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The first order correction to harmonic measure for random walks of rotationally invariant step distribution

LONGMIN WANG[1,*], KAINAN XIANG[2] and LANG ZOU[1,**]

[1] *School of Mathematical Sciences, LPMC, Nankai University. 94 Weijin Road, Tianjin 300071, P.R. China. E-mail:* *wanglm@nankai.edu.cn; **lang.zou.proba@gmail.com
[2] *School of Mathematics and Computational Science, Xiangtan University Xiangtan City 210000, Hunan Province, P. R. China. E-mail:* kainan.xiang@xtu.edu.cn

Let $D \subset \mathbb{R}^d$ $(d \geq 2)$ be an open simply-connected bounded domain with smooth boundary ∂D and $\mathbf{0} = (0, \ldots, 0) \in D$. Fix any rotationally invariant probability μ on closed unit ball $\{z \in \mathbb{R}^d : |z| \leq 1\}$ with $\mu(\{\mathbf{0}\}) < 1$. Let $\{S_n^\mu\}_{n=0}^\infty$ be the random walk with step-distribution μ starting at $\mathbf{0}$. Denote by $\omega_\delta(\mathbf{0}, \mathrm{d}z; D)$ the discrete harmonic measure for $\{\delta S_n^\mu\}_{n=0}^\infty$ $(\delta > 0)$ exiting from D, which is viewed as a probability on ∂D by projecting suitably the first exiting point to ∂D. Denote by $\omega(\mathbf{0}, \mathrm{d}z; D)$ the harmonic measure for the d-dimensional standard Brownian motion exiting from D. Then in the weak convergence topology,

$$\lim_{\delta \to 0} \frac{1}{\delta}\big[\omega_\delta(\mathbf{0}, \mathrm{d}z; D) - \omega(\mathbf{0}, \mathrm{d}z; D)\big] = c_\mu \rho_D(z)\,|\mathrm{d}z|,$$

where $\rho_D(\cdot)$ is a smooth function depending on D but not on μ, c_μ is a constant depending only on μ, and $|\mathrm{d}z|$ is the Lebesgue measure with respect to ∂D. Additionally, $\rho_D(z)$ is determined by the following equation: For any smooth function g on ∂D,

$$\int_{\partial D} g(z)\rho_D(z)\,|\mathrm{d}z| = \int_{\partial D} \frac{\partial f}{\partial \mathbf{n}_z}(z) H_D(\mathbf{0}, z)\,|\mathrm{d}z|,$$

where f is the harmonic function in D with boundary values given by g, $H_D(\mathbf{0}, z)$ is the Poisson kernel and derivative $\frac{\partial f}{\partial \mathbf{n}_z}$ is with respect to the inward unit normal $\mathbf{n}_z$ at $z \in \partial D$.

Keywords: discrete harmonic measure; first order correction; harmonic measure; random walk

1. Introduction

We study in this paper the universality for the first order correction between discrete and continuous harmonic measures in $\mathbb{R}^d$ with $d \geq 2$, which was initiated by Kennedy [12] and Jiang and Kennedy [10] in $\mathbb{R}^2$.

To begin, denote $\mathbf{0} = (0, \ldots, 0) \in \mathbb{R}^d$, $x = (x_1, \ldots, x_d)$ for any $x \in \mathbb{R}^d$. Let $\{X_i\}_{i=0}^\infty$ be an i.i.d. sequence of random variables in $\mathbb{R}^d$ with common distribution μ such that

$$\begin{cases} \mathbb{E}[X_i] = \mathbf{0}, \qquad \mathbb{E}\big[X_i^{(n)} X_i^{(m)}\big] = 0, \qquad n \neq m, \\ \mathbb{E}\big[\big(X_i^{(1)}\big)^2\big] = \cdots = \mathbb{E}\big[\big(X_i^{(d)}\big)^2\big] = \kappa \in (0, \infty), \end{cases} \tag{1.1}$$

where $X_i = (X_i^{(1)}, \dots, X_i^{(d)})$. For any $s_0 \in \mathbb{R}^d$, we define the random walk $\{S_n^\mu\}_{n\geq 0}$ with step distribution μ starting at s_0 by

$$S_0^\mu = s_0, \qquad S_n^\mu = s_0 + X_1 + \cdots + X_n, \qquad n \geq 1.$$

The rescaled process $\{\delta(S^\mu_{\lfloor \delta^{-2}t \rfloor} - s_0)\}_{t\geq 0}$ converges in law to $\{B(\kappa t)\}_{t\geq 0}$ as $\delta \to 0$, where $\lfloor x \rfloor$ is the integer part of $x \in \mathbb{R}$ and $\{B(t)\}_{t\geq 0}$ is the d-dimensional standard Brownian motion in $\mathbb{R}^d$ starting at $\mathbf{0}$.

Let $D \subset \mathbb{R}^d$ be an open simply-connected bounded domain with smooth boundary ∂D and $\mathbf{0} \in D$. It is known that D is regular (Gilbarg and Trudinger [7], p. 27 and Karatzas and Shreve [11], pp. 245–250), in the sense that,

$$\mathbb{P}_z\{\tau_D = 0\} = 1 \qquad \forall z \in \partial D;$$

where $\tau_D = \inf\{t \geq 0 : B(t) \notin D\}$ and $\mathbb{P}_z$ is the law of $\{B(t)\}_{t\geq 0}$ starting from z. Let $\omega(x, \mathrm{d}z; D)$ be the continuous harmonic measure for $\{B(t)\}_{t\geq 0}$ exiting from D when staring at $x \in D$, that is,

$$\omega(x, \mathrm{d}z; D) = \mathbb{P}_x\big(B(\tau_D) \in \mathrm{d}z\big).$$

It is known that for fixed $x \in D$, $\omega(x, \mathrm{d}z; D)$ is absolutely continuous with respect to $|\mathrm{d}z|$, the Lebesgue measure on ∂D. More precisely,

$$\omega(x, \mathrm{d}z; D) = H_D(x, z)\,|\mathrm{d}z|,$$

where $H_D(x, z)$ is the Poisson kernel in D and may be defined as the derivative of the Green function $G_D(x, z)$ in the direction $\mathbf{n}_z$, the inward unit normal at $z \in \partial D$, that is,

$$H_D(x, z) = \frac{\partial G_D(x, z)}{\partial \mathbf{n}_z}.$$

Since ∂D is smooth, we have that $H_D(x, z)$ is smooth in $z \in \partial D$ (cf. [9] and Gilbarg and Trudinger [7]). For more details on harmonic measures, please refer to Axler, Bourdon and Wade [1], Garnett and Marshall [5], Karatzas and Shreve [11], Möters and Peres [15].

Now we back to the discrete setting. Without loss of generality, in the rest of this paper we will always assume $S_0^\mu = s_0 = \mathbf{0}$, unless otherwise specified. Let

$$T_D = T_D(\delta) = \inf\big\{n \geq 0 : \delta S_n^\mu \notin D\big\}.$$

Define the discrete harmonic measure $\hat{\omega}_\delta(\mathbf{0}, \mathrm{d}z; D)$ for $\{\delta S_n^\mu\}_{n\geq 0}$ exiting from D by

$$\hat{\omega}_\delta(\mathbf{0}, \Gamma; D) = \mathbb{P}\big(\overline{\delta S^\mu_{T_D}} \in \Gamma\big) \qquad \forall \text{ measurable } \Gamma \subseteq \partial D, \tag{1.2}$$

where $\overline{\delta S^\mu_{T_D}}$ is the point on ∂D with the smallest distance to $\delta S^\mu_{T_D}$. Note that the choice for $\overline{\delta S^\mu_{T_D}}$ is unique when δ is sufficiently small. Indeed, the following map

$$\varphi_a : z \in \partial D \to z - a\mathbf{n}_z \in \partial D(a) := \{y - a\mathbf{n}_y : y \in \partial D\}$$

is smooth for any $a \in [0,1]$, and $\varphi_a(z)$ is smooth in $(z,a) \in \partial D \times [0,1]$; and the Jacobian determinant $J_a(z)$ of φ_a is smooth in $z \in \partial D$ for any $a \in [0,1]$ and $J_a(z)$ is smooth in $(z,a) \in \partial D \times [0,1]$. Thus from $J_0(z) \equiv 1$, for small enough $a > 0$, $J_a(z) > 0$, $z \in \partial D$, namely $\varphi_a(\cdot)$is injective; which implies that for any $x \notin \partial D$ with $\mathrm{dist}(x, \partial D) \le a$, there is a unique $y \in \partial D$ satisfying $\mathrm{dist}(x, \partial D) = |x - y|$. Here $\mathrm{dist}(x, \partial D) := \inf_{z \in \partial D} |x - z|$ is the distance between x and ∂D.

Note that $\hat{\omega}_\delta(\mathbf{0}, \cdot; D)$ converges weakly to $\omega(\mathbf{0}, \cdot; D)$ as $\delta \downarrow 0$. Our primary motivation is to study the following conjecture of universality problem which was pioneered by Kennedy [12].

$$\lim_{\delta \to 0} \frac{1}{\delta}\big[\hat{\omega}_\delta(\mathbf{0}, \mathrm{d}z; D) - \omega(\mathbf{0}, \mathrm{d}z; D)\big] = c_\mu \rho_D(z)\,|\mathrm{d}z| \qquad \text{(in the weak convergence topology)},$$

where $c_\mu > 0$ is a constant depending on μ but not on D, and $\rho_D(z)$ is a measurable function on ∂D independent of μ.

The above universality may not hold for the random walk $\{\delta S_n^\mu\}_{n \ge 0}$ whose μ is not rotationally invariant. The reason is that the standard d-dimensional Brownian motion starting at $\mathbf{0}$ is rotationally invariant. Thus, it is natural to use the following $\omega_\delta(\mathbf{0}, \cdot; D)$ to replace $\hat{\omega}_\delta(\mathbf{0}, \cdot; D)$:

$$\omega_\delta(\mathbf{0}, \mathrm{d}z; D) = \int_{\mathrm{O}(d)} \hat{\omega}_{\delta,\alpha}(\mathbf{0}, \mathrm{d}z; D)\,\mathrm{d}\widetilde{m}(\alpha);$$

where $\widetilde{m}$ is the normalized Haar measure on $\mathrm{O}(d)$, the group of $d \times d$ orthogonal matrices, and $\hat{\omega}_{\delta,\alpha}(\mathbf{0}, \mathrm{d}z; D)$ is the image of $\hat{\omega}_\delta(\mathbf{0}, \mathrm{d}z; D)$ under rotation $\alpha \in \mathrm{O}(d)$. For a detailed discussion on the above replacement in $\mathbb{R}^2$, refer to Kennedy [12].

Now we restate the universality problem as follows.

Conjecture 1.1. Assume that $D \subset \mathbb{R}^d$ is an open simply-connected bounded domain with $\mathbf{0} \in D$ and ∂D is smooth. Let g be any smooth function on ∂D. Then

$$\lim_{\delta \to 0} \frac{1}{\delta}\left(\int_{\partial D} g(z)\omega_\delta(\mathbf{0}, \mathrm{d}z; D) - \int_{\partial D} g(z)\omega(\mathbf{0}, \mathrm{d}z; D)\right) = c_\mu \int_{\partial D} g(z)\rho_D(z)\,|\mathrm{d}z|, \tag{1.3}$$

where $c_\mu > 0$ is a constant depending only on μ, and $\rho_D(z)$ is a measurable function on ∂D independent of μ.

Moreover, $\rho_D(z)$ is determined by the following equation:

$$\int_{\partial D} g(z)\rho_D(z)\,|\mathrm{d}z| = \int_{\partial D} \frac{\partial f}{\partial \mathbf{n}_z}(z) H_D(\mathbf{0}, z)\,|\mathrm{d}z|, \tag{1.4}$$

where f is the harmonic function in D with boundary values given by g, and derivative $\frac{\partial f}{\partial \mathbf{n}_z}$ is with respect to the inward unit normal $\mathbf{n}_z$ at $z \in \partial D$.

For any $r > 0$ and $z_0 \in \mathbb{R}^d$, write

$$\mathbb{B}^d(z_0, r) = \big\{|z - z_0| < r : z \in \mathbb{R}^d\big\}, \qquad \overline{\mathbb{B}^d(z_0, r)} = \big\{|z - z_0| \le r : z \in \mathbb{R}^d\big\}.$$

Let $\partial\mathbb{B}^d(z_0, r)$ be the boundary of $\mathbb{B}^d(z_0, r)$. For convenience, put

$$\mathbb{B}_r^d := \mathbb{B}^d(\mathbf{0}, r), \qquad \mathbb{B}^d := \mathbb{B}^d(\mathbf{0}, 1).$$

When μ is the uniform probability measure on $\mathbb{B}^2$, Jiang and Kennedy [10] proved Conjecture 1.1 in $\mathbb{R}^2$. We attribute the above conjecture to Kennedy [12] and Jiang and Kennedy [10], though in Kennedy [12] this conjecture was stated for three kinds of random walk models (simple random walk, the nearest neighbor random walk not allowed to backtrack, the smart kinetic walk) on square, triangular and hexagonal planar lattices; and we are responsible for any possible mistake.

In the rest of the paper, we assume that μ is a rotationally invariant probability measure on $\overline{\mathbb{B}^d}$ such that $\mu(\{\mathbf{0}\}) < 1$. Under this setting, there is a probability measure ν on $[0, 1]$ with $\nu(\{\mathbf{0}\}) < 1$ such that

$$\int f \,\mathrm{d}\mu = \int_0^1 \int_{\mathbb{S}^{d-1}} f(r\alpha)\,\mathrm{d}\nu(r)\,\mathrm{d}m(\alpha) \tag{1.5}$$

for all positive measurable function f, where m is the uniform distribution on the unit sphere $\mathbb{S}^{d-1}$. Hence, the rotational invariant property of μ implies that

$$\omega_\delta(\mathbf{0}, \cdot; D) = \hat{\omega}_\delta(\mathbf{0}, \cdot; D) = \hat{\omega}_{\delta,\alpha}(\mathbf{0}, \cdot; D).$$

Let

$$\mathbb{H}^d = \{(x_1, x_2, \dots, x_d) \in \mathbb{R}^d : x_d > 0\}, \qquad \partial\mathbb{H}^d = \{(x_1, x_2, \dots, x_d) \in \mathbb{R}^d : x_d = 0\},$$

and $T_{\mathbb{H}^d} := \inf\{n \geq 0, S_n^\mu \notin \mathbb{H}^d\}$. Define $h^\mu(\ell)$ on $[0, 1]$ by

$$\begin{aligned} h^\mu(\ell) = \int_{[\ell,1]} \frac{\Gamma(d/2)}{\sqrt{\pi}\,\Gamma(\frac{d-1}{2})} \bigg[& \frac{(r^2-\ell^2)^{(d-1)/2}}{(d-1)r^{d-2}} \\ & + {}_2F_1\left(\frac{1}{2}, \frac{3-d}{2}; \frac{3}{2}; \frac{\ell^2}{r^2}\right)\frac{\ell^2}{r}\bigg]\,\mathrm{d}\nu(r) - \frac{\ell}{2}\nu([\ell, 1]), \end{aligned} \tag{1.6}$$

where ${}_2F_1(a, b; c; z)$ is the hypergeometric function given by

$${}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!},$$

and $(x)_n = x(x+1)\cdots(x+n-1)$ is the Pochhammer symbol. For hypergeometric functions, see Gasper and Rahman [6]. Let

$$c_\mu = \frac{2}{\kappa} \int_0^1 \left(\ell + \mathbb{E}^\ell\left[V\left(S_{T_{\mathbb{H}^d}}^\mu\right)\right]\right) h^\mu(\ell)\,\mathrm{d}\ell, \tag{1.7}$$

where $\mathbb{E}^\ell$ is the expectation with respect to the distribution of $\{S_n^\mu\}_{n\geq 0}$ starting at $s_0 = (0, \dots, 0, \ell) \in \mathbb{H}^d$, and

$$V\left(S_{T_{\mathbb{H}^d}}^\mu\right) = \left|S_{T_{\mathbb{H}^d}}^\mu - \overline{S_{T_{\mathbb{H}^d}}^\mu}\right|.$$

We state our main result as follows.

Theorem 1.2. *Assume that μ is a rotationally invariant probability on $\overline{\mathbb{B}^d}$ and $\mu(\{\mathbf{0}\}) < 1$. Then Conjecture 1.1 holds for constant c_μ specified in (1.7) and smooth function $\rho_D(z)$ specified by*

$$\rho_D(z) = \lim_{\delta \to 0} \frac{1}{\delta} \left\{ \int_{\partial D} H_D(\mathbf{0}, \zeta) H_D(z + \delta \mathbf{n}_z, \zeta) \, |\mathrm{d}\zeta| - H_D(\mathbf{0}, z) \right\}, \qquad z \in \partial D. \tag{1.8}$$

The limit of (1.8) does exist, see the proof of Lemma 2.12.

Remark 1.3. (i) The most difficult situation for Conjecture 1.1 is the case that μ is not a rotationally invariant probability measure. Refer to Kennedy [12] and Dai [4] for several very interesting discrete random walks.

(ii) As pointed out by Kennedy [12], there is another very natural way to define the "exit" point in ∂D when the random walk exits D: By linearly interpolating between the steps of the random walk so that it becomes a piece-wise linear curve in $\mathbb{R}^d$, we can consider the first point where this curve intersects ∂D as the exit point. In this setting, Theorem 1.2 also holds.

The paper is organized as follows. In Section 2, we recall firstly some preliminary facts on harmonic measures and random walks; then after giving a series of lemmas on discrete and continuous harmonic measures, we prove Theorem 1.2. In Section 3, we take several examples for Theorem 1.2.

2. Proof of Theorem 1.2

In Section 2.1, we recall some preliminary knowledge. Then in Section 2.2, we prove a series of lemmas which will be used to prove Theorem 1.2. Finally in Section 2.3, we finish the proof of Theorem 1.2.

Recall in $\mathbb{R}^2$, by using the conformal mapping from D to $\mathbb{B}^2$, Kennedy computed $\rho_D(z)$ in Kennedy [12] Section 3 and Jiang and Kennedy deduced a formula for $\rho_D(z)$ in Jiang and Kennedy [10] Proposition 3. In this paper, we have found a more general approach without resorting the conformal mapping to compute explicitly the function $\rho_D(z)$ for d-dimensional ($d \geq 2$) domain D. This together with Jiang and Kennedy [10] lead that we can extend main result of Jiang and Kennedy [10] to higher dimensional $\mathbb{R}^d$ and any non-degenerate rotationally invariant step-distribution on $\overline{\mathbb{B}^d}$. Our proof is similar to that of Jiang and Kennedy [10], but we need some new insights in higher dimensional $\mathbb{R}^d$ which scatter in proofs of lemmas in Section 2.2.

2.1. Preliminaries

For small $\delta > 0$, let

$$\begin{aligned} D_2 &= \left\{ z \in D : \operatorname{dist}(z, \partial D) < \delta \right\}, \\ D_3 &= \left\{ z \in \mathbb{R}^d \setminus D : \operatorname{dist}(z, \partial D) < \delta \right\}, \end{aligned} \tag{2.1}$$

$$D_1 = D \setminus D_2, \qquad D_+ = D \cup D_3. \tag{2.2}$$

For any continuous function g on ∂D, consider the following Dirichlet problem:

$$\begin{cases} \triangle f(z) = 0, & z \in D, \\ f(z) = g(z), & z \in \partial D. \end{cases} \tag{2.3}$$

Recall that D is regular. The unique solution to (2.3) can be written as

$$f(z) = \int_{\partial D} g(\xi)\omega(z, \mathrm{d}\xi; D). \tag{2.4}$$

Furthermore, if g is smooth on ∂D, then $f \in 3C^\infty(\overline{D})$ (Gilbarg and Trudinger [7], p. 111 Theorem 6.19) and therefore f has a C^∞-extension to a larger domain $\overline{D_+}$, which is still denoted by f by abusing notations.

Clearly the generator $\triangle_\delta$ for the random walk $\{\delta S_n^\mu\}_{n\geq 0}$ is given by

$$\triangle_\delta f(z) = \int_{\overline{\mathbb{B}^d}} \big[f(z+\delta\xi) - f(z)\big]\,\mathrm{d}\mu(\xi), \tag{2.5}$$

for any bounded measurable function f on $\mathbb{R}^d$. Consider the following discrete Dirichlet problem:

$$\begin{cases} \triangle_\delta f_\delta(z) = 0, & z \in D, \\ f_\delta(z) = g(\widetilde{z}), & z \in D_3; \end{cases} \tag{2.6}$$

where D_3 is given by (2.1), and $\widetilde{z} \in \partial D$ satisfies $|\widetilde{z} - z| = \min\{|\zeta - z| : \zeta \in \partial D\}$. It is easy to see that the function f_δ defined by

$$f_\delta(z) = \int_{\partial D} g(\xi)\omega_\delta(z, \mathrm{d}\xi; D) \tag{2.7}$$

is the unique solution to (2.6) (the uniqueness follows from the maximum principle).

Let $\mathbb{P}^{x,\delta}$ be the law of $\{\delta S_n^\mu\}_{n\geq 0}$ with $\delta S_0^\mu = x$, and $\mathbb{E}^{x,\delta}$ the corresponding expectation. Define the following probability on $\overline{\mathbb{B}^d}$: For any measurable subset A of $\overline{\mathbb{B}^d}$,

$$\widehat{\mu}(A) = \frac{1}{1 - \mu(\{\mathbf{0}\})}\mu\big(A \setminus \{\mathbf{0}\}\big).$$

Note that the random walks $\{\delta S_n^\mu\}_{n\geq 0}$ and $\{\delta S_n^{\widehat{\mu}}\}_{n\geq 0}$ have the same discrete harmonic measure. Thus in the rest of this paper, we assume that

$$\mu\big(\{\mathbf{0}\}\big) = 0.$$

Notice that μ is rotationally invariant on $\overline{\mathbb{B}^d}$ $(d \geq 2)$ with $\mu(\{\mathbf{0}\}) = 0$. By Zabczyk [17], the k-fold convolution

$$\underbrace{\mu * \mu * \cdots * \mu}_{k}$$

with $k \geq 2$ is absolutely continuous with respect to the d-dimensional Lebesgue measure. Define the transition probability density for the random walk $\{\delta S_n^\mu\}_{n\geq 0}$:

$$\begin{aligned} p_\delta(0,x,y) &= \delta(x,y), \\ p_\delta(n,x,y) &= \lim_{\varepsilon\to 0}\frac{\mathbb{P}^{x,\delta}(|\delta S_n^\mu - y| \leq \varepsilon)}{2\pi^{d/2}\varepsilon^d/(d\Gamma(d/2))}, \qquad n\geq 2, \end{aligned} \tag{2.8}$$

where $\delta(x,y)$ is the Dirac delta function giving unit mass to x. In general, for $n=1$, the limit above does not exist and so $p_\delta(1,x,y)$ may be a distribution rather than a function.

Likewise, define the transition probability density for $\{\delta S_n^\mu\}_{n\geq 0}$ killed on exiting D as follows: For any $x \in D$ and $y \in \mathbb{R}^d$,

$$\begin{aligned} p_{D,\delta}(0,x,y) &= \delta(x,y), \\ p_{D,\delta}(n,x,y) &= \lim_{\varepsilon\to 0}\frac{\mathbb{P}^{x,\delta}(|\delta S_n^\mu - y| \leq \varepsilon, n < T_D)}{2\pi^{d/2}\varepsilon^d/(d\Gamma(d/2))}, \qquad n\geq 2. \end{aligned}$$

Here $p_{D,\delta}(n,x,y)$ does exist by (2.8) for $n \geq 2$.

2.2. Some lemmas

By the Markov property for $\{\delta S_n^\mu\}_{n\geq 0}$, we may rewrite $p_{D,\delta}(n,x,y)$ as

$$p_{D,\delta}(n,x,y) = \int_{\overline{\mathbb{B}^d}} p_{D,\delta}(n-1,x,y+\delta\xi)\,\mathrm{d}\mu(\xi), \qquad x,y\in D, n\geq 3. \tag{2.9}$$

The killed discrete Green function is defined by

$$G_\delta(x,y) = \sum_{n=0}^{\infty} p_{D,\delta}(n,x,y), \qquad x\in D, y\in D\setminus\overline{\mathbb{B}^d(x,2\delta)},$$

where $p_{D,\delta}(1,x,y)$ is viewed as 0 due to $|y-x| > 2\delta$. In fact in the above definition, $p_{D,\delta}(n,x,y)$ with $n \leq 2$ make no contribution to $G_\delta(x,y)$. Recall D_3 from (2.1).

Lemma 2.1. *For any fixed $x \in D$,*

$$\begin{cases} \triangle_\delta G_\delta(x,y) = 0, & y\in D\setminus\overline{\mathbb{B}^d(x,2\delta)}, \\ G_\delta(x,y) = 0, & y \in D_3. \end{cases} \tag{2.10}$$

Proof. The boundary condition is obvious. By (2.9) and the Fubini theorem, for any $y \in D \setminus \overline{\mathbb{B}^d(x,2\delta)}$,

$$G_\delta(x,y) = \sum_{n=0}^{2} p_{D,\delta}(n,x,y) + \sum_{n=3}^{\infty}\int_{\overline{\mathbb{B}^d}} p_{D,\delta}(n-1,x,y+\delta\xi)\,\mathrm{d}\mu(\xi)$$

$$= \int_{\overline{\mathbb{B}^d}} \sum_{n=3}^{\infty} p_{D,\delta}(n-1, x, y+\delta\xi)\,\mathrm{d}\mu(\xi) = \int_{\overline{\mathbb{B}^d}} G_\delta(x, y+\delta\xi)\,\mathrm{d}\mu(\xi),$$

where we used the facts that $p_{D,\delta}(0, x, y+\delta\xi) = p_{D,\delta}(1, x, y+\delta\xi) = 0$ for $y \in D \setminus \overline{\mathbb{B}^d(x, 2\delta)}$ and $\xi \in \overline{\mathbb{B}^d}$. Hence,

$$\triangle_\delta G_\delta(x, y) = \int_{\overline{\mathbb{B}^d}} G_\delta(x, y+\delta\xi)\,\mathrm{d}\mu(\xi) - G_\delta(x, y) = 0, \qquad y \in D \setminus \overline{\mathbb{B}^d(x, 2\delta)}. \qquad \square$$

Similarly to Jiang and Kennedy [10], Lemma 3, we can prove the following lemma.

Lemma 2.2. *In the settings of Theorem 1.2, for smooth function g on ∂D,*

$$f_\delta(\mathbf{0}) - f(\mathbf{0}) = \int_{D_2} G_\delta(\mathbf{0}, z)\triangle_\delta f(z)\,\mathrm{d}z. \tag{2.11}$$

Define potential kernel $a_\delta(x)$ for the random walk $\{\delta S_n^\mu\}_{n\geq 0}$ by

$$a_\delta(x) = \sum_{n=3}^{\infty} p_\delta(n, \mathbf{0}, x), \qquad x \in \mathbb{R}^d \tag{2.12}$$

for $d \geq 3$ and by

$$a_\delta(x) = \sum_{n=3}^{\infty} \big[p_\delta(n, \mathbf{0}, \mathbf{0}) - p_\delta(n, \mathbf{0}, x)\big], \qquad x \in \mathbb{R}^2 \tag{2.13}$$

for $d = 2$.

For convenience, when $\delta = 1$, we define

$$a(x) := a_1(x), \qquad p(n, x, y) := p_1(n, x, y).$$

Lemma 2.3. *Assume $d \geq 3$. Then $a(x)$ is well defined, and*

$$a(x) = \frac{\Gamma(\frac{d-2}{2})}{2\kappa\pi^{d/2}} \frac{1}{|x|^{d-2}} + O\big(|x|^{-d}\big), \qquad |x| \to \infty, \tag{2.14}$$

where the constant in big O term only depends on μ.

Proof. Let $X = (X^{(1)}, X^{(2)}, \ldots, X^{(d)})$ be a random variable with distribution μ, and $i = \sqrt{-1}$. Let

$$\phi(\theta) = \mathbb{E}\big(e^{i\theta\cdot X}\big) = \mathbb{E}\left(\exp\left\{i \sum_{j=1}^{d} \theta_j X^{(j)}\right\}\right)$$

be the characteristic function of X, where $\theta = (\theta_1, \dots, \theta_d) \in \mathbb{R}^d$. By the rotational invariance of μ, for any $n \geq 1$,

$$\mathbb{E}\big[(\theta \cdot X)^n\big] = \mathbb{E}\big[\big\{\big(0, \dots, 0, |\theta|\big) \cdot \big(X^{(1)}, \dots, X^{(d)}\big)\big\}^n\big] = |\theta|^n \mathbb{E}\big[\big(X^{(d)}\big)^n\big];$$

and $\phi(\theta)$ is rotationally invariant in θ, namely a function in $|\theta|$:

$$\phi(\theta) = \chi\big(|\theta|\big). \tag{2.15}$$

Note $|\theta \cdot X| \leq |\theta|$. Then as $\theta \to \mathbf{0}$,

$$\mathbb{E}\left[\sum_{n=3}^{\infty} \frac{(-1)^n}{(2n)!} (\theta \cdot X)^{2n}\right] = \sum_{n=3}^{\infty} \frac{(-1)^n}{(2n)!} \mathbb{E}\big[(\theta \cdot X)^{2n}\big] = \sum_{n=3}^{\infty} \frac{(-1)^n}{(2n)!} |\theta|^{2n} \mathbb{E}\big[\big(X^{(d)}\big)^{2n}\big] = O\big(|\theta|^6\big).$$

Since the Taylor series at 0 of $\cos(x)$ is $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$, we have that

$$\begin{aligned} \phi(\theta) &= \mathbb{E}\big(\cos(\theta \cdot X) + i \sin(\theta \cdot X)\big) = \mathbb{E}\big(\cos(\theta \cdot X)\big) \\ &= \mathbb{E}\left(1 - \frac{(\theta \cdot X)^2}{2} + \frac{(\theta \cdot X)^4}{24} + \sum_{n=3}^{\infty} \frac{(-1)^n}{(2n)!} (\theta \cdot X)^{2n}\right) \\ &= 1 - \frac{\kappa |\theta|^2}{2} + \frac{1}{24} \mathbb{E}\big[\big(X^{(d)}\big)^4\big] |\theta|^4 + O\big(|\theta|^6\big), \qquad \theta \to \mathbf{0}. \end{aligned} \tag{2.16}$$

From (2.27) in Hughes [8], the function $\chi(\cdot)$ has the following form:

$$\chi(q) = \Gamma(d/2) \int_0^1 \left(\frac{2}{rq}\right)^{d/2-1} J_{d/2-1}(rq)\, \mathrm{d}\nu(r), \qquad q \geq 0. \tag{2.17}$$

Here $J_v(z)$ is the Bessel function of the first kind of order v (see Watson [16]), which is defined by the series

$$J_v(z) = \left(\frac{1}{2} z\right)^v \sum_{k=0}^{\infty} \frac{(-\frac{1}{4} z^2)^k}{k! \Gamma(v + k + 1)}.$$

Clearly, for any $n \geq 3$, $p(n, \mathbf{0}, x)$ is rotationally invariant in x. Let

$$p(n, \mathbf{0}, x) = \psi_n\big(|x|\big).$$

Then by (2.34) in Hughes [8],

$$\psi_n(\rho) = \frac{2}{(4\pi)^{d/2}} \int_0^{\infty} \left(\frac{2}{q\rho}\right)^{d/2-1} J_{d/2-1}(q\rho) q^{d-1} \chi^n(q)\, \mathrm{d}q, \tag{2.18}$$

where $\chi(q)$ is given by (2.17). By (2.15) and (2.16), and $\ln(1 + \varepsilon) = \varepsilon + O(\varepsilon^2)$ $(\varepsilon \to 0)$, we get that

$$\chi^n(q) = e^{n \ln(\chi(q))} \sim e^{-\frac{\kappa}{2} n q^2} \qquad \text{as } q \to 0. \tag{2.19}$$

Recall the following definite integral in Watson [16] page 394:

$$\int_0^\infty q^{v+1} J_v(\beta q) e^{-\alpha^2 q^2}\,\mathrm{d}q = \frac{\beta^v}{(2\alpha^2)^{v+1}} e^{-\beta^2/(4\alpha^2)}. \tag{2.20}$$

Thus the large n behaviour of the integral in (2.18) is governed by the small q behaviour of $\chi(q)$, and by (2.19)–(2.20),

$$p(n,\mathbf{0},x) = \psi_n\big(|x|\big) \sim \frac{1}{(2\pi\kappa n)^{d/2}} e^{-\frac{|x|^2}{2n\kappa}}. \tag{2.21}$$

For any $n \geq 1$ and $x \in \mathbb{R}^d$, let

$$\overline{p}(n,\mathbf{0},x) = \frac{1}{(2\pi\kappa n)^{d/2}} e^{-\frac{|x|^2}{2n\kappa}}. \tag{2.22}$$

By Lawler and Limic [14], Lemma 4.3.2, we have that for any $b > 1$, as $r \to \infty$,

$$\sum_{n=3}^{\infty} n^{-b} e^{-r/n} = \frac{\Gamma(b-1)}{r^{b-1}} + O\left(\frac{1}{r^{b+1}}\right). \tag{2.23}$$

Plugging $b = d/2$, $r = \frac{|x|^2}{2\kappa}$ into (2.23), we obtain that as $|x| \to \infty$,

$$\sum_{n=3}^{\infty} \overline{p}(n,\mathbf{0},x) = \frac{\Gamma(\frac{d-2}{2})}{2\kappa\pi^{d/2}} \frac{1}{|x|^{d-2}} + O\left(\frac{1}{|x|^{d+2}}\right). \tag{2.24}$$

Let $E(n,x) = |p(n,\mathbf{0},x) - \overline{p}(n,\mathbf{0},x)|$. Similarly to the proof of Lawler [13], Lemma 1.5.2, we can verify that

$$\sum_{n=3}^{\infty} E(n,x) = O\left(\frac{1}{|x|^d}\right), \tag{2.25}$$

where the constant in $O(\cdot)$ does not depend on x but on μ.

Now from (2.24)–(2.25), we see the lemma holds immediately. □

Lemma 2.4. *When $d = 2$, $a(x)$ is well defined, and there exists a constant C_0 depending on μ such that as $|x| \to \infty$,*

$$a(x) = \frac{1}{\kappa\pi} \ln|x| + C_0 + O\big(|x|^{-2}\big), \tag{2.26}$$

where the constant in big O term only depends on μ.

Proof. We rewrite

$$a(x) = \sum_{3 \leq n \leq |x|^2/\kappa} \big[p(n,\mathbf{0},\mathbf{0}) - p(n,\mathbf{0},x)\big] + \sum_{n > |x|^2/\kappa} \big[p(n,\mathbf{0},\mathbf{0}) - p(n,\mathbf{0},x)\big]. \tag{2.27}$$

For any $n \ge 1$ and $x \in \mathbb{R}^2$, let

$$\overline{p}(n, \mathbf{0}, x) = \frac{1}{2\pi\kappa n} e^{-\frac{|x|^2}{2n\kappa}}.$$

It is easy to see that (2.16)–(2.21) are also true for $d = 2$.

Let $E(n, x) = |p(n, \mathbf{0}, x) - \overline{p}(n, \mathbf{0}, x)|$. Similarly to the proof of Theorem 1.2.1 in Lawler [13], we can check that

$$E(n, x) = O\left(\frac{1}{n^2}\right), \qquad n \to \infty,$$

where the constant in $O(\cdot)$ does not depend on x but on μ.

Hence

$$p(n, \mathbf{0}, \mathbf{0}) = \frac{1}{2\pi\kappa} \frac{1}{n} + O\left(\frac{1}{n^2}\right), \qquad n \ge 3;$$

$$\sum_{3 \le n \le |x|^2/\kappa} p(n, \mathbf{0}, \mathbf{0}) = \sum_{3 \le n \le |x|^2/\kappa} \frac{1}{2\pi\kappa} \frac{1}{n} + \sum_{3 \le n \le |x|^2/\kappa} O\left(\frac{1}{n^2}\right).$$

Recall that

$$\sum_{3 \le n \le |x|^2/\kappa} \frac{1}{n} = 2 \ln |x| + \gamma - \ln \kappa - \frac{3}{2} + O\left(|x|^{-2}\right),$$

where γ is Euler's constant. So for some constant C_1 depending on μ,

$$\sum_{3 \le n \le |x|^2/\kappa} p(n, \mathbf{0}, \mathbf{0}) = \frac{1}{\kappa\pi} \ln |x| + C_1 + O\left(|x|^{-2}\right).$$

By the martingale maximal inequality, there exist $\beta > 0$, $c > 0$ depending on μ such that for all n and all $s > 0$,

$$\mathbb{P}\left\{\max_{0 \le j \le n} \left|S_j^\mu\right| \ge s\sqrt{n}\right\} \le c e^{-\beta s^2}.$$

This implies that $\sum_{n<|x|} p(n, \mathbf{0}, x)$ decays faster than any power of $|x|$ as $|x| \to \infty$, particularly

$$\sum_{n<|x|} p(n, \mathbf{0}, x) = o\left(|x|^{-2}\right).$$

Similarly to the proof of Theorem 4.4.4 in Lawler and Limic [14], we can show that for some constant c depending on μ,

$$\sum_{|x| \le n \le |x|^2/\kappa} \left|p(n, \mathbf{0}, x) - \overline{p}(n, \mathbf{0}, x)\right| \le c|x|^{-2}.$$

Therefore, as $|x| \to \infty$,

$$\begin{aligned}\sum_{|x|\le n\le |x|^2/\kappa} p(n,\mathbf{0},x) &= \sum_{|x|\le n\le |x|^2/\kappa} \overline{p}(n,\mathbf{0},x) + O\big(|x|^{-2}\big)\\ &= \sum_{|x|\le n\le |x|^2/\kappa} \frac{1}{2\pi n\kappa} e^{-\frac{|x|^2}{2n\kappa}} + O\big(|x|^{-2}\big)\\ &= \frac{1}{2\pi\kappa}\int_{|x|}^{\frac{|x|^2}{\kappa}} \frac{1}{t} e^{-\frac{|x|^2}{2\kappa t}}\,\mathrm{d}t + O\big(|x|^{-2}\big)\\ &= \frac{1}{2\pi\kappa}\int_1^{\infty} \frac{1}{y} e^{-y/2}\,\mathrm{d}y + O\big(|x|^{-2}\big).\end{aligned}$$

So far we have proved that

$$\sum_{n\le |x|^2/\kappa} \big[p(n,\mathbf{0},\mathbf{0}) - p(n,\mathbf{0},x)\big] = \frac{1}{\kappa\pi}\ln|x| + C' + O\big(|x|^{-2}\big), \qquad |x|\to\infty, \tag{2.28}$$

holds for some constant C' depending on μ.

Likewise, consider the case when $n > |x|^2/\kappa$, similarly to the proof of Theorem 4.4.4 in Lawler and Limic [14], we can verify that

$$\begin{aligned}&\sum_{n>|x|^2/\kappa} \big[p(n,\mathbf{0},\mathbf{0}) - p(n,\mathbf{0},x)\big]\\ &\quad= \frac{1}{2\pi\kappa}\int_0^1 \frac{1}{y}\big(1-e^{-y/2}\big)\,\mathrm{d}y + O\big(|x|^{-2}\big), \qquad |x|\to\infty.\end{aligned} \tag{2.29}$$

By (2.28) and (2.29), we finish proving the lemma. □

Let $G_D(\mathbf{0},\cdot)$ be the Green function for the killed Brownian motion in $\overline{D}$ starting at $\mathbf{0}$. Then $G_D(\mathbf{0},z)$ is the unique harmonic function on $D\setminus\{\mathbf{0}\}$ such that $G_D(\mathbf{0},z)\to 0$ as $z\to\partial D$. When $d\ge 3$, let $\varphi(z)$ be the harmonic function in $z\in D$ satisfying that

$$\begin{cases}\triangle\varphi(z) = 0, & z\in D,\\ \varphi(z) = -\dfrac{1}{4}\pi^{-d/2}\Gamma\left(\dfrac{d-2}{2}\right)\dfrac{1}{|z|^{d-2}}, & z\in\partial D.\end{cases} \tag{2.30}$$

It is well known that for $d\ge 3$,

$$G_D(\mathbf{0},z) = \frac{1}{4}\pi^{-d/2}\Gamma\left(\frac{d-2}{2}\right)\frac{1}{|z|^{d-2}} + \varphi(z), \qquad z\in\overline{D}\setminus\{\mathbf{0}\};$$

and $\frac{1}{4}\pi^{-d/2}\Gamma(\frac{d-2}{2})\frac{1}{|z|^{d-2}}$ is harmonic in $z\in\mathbb{R}^d\setminus\{\mathbf{0}\}$, and thus $\varphi(z)$ can be extended to a harmonic function in the domain $\overline{D_+}$ for small enough $\delta>0$.

Lemma 2.5. *There exists a constant $C > 0$ depending on μ but not on δ such that*

$$\left|\delta^2 G_\delta(\mathbf{0}, z) - \frac{2}{\kappa} G_D(\mathbf{0}, z)\right| \le C\delta \tag{2.31}$$

holds uniformly in $z \in D$ with $|z| > \delta^{1/d}$.

Proof. When $d = 2$, proof of the lemma is similar to that of Lemma 5 in Jiang and Kennedy [10].
Assume $d \ge 3$, and δ is small enough such that $\mathbb{B}^d_{\delta^{1/d}} \subseteq D$. Let

$$H_\delta(z) = \delta^2 \sum_{n=0}^{2} p_\delta(n, \mathbf{0}, z) + \delta^{2-d} a(z/\delta), \qquad z \in \mathbb{R}^d \setminus \overline{\mathbb{B}^d_{2\delta}}.$$

Then by Lemma 2.3, we have that

$$H_\delta(z) = \frac{\Gamma(\frac{d-2}{2})}{2\kappa\pi^{d/2}} \frac{1}{|z|^{d-2}} + O\big(\delta^2/|z|^d\big), \qquad z \in D \setminus \mathbb{B}^d_{\delta^{1/d}}.$$

Define the following function on $\mathbb{R}^d$:

$$e_\delta(z) = \begin{cases} H_\delta(z) - \delta^2 G_\delta(\mathbf{0}, z), & z \in \mathbb{R}^d \setminus \overline{\mathbb{B}^d_{2\delta}}, \\ \delta^{2-d} a(z/\delta) - \delta^2 \sum_{n=3}^{\infty} p_{D,\delta}(n, \mathbf{0}, z), & z \in \overline{\mathbb{B}^d_{2\delta}}. \end{cases}$$

Then by (2.10), we obtain that

$$\begin{cases} \triangle_\delta e_\delta(z) = 0, & z \in D, \\ e_\delta(z) = \frac{\Gamma(\frac{d-2}{2})}{2\kappa\pi^{d/2}} \frac{1}{|z|^{d-2}} + O\big(\delta^2/|z|^d\big), & z \in D_3. \end{cases} \tag{2.32}$$

Recall that $\varphi(z)$ can be extended to a harmonic function in the domain $\overline{D_+}$ for small enough $\delta > 0$. Therefore, for sufficiently small $\delta > 0$,

$$\begin{cases} \triangle_\delta \varphi(z) = 0, & z \in D, \\ \varphi(z) = -\frac{1}{4}\pi^{-d/2}\Gamma\left(\frac{d-2}{2}\right)\frac{1}{|z|^{d-2}} + O(\delta), & z \in D_3. \end{cases} \tag{2.33}$$

Combining with (2.32)–(2.33), we get that

$$\begin{cases} \triangle_\delta\left[e_\delta(z) + \frac{2}{\kappa}\varphi(z)\right] = 0, & z \in D, \\ e_\delta(z) + \frac{2}{\kappa}\varphi(z) = O(\delta), & z \in D_3. \end{cases}$$

Then the maximum principle for $\triangle_\delta$ implies that

$$e_\delta(z) + \frac{2}{\kappa}\varphi(z) = O(\delta), \qquad z \in D.$$

Therefore, when δ is small enough, for any $z \in D$ with $|z| > \delta^{1/d}$,

$$\begin{aligned}\delta^2 G_\delta(\mathbf{0}, z) = H_\delta(z) - e_\delta(z) &= \frac{2}{\kappa}\frac{\Gamma(\frac{d-2}{2})}{4\pi^{d/2}}\frac{1}{|z|^{d-2}} + \frac{2}{\kappa}\varphi(z) + O(\delta) \\ &= \frac{2}{\kappa}G_D(\mathbf{0}, z) + O(\delta).\end{aligned}$$

□

Proofs of the following Lemmas 2.6–2.7 are similar to those of Lemmas 6 and 8 in Jiang and Kennedy [10], respectively.

Lemma 2.6. *As $\delta \to 0$, for $\ell \in [0, \delta]$, uniformly in $x \in \partial D$,*

$$G_D(\mathbf{0}, x + \ell\mathbf{n}_x) = \ell H_D(\mathbf{0}, x) + O(\delta^2).$$

Note D_2 is specified in (2.1).

Lemma 2.7. *Let $F(z)$ be a Lebesgue measurable bounded function on D_2. Then the following holds for sufficiently small $\delta > 0$:*

$$\int_{D_2} F(z)\,\mathrm{d}z = \big(1 + O(\delta)\big)\int_{\partial D}\int_0^\delta F(x + \ell\mathbf{n}_x)\,\mathrm{d}\ell\,|\mathrm{d}x|,$$

where $O(\delta)$ only depends on D.

Lemma 2.8. *Define the following function in (ℓ, δ) with $0 \le \ell \le \delta$:*

$$\begin{aligned}h^\mu(\ell, \delta) = \int_{[\ell/\delta, 1]}\Bigg\{\frac{\Gamma(d/2)}{\sqrt{\pi}\Gamma(\frac{d-1}{2})}\Bigg[&\frac{(\delta^2 r^2 - \ell^2)^{(d-1)/2}}{(d-1)(\delta r)^{d-2}} \\ &+ {}_2F_1\bigg(\frac{1}{2}, \frac{3-d}{2}; \frac{3}{2}; \frac{\ell^2}{\delta^2 r^2}\bigg)\frac{\ell^2}{\delta r}\Bigg] - \frac{\ell}{2}\Bigg\}\,\mathrm{d}\nu(r).\end{aligned}$$

Let f be given by (2.3) with smooth function g. Then as $\delta \to 0$, for $\ell \in [0, \delta]$,

$$\triangle_\delta f(x + \ell\mathbf{n}_x) = h^\mu(\ell, \delta)\frac{\partial f(x)}{\partial \mathbf{n}_x} + O(\delta^2),$$

and $O(\delta^2)$ only depends on f, D and μ.

Proof. For $\xi = (x_1, x_2, \dots, x_d) \in \mathbb{R}^d$, we introduce the d-dimensional spherical polar coordinates transform:

$$\begin{cases} x_1 = r\sin(\varphi_1)\cdots\sin(\varphi_{d-2})\sin(\varphi_{d-1}), \\ x_2 = r\sin(\varphi_1)\cdots\sin(\varphi_{d-2})\cos(\varphi_{d-1}), \\ \vdots \\ x_{d-1} = r\sin(\varphi_1)\cos(\varphi_2), \\ x_d = r\cos(\varphi_1), \end{cases} \tag{2.34}$$

where $0 \le r \le 1, 0 \le \varphi_{d-1} \le 2\pi$, $0 \le \varphi_i \le \pi$, $1 \le i \le d-2$. Then the corresponding Jacobian determinant J_d satisfies that

$$J_d = \frac{\partial(x_1, x_2, \dots, x_{d-1}, x_d)}{\partial(\varphi_1, \varphi_2, \dots, \varphi_{d-1}, r)} = r^{d-1}(\sin\varphi_1)^{d-2}(\sin\varphi_2)^{d-3}\cdots\sin\varphi_{d-2}. \tag{2.35}$$

For the convenience of calculation, we rewrite $\mathrm{d}\mu(\xi)$ as the form of spherical coordinates:

$$\begin{aligned} &\mathrm{d}\mu(\xi) = \mathrm{d}\mu(r, \varphi_1, \dots, \varphi_{d-1}) = \frac{\Gamma(d/2)}{2\pi^{d/2}r^{d-1}}\,\mathrm{d}\nu(r)\,\mathrm{d}\varphi_1\cdots\mathrm{d}\varphi_{d-1}, \\ &(r, \varphi_1, \dots, \varphi_{d-1}) \in [0,1]\times[0,\pi)^{d-2}\times[0,2\pi). \end{aligned} \tag{2.36}$$

Case 1. $d = 2$. From (2.36), $\mathrm{d}\mu(r, \varphi_1) = \frac{1}{2\pi r}\,\mathrm{d}\nu(r)\,\mathrm{d}\varphi_1$. Let

$$h^\mu(\ell, \delta) = \int_{[\ell/\delta, 1]} \frac{1}{\pi}\left[\sqrt{\delta^2 r^2 - \ell^2} - \ell\arccos\left(\frac{\ell}{\delta r}\right)\right]\mathrm{d}\nu(r).$$

Similarly to Lemma 9 of Jiang and Kennedy [10], we can prove that as $\delta \to 0$, for $\ell \in [0, \delta]$,

$$\triangle_\delta f(x + \ell\mathbf{n}_x) = h^\mu(\ell, \delta)\frac{\partial f(x)}{\partial \mathbf{n}_x} + O(\delta^2).$$

Case 2. $d \ge 3$. For any $\xi \in \overline{\mathbb{B}^d}$, write $\rho := \rho(r, \varphi_1, \varphi_2, \dots, \varphi_{d-1})$ for ξ in the spherical polar coordinates. Notice (2.36). Then

$$\begin{aligned} &\triangle_\delta f(x + \ell\mathbf{n}_x) \\ &\quad = \int_{\overline{\mathbb{B}^d}} \big[f(x + \ell\mathbf{n}_x + \delta\xi) - f(x + \ell\mathbf{n}_x)\big]\,\mathrm{d}\xi \\ &\quad = \int_0^1 \int_0^{2\pi} \int_0^\pi \cdots \int_0^\pi \big[f(x + \ell\mathbf{n}_x + \delta\rho) - f(x + \ell\mathbf{n}_x)\big] J_d\,\mathrm{d}\mu(r, \varphi_1, \dots, \varphi_{d-1}) \\ &\quad = \int_{[0,\ell/\delta)} \int_0^{2\pi} \int_0^\pi \cdots \int_0^\pi \big[f(x + \ell\mathbf{n}_x + \delta\rho) - f(x + \ell\mathbf{n}_x)\big] J_d\,\mathrm{d}\mu(r, \varphi_1, \dots, \varphi_{d-1}) \\ &\qquad + \int_{[\ell/\delta, 1]} \int_0^{2\pi} \int_0^\pi \cdots \int_0^\pi \big[f(x + \ell\mathbf{n}_x + \delta\rho) - f(x + \ell\mathbf{n}_x)\big] J_d\,\mathrm{d}\mu(r, \varphi_1, \dots, \varphi_{d-1}) \end{aligned}$$

$$
\begin{aligned}
&= \int_{[\ell/\delta,1]} \int_0^{2\pi} \int_0^{\pi} \cdots \int_0^{\pi} \big[f(x + \ell \mathbf{n}_x + \delta\rho) - f(x + \ell\mathbf{n}_x) \big] J_d \, \mathrm{d}\mu(r, \varphi_1, \ldots, \varphi_{d-1}) \quad (2.37) \\
&= \int_{[\ell/\delta,1]} \int_0^{2\pi} \int_0^{\pi} \cdots \int_0^{\pi/2+\arcsin(\frac{\ell}{\delta r})} \big[f(x + \ell \mathbf{n}_x + \delta\rho) - f(x + \ell\mathbf{n}_x) \big] \\
&\quad \times J_d \, \mathrm{d}\mu(r, \varphi_1, \ldots, \varphi_{d-1}) \\
&\quad + \int_{[\ell/\delta,1]} \int_0^{2\pi} \int_0^{\pi} \cdots \int_0^{\arccos(\frac{\ell}{\delta r})} \big[f(x + \ell \mathbf{n}_x - \delta\rho) - f(x + \ell\mathbf{n}_x) \big] \\
&\quad \times J_d \, \mathrm{d}\mu(r, \varphi_1, \ldots, \varphi_{d-1}) \\
&= I_1(x,\ell) + I_2(x,\ell).
\end{aligned}
$$

The smoothness and boundedness of ∂D implies that there are positive constants $C_1, C_2, \ldots, C_d$ such that for any $x \in \partial D$,

$$
\partial D \cap \overline{\mathbb{B}^d(x,\delta)} \subset [-C_1\delta, C_1\delta] \cdot \mathbf{t}_x^1 + [-C_2\delta, C_2\delta] \cdot \mathbf{t}_x^2 + \cdots + \big[-C_d\delta^2, C_d\delta^2\big] \cdot \mathbf{n}_x,
$$

where $\{\mathbf{t}_x^1, \mathbf{t}_x^2, \ldots, \mathbf{t}_x^{d-1}\}_{x\in\partial D}$ is a moving tangent orthonormal frame of ∂D.

Since f has a C^2-extension, which is still denoted by f, to $\overline{D_+}$, by the Taylor expansion of f at $x \in \partial D$ with respect to coordinate directions $\mathbf{n}_x$ and $\mathbf{t}_x^1, \ldots, \mathbf{t}_x^{d-1}$, we get that

$$
\begin{aligned}
I_1(x,\ell) = \frac{\partial f(x)}{\partial \mathbf{n}_x} \int_{[\ell/\delta,1]} \int_0^{2\pi} \int_0^{\pi} \cdots \int_0^{\pi/2+\arcsin(\frac{\ell}{\delta r})} \delta r \cos(\varphi_1) \\
\times J_d \, \mathrm{d}\mu(r, \varphi_1, \ldots, \varphi_{d-1}) + O\big(\delta^2\big).
\end{aligned}
$$

Recall

$$
\int_0^{2\pi} \int_0^{\pi} \cdots \int_0^{\pi} (\sin\varphi_2)^{d-3} \cdots \sin\varphi_{d-2} \, \mathrm{d}\varphi_2 \cdots \mathrm{d}\varphi_{d-2} \, \mathrm{d}\varphi_{d-1} = \frac{2\pi^{(d-1)/2}}{\Gamma(\frac{d-1}{2})}.
$$

Hence by (2.36),

$$
\begin{aligned}
&I_1(x,\ell) \\
&\quad = \frac{\partial f(x)}{\partial \mathbf{n}_x} \int_{[\ell/\delta,1]} \int_0^{\pi/2+\arcsin(\frac{\ell}{\delta r})} \frac{2\pi^{(d-1)/2}}{\Gamma(\frac{d-1}{2})} \frac{\Gamma(d/2)}{2\pi^{d/2} r^{d-1}} \delta r^d \\
&\qquad \times \cos(\varphi_1) \sin^{d-2}(\varphi_1) \, \mathrm{d}\varphi_1 \, \mathrm{d}\nu(r) + O\big(\delta^2\big) \\
&\quad = \frac{\partial f(x)}{\partial \mathbf{n}_x} \int_{[\ell/\delta,1]} \frac{\Gamma(d/2)}{\sqrt{\pi}\,\Gamma(\frac{d-1}{2})} \frac{(\delta^2 r^2 - \ell^2)^{(d-1)/2}}{(d-1)(\delta r)^{d-2}} \, \mathrm{d}\nu(r) + O\big(\delta^2\big).
\end{aligned}
$$

Likewise, we have that

$$I_2(x,\ell) = \frac{\partial f(x)}{\partial \mathbf{n}_x}\int_{[\ell/\delta,1]}\int_0^{\pi}\cdots\int_0^{\arccos(\frac{\ell}{\delta r})} -\ell J_d \frac{\Gamma(d/2)}{2\pi^{d/2}(\delta r)^{d-1}}\,\mathrm{d}\varphi_1\cdots\mathrm{d}\varphi_{d-2}\,\mathrm{d}\nu(r) + O(\delta^2)$$

$$= \frac{\partial f(x)}{\partial \mathbf{n}_x}\int_{[\ell/\delta,1]}\int_0^{\arccos(\frac{\ell}{\delta r})} -\ell \frac{\Gamma(d/2)}{\sqrt{\pi}\,\Gamma(\frac{d-1}{2})}\sin^{d-2}(\varphi_1)\,\mathrm{d}\varphi_1\,\mathrm{d}\nu(r) + O(\delta^2).$$

From Brychkov [2],

$$\int \sin^{d-2}(\varphi_1)\,\mathrm{d}\varphi_1 = -\cos(\varphi_1)\,{}_2F_1\left(\frac{1}{2},\frac{3-d}{2};\frac{3}{2};\cos^2(\varphi_1)\right) + C, \qquad 0\le\varphi_1\le\pi,$$

$${}_2F_1\left(\frac{1}{2},\frac{3-d}{2};\frac{3}{2};1\right) = \frac{\sqrt{\pi}\,\Gamma(\frac{d-1}{2})}{2\Gamma(d/2)}.$$

Therefore, the lemma follows by a simple computation. □

Lemma 2.9. *Fix $a\in(0,\infty)$ and $\theta\in[\frac{1}{8}\pi,\pi]$. Let*

$$\Gamma = \Gamma(a,\theta) = \{x=(x_1,\dots,x_d)\in\partial\mathbb{B}_a^d : x_d \ge a\cos\theta\}.$$

Then there exists a constant C, depending on μ, but being independent of a and θ, such that when δ is small enough,

$$\omega_\delta(u,\Gamma;\mathbb{B}_a^d) \ge C > 0 \qquad \textit{for all } u\in\mathbb{B}_{a/2}^d.$$

Proof. The proof is similar to that of Lemma 2.10 in Chelkak and Smirnov [3]. Write $x=(x_1,\dots,x_d)$ in the d-dimensional spherical polar coordinates transform given by (2.34). For any $\rho\in(0,1/4)$, let

$$\Pi_1(\rho) = \{x\in\mathbb{R}^d : r=|x|\in(1-\rho,1+\rho), \varphi_1\in[0,\pi/8]\},$$
$$\Pi_2(\rho) = \{x\in\mathbb{R}^d : r=|x|\in(1-\rho,1+\rho), \varphi_1\in(\pi/8,\pi]\};$$

and $h_\rho(\cdot)$ be the continuous harmonic measure of $\{x\in\mathbb{R}^d : |x|=1+\rho, \varphi_1\in[0,\frac{1}{9}\pi]\}$, and

$$c_1 := \frac{1}{2}\liminf_{\rho\to 0}\min_{|x|\le 1/2}\{h_\rho(x)\} > 0.$$

Choose ρ small enough such that

$$\min_{|x|\le\frac{1}{2}+\rho}\{h_\rho(x)\}\ge 3c_1/2 \quad\text{and}\quad \max_{x\in\Pi_2(\rho)}\{h_\rho(x)\}\le c_1/2.$$

Then choose constant $c_2>0$ sufficiently small such that $c_2(1+\rho)^2\le\frac{1}{2}c_1$. Let

$$\phi_0(x) := h_\rho(x) - c_1 + c_2|x|^2, \qquad x\in\overline{\mathbb{B}_{1+\rho}^d}.$$

Then smooth function $\phi_0 : \mathbb{B}^d_{1+\rho} \to \mathbb{R}$ satisfies that

(i) $\phi_0(x) \le 1$ for all $x \in \Pi_1(\rho)$;
(ii) $\phi_0(x) \le 0$ for any $x \in \Pi_2(\rho)$;
(iii) ϕ_0 is subharmonic, and $\Delta\phi_0 \ge 2dc_2$;
(iv) $\phi_0(x) \ge c_1/2$ for all $x \in \mathbb{B}^d_{1/2+\rho}$.

Let

$$\phi(x) = \phi_0\left(\frac{x}{a}\right), \qquad x \in \overline{\mathbb{B}^d_a}.$$

Then $\phi \le 1$ on Γ and $\phi \le 0$ on $\partial\mathbb{B}^d_a \setminus \Gamma$.

Notice $\lim_{\delta\to 0} \sup_{x\in\overline{\mathbb{B}^d_a}} |(\frac{\kappa}{2}\Delta\phi - \Delta_\delta\phi)(x)| = 0$. By (iii), for small enough δ, ϕ is Δ_δ-subharmonic in $\mathbb{B}^d_a$. By the maximum principle, for some constant $C > 0$ which depends on μ but is independent of a and θ, when δ is sufficiently small,

$$\omega_\delta\big(x, \Gamma; \mathbb{B}^d_a\big) \ge \phi(x) \ge C > 0, \qquad x \in \mathbb{B}^d_{a/2}. \qquad \square$$

Lemma 2.10. *There exist two constants $\beta > 0$ and $C > 0$ depending on μ such that for any simply-connected bounded regular domain $\mathcal{D}$, $u \in \mathcal{D}$ and $V \subseteq \partial\mathcal{D}$, and sufficiently small $\delta > 0$,*

$$\mathbb{P}^{u,\delta}\big(\overline{\delta S^\mu_{T_\mathcal{D}}} \in V\big) \le C\left[\frac{\operatorname{dist}(u, \partial\mathcal{D})}{\operatorname{dist}_\mathcal{D}(u; V)}\right]^\beta,$$

where $\overline{\delta S^\mu_{T_\mathcal{D}}}$ is the orthogonal projection of $\delta S^\mu_{T_\mathcal{D}}$ onto $\partial\mathcal{D}$, and

$$\operatorname{dist}_\mathcal{D}(u; V) = \inf\big\{R > 0 : u \text{ and } V \text{ are connected in } \mathcal{D} \cap \mathbb{B}^d(u, R)\big\}.$$

Proof. The proof is similar to that of Proposition 2.11 in Chelkak and Smirnov [3]. Let $a = \operatorname{dist}(u, \partial\mathcal{D})$ and $r = \operatorname{dist}_\mathcal{D}(u; V)$. Recall that $\omega_\delta(u, V; \mathcal{D})$ is equal to the probability that the first exit point of $\{\delta S^\mu_n\}_{n\ge 0}$ ($\delta S^\mu_0 = u$) from $\mathcal{D}$ whose orthogonal projection onto $\partial\mathcal{D}$ is in V. By Lemma 2.9, without loss of generality, we assume $a < r/2$. It is easy to show that for each $a \le r' \le \frac{r}{2}$ the probability to cross the annulus $\mathbb{B}^d(u, 2r') \setminus \mathbb{B}^d(u, r')$ inside $\mathcal{D}$ without touching its boundary is bounded above by some constant $p > 0$ depending on μ that does not depend on r' and $\mathcal{D}$. Hence,

$$\omega_\delta(u, V; \mathcal{D}) \le p^{\log_2(r/a)-1} = p^{-1} \cdot (a/r)^{-\log_2 p},$$

so the lemma holds true with the exponent $\beta = -\log_2 p$. $\square$

Note Lemma 2.5. The proof of the following Lemma 2.11 is similar to that of Jiang and Kennedy [10] Proposition 2.

Lemma 2.11. *Given any $\varepsilon \in (0, 1/2)$. Then as $\delta \to 0$, for any $\ell \in [0, \delta]$ and $x \in \partial D$,*

$$\delta^2 G_\delta(\mathbf{0}, z) - \frac{2}{\kappa} G_D(\mathbf{0}, z) - \frac{2}{\kappa} H_D(\mathbf{0}, x)\mathbb{E}^\ell\big[V\big(S^\mu_{T_{\mathbb{H}^d}}\big)\big] = O\big(\delta^{1+\varepsilon\beta}\big) + O\big(\delta^{2-2\varepsilon}\big),$$

where $z = x + \ell\mathbf{n}_x \in D_2$ and the big O terms depend on D and μ.

Lemma 2.12. *Let f be the harmonic function in D with boundary values given by the smooth function g. Then (1.4) holds for C^∞-function $\rho_D(z)$ specified in (1.8).*

Proof. Let $h(\cdot)$ be the harmonic function on D with boundary values given by the smooth function $H_D(\mathbf{0},\cdot)$. Then $h \in C^\infty(\overline{D})$. Since $\triangle f(z) = 0$, $z \in D$, by the divergence theorem,

$$\int_{\partial D} H_D(\mathbf{0}, z) \frac{\partial f}{\partial \mathbf{n}_z}(z)\,|\mathrm{d}z| = -\int_D \nabla h(z) \cdot \nabla f(z) + h(z) \cdot \triangle f(z)\,\mathrm{d}z$$
$$= -\int_D \nabla h(z) \cdot \nabla f(z)\,\mathrm{d}z.$$

Moreover, note $\triangle h(z) = 0$, $z \in D$, by the divergence theorem again,

$$\int_{\partial D} H_D(\mathbf{0}, z) \frac{\partial f}{\partial \mathbf{n}_z}(z)\,|\mathrm{d}z| = -\int_D \triangle h(z) \cdot f(z) + \nabla h(z) \cdot \nabla f(z)\,\mathrm{d}z$$
$$= \int_{\partial D} g(z) \frac{\partial h}{\partial \mathbf{n}_z}(z)\,|\mathrm{d}z|.$$

The mean value and regularity property of harmonic function $h(\cdot)$ implies that for any $z \in \partial D$,

$$\frac{\partial h}{\partial \mathbf{n}_z}(z) = \lim_{\delta \to 0} \frac{1}{\delta}\big[h(z + \delta \mathbf{n}_z) - h(z)\big]$$
$$= \lim_{\delta \to 0} \frac{1}{\delta}\left\{\int_{\partial D} H_D(\mathbf{0}, \zeta) H_D(z + \delta \mathbf{n}_z, \zeta)\,|\mathrm{d}\zeta| - H_D(\mathbf{0}, z)\right\}.$$

For any $z \in \partial D$, let

$$\rho_D(z) = \frac{\partial h}{\partial \mathbf{n}_z}(z).$$

Thus (1.4) holds for C^∞-function $\rho_D(z)$ specified in (1.8). □

2.3. Proof of Theorem 1.2

Recall (2.4) and (2.7). By Lemmas 2.2 and 2.7, we get that

$$f_\delta(\mathbf{0}) - f(\mathbf{0}) = \int_{D_2} G_\delta(\mathbf{0}, z) \triangle_\delta f(z)\,\mathrm{d}z$$
$$= \big(1 + O(\delta)\big) \int_{\partial D} \int_0^\delta G_\delta(\mathbf{0}, x + \ell \mathbf{n}_x) \triangle_\delta f(x + \ell \mathbf{n}_x)\,\mathrm{d}\ell\,|\mathrm{d}x|.$$

By Lemma 2.8,

$$\Phi(\delta) := \int_{\partial D} \int_0^\delta G_\delta(\mathbf{0}, x + \ell \mathbf{n}_x) \triangle_\delta f(x + \ell \mathbf{n}_x)\,\mathrm{d}\ell\,|\mathrm{d}x|$$

$$= \int_{\partial D} \frac{\partial f(x)}{\partial \mathbf{n}_x} \int_0^\delta G_\delta(\mathbf{0}, x + \ell \mathbf{n}_x)\big[h^\mu(\ell, \delta) + O\big(\delta^2\big)\big] \,\mathrm{d}\ell \,|\mathrm{d}x|.$$

Combining with Lemma 2.11, we have that

$$\Phi(\delta) = \int_{\partial D} \frac{\partial f(x)}{\partial \mathbf{n}_x} \int_0^\delta \frac{2}{\kappa \delta^2}\big[G_D(\mathbf{0}, x + \ell \mathbf{n}_x) + H_D(\mathbf{0}, x)\mathbb{E}^\ell\big[V\big(S^\mu_{T_{\mathbb{H}^d}}\big)\big] + o(\delta)\big] \\ \times \big[h^\mu(\ell, \delta) + O\big(\delta^2\big)\big] \,\mathrm{d}\ell \,|\mathrm{d}x|.$$

By Lemma 2.6,

$$\Phi(\delta) = \int_{\partial D} \frac{\partial f(x)}{\partial \mathbf{n}_x} \int_0^\delta \frac{2}{\kappa \delta^2}\big[\ell H_D(\mathbf{0}, x) + H_D(\mathbf{0}, x)\mathbb{E}^\ell\big[V\big(S^\mu_{T_{\mathbb{H}^d}}\big)\big] + o(\delta)\big] \\ \times \big[h^\mu(\ell, \delta) + O\big(\delta^2\big)\big] \,\mathrm{d}\ell \,|\mathrm{d}x|.$$

A straightforward calculation gives that

$$\Phi(\delta) = c_\mu \delta \int_{\partial D} \frac{\partial f(x)}{\partial \mathbf{n}_x} H_D(\mathbf{0}, x)|\,\mathrm{d}x| + o(\delta),$$

where c_μ is given by (1.7). Therefore,

$$f_\delta(\mathbf{0}) - f(\mathbf{0}) = \delta c_\mu \int_{\partial D} \frac{\partial f(x)}{\partial \mathbf{n}_x} H_D(\mathbf{0}, x)\,|\mathrm{d}x| + O\big(\delta^2\big),$$

combining with Lemma 2.12, then (1.3), (1.4) and (1.8) hold. So far we have completed proving Theorem 1.2. □

3. Several examples for Theorem 1.2

In this section, we consider a class of random walks $\{\delta S^\mu_n\}_{n \geq 0}$ for Theorem 1.2 such that μ has the form specified by (1.5) with

$$\nu(r) := \nu\big([0, r]\big) = 1 - \big[1 - r^{1/\alpha}\big]^{1/\beta}, \qquad r \in [0, 1], \tag{3.1}$$

where $\alpha \geq 0$ and $\beta > 0$. A tedious and straightforward calculation shows that κ in (1.1) is given by

$$\kappa = \kappa(d, \alpha, \beta) = \frac{\Gamma(d/2)\Gamma(2 + 2\alpha)\Gamma(1/\beta)}{2\beta(1 + 2\alpha)\Gamma(d/2 + 1)\Gamma(1/\beta + 2\alpha + 1)}. \tag{3.2}$$

Let $d = 2, \beta = 1, \alpha = \frac{1}{2}$, then $\nu(r) = r^2, r \in [0, 1]$, $\kappa = \frac{1}{4}$, and the random walk $\{\delta S^\mu_n\}_{n \geq 0}$ is exactly the case of continuous-state random walk in Jiang and Kennedy [10].

The following corollary is a consequence of some tedious but straightforward calculuses.

Corollary 3.1. *In* (3.1), *let* d, β, α *be as follows.*

(i) *When* $d = 2, \beta = 1$ *and* $\alpha = 0$, *that is*, $\nu(r) = I_{\{r=1\}}$, $\kappa = \frac{1}{2}$, *then step-distribution of random walk* $\{S_n^\mu\}_{n\geq 0}$ *is the uniform distribution on the unit circle*, *and*

$$c_\mu = \frac{4}{9\pi} + \frac{4}{\pi}\int_0^1 \left[\sqrt{1-\ell^2} - \ell \arccos(\ell)\right]\mathbb{E}^\ell\left[V\left(S^\mu_{T_{\mathbb{H}^2}}\right)\right]\mathrm{d}\ell.$$

(ii) *When* $d = 3, \beta = 1$ *and* $\alpha = 0$, *that is*, $\nu(r) = I_{\{r=1\}}$, $\kappa = \frac{1}{3}$, *then step-distribution of random walk* $\{S_n^\mu\}_{n\geq 0}$ *is the uniform distribution on the unit spherical surface in* $\mathbb{R}^3$, *and*

$$c_\mu = \frac{1}{8} + \int_0^1 \left(\frac{3}{2} - 3\ell + \frac{3\ell^2}{2}\right)\mathbb{E}^\ell\left[V\left(S^\mu_{T_{\mathbb{H}^3}}\right)\right]\mathrm{d}\ell.$$

(iii) *When* $d = 3, \beta = 1$ *and* $\alpha = 1/3$, *i.e.* $\nu(r) = r^3, r \in [0, 1]$, $\kappa = \frac{1}{5}$, *then step-distribution of random walk* $\{S_n^\mu\}_{n\geq 0}$ *is the uniform distribution on the closed* 3-*dimensional unit ball*, *and*

$$c_\mu = \frac{5}{48} + \frac{5}{8}\int_0^1 (3+\ell)(1-\ell)^3\mathbb{E}^\ell\left[V\left(S^\mu_{T_{\mathbb{H}^3}}\right)\right]\mathrm{d}\ell.$$

(iv) *When* $d = 4, \beta = 1$ *and* $\alpha = 0$, *that is*, $\nu(r) = I_{\{r=1\}}$, $\kappa = \frac{1}{4}$, *then step-distribution of random walk* $\{S_n^\mu\}_{n\geq 0}$ *is the uniform distribution on the unit spherical surface in* $\mathbb{R}^4$, *and*

$$c_\mu = \frac{16}{45\pi} + \int_0^1 \left(\frac{16}{3\pi}\left(1-\ell^2\right)^{3/2} - \frac{8\ell}{\pi}\arccos(\ell) + \frac{8\ell^2}{\pi}\sqrt{1-\ell^2}\right)\mathbb{E}^\ell\left[V\left(S^\mu_{T_{\mathbb{H}^4}}\right)\right]\mathrm{d}\ell.$$

(v) *When* $d = 4, \beta = 1$ *and* $\alpha = 1/4$, *that is*, $\nu(r) = r^4, r \in [0, 1]$, $\kappa = \frac{1}{6}$, *then step-distribution of random walk* $\{S_n^\mu\}_{n\geq 0}$ *is the uniform distribution on the closed* 4-*dimensional unit ball*, *and*

$$c_\mu = \frac{32}{105\pi} + \int_0^1 \left(\frac{32}{5\pi}\left(1-\ell^2\right)^{5/2} - \frac{4}{\pi}\ell^2\left(2\ell^2-5\right)\sqrt{1-\ell^2} - \frac{12}{\pi}\ell\arccos(\ell)\right)\mathbb{E}^\ell\left[V\left(S^\mu_{T_{\mathbb{H}^4}}\right)\right]\mathrm{d}\ell.$$

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