, n} \sigma_{is}^2 Z_{is}^2 > \epsilon a_{np}^2\right) \\ &= I_{i1} + I_{i2}. \end{aligned}$$

In view of the uniform convergence theorem for regularly varying functions and since we assume σ to be bounded we have

$$\lim_{\epsilon \downarrow 0} \limsup_{n \rightarrow \infty} \sup_{i=1, \dots, p} \frac{I_{i1}}{\sum_{t=1}^n \sigma_{it}^\alpha \mathbb{P}(Z^2 > a_{np}^2 (1 - \epsilon))} \leq 1 \quad \text{a.s.} \tag{6.11}$$

For $\delta > 0$, we define the counting variable $T_i(\delta) = \sum_{t=1}^n \mathbf{1}(\sigma_{it}^2 Z_{it}^2 > \delta a_{np}^2)$ and consider the disjoint partition

$$\{T_i(\delta) \geq 2\}, \quad \{T_i(\delta) = 1\}, \quad \{T_i(\delta) = 0\}.$$

We have by the same argument as for I_{i1} ,

$$\limsup_{n \rightarrow \infty} \sup_{i=1, \dots, p} \frac{\mathbb{P}_\sigma(T_i(\delta) \geq 2)}{(\sum_{t=1}^n \sigma_{it}^\alpha \mathbb{P}(Z^2 > \delta a_{np}^2))^2} = c(\delta) \quad \text{a.s.},$$

for some constant $c(\delta)$ and therefore the contribution of the set $\{T_i(\delta) \geq 2\}$ is negligible. Moreover,

$$\begin{aligned} &\mathbb{P}_\sigma\left(T_i(\delta) = 1, S_i - \max_{t=1, \dots, n} \sigma_{it}^2 Z_{it}^2 > \epsilon a_{np}^2\right) \\ &\leq \sum_{t=1}^n \mathbb{P}_\sigma(\sigma_{it}^2 Z_{it}^2 > \delta a_{np}^2, S_i - \sigma_{it}^2 Z_{it}^2 > \epsilon a_{np}^2) \\ &= \sum_{t=1}^n \mathbb{P}_\sigma(\sigma_{it}^2 Z_{it}^2 > \delta a_{np}^2) \mathbb{P}_\sigma(S_i - \sigma_{it}^2 Z_{it}^2 > \epsilon a_{np}^2) = o(1) c \mathbb{P}(Z^2 > a_{np}^2) \sum_{t=1}^n \sigma_{it}^\alpha, \end{aligned}$$

where $o(1)$ does not depend on i . Here we used the same argument as for (6.11). As regards the set $\{T_i(\delta) = 0\}$, we have

$$\begin{aligned} & \mathbb{P}_\sigma\left(T_i(\delta) = 0, S_i - \max_{t=1, \dots, n} \sigma_{it}^2 Z_{it}^2 > \epsilon a_{np}^2\right) \\ & \leq \mathbb{P}_\sigma\left(\max_{t=1, \dots, n-1} \sigma_{it}^2 Z_{it}^2 \leq \delta a_{np}^2, S_i - \sigma_{in}^2 Z_{in}^2 > \epsilon a_{np}^2\right) \\ & \leq \mathbb{P}_\sigma\left(a_{np}^{-2} \sum_{t=1}^n \sigma_{it}^2 Z_{it}^2 \mathbf{1}(\sigma_{it}^2 Z_{it}^2 \leq \delta a_{np}^2) > \epsilon\right) = I_{i3}. \end{aligned}$$

Since $\sigma^2 \leq M$ and $p \rightarrow \infty$ we have by Karamata's theorem

$$a_{np}^{-2} \sum_{t=1}^n \mathbb{E}_\sigma[\sigma_{it}^2 Z^2 \mathbf{1}(\sigma_{it}^2 Z_{it}^2 \leq \delta a_{np}^2)] \leq a_{np}^{-2} n M \mathbb{E}[Z^2 \mathbf{1}(MZ^2 \leq \delta a_{np}^2)] \rightarrow 0. \tag{6.12}$$

Hence for large n ,

$$I_{i3} \leq \mathbb{P}_\sigma\left(a_{np}^{-2} \sum_{t=1}^n (\sigma_{it}^2 Z_{it}^2 \mathbf{1}(\sigma_{it}^2 Z_{it}^2 \leq \delta a_{np}^2)) - \mathbb{E}_\sigma[\sigma_{it}^2 Z^2 \mathbf{1}(\sigma_{it}^2 Z^2 \leq \delta a_{np}^2)] > \epsilon/2\right).$$

An application of the Fuk–Nagaev inequality (see Petrov [33], page 78, 2.6.5) yields for $r \geq 2$, $c_1, c_2 > 0$,

$$\begin{aligned} I_{i3} & \leq a_{np}^{-2r} c_1 \sum_{t=1}^n \mathbb{E}_\sigma[|\sigma_{it} Z|^{2r} \mathbf{1}(\sigma_{it}^2 Z^2 \leq \delta a_{np}^2)] \\ & \quad + \exp\left(-c_2 a_{np}^4 / \sum_{t=1}^n \text{var}(\sigma_{it}^2 Z_{it}^2 \mathbf{1}(\sigma_{it}^2 Z_{it}^2 \leq \delta a_{np}^2) \mid \mathcal{F}_\sigma)\right). \end{aligned}$$

An argument similar to (6.12) shows that

$$\limsup_{n \rightarrow \infty} \sup_{i=1, \dots, p} \frac{I_{i3}}{\sum_{t=1}^n \sigma_{it}^\alpha \mathbb{P}(Z^2 > a_{np}^2)} = 0 \quad \text{a.s.}$$

Summarizing the previous bounds and observing that all of them are uniform in i , we proved for given ϵ and sufficiently large n that, with probability 1,

$$\sum_{i=1}^p \mathbb{P}_\sigma(S_i/a_{np}^2 > x) \leq (1 + \epsilon) \mathbb{P}(Z^2 > a_{np}^2) \sum_{i=1}^p \sum_{t=1}^n \sigma_{it}^\alpha.$$

Next, we show the corresponding lower bound. In view of the uniform convergence theorem for regularly varying functions and since we assume σ to be bounded we have for $x = 1$ and

$\epsilon > 0$,

$$\begin{aligned} \mathbb{P}_\sigma(S_i/a_{np}^2 > x) &\geq \mathbb{P}_\sigma\left(\max_{t=1,\dots,n} \sigma_{it}^2 Z_{it}^2 > (1 + \epsilon)a_{np}^2\right) \\ &\geq \sum_{t=1}^n \mathbb{P}_\sigma(\sigma_{it}^2 Z_{it}^2 > (1 + \epsilon)a_{np}^2) \\ &\quad - \sum_{1 \leq s < t \leq n} \mathbb{P}_\sigma(\sigma_{it}^2 Z^2 > (1 + \epsilon)a_{np}^2) \mathbb{P}_\sigma(\sigma_{is}^2 Z^2 > (1 + \epsilon)a_{np}^2) \\ &= \sum_{t=1}^n \sigma_{it}^\alpha \mathbb{P}(Z^2 > a_{np}^2) (1 + \epsilon)^{-\alpha/2} (1 + o(1)). \end{aligned}$$

Since this bound is uniform in i , we conclude that, for given $\epsilon > 0$ and sufficiently large n ,

$$\sum_{i=1}^p \mathbb{P}_\sigma(S_i/a_{np}^2 > x) \geq (1 - \epsilon) \mathbb{P}(Z^2 > a_{np}^2) \sum_{i=1}^p \sum_{t=1}^n \sigma_{it}^\alpha.$$

This proves the lemma in the case $\alpha \in (0, 2)$.

In the case $\alpha \in (2, 4)$, first replace the points $(S_i - c_n)/a_{np}^2$ by $a_{np}^{-2} \sum_{t=1}^n \sigma_{it}^2 (Z_{it}^2 - \mathbb{E}[Z^2])$. The argument is similar to the one after Lemma 6.1. Now one can follow the lines of the proof in the case $\alpha \in (0, 2)$. We omit details. \square

7. Proof of Theorem 4.3

The proof is similar to the proof of Theorem 3.5 in [21] and to the proof of Theorem 2.1. We will sketch the proof, illustrating the differences one has to pay attention to. We restrict ourselves to the case $\alpha \in (0, 8/3) \setminus \{2\}$; the case $\alpha \in [8/3, 4)$ can be handled in a way similar to Theorem 2.1.

Indeed, the proof is even simpler because the field $(\sigma_{it}^{(n)})$ is i.i.d.

Since the Frobenius norm $\|\cdot\|_F$ is an upper bound of the spectral norm we have

$$\begin{aligned} &b_n^{-4} \|\mathbf{S} - \text{diag}(\mathbf{S})\|_2^2 \\ &\leq b_n^{-4} \|\mathbf{S} - \text{diag}(\mathbf{S})\|_F^2 \\ &= b_n^{-4} \sum_{i,j=1; i \neq j}^p \sum_{t=1}^n (X_{it}^{(n)})^2 (X_{jt}^{(n)})^2 + b_n^{-4} \sum_{i,j=1; i \neq j}^p \sum_{t_1, t_2=1; t_1 \neq t_2}^n X_{i,t_1}^{(n)} X_{j,t_1}^{(n)} X_{i,t_2}^{(n)} X_{j,t_2}^{(n)} \\ &= b_n^{-4} \sum_{i,j=1; i \neq j}^p \sum_{t=1}^n (X_{it}^{(n)})^2 (X_{jt}^{(n)})^2 [\mathbf{1}((X_{it}^{(n)})^2 (X_{jt}^{(n)})^2 > b_n^4) + \mathbf{1}((X_{it}^{(n)})^2 (X_{jt}^{(n)})^2 \leq b_n^4)] \\ &\quad + I_2^{(n)} \\ &= I_{11}^{(n)} + I_{12}^{(n)} + I_2^{(n)}. \end{aligned}$$

Thus it suffices to show that each of the expressions on the right-hand side converges to zero in probability. By (4.4) and the Potter bounds for regularly varying functions we have for any $\epsilon > 0$ and $n \rightarrow \infty$,

$$\mathbb{P}(I_{11}^{(n)} > \epsilon) \leq p^2 n \mathbb{P}((X_1^{(n)})^2 (X_2^{(n)})^2 > b_n^4) \sim p^2 n (\mathbb{E}[(\sigma^{(n)})^\alpha])^2 \mathbb{P}(|Z_1 Z_2| > b_n^2) \rightarrow 0.$$

Here we also used that $\mathbb{P}(|Z_1 Z_2| > x)$ is regularly varying with index α .

Assume first $\alpha \in (0, 2)$. Applications of Markov’s inequality, Karamata’s theorem and the Potter bounds yield

$$\begin{aligned} \mathbb{P}(I_{12}^{(n)} > \epsilon) &\leq c \frac{p^2 n}{b_n^4} \mathbb{E}[|X_1^{(n)} X_2^{(n)}|^2 \mathbf{1}(|X_1^{(n)} X_2^{(n)}| \leq b_n^2)] \\ &= c p^2 n \sum_{i,j=1}^m q_i^{(n)} q_j^{(n)} \frac{s_i^2 s_j^2 \mathbb{E}[(Z_1 Z_2)^2 \mathbf{1}(s_i s_j |Z_1 Z_2| \leq b_n^2)]}{b_n^4 \mathbb{P}(s_i s_j |Z_1 Z_2| > b_n^2)} \mathbb{P}(s_i s_j |Z_1 Z_2| > b_n^2) \\ &\sim c p^2 n \sum_{i,j=1}^m q_i^{(n)} q_j^{(n)} s_i^\alpha s_j^\alpha \mathbb{P}(|Z_1 Z_2| > b_n^2) \\ &= c p^2 n (\mathbb{E}[(\sigma^{(n)})^\alpha])^2 \mathbb{P}(|Z_1 Z_2| > b_n^2) \rightarrow 0, \quad n \rightarrow \infty. \end{aligned}$$

If $\alpha \in (2, 8/3)$, we have $\mathbb{E}[Z^2] < \infty$. Hence,

$$\mathbb{P}(I_{12}^{(n)} > \epsilon) \leq c \frac{p^2 n}{b_n^4} \mathbb{E}[|(X_1^{(n)} X_2^{(n)})|^2] = c \frac{p^2 n}{b_n^4} (\mathbb{E}[(\sigma^{(n)})^2])^2 \rightarrow 0.$$

Here we also used the fact that all moments of $\sigma^{(n)}$ are of the same size; see Remark 4.2.

For $\alpha \in (0, 2)$, the probability $P_2^{(n)} = \mathbb{P}(I_2^{(n)} > \epsilon)$ can be handled analogously; we omit details. We turn to $P_2^{(n)}$ in the case $\alpha \in (2, 8/3)$. In particular, we have $\mathbb{E}[Z] = 0$ and $\mathbb{E}[Z^2] < \infty$. With Chebychev’s inequality, also using the independence and the fact that $\mathbb{E}[X^{(n)}] = 0$, we find that

$$\begin{aligned} P_2^{(n)} &\leq c \frac{1}{b_n^8} \mathbb{E} \left[\left(\sum_{i,j=1; i \neq j}^p \sum_{t_1, t_2=1; t_1 \neq t_2}^n X_{i,t_1}^{(n)} X_{j,t_1}^{(n)} X_{i,t_2}^{(n)} X_{j,t_2}^{(n)} \right)^2 \right] \\ &\leq c \frac{(pn)^2}{b_n^8} (\mathbb{E}[(\sigma^{(n)})^2])^4 \rightarrow 0, \quad n \rightarrow \infty. \end{aligned}$$

This finishes the proof.

8. Proof of Theorem 4.4

In what follows, we will write S for a generic element of the sequence of diagonal entries (S_i) . Since we have

$$b_n^{-2} \max_{i=1, \dots, p} \|(\lambda_i(\mathbf{S}) - c_n) - (\lambda_i(\text{diag}(\mathbf{S})) - c_n)\|_2 \xrightarrow{\mathbb{P}} 0, \quad n \rightarrow \infty,$$

a continuous mapping argument shows that it suffices to show the point process convergence

$$\tilde{N}_n = \sum_{i=1}^p \varepsilon_{b_n^{-2}(S_i - c_n)} \xrightarrow{d} N, \quad n \rightarrow \infty.$$

Since the points (S_i) are independent it suffices to show that for $x > 0$,

$$\mathbb{E}[\tilde{N}_n(x, \infty)] = p\mathbb{P}(S > xb_n^2 + c_n) \rightarrow \mathbb{E}[N(x, \infty)] = x^{-\alpha/2}, \quad (8.1)$$

$$\mathbb{E}[\tilde{N}_n(-\infty, x)] = p\mathbb{P}(S < -xb_n^2 + c_n) \rightarrow \mathbb{E}[N(-\infty, -x)] = 0. \quad (8.2)$$

We restrict ourselves to prove (8.1); the proof of (8.2) is analogous. For generic sequences (Z_t) and $(\sigma_t^{(n)})$, we have the representation

$$S = \sum_{j=1}^m s_j^2 \sum_{t=1}^n Z_t^2 \mathbf{1}(\sigma_t^{(n)} = s_j) = \sum_{j=1}^m s_j^2 \sum_{t \in A_j} Z_t^2,$$

where $A_j = A_j^{(n)} = \{1 \leq t \leq n : \sigma_t^{(n)} = s_j\}$. Write M_j for the cardinality of A_j . Then we have the representation

$$S \stackrel{d}{=} \sum_{j=1}^m s_j^2 T_j, \quad \text{where } T_j = \sum_{t=1}^{M_j} Z_{jt}^2,$$

and (M_j) and $(Z_{jt})_{t=1,2,\dots;j=1,\dots,m}$ are independent. We observe that M_j is binomially distributed with mean $\mathbb{E}[M_j] = nq_j^{(n)}$. The next lemma concludes the proof of Theorem 4.4.

Lemma 8.1. *Assume the conditions of Theorem 4.4. Then (8.1) holds.*

Proof. Define $\tau_j = \lim_{n \rightarrow \infty} nq_j^{(n)}$, $j = 1, \dots, m$. We will consider two cases:

1. At least one τ_j is infinite.
2. All τ_j are finite.

Throughout we assume $\alpha \in (0, 2)$; the case $\alpha \in (2, 4)$ is analogous, taking into account the centering c_n for S .

We start with the case that $\tau_k = \infty$. If $0 < \tau_j < \infty$ for some $j \neq k$ we will show that $s_j^2 T_j$ does not contribute to $\lim_{n \rightarrow \infty} p\mathbb{P}(S > xb_n^2)$. In this case, $M_j \xrightarrow{d} Y_j \sim \text{Pois}(\tau_j)$ and $\mathbb{E}[e^{hM_j}] \rightarrow$

$\mathbb{E}[e^{hY_j}]$, $h > 0$. We have by Markov's inequality for positive h, ϵ ,

$$\begin{aligned} p\mathbb{P}(s_j^2 T_j > \epsilon b_n^2) &= p\mathbb{P}(Z^2 > b_n^2) \sum_{k=1}^{\infty} \mathbb{P}(M_j = k) \frac{\mathbb{P}(\sum_{t=1}^k Z_t^2 > \epsilon b_n^2)}{\mathbb{P}(Z^2 > b_n^2)} \\ &\leq p\mathbb{P}(Z^2 > b_n^2) \mathbb{E}[e^{hM_j}] \sum_{k=1}^{\infty} e^{-hk} \frac{\mathbb{P}(\sum_{t=1}^k Z_t^2 > \epsilon b_n^2)}{\mathbb{P}(Z^2 > b_n^2)} \\ &\sim \epsilon^{-\alpha/2} \frac{1}{n\mathbb{E}[(\sigma^{(n)})^\alpha]} \underbrace{pn\mathbb{E}[(\sigma^{(n)})^\alpha]}_{\sim 1} \mathbb{P}(Z^2 > b_n^2) \mathbb{E}[e^{hY_j}] \sum_{k=1}^{\infty} k e^{-hk} \rightarrow 0. \end{aligned}$$

Here we also used the subexponential property of the distribution of Z^2 (see Theorem A3.20 in Embrechts et al. [17]).

Therefore, we assume for the rest of the proof of case (1) that $\tau_j = \infty$ for all $1 \leq j \leq m$. We have for small $\epsilon > 0$,

$$\mathbb{P}(S > x b_n^2) \leq \sum_{j=1}^m \mathbb{P}(s_j^2 T_j > x b_n^2 (1 - \epsilon)) + \mathbb{P}\left(\bigcap_{k=1}^m |S - s_k^2 T_k| > \epsilon x b_n^2\right) = I_1 + I_2. \tag{8.3}$$

First, we deal with I_1 . We notice that $b_n^2 / (nq_j^{(n)})^{2/\alpha} \rightarrow \infty$. Our goal is to apply classical large deviation results (see Theorem A.1 in [21]) after replacing M_j by $\mathbb{E}[M_j]$. We have for small δ ,

$$\begin{aligned} J_j &= \mathbb{P}(s_j^2 T_j > x b_n^2 (1 - \epsilon)) \\ &= \mathbb{P}(s_j^2 T_j > x b_n^2 (1 - \epsilon), |M_j - \mathbb{E}[M_j]| \leq \delta \mathbb{E}[M_j]) \\ &\quad + \mathbb{P}(s_j^2 T_j > x b_n^2 (1 - \epsilon), |M_j - \mathbb{E}[M_j]| > \delta \mathbb{E}[M_j]) = J_{j1} + J_{j2}. \end{aligned}$$

We have

$$J_{j2} \leq \mathbb{P}(|M_j - \mathbb{E}[M_j]| > \delta \mathbb{E}[M_j]) = \mathbb{P}(M_j > (1 + \delta)\mathbb{E}[M_j]) + \mathbb{P}(M_j < (1 - \delta)\mathbb{E}[M_j]).$$

An application of Markov's exponential inequality yields for $h = \log(1 + \delta)$ and small $\delta > 0$,

$$\begin{aligned} \mathbb{P}(M_j > (1 + \delta)\mathbb{E}[M_j]) &\leq e^{-h(1+\delta)nq_j^{(n)}} (1 - q_j^{(n)}(1 - e^h))^n \\ &\leq e^{-nq_j^{(n)}(h(1+\delta) + (1 - e^h))} \\ &= e^{-nq_j^{(n)}((1+\delta)\log(1+\delta) - \delta)} \\ &\leq e^{-0.5\delta^2 nq_j^{(n)}}. \end{aligned}$$

A similar argument shows that for small $\delta > 0$,

$$\mathbb{P}(M_j < (1 - \delta)\mathbb{E}[M_j]) \leq e^{-0.5\delta^2 nq_j^{(n)}}.$$

In view of condition (4.7), we have

$$pJ_{j2} \leq 2pe^{-0.5\delta^2 nq_j^{(n)}} \rightarrow 0, \quad n \rightarrow \infty.$$

We also have in view of Theorem A.1 in [21]

$$\begin{aligned} J_{j1} &\leq \mathbb{P}\left(s_j^2 \sum_{t=1}^{(1+\delta)\mathbb{E}[M_j]} Z_t^2 > xb_n^2(1-\epsilon)\right) \sim (1+\delta)\mathbb{E}[M_j]\mathbb{P}(s_j^2 Z^2 > xb_n^2(1-\epsilon)) \\ &\sim x^{-\alpha/2} \frac{1+\delta}{(1-\epsilon)^{\alpha/2}} ns_j^\alpha q_j^{(n)} \mathbb{P}(|Z| > b_n), \\ J_{j1} &\geq \mathbb{P}\left(s_j^2 \sum_{t=1}^{(1-\delta)\mathbb{E}[M_j]} Z_t^2 > xb_n^2(1-\epsilon)\right) \sim x^{-\alpha/2} \frac{1-\delta}{(1-\epsilon)^{\alpha/2}} ns_j^\alpha q_j^{(n)} \mathbb{P}(|Z| > b_n). \end{aligned}$$

Letting $\delta \downarrow 0$ and recalling the definition of b_n , we conclude that

$$\lim_{\epsilon \rightarrow 0} \limsup_{n \rightarrow \infty} pI_1 = \lim_{\epsilon \rightarrow 0} \limsup_{n \rightarrow \infty} p \sum_{j=1}^m J_j = x^{-\alpha/2}.$$

Our next goal is to show that $pI_2 \rightarrow 0$. Consider a disjoint partition for small $\delta > 0$ and $j = 1, \dots, m$,

$$\begin{aligned} B_1 &= \bigcup_{1 \leq i < j \leq m} \{s_i^2 T_i > \delta b_n^2, s_j^2 T_j > \delta b_n^2\}, \\ B_2 &= \bigcup_{j=1}^m \{s_j^2 T_j > \delta b_n^2, s_i^2 T_i \leq \delta b_n^2, i \neq j, i = 1, \dots, m\}, \\ B_3 &= \left\{ \max_{j \leq m} s_j^2 T_j \leq \delta b_n^2 \right\}. \end{aligned}$$

We have

$$p\mathbb{P}(B_1) \leq p \sum_{1 \leq i < j \leq m} \mathbb{P}(s_i^2 T_i > \delta b_n^2, s_j^2 T_j > \delta b_n^2).$$

To show that the right-hand side converges to 0, we proceed as for J_j . For $i < j$ we replace the random indices M_i and M_j in T_i and T_j by their corresponding expectations. We omit further details. Abusing notation here and in what follows, we denote the resulting modified quantities by the same symbols T_i and T_j . After this operation, T_i and T_j are independent and we can treat their tail probabilities in the same way as for J_j , yielding $\lim_{n \rightarrow \infty} p\mathbb{P}(B_1) = 0$.

Next, we observe that

$$\mathbb{P}(\{|S - s_j^2 T_j| > \epsilon b_n^2, j \leq m\} \cap B_2) \leq \sum_{j=1}^m \mathbb{P}(|S - s_j^2 T_j| > \epsilon b_n^2, s_j^2 T_j > \delta b_n^2).$$

Now proceed as for J_j : replace all M_j by $\mathbb{E}[M_j]$ in each probability in the sum. Then the modified sums $S - s_j^2 T_j$ and $s_j^2 T_j$ become independent. Using the independence, we see that

$$\begin{aligned} & \limsup_{n \rightarrow \infty} p \mathbb{P}(\{|S - s_j^2 T_j| > \epsilon b_n^2, j \leq m\} \cap B_2) \\ &= \limsup_{n \rightarrow \infty} \frac{1}{p} \sum_{j=1}^m (p \mathbb{P}(|S - s_j^2 T_j| > \epsilon b_n^2))(p \mathbb{P}(s_j^2 T_j > \delta b_n^2)) = 0. \end{aligned}$$

Finally, we deal with

$$\begin{aligned} \mathbb{P}(\{|S - s_j^2 T_j| > \epsilon b_n^2, j \leq m\} \cap B_3) &\leq \mathbb{P}\left(b_n^{-2} \sum_{j=1}^m s_j^2 T_j \mathbf{1}(s_j^2 T_j \leq \delta b_n^2) > \epsilon\right) \\ &\leq \sum_{j=1}^m \mathbb{P}(b_n^{-2} s_j^2 T_j \mathbf{1}(s_j^2 T_j \leq \delta b_n^2) > \epsilon/m). \end{aligned}$$

Since we can choose δ independently from ϵ , we can take $\delta < \epsilon/m$, making the right-hand side vanish. Combining all the previous bounds, we finally arrived at

$$\limsup_{n \rightarrow \infty} p \mathbb{P}(S > x b_n^2) \leq x^{-\alpha/2}, \quad x > 0,$$

in the case $\alpha \in (0, 2)$. In the case $\alpha \in (2, 4)$ we have to center the quantities S and T_j . Then the same ideas of the proof apply, in particular the large deviations results of Theorem A.1 in [21]. We omit details.

Next consider, for $\alpha \in (0, 2)$,

$$\begin{aligned} \mathbb{P}(S > x b_n^2) &\geq \mathbb{P}(s_j^2 T_j > x b_n^2(1 + \epsilon), |S - s_j^2 T_j| \leq \epsilon b_n^2 \text{ for some } j \leq m) \\ &\geq \sum_{j=1}^m \mathbb{P}(s_j^2 T_j > x b_n^2(1 + \epsilon), |S - s_j^2 T_j| \leq \epsilon b_n^2) \\ &\quad - \sum_{1 \leq i < j \leq m} \mathbb{P}(s_i^2 T_i > x b_n^2(1 + \epsilon), s_j^2 T_j > x b_n^2(1 + \epsilon)). \end{aligned} \tag{8.4}$$

We proceed as before: we replace the numbers M_j by their expectations. After this operation the modified sums $s_j^2 T_j$, $S - s_j^2 T_j$ and $s_i^2 T_i$ for $i \neq j$ become independent. Moreover, $\mathbb{P}(|S - s_j^2 T_j| \leq \epsilon b_n^2) \rightarrow 1$. Hence for fixed small $\delta > 0$ and large n ,

$$\mathbb{P}(S > x b_n^2) \geq (1 - \delta) \sum_{j=1}^m \mathbb{P}(s_j^2 T_j > x b_n^2(1 + \epsilon)).$$

Applying Theorem A.1 in [21] and letting ϵ, δ go to zero, we proved that

$$\liminf_{n \rightarrow \infty} p \mathbb{P}(S > x b_n^2) \geq x^{-\alpha/2}, \quad x > 0.$$

Our next goal is to consider case (2) in which $0 < \tau_j < \infty$ for all j . We will show that

$$\begin{aligned}
 p\mathbb{P}(S > xb_n^2) &\sim p \sum_{j=1}^m \mathbb{P}(s_j^2 T_j > xb_n^2) \\
 &\sim p \sum_{j=1}^m s_j^\alpha \mathbb{E}[M_j] \mathbb{P}(Z^2 > xb_n^2) \\
 &= pn \mathbb{E}[(\sigma^{(n)})^\alpha] \mathbb{P}(Z^2 > xb_n^2) \rightarrow x^{-\alpha/2}.
 \end{aligned}
 \tag{8.5}$$

We have $M_j \xrightarrow{d} Y_j \sim \text{Pois}(\tau_j)$ as $n \rightarrow \infty$, in particular $\mathbb{P}(M_j = k) \rightarrow \pi_k^{(j)} = \mathbb{P}(Y_j = k)$ and $\mathbb{P}(M_j = k) \leq ce^{-hk}$, $k \geq 1$, $h > 0$; see [17] page 41, equation (1.31). Keeping this in mind, subexponentiality of the distribution of Z^2 yields

$$\begin{aligned}
 \frac{\mathbb{P}(s_j^2 T_j > xb_n^2)}{\mathbb{P}(Z^2 > b_n^2)} &= \sum_{k=1}^{\infty} \mathbb{P}(M_j = k) \frac{\mathbb{P}(s_j^2 \sum_{t=1}^k Z_t^2 > xb_n^2)}{\mathbb{P}(Z^2 > b_n^2)} \\
 &\rightarrow s_j^\alpha \sum_{k=1}^{\infty} \pi_k^{(j)} k = s_j^\alpha \mathbb{E}[M_j], \quad n \rightarrow \infty.
 \end{aligned}
 \tag{8.6}$$

For the upper bound in (8.5) we recall the inequality (8.3). In view of (8.6) and regular variation of Z^2 , for the upper bound it remains to show that

$$\frac{\mathbb{P}(s_j^2 T_j > b_n^2, j = 1, \dots, m)}{\mathbb{P}(Z^2 > b_n^2)} \rightarrow 0.
 \tag{8.7}$$

We show (8.7) only for $m = 2$. We have

$$\begin{aligned}
 &\mathbb{P}(s_1^2 T_1 > b_n^2, s_2^2 T_2 > b_n^2) \\
 &= \sum_{k,l=1}^{\infty} \mathbb{P}(M_1 = k, M_2 = l) \mathbb{P}\left(s_1^2 \sum_{t=1}^k Z_t^2 > b_n^2\right) \mathbb{P}\left(s_2^2 \sum_{t=1}^l Z_t^2 > b_n^2\right) \\
 &\leq \sum_{k=1}^{\infty} \sqrt{\mathbb{P}(M_1 = k)} \mathbb{P}\left(\sum_{t=1}^k Z_t^2 > b_n^2\right) \sum_{l=1}^{\infty} \sqrt{\mathbb{P}(M_2 = l)} \mathbb{P}\left(\sum_{t=1}^l Z_t^2 > b_n^2\right).
 \end{aligned}$$

The same arguments which established (8.6) show that the right-hand side is of the order $O((\mathbb{P}(Z^2 > b_n^2))^2)$. This proves (8.7).

The lower bound in (8.5) follows by similar arguments, taking into account the inequality (8.4). □

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