

Sticky processes, local and true martingales

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We prove that for a so-called sticky process S there exists an equivalent probability Q and a Q -martingale \tilde{S} that is arbitrarily close to S in $L^p(Q)$ norm. For continuous S , \tilde{S} can be chosen arbitrarily close to S in supremum norm. In the case where S is a local martingale we may choose Q arbitrarily close to the original probability in the total variation norm. We provide examples to illustrate the power of our results and present an application in mathematical finance.

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1. Introduction

By their very definition, local martingales are “almost” martingales. Moreover, in discrete time every local martingale is a martingale under an equivalent change of measure and the new measure can be chosen to be arbitrarily close to the original one in the total variation norm, even on an infinite horizon, see, for example, Theorem 2.2.2 in Kabanov and Safarian [17].

In continuous time, such a strong result does not hold. For example, the inverse of the three dimensional Bessel process is a local martingale and it is not a martingale under any equivalent change of probability measure. We may ask, however, whether there is a process “near” the given local martingale which becomes a martingale under an equivalent probability.

It turns out that such a result holds provided that the given local martingale satisfies the natural condition of *stickiness*: for sticky local martingales a martingale (modulo a change of measure to some $Q \sim P$) that stays in any small neighborhood of it under the $L^p(Q)$ norm can be found, and Q can even be chosen to be as close as one wants to P in total variation norm, see Corollary 5.2 below for this result.

A process is sticky if, starting from any stopping time on, it is never certain to exit a small ball in a given time horizon no matter how small the ball is. This condition was first used in the paper Guasoni [10] in the context of finance and according to the Proposition 3.1 of Guasoni [10] all regular strong Markov processes are sticky. This includes, for example, most Lévy processes, see Section 3 for further details. Other than this, stochastic processes with the conditional full support (henceforth, CFS) property are also sticky. The CFS property (see Remark 2.3 below for its definition) was introduced in the paper Guasoni et al. [13] and a large class of stochastic processes, including fractional Brownian motion (fBm), enjoys this property, see [3,9,15,21] for example.

In Guasoni et al. [13], it was shown that processes with CFS can be approximated arbitrarily closely under the supremum norm by semi-martingales that admit equivalent martingale mea-

tures. In the subsequent paper Bender et al. [2], the same result was obtained for continuous path processes that are merely sticky. In these papers, such approximation was possible because the stochastic processes were assumed to be continuous.

For the case of jump processes, approximation under the supremum norm, however, seems difficult if not impossible. In this note we show, along with our result on local martingales, that càdlàg sticky processes can be approximated by martingales (modulo a change of measure to some $Q \sim P$) arbitrarily closely under the $L^p(Q)$ norm.

The paper is organized as follows. In Section 2, we recall the stickiness condition. In Section 3, we provide examples of sticky processes. In Section 4, we prove that sticky processes can be approximated “arbitrarily closely” by martingales in the sense explained above, see Theorem 4.1 and Corollary 4.3. In Section 5, we show that, in the case of local martingales, one can choose the new probability measure arbitrarily close to the original one in the total variation norm, see Theorem 5.1 and Corollary 5.2. In Section 6, we explain the relevance of our results to mathematical finance. Finally, some technical details are relegated to Section 7.

2. Sticky processes

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $S = (S_t)_{t \in [0, T]}$ be a càdlàg \mathbb{R}^d -valued process adapted to a filtration $\mathbb{F} = (\mathcal{F}_t)_{t \in [0, T]}$ satisfying the usual assumptions (i.e., \mathbb{F} is right continuous and \mathcal{F}_0 contains all of the \mathbb{P} null sets of \mathcal{F}). In this paper, for generality’s sake, we do not assume that \mathcal{F}_0 is a trivial σ -algebra. It can contain sets other than just the null and full measure sets.

We say that the process S is *sticky* with respect to the filtration \mathbb{F} if, for any stopping time τ of \mathbb{F} and any \mathcal{F}_τ -measurable strictly positive random variable κ , the following condition is satisfied

$$\mathbb{P}\left(\sup_{u \in [\tau, T]} |S_u - S_\tau| < \kappa \mid \mathcal{F}_\tau\right) > 0 \quad \text{a.s.} \tag{1}$$

Here $|\cdot|$ is the Euclidean norm of \mathbb{R}^d . This definition is clearly equivalent to Definition 2.2 of Guasoni [10] where κ is assumed to be any deterministic number. In Lemma 3.1 of Bender et al. [2] it was shown that, for processes with continuous paths, stickiness is equivalent to

$$\mathbb{P}\left(\sup_{u \in [t, T]} |S_u - S_t| < \kappa \mid \mathcal{F}_t\right) > 0 \quad \text{a.s.,} \tag{2}$$

for any deterministic time point $0 \leq t \leq T$ and any strictly positive and \mathcal{F}_t -measurable random variable κ . Lemma 3.1 of Bender et al. [2] is also true for càdlàg processes, this is the content of Lemma 2.1 below.

Lemma 2.1. *A càdlàg process S is sticky iff it satisfies (2) for any deterministic $t \in [0, T]$.*

Proof. One direction is trivial. To show the other one, let $0 \leq \tau \leq T$ be any stopping time of \mathbb{F} . Let κ be any strictly positive \mathcal{F}_τ -measurable random variable. Take any $A \in \mathcal{F}_\tau$ with $P(A) > 0$. We would like to show that

$$P\left(A \cap \left\{ \sup_{t \in [\tau, T]} |S_t - S_\tau| < \kappa \right\}\right) > 0.$$

Without loss of generality assume that $\tau < T$ on A . There exists a (deterministic) rational number $r > 0$ such that

$$A_r := A \cap \left\{ \sup_{t \in [\tau, r]} |S_t - S_\tau| < \frac{\kappa}{2} \right\} \cap \{\tau \leq r\}$$

has positive probability. This can be seen from the following, obvious relation:

$$A = \bigcup_{r \in [0, T] \cap \mathbb{Q}} \left(\{\tau \leq r\} \cap \left\{ \sup_{t \in [\tau, r]} |S_t - S_\tau| < \frac{\kappa}{2} \right\} \cap A \right),$$

which holds since S is right-continuous. Observe that $A_r \in \mathcal{F}_r$ so (2) implies that

$$P \left(A_r \cap \left\{ \sup_{t \in [r, T]} |S_t - S_r| < \frac{\kappa}{2} \right\} \right) > 0.$$

Now the claim follows from

$$A_r \cap \left\{ \sup_{t \in [r, T]} |S_t - S_r| < \frac{\kappa}{2} \right\} \subset A \cap \left\{ \sup_{t \in [\tau, T]} |S_t - S_\tau| < \kappa \right\}. \quad \square$$

Remark 2.2. For Markov processes S stickiness reduces to checking

$$P \left(\sup_{u \in [t, T]} |S_u - S_t| < \kappa \mid S_t \right) > 0$$

for almost all ω and all $\kappa > 0, 0 \leq t < T$. For processes with independent increments, it boils down to $P(\sup_{u \in [t, T]} |S_u - S_t| < \kappa)$ being positive for all $\kappa > 0, 0 \leq t < T$. It follows thus from Simon [25] that most Lévy processes have the stickiness property, see Example 3.3 below. See also Aurzada and Dereich [1] for more results on the related theory of “small deviations”.

Remark 2.3. Processes with the CFS property in any open domain are sticky. We recall the CFS property here. Let O be a non-empty open subset of \mathbb{R}^d and let $C[a, b](O)$ denote the metric space of O -valued continuous functions on the interval $[a, b]$ equipped with the metric coming from the supremum norm. For $x \in O$, set $C_x[a, b](O) := \{f \in C[a, b](O) : f(a) = x\}$. We say that S has *conditional full support* in O (CFS- O) if S has continuous trajectories in O and for all $0 \leq t < T$,

$$\text{supp } P(S|_{[t, T]} \in \cdot \mid \mathcal{F}_t) = C_{S_t}[t, T](O).$$

Here $P(S|_{[t, T]} \in \cdot \mid \mathcal{F}_t)$ denotes the \mathcal{F}_t -conditional distribution of the $C[t, T](O)$ -valued random variable $S|_{[t, T]}$. When $O = \mathbb{R}^d$ we simply write CFS instead of CFS- O .

Remark 2.4. Stickiness is invariant under composition with continuous functions and, in the case of S with continuous trajectories, under bounded time changes, as shown in Sayit and Viens [24]. This helps to generate a large class of sticky processes. For example, the process $|B_t|^{\frac{1}{3}}$, where B_t is a one dimensional Brownian motion, is not a semimartingale according to Theorem 72 on page 221 of Protter [23] though it is sticky. See Section 3 for further examples.

In the recent paper Bender et al. [2], it was shown that if a continuous path process is sticky then for any $\varepsilon > 0$ there exists a semi-martingale \tilde{S} that admits an equivalent martingale measure such that

$$\sup_{t \in [0, T]} |S_t - \tilde{S}_t| < \varepsilon \tag{3}$$

holds almost surely. To prove their main result, they constructed a discrete time stochastic sequence that is sufficiently close to the stochastic sequence obtained by stopping the process at each ε -increments and that, in the meantime, satisfies the conditions of Theorem 2.1 in Kabanov and Stricker [19]. They were able to show that the sets $C_n^i, n \in N, i = 1, 2, \dots, 2d + 1$ defined in their paper have positive conditional probabilities, see Lemma 3.3 of that paper. A closer look reveals that the continuous path property of the stochastic processes plays a key role in the proof of this Lemma 3.3. In the presence of jumps, we can not obtain the same property for the sets C_n^i as in their Lemma 3.3. However, we are able to prove a similar result for sticky jump processes under an additional assumption which will be stated below. Also, in the presence of jumps, we can only control the *moments* of the supremum in (3). The following is the assumption that we will need in the proof of our main results.

Assumption 2.5. The probability space supports a d -dimensional Brownian motion $B_t, t \in [0, T]$ with its augmented natural filtration $\mathbb{G} = (\mathcal{G}_t)_{t \in [0, T]}$ such that \mathcal{G}_T is independent of \mathcal{F}_T .

Remark 2.6. Such an assumption often appears in stochastic analysis, e.g. recall the theorem asserting that a continuous martingale is a time-changed Brownian motion. In the present setting, we use this extra Brownian motion to construct a new sticky process which is as close as we want to the original process and has a sufficiently rich collection of paths. We then use this new sticky process to construct the $Q \sim P$ and \tilde{S} we want, see Theorem 4.1 below.

3. Examples

In this section, we give some examples of sticky processes. As stickiness is invariant under various transformations with continuous functions, identifying the stickiness property for stochastic processes, even when they admit martingale measures, is useful. Most Lévy processes are known to admit equivalent martingale measures, see Proposition 9.9 on page 315 of [4], for example. However, their transformations under continuous functions may lose even the semi-martingale property as discussed in Remark 2.4.

Example 3.1. Let W denote a d -dimensional Brownian motion. Let $b : \mathbb{R}^d \rightarrow \mathbb{R}^d$ be locally bounded and $v : \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}$ be continuous with $v(x)$ non-singular for all $x \in \mathbb{R}^d$. If the stochastic differential equation

$$dX_t = b(X_t) dt + v(X_t) dW_t, \quad X_0 = x,$$

has a weak solution, unique in law, for all $x \in \mathbb{R}^d$, then any solution satisfies CFS, a fortiori, stickiness, as shown in Guasoni and Rásonyi [11].

CFS also holds for many non-semimartingales: fractional Brownian motion and other Gaussian processes, see [3,9,13].

Example 3.2. Let's look at the case of a skew Brownian motion X_t which is defined to be the solution of the following equation

$$X_t = W_t + \beta L_t^0,$$

where W_t is a one-dimensional Brownian motion, L_t^0 is local time of the unknown process X_t at time 0, and β is a constant with $|\beta| < 1$, see Harrison and Shepp [14] for further details. Since the local time L_t^0 generates a measure singular to the Lebesgue measure, X_t does not admit any local martingale measure. Let $\alpha = (\beta + 1)/2$ and define the strictly monotone continuous function s_α as $s_\alpha = (1 - \alpha)x$ for $x \geq 0$, αx for $x < 0$. Let $Y_t = s_\alpha(X_t)$. It was shown in Harrison and Shepp [14] that Y_t satisfies $dY_t = f(Y_t) dW_t$, where $f(x) = 1 - \alpha$ for $x > 0$, $\frac{1}{2}$ for $x = 0$, and α for $x < 0$. Since f is non-singular and bounded, from the results of Stroock and Varadhan [26] we can conclude that Y_t has full support on the space of continuous functions for any initial value. Consequently, as Y_t is Markovian, it has CFS and hence it is sticky. It is clear that Y_t is a martingale as f is bounded. The process X_t inherits the stickiness property from Y_t as it can be written as a composition of Y_t with a strictly monotone continuous function (the inverse function of s_α).

Example 3.3. Let us turn to processes with jumps now. For simplicity, we assume $d = 1$. Let L be a Lévy process. Then it has the following decomposition

$$L_t = ct + \sigma B_t + \int_{|\theta| < 1} \theta \tilde{N}(t, d\theta) + \int_{|\theta| \geq 1} \theta N(t, d\theta), \tag{4}$$

for some constants $c, \sigma \in \mathbb{R}$. Here ν is the Lévy measure of L , N the Poisson random measure of L , and $\tilde{N}(dt, d\theta) = N(dt, d\theta) - \nu(d\theta) dt$ is its compensated version. B is an independent Brownian motion from N . It is shown in Simon [25] that L satisfies the stickiness property provided that $\sigma^2 \neq 0$ or $\int_{-1}^1 |x| \nu(dx) = \infty$. If $\sigma^2 = 0$ and $\int_{-1}^1 |x| \nu(dx) < \infty$, then L satisfies stickiness if $h := c - \int_{-1}^1 |x| \nu(dx) = 0$ or $h > 0$ (resp. $h < 0$) and, for all $\epsilon > 0$, $\nu((-\epsilon, 0)) > 0$ (resp. $\nu((0, \epsilon)) > 0$).

Example 3.4. Let X satisfy CFS and let L be a sticky Lévy process such that they are independent. Then $S_t := f(X_t, L_t)$ is also sticky for any continuous function $f : \mathbb{R}^{d+1} \rightarrow \mathbb{R}$, by Proposition 1 of Sayit and Viens [24]. For example, one can replace the Brownian motion B_t in (4) by a fractional Brownian motion B_t^H that is independent from N_t , and obtain a sticky process which is not a semi-martingale.

Remark 3.5. We expect that solutions of Lévy process-driven stochastic differential equations are also sticky under mild conditions. It is outside the scope of the present paper to pursue related investigations.

4. Main result

As explained in the above section, a large class of stochastic processes enjoy the stickiness property. Our main goal in this section is to show that martingales (under an equivalent change of measure) live “near” to them e.g. in the L^p norm. In the following theorem, we state this result and present its proof after some preparations.

Theorem 4.1. *Let $g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a convex function with $g(0) = 0$ and let $\chi > 0$ be any fixed number. Let S be a càdlàg process which is sticky with respect to \mathbb{F} . Let Assumption 2.5 be in force. Let $\mathcal{H}_t = \mathcal{F}_t \vee \mathcal{G}_t$ for each $t \in [0, T]$. Then the process S is sticky with respect to $\mathbb{H} = (\mathcal{H}_t)_{t \in [0, T]}$ and there exists $Q \sim P$ and a d -dimensional Q -martingale \tilde{S} (with respect to \mathbb{H}) such that $\tilde{S}_0 = S_0$ and*

$$E_Q g\left(\sup_{t \in [0, T]} |S_t - \tilde{S}_t|\right) < \chi. \tag{5}$$

If S has continuous trajectories, then even

$$\sup_{t \in [0, T]} |S_t - \tilde{S}_t| < \chi \tag{6}$$

holds a.s.

Example 4.2. In general, it is not possible to replace Q by the physical measure P in (5) above. This is shown by a simple example: let $T := 1$, $S_t := 0$, $t < 1$, and let S_1 be uniform on $[0, 1]$. We take \mathbb{F} to be the natural filtration of S . Set $g(x) := |x|$ and choose $\chi := 1/4$.

The process S is trivially sticky. Arguing by contradiction, suppose that there is \tilde{S}_1 such that $\chi > E \sup_{t \in [0, 1]} |E[\tilde{S}_1 | \mathcal{H}_t] - S_t|$. Then also $E|E[\tilde{S}_1 | \mathcal{H}_0] - 0| = |E\tilde{S}_1| < \chi$, as $S_0 = 0$ and \mathcal{H}_0 is trivial. On the other hand, $\chi > E \sup_{t \in [0, 1]} |E[\tilde{S}_1 | \mathcal{F}_t] - S_t| \geq E|\tilde{S}_1 - S_1|$. Noting that $ES_1 = 1/2$, this would mean $E\tilde{S}_1 > 1/4$ while we have just seen that $E\tilde{S}_1 < 1/4$, a contradiction.

Corollary 4.3. *Let $\chi > 0$ be any fixed number. Let S be a càdlàg process which is sticky with respect to \mathbb{F} . Let Assumption 2.5 be in force. Let $\mathcal{H}_t = \mathcal{F}_t \vee \mathcal{G}_t$ for each $t \in [0, T]$. For each $p \geq 1$ there exists $Q \sim P$ and a d -dimensional Q -martingale \tilde{S} (with respect to \mathbb{H}) such that*

$$E_Q \sup_{t \in [0, T]} |S_t - \tilde{S}_t|^p < \chi. \tag{7}$$

Proof. Indeed, let $g(x) := x^p$, $x \geq 0$, and apply Theorem 4.1. □

Remark 4.4. In the case where S is a continuous process, Theorem 4.1 was proved in Bender et al. [2] in a slightly different form. In that paper, S is assumed to be positive and \tilde{S} is shown to satisfy

$$\sup_{t \in [0, T]} |S_t / \tilde{S}_t - 1| < \chi \quad \text{a.s.} \tag{8}$$

Minor modifications of that argument would work for not necessarily positive, continuous S and they would lead to (6) instead of (8), *without* using Assumption 2.5. Thus, the novelty of Theorem 4.1 lies in treating the case of discontinuous processes, at the price of requiring Assumption 2.5. We do not know whether this assumption could be dropped.

The following lemma will be a key ingredient for the proof of Theorem 4.1. We now consider a discrete-time filtration $(\mathcal{K}_n)_{n \in \mathbb{N}}$. We introduce some notation that will be used in the sequel. For an \mathbb{R}^d -valued random variable X , let $\mathcal{D}(X)$ be the smallest affine subspace containing the support of $\text{Law}(X)$. Let $\mathcal{S}(X)$ be the relative interior of the convex hull of the support of $\text{Law}(X)$. The meanings of $\mathcal{D}(\mu)$, $\mathcal{S}(\mu)$ are analogous for a probability μ on \mathbb{R}^d . We denote by $B(x, r)$ the closed ball of radius $r \geq 0$ around $x \in \mathbb{R}^d$.

Lemma 4.5. *Fix any $\varepsilon > 0$ and assume that $w : \mathbb{R}^d \rightarrow \mathbb{R}_+$ is a continuous function with $w(0) = 0$ and $w(x) \geq |x|$. Let $(M_n)_{n \in \mathbb{N}}$ be a discrete-time process adapted to $(\mathcal{K}_n)_{n \in \mathbb{N}}$. Assume that $0 \in \mathcal{S}(Q_n(\cdot, \omega))$ a.s. and, for all $\varepsilon > 0$, $Q_n(B(0, \varepsilon), \omega) > 0$ a.s., where $Q_n(\cdot, \cdot)$ is the conditional law of $M_n - M_{n-1}$ with respect to \mathcal{K}_{n-1} , $n \geq 1$. Assume that there exists a random variable M_∞ and $A_n \in \mathcal{K}_n$ such that 1_{A_n} increases to 1 a.s. when $n \rightarrow \infty$ and*

$$\{M_k = M_\infty, k \geq n\} \supset A_n \supset \{|M_n - M_{n-1}| < \varepsilon\} \tag{9}$$

for all n . Then there is a $Q \sim P$ such that $M_n, n \in \mathbb{N} \cup \{\infty\}$, is a uniformly integrable Q -martingale and

$$E_Q \left[\sum_{n=1}^{\infty} w(M_n - M_{n-1}) \right] < \varepsilon. \tag{10}$$

Proof. By applying Lemma 7.2 with the choices

$$X := M_n - M_{n-1}, \quad \mathcal{K} := \mathcal{K}_{n-1}, \quad \eta := \varepsilon/2^n,$$

we obtain $j_n(y, \omega)$ for each $n \geq 1$. Define

$$Z_n(\omega) := j_n(M_n(\omega) - M_{n-1}(\omega), \omega).$$

Set $dQ/dP := \prod_{n=1}^{\infty} Z_n$. Note that, by the last statement of Lemma 7.2 and by (9), we have that $Z_k = 1$ for all $k \geq n + 1$ on A_n . Hence, for almost all ω , only finitely many $Z_n(\omega)$ differ from 1. So the infinite product converges almost surely. We claim that $Q(\Omega) = 1$. Indeed, by monotone convergence, we have

$$\begin{aligned} E \frac{dQ}{dP} &= \lim_{n \rightarrow \infty} E \left[1_{A_n} \frac{dQ}{dP} \right] = \lim_{n \rightarrow \infty} E[1_{A_n} Z_n \cdots Z_1] \\ &\geq 1 - \limsup_{n \rightarrow \infty} E[1_{A_n^c} Z_n \cdots Z_1]. \end{aligned}$$

By (9), $A_n^c \subset \{|M_n - M_{n-1}| \geq \varepsilon\}$, and by Lemma 7.2, we have

$$E[Z_n 1_{\{|M_n - M_{n-1}| \geq \varepsilon\}} | \mathcal{K}_{n-1}] < \varepsilon/2^n.$$

It follows that

$$\begin{aligned} E[1_{A_n^c} Z_n \cdots Z_1] &= E[E[1_{A_n^c} Z_n | \mathcal{K}_{n-1}] Z_{n-1} \cdots Z_1] \\ &\leq (\varepsilon/2^n) E[Z_{n-1} \cdots Z_1] = \varepsilon/2^n \rightarrow 0, \end{aligned}$$

as $n \rightarrow \infty$, showing that $Q(\Omega) \geq 1$. Fatou's lemma ensures $Q(\Omega) \leq 1$.

Now it remains to show that M_n is a uniformly integrable martingale under Q . The martingale property of M_n , $n \in \mathbb{N}$ under Q is clear from the construction of Q . Since $w(x) \geq |x|$, (10) implies that M_n converges to M_∞ in $L^1(Q)$ hence M_n , $n \in \mathbb{N} \cup \{\infty\}$ is a uniformly integrable martingale under Q . \square

Remark 4.6. Assume that $w(x) \geq |x|^\kappa$, $x \in \mathbb{R}^d$ with some $\kappa \geq 1$. Then a trivial modification of the proof of Lemma 4.5 yields not only (10) but also

$$\sum_{n=1}^{\infty} E_Q^{1/\kappa} |M_n - M_{n-1}|^\kappa < \varepsilon,$$

which implies

$$E_Q^{1/\kappa} \left[\sup_n |M_n|^\kappa \right] < \infty,$$

whenever $E|M_0|^\kappa < \infty$, in particular, when M_0 is constant.

Proposition 4.7. Assume that S is sticky with respect to \mathbb{F} . Let $g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be any convex function with $g(0) = 0$. Assume that for the sequence $(S_{\tau_n})_{n \geq 0}$ we have $0 \in \mathcal{S}(P(S_{\tau_{n+1}} - S_{\tau_n} \in \cdot | \mathcal{F}_{\tau_n}))$ almost surely, where the stopping times τ_n are recursively defined by

$$\tau_0 = 0, \quad \tau_{n+1} := \inf\{t > \tau_n : |S_t - S_{\tau_n}| \geq \varepsilon\} \wedge T,$$

for some $\varepsilon > 0$. Then there exists $Q \sim P$ and a d -dimensional Q -martingale \tilde{S} with respect to the filtration \mathbb{F} such that $\tilde{S}_0 = S_0$, $\tilde{S}_T = S_T$, and

$$E_Q g \left(\sup_{t \in [0, T]} |S_t - \tilde{S}_t| \right) < g(2\varepsilon) + 2\sqrt{\varepsilon},$$

where the latter expression can be made arbitrarily small when $\varepsilon \rightarrow 0$. If S has continuous trajectories, then even

$$\sup_{t \in [0, T]} |S_t - \tilde{S}_t| < 2\varepsilon \tag{11}$$

holds almost surely. If S is (strictly) positive, then so is \tilde{S} .

Remark 4.8. Comparing Proposition 4.7 to Theorem 4.1, the former does not require Assumption 2.5 and it provides \tilde{S} satisfying $S_T = \tilde{S}_T$ but this comes at the price of a hypothesis involving

the τ_n . Still, Proposition 4.7 improves on previous results *even* in the case of continuous S . Indeed, if S has the CFS property then the conditions of Proposition 4.7 are easily seen to hold, by an argument similar to Lemma A.2 of Guasoni et al. [13]. Therefore Proposition 4.7 strengthens the conclusion of Theorem 2.11 of Guasoni et al. [13] (see also Theorem 2.1 of the same paper): we get \tilde{S} as in (11) but satisfying $S_0 = \tilde{S}_0$ and $S_T = \tilde{S}_T$ a.s. as well.

Proof of Proposition 4.7. The idea here is to apply Lemma 4.5 to S sampled at the stopping times τ_n . The Q constructed is such that all the increments $S_{\tau_n} - S_{\tau_{n-1}}$ will be “small” but $S_{\tau_n} = M_n$, $n \in \mathbb{N}$ is a Q -martingale. Then \tilde{S} will be just the continuous-time Q -martingale with terminal value $S_T = M_\infty$ and, by the choice of Q , $\sup_{t \in [0, T]} |S_t - \tilde{S}_t|$ will also be “small”.

Note that g is necessarily continuous (even at 0). Set $M_n := S_{\tau_n}$ and $\mathcal{K}_n := \mathcal{F}_{\tau_n}$ for all $n \in \mathbb{N}$. Using the notations of Lemma 4.5, the conditions of the present proposition imply that $0 \in \mathcal{S}(Q_n(\cdot, \omega))$ almost surely. The stickiness property guarantees that, for any small real number $\zeta > 0$ and all $n \geq 1$, $Q_n(B(0, \zeta), \omega) > 0$ almost surely. Define $A_n := \{\tau_n = T\} \in \mathcal{K}_n$. As S has càdlàg paths, for almost all ω , the increasing sequence $\tau_n(\omega)$ can not have a limit strictly less than T . This shows that $\tau_n(\omega) = T$ for all $n \geq m(\omega)$ for some $m(\omega) \in \mathbb{N}$ almost surely. Therefore, 1_{A_n} increases to 1 almost surely.

Set $M_\infty := S_T$. From the definition of τ_n we have $\{|M_n - M_{n-1}| < \varepsilon\} \subset A_n$ and therefore (9) holds. Using Lemma 4.5 with the choice $w(x) := g^2(2|x|) + |x|$ we obtain Q . Now define $\tilde{S}_t := E_Q[S_T | \mathcal{F}_t]$, $t \in [0, T]$ (we take a càdlàg version of this Q -martingale). This definition makes sense since S_T is Q -integrable by $|x| \leq w(x)$ and (10). We clearly have $\tilde{S}_0 = S_0$ and $\tilde{S}_T = S_T$. It remains to estimate

$$\sup_{t \in [0, T]} |S_t - \tilde{S}_t|.$$

Fix t, n for a moment and let us work on the event $B_n := \{\tau_n \leq t < \tau_{n+1}\}$ till further notice. We have

$$|S_t - \tilde{S}_t| = |S_{t \wedge \tau_{n+1}} - \tilde{S}_{t \wedge \tau_{n+1}}| = |S_{t \wedge \tau_{n+1}} - E_Q[\tilde{S}_{\tau_{n+1}} | \mathcal{F}_{t \wedge \tau_{n+1}}]|,$$

by the Q -martingale property of \tilde{S} . We further have

$$\begin{aligned} |E_Q[S_{t \wedge \tau_{n+1}} - \tilde{S}_{\tau_{n+1}} | \mathcal{F}_{t \wedge \tau_{n+1}}]| &\leq E_Q[|S_{t \wedge \tau_{n+1}} - \tilde{S}_{\tau_n}| + |S_{\tau_n} - \tilde{S}_{\tau_{n+1}}| | \mathcal{F}_{t \wedge \tau_{n+1}}] \\ &\leq E_Q[\varepsilon + |M_{n+1} - M_n| | \mathcal{F}_{t \wedge \tau_{n+1}}], \end{aligned} \tag{12}$$

which follows from the definitions of B_n, τ_n , and

$$\tilde{S}_{\tau_k} = E_Q[S_T | \mathcal{F}_{\tau_k}] = M_k = S_{\tau_k}$$

for both $k = n$ and $k = n + 1$. Hence, we get

$$\begin{aligned} g(|S_t - \tilde{S}_t|) &\leq g(\varepsilon + E_Q[|M_{n+1} - M_n| | \mathcal{F}_{t \wedge \tau_{n+1}}]) \\ &\leq \frac{1}{2}(g(2\varepsilon) + g(E_Q[2|M_{n+1} - M_n| | \mathcal{F}_{t \wedge \tau_{n+1}}])) \\ &\leq g(2\varepsilon) + E_Q[g(2|M_{n+1} - M_n|) | \mathcal{F}_{t \wedge \tau_{n+1}}], \end{aligned}$$

by the convexity of g . Noting that g is necessarily non-decreasing, we get

$$\begin{aligned} g(|S_t - \tilde{S}_t|) &\leq g(2\varepsilon) + E_Q \left[g \left(2 \sup_n |M_{n+1} - M_n| \right) \middle| \mathcal{F}_{t \wedge \tau_{n+1}} \right] \\ &\leq g(2\varepsilon) + E_Q [L_T | \mathcal{F}_{t \wedge \tau_{n+1}}] \\ &\leq g(2\varepsilon) + \sup_{s \in [0, T]} L_s, \end{aligned}$$

for the positive Q -martingale $L_s := E_Q [g(2 \sup_n |M_{n+1} - M_n|) | \mathcal{F}_s]$, $s \in [0, T]$. The right-hand side here, however, does not depend either on t or on n so this estimate, in fact, holds a.s. on $\Omega = \bigcup_n B_n$. Hence,

$$\begin{aligned} E_Q g \left(\sup_{t \in [0, T]} |S_t - \tilde{S}_t| \right) &\leq g(2\varepsilon) + E_Q \left[\sup_{s \in [0, T]} L_s \right] \\ &\leq g(2\varepsilon) + E_Q^{1/2} \left[\sup_{s \in [0, T]} L_s^2 \right] \\ &\leq g(2\varepsilon) + 2E_Q^{1/2} L_T^2 \\ &\leq g(2\varepsilon) + 2E_Q^{1/2} \left[\sum_{n=0}^{\infty} w(M_{n+1} - M_n) \right] \\ &\leq g(2\varepsilon) + 2\sqrt{\varepsilon}, \end{aligned}$$

using Doob's inequality and Lemma 4.5. Positivity of \tilde{S} is clear since $\tilde{S}_t = E_Q [S_T | \mathcal{F}_t]$ and S_T is positive. If S is continuous, then $|S_{\tau_n} - S_{\tau_{n-1}}| \leq \varepsilon$ for all n , so we can deduce (11) directly from (12). \square

Lemma 4.9. *Let X and Y be two independent càdlàg processes. Let $\mathbb{F} = (\mathcal{F}_t)_{t \in [0, T]}$ and $\mathbb{G} = (\mathcal{G}_t)_{t \in [0, T]}$ be independent, complete, right-continuous filtrations to which X and Y are adapted, respectively. Let $\mathcal{H}_t = \mathcal{F}_t \vee \mathcal{G}_t$ for all $t \in [0, T]$. Then $(\mathcal{H}_t)_{t \in [0, T]}$ is a complete, right-continuous filtration. If X is sticky with respect to \mathbb{F} and Y is sticky with respect to \mathbb{G} , then all of $X, Y, X \pm Y$ and (X, Y) are sticky with respect to the filtration $\mathbb{H} = (\mathcal{H}_t)_{t \in [0, T]}$.*

Proof. First, observe that \mathbb{H} is a complete filtration as both \mathbb{F} and \mathbb{G} are complete. Therefore, it is enough to prove that \mathbb{H} is right-continuous and, to this end, it is enough to prove $E[Z | \mathcal{H}_t] = E[Z | \mathcal{H}_{t+}]$ for any \mathcal{H}_T -measurable nonnegative random variable Z and for all $0 \leq t < T$. By the monotone class theorem, it is enough to prove this equality for $Z = UV$ where $U \geq 0$ is \mathcal{F}_T -measurable and $V \geq 0$ is \mathcal{G}_T -measurable. However, Lemma 7.4 implies that

$$\begin{aligned} \lim_{h \rightarrow 0} E[UV | \mathcal{H}_{t+h}] &= \lim_{h \rightarrow 0} E[U | \mathcal{F}_{t+h}] E[V | \mathcal{G}_{t+h}] \\ &= E[U | \mathcal{F}_t] E[V | \mathcal{G}_t] = E[UV | \mathcal{H}_t], \end{aligned}$$

by the right-continuity of $\mathcal{F}_t, \mathcal{G}_t, t \in [0, T]$. This shows right-continuity of $\mathcal{H}_t, t \in [0, T]$.

To show the second claim in the lemma it is sufficient to show that X_t, Y_t are sticky for \mathbb{H} . The stickiness of $X_t \pm Y_t$ with respect to \mathbb{H} then follows from Proposition 1 of Sayit and Viens [24] (continuous functions of sticky processes are sticky). We only show that X is sticky for \mathbb{H} , the argument for Y being identical. Since X_t is a right-continuous process we need to check

$$P\left(\sup_{t \in [s, T]} |X_t - X_s| < \kappa \mid \mathcal{H}_s\right) > 0 \quad \text{a.s.,}$$

for any $\kappa > 0$ and any deterministic $s \in [0, T]$ (see Lemma 2.1 above). This follows by Lemma 7.4 from

$$P\left(\sup_{t \in [s, T]} |X_t - X_s| < \varepsilon \mid \mathcal{F}_s \vee \mathcal{G}_s\right) = P\left(\sup_{t \in [s, T]} |X_t - X_s| < \varepsilon \mid \mathcal{F}_s\right) > 0 \quad \text{a.s.,}$$

as $\mathcal{F}_s \vee \sigma(X)$ is independent from \mathcal{G}_s and X is sticky for \mathbb{F} .

To see the last statement, apply Lemma 7.4 to obtain

$$\begin{aligned} &P\left(\sup_{t \in [s, T]} |X_t - X_s| < \kappa, \sup_{t \in [s, T]} |Y_t - Y_s| < \kappa \mid \mathcal{H}_s\right) \\ &= P\left(\sup_{t \in [s, T]} |X_t - X_s| < \kappa \mid \mathcal{F}_s\right) P\left(\sup_{t \in [s, T]} |Y_t - Y_s| < \kappa \mid \mathcal{G}_s\right) > 0, \end{aligned}$$

by the stickiness of X, Y with their respective filtrations. □

Now, using the previous arguments, it is possible to establish Theorem 4.1, too. Before presenting the proof, we make some important observations.

Remark 4.10. Let B_t be a Brownian motion with respect to a filtration \mathcal{L}_t and let $0 \leq \theta < T$ be an arbitrary deterministic time. Then, by Theorem 6.1 in Chapter 2 of Karatzas and Shreve [20], $B_{s+\theta} - B_\theta, s \geq 0$, is a Brownian motion independent of \mathcal{L}_θ . Let $C_0[\theta, T]$ denote the space of \mathbb{R}^d -valued continuous functions on $[\theta, T]$ which are 0 at θ .

Let us first note that the mapping $f \rightarrow \sup_{s \in [\theta, T]} |B_s(\omega) - B_\theta(\omega) - f_s|, f \in C_0[\theta, T]$, is continuous for a.e. $\omega \in \Omega$ and hence it is jointly measurable in (ω, f) . It follows that $\{\sup_{s \in [\theta, T]} |B_s(\omega) - B_\theta(\omega) - G_s| < \epsilon\}$ and $\{\sup_{s \in [\theta, T]} |B_s(\omega) - B_\theta(\omega) - G_s| \leq \epsilon\}$ are events for each $\epsilon > 0$, where G is a random element of $C_0[\theta, T]$.

Now define $q(\epsilon, f) := P(\sup_{s \in [\theta, T]} |B_s - B_\theta - f_s| < \epsilon)$ and notice that $q(\epsilon, f) > 0$ for all $f \in C_0[\theta, T]$ as $B_t - B_\theta, t \in [\theta, T]$, has full support on $C_0[\theta, T]$. Fatou's lemma for events shows that $f \rightarrow q(\epsilon, f)$ is lower semicontinuous. Notice that, if G^n are \mathcal{L}_θ -measurable $C_0[\theta, T]$ -valued random variables taking only countable many values, then

$$P\left(\sup_{s \in [\theta, T]} |B_s - B_\theta - G_s^n| < \epsilon \mid \mathcal{L}_\theta\right) = q(\epsilon, G^n).$$

Now let G be an arbitrary \mathcal{L}_θ -measurable random element in $C_0[\theta, T]$. Choose a sequence $G^n, n \in \mathbb{N}$, of discrete \mathcal{L}_θ -measurable random elements in $C_0[\theta, T]$ such that G^n tend to G almost

surely. Lower semicontinuity of $q(\epsilon, \cdot)$ and Fatou’s lemma for events imply that

$$\begin{aligned}
 0 &< q(\epsilon, G) \\
 &\leq \liminf_n q(\epsilon, G^n) \\
 &= \liminf_n P\left(\sup_{s \in [\theta, T]} |B_s - B_\theta - G_s^n| < \epsilon | \mathcal{L}_\theta\right) \\
 &\leq \liminf_n P\left(\sup_{s \in [\theta, T]} |B_s - B_\theta - G_s^n| \leq \epsilon | \mathcal{L}_\theta\right) \\
 &\leq \limsup_n P\left(\sup_{s \in [\theta, T]} |B_s - B_\theta - G_s^n| \leq \epsilon | \mathcal{L}_\theta\right) \\
 &\leq P\left(\sup_{s \in [\theta, T]} |B_s - B_\theta - G_s| \leq \epsilon | \mathcal{L}_\theta\right). \tag{13}
 \end{aligned}$$

Proof of Theorem 4.1. We wish to apply Proposition 4.7 but S does not necessarily satisfy $0 \in \mathcal{S}(P(S_{\tau_{n+1}} - S_{\tau_n} \cdot | \mathcal{F}_{\tau_n}))$. To fix this, we perturb S by an independent “small noise” W such that, for $Y_t := S_t + W_t$, $0 \in \mathcal{S}(P(Y_{\tau_{n+1}} - Y_{\tau_n} \cdot | \mathcal{F}_{\tau_n}))$ holds. As Y is close to S , the $\tilde{\mathcal{S}}$ constructed for Y by Proposition 4.7 will also be close to S .

Fix any $\epsilon > 0$. Let $B_t = (B_t^1, B_t^2, \dots, B_t^d)$ be the Brownian motion of Assumption 2.5. We remark that B clearly has the CFS property. Let $\pi : (-\infty, +\infty) \rightarrow (-\epsilon, +\epsilon)$ be a bijective and Lipschitz-continuous (deterministic) function. Let $F : \mathbb{R}^d \rightarrow (-\epsilon, +\epsilon)^d$ be defined by $F(x^1, \dots, x^d) := (\pi(x^1), \dots, \pi(x^d))$. Denote by L a Lipschitz constant for the mapping F . Now set

$$W_t := (W_t^1, W_t^2, \dots, W_t^d) := F(B_t^1, B_t^2, \dots, B_t^d).$$

Define $Y_t = S_t + W_t$. From Lemma 4.9 above, Y is sticky for the filtration \mathbb{H} .

For each positive integer $n \geq 1$, define

$$\tau_n = \inf\{t \geq \tau_{n-1} : |Y_t - Y_{\tau_{n-1}}| \geq \epsilon\}, \quad \tau_0 = 0.$$

These are stopping times with respect to the filtration \mathbb{H} . We would like to show that $\Delta_n := Y_{\tau_n} - Y_{\tau_{n-1}}$ satisfies

$$0 \in \mathcal{S}(P(\Delta_n \in \cdot | \mathcal{H}_{\tau_{n-1}})) \tag{14}$$

almost surely, for each n . Fixing n , from now on we are working on the set $\{\tau_{n-1} < T\}$ (since (14) is trivial on $\{\tau_{n-1} = T\}$). We will write $\tau := \tau_{n-1}$ henceforth.

Let $0 < \eta < \epsilon/2$ be an \mathcal{H}_τ -measurable random variable such that $B(W_\tau, \eta) \subset (-\epsilon, \epsilon)^d$. Working separately on events of the form $\{\eta \geq 1/j\}$, $j \in \mathbb{N}$ we may and will assume that η is a constant.

Fix $x \in B(0, \eta/2) \cap \mathbb{Q}^d$. It suffices to show that

$$x \in \text{supp } P(\Delta_n \in \cdot | \mathcal{H}_\tau) \quad \text{a.s.} \tag{15}$$

on $\{\tau < T\}$ since this implies that, outside a null set of ω 's, $\text{supp } P(\Delta_n \in \cdot | \mathcal{H}_\tau)(\omega)$ contains $B(0, \eta/2) \cap \mathbb{Q}^d$ hence, being a closed set, also the whole of $B(0, \eta/2)$. The statement (15) will follow if, for each $l \in \mathbb{N}$,

$$\begin{aligned} &P(\Delta_n \in B(x, (1 + \eta)/l) | \mathcal{H}_\tau) \\ &\geq P(\tau_n = T, \Delta_n \in B(x, (1 + \eta)/l) | \mathcal{H}_\tau) > 0 \end{aligned} \tag{16}$$

almost surely on $\{\tau < T\}$.

Fix $l \in \mathbb{N}$. We will now prove (16). To this end, define

$$J_t = B_\tau(\omega) - F^{-1}\left(\frac{t - \tau}{T - \tau}x + F(B_\tau(\omega))\right), \quad \text{for } t \in [\tau(\omega), T],$$

and $J_t = 0, t < \tau(\omega)$. This definition makes sense since $|\frac{t-\tau}{T-\tau}x| \leq \eta/2$ and, by the choice of $\eta, \frac{t-\tau}{T-\tau}x + F(B_\tau) \in (-\varepsilon, \varepsilon)^d$. Furthermore, define the events

$$\begin{aligned} D(l) &:= \left\{ \sup_{t \in [\tau, T]} |S_t - S_\tau| < \eta/l \right\}, \\ K(l) &:= \left\{ \sup_{t \in [\tau, T]} |W_t - W_\tau| < \eta, |W_T - W_\tau - x| \in B(0, 1/l) \right\}, \\ H(l) &:= \left\{ \sup_{t \in [\tau, T]} |B_t - B_\tau + J_t| < \frac{1}{L} \min\{1/l, \eta/2\} \right\}. \end{aligned}$$

We first show that

$$P(D(l) \cap H(l) | \mathcal{H}_\tau) > 0 \quad \text{a.s.} \tag{17}$$

on $\{\tau < T\}$. For this, it is sufficient to show that for any $A \in \mathcal{H}_\tau, A \subset \{\tau < T\}$ and $P(A) > 0$, the relation $P(A \cap D(l) \cap H(l)) > 0$ holds. Fix such an A and a deterministic number $0 < \epsilon_0 < \min\{\eta/l, \frac{1}{L} \min\{1/l, \eta/2\}\}$.

As J_t is an \mathcal{H}_τ -measurable continuous process with $J_\tau = 0$, there exists a deterministic number $\theta \leq T$ such that the event $A_1 = A \cap \{\sup_{t \in [\tau, \theta]} |J_t| \leq \frac{\epsilon_0}{6}\} \cap \{\tau < \theta\}$ has positive probability. Note that $A_1 \in \mathcal{H}_\theta \cap \mathcal{H}_\tau$. The joint stickiness of the process (S_t, B_t) , see Lemma 4.9, shows that the event

$$A_2 = A_1 \cap \left\{ \sup_{t \in [\tau, \theta]} |S_t - S_\tau| \leq \frac{\epsilon_0}{2} \right\} \cap \left\{ \sup_{t \in [\tau, \theta]} |B_t - B_\tau| \leq \frac{\epsilon_0}{6} \right\}$$

has positive probability. Now observe that $A_2 \cap d(l) \subset D(l)$ where

$$d(l) =: \left\{ \sup_{t \in [\theta, T]} |S_t - S_\theta| \leq \frac{\epsilon_0}{2} \right\}.$$

We also claim $A_2 \cap h(l) \subset H(l)$, where

$$h(l) =: \left\{ \sup_{t \in [\theta, T]} |B_t - B_\theta + J_t - J_\theta| \leq \frac{\epsilon_0}{3} \right\}.$$

Indeed,

$$\begin{aligned}
 \sup_{t \in [\tau, T]} |B_t - B_\tau + J_t| &\leq \sup_{t \in [\tau, \theta]} |B_t - B_\tau + J_t| + \sup_{t \in [\theta, T]} |B_t - B_\tau + J_t| \\
 &\leq \sup_{t \in [\tau, \theta]} |B_t - B_\tau| + \sup_{t \in [\tau, \theta]} |J_t| + \sup_{t \in [\theta, T]} |B_t - B_\theta + J_t - J_\theta| \\
 &\quad + |B_\theta + J_\theta - B_\tau| \\
 &< \epsilon_0/6 + \epsilon_0/6 + \epsilon_0/3 + |B_\theta - B_\tau| + |J(\theta)| \\
 &\leq \epsilon_0
 \end{aligned}$$

on $A_2 \cap h(l)$ so $A_2 \cap h(l) \subset H(l)$.

We conclude that

$$A_2 \cap d(l) \cap h(l) \subset A \cap D(l) \cap H(l). \tag{18}$$

Therefore it is sufficient to show that the left-hand side of (18) has positive probability. Since $A_2 \in \mathcal{H}_\theta$, it is sufficient to show that

$$P(d(l) \cap h(l) | \mathcal{H}_\theta) > 0 \quad \text{a.s.} \tag{19}$$

Define $\mathcal{L}_t := \mathcal{G}_t \vee \mathcal{F}_T$. We can write (19) as follows:

$$\begin{aligned}
 P(d(l) \cap h(l) | \mathcal{H}_\theta) &= E[E[1_{d(l) \cap h(l)} | \mathcal{L}_\theta] | \mathcal{H}_\theta] \\
 &= E[1_{d(l)} E[1_{h(l)} | \mathcal{L}_\theta] | \mathcal{H}_\theta].
 \end{aligned}$$

Notice that $G_s := J_s - J_\theta$, $s \in [\theta, T]$, is a $\mathcal{H}_\theta \subset \mathcal{L}_\theta$ -measurable random element in $C_0[\theta, T]$ so $E[1_{h(l)} | \mathcal{L}_\theta] \geq q(\epsilon_0/3, G) > 0$ a.s. by (13) in Remark 4.10 above. It follows that

$$P(d(l) \cap h(l) | \mathcal{H}_\theta) \geq q(\epsilon_0/3, G) P(d(l) | \mathcal{H}_\theta) > 0,$$

by the stickiness of S with respect to \mathbb{F} and by Lemma 4.9. We conclude that (17) holds.

We will now show that

$$H(l) \cap D(l) \subset K(l) \cap D(l) \subset \{\tau_n = T, \Delta_n \in B(x, (1 + \eta)/l)\},$$

which will entail (16), in view of (17).

The second containment is trivial since $\sup_{t \in [\tau, T]} |W_t - W_\tau| < \eta$ and $\sup_{t \in [\tau, T]} |S_t - S_\tau| < \eta/l$ entail $\sup_{t \in [\tau, T]} |Y_t - Y_\tau| < \varepsilon$ by $\eta < \varepsilon/2$ which implies $\tau_n = T$. Obviously, $|W_T - W_\tau - x| \in B(0, 1/l)$ together with $\sup_{t \in [\tau, T]} |S_t - S_\tau| < \eta/l$ imply $\Delta_n \in B(x, (1 + \eta)/l)$.

For the first containment, the Lipschitz property of F and $W = F(B)$ clearly imply that

$$\left| W_t - \frac{t - \tau}{T - \tau} x - W_\tau \right| \leq L \left| B_t - F^{-1} \left(\frac{t - \tau}{T - \tau} x + F(B_\tau) \right) \right| < \min \left\{ \frac{1}{l}, \frac{\eta}{2} \right\},$$

on $H(l)$. For $t = T$ this gives $|W_T - W_\tau - x| < 1/l$ whereas

$$\sup_{t \in [\tau, T]} |W_t - W_\tau| \leq \sup_{t \in [\tau, T]} \left| W_t - W_\tau - \frac{t - \tau}{T - \tau} x \right| + \sup_{t \in [\tau, T]} \left| \frac{t - \tau}{T - \tau} x \right| < \eta/2 + \eta/2 = \eta,$$

showing $H(l) \subset K(l)$.

Now apply Proposition 4.7 to the process Y with the convex function $x \rightarrow g(2x)$ to obtain \tilde{S} . We get

$$\begin{aligned} E_Q g\left(\sup_{t \in [0, T]} |S_t - \tilde{S}_t|\right) &\leq \frac{1}{2} E_Q g\left(2 \sup_{t \in [0, T]} |Y_t - \tilde{S}_t|\right) + \frac{1}{2} E_Q g\left(2 \sup_{t \in [0, T]} |Y_t - S_t|\right) \\ &\leq \frac{g(4\varepsilon) + 2\sqrt{\varepsilon}}{2} + \frac{1}{2} g(2\varepsilon), \end{aligned}$$

which can be made smaller than χ when $\varepsilon \rightarrow 0$. This completes the proof. □

Remark 4.11. By Remark 4.8, Proposition 4.7 applies to Example 3.1. If $b = 0$ in Example 3.1, then even Theorem 5.1 below applies and one can approximate many local martingales with true ones.

Let us now recall Example 3.2. Since Y_t is a martingale and the inverse function of s_α is strictly monotone, the process X_t satisfies $0 \in \mathcal{S}(P(X_\tau - X_\theta \in \cdot | \mathcal{F}_\theta))$ almost surely even for all stopping times $\tau \geq \theta$. So Proposition 4.7 applies to the case of skew Brownian motion.

Theorem 4.1 applies to the large class of processes presented in Example 3.4.

Remark 4.12. At first sight, the argument for Proposition 4.7 looks just a variant of that of Theorem 1.2 in Guasoni et al. [13], see also Kabanov and Stricker [19] and Section 3.6.8 in Kabanov and Safarian [17]. Fine details, however, do differ significantly. Not only does Proposition 4.7 cover the case of processes with jumps, too, but it is sharper even in the case of continuous processes, as we have already pointed out in Remark 4.8.

5. Local martingales

We denote by $\|\cdot\|_{\text{tv}}$ the total variation norm for finite signed measures on (Ω, \mathcal{F}) .

Theorem 5.1. *Let $g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be convex with $g(0) = 0$ and let $\chi > 0$. Let Assumption 2.5 be in force. Assume that S is a sticky local martingale. Then there exists $Q \sim P$ with $\|Q - P\|_{\text{tv}} < \chi$ and a d -dimensional Q -martingale \tilde{S} with respect to the enlarged filtration \mathbb{H} such that $E_Q g(\sup_{t \in [0, T]} |S_t - \tilde{S}_t|) < \chi$. If S has continuous trajectories, then even $\sup_{t \in [0, T]} |S_t - \tilde{S}_t| < \chi$ holds a.s. Finally, S remains a local martingale with respect to \mathbb{H} , too.*

Proof. Let $\sigma_n, n \in \mathbb{N}$ be the stopping times increasing to ∞ such that $S_{t \wedge \sigma_n}, t \in [0, T]$ is a martingale for each n . Fix k for the moment. We will apply the proof of Theorem 4.1 (which relies on Proposition 4.7), starting from $\sigma_k \wedge T$, that is, using the sequence

$$\tau_0(k) := \sigma_k \wedge T, \quad \tau_{n+1}(k) := \inf\{t > \tau_n(k) : |Y_t - Y_{\tau_n}| \geq \varepsilon\} \wedge T,$$

where $Y_t := S_t + W_t$ and W_t is as in the proof of Theorem 4.1.

Apply the argument of Proposition 4.7 starting from $\sigma_k \wedge T$ instead of 0, using Lemma 4.5 for $M_n := Y_{\tau_n(k)}$. Choosing ε small enough, we get that there is $Q(k) \sim P$ such that

$$E_{Q(k)}g\left(\sup_{\sigma_k \leq t \leq T} |S_t - \tilde{S}_t|\right) < \chi \quad \text{and} \quad dQ(k)/dP = \prod_{j=1}^{\infty} Z_j^{(k)},$$

where we define $\tilde{S}_t = E_{Q(k)}[S_T | \mathcal{H}_t]$, for all $t \in [0, T]$. Here $Z_j^{(k)}$ is $\mathcal{H}_{\tau_j(k)}$ -measurable for each $j \in \mathbb{N}$, corresponding to the Z_j appearing in the proof of Lemma 4.5. We now check that $S_t = \tilde{S}_t$ a.s. on $\{t \leq \sigma_k \wedge T\}$. Indeed, on this set

$$\begin{aligned} S_t - \tilde{S}_t &= E_{Q(k)}[S_{t \wedge \sigma_k} - \tilde{S}_{t \wedge \sigma_k} | \mathcal{H}_{t \wedge \sigma_k}] \\ &= E_{Q(k)}[S_{t \wedge \sigma_k} - S_{T \wedge \sigma_k} | \mathcal{H}_{t \wedge \sigma_k}] \\ &= \frac{E[(dQ(k)/dP)[S_{t \wedge \sigma_k} - S_{T \wedge \sigma_k}] | \mathcal{H}_{t \wedge \sigma_k}]}{E[dQ(k)/dP | \mathcal{H}_{t \wedge \sigma_k}]} \\ &= \frac{E[E[(dQ(k)/dP)[S_{t \wedge \sigma_k} - S_{T \wedge \sigma_k}] | \mathcal{H}_{T \wedge \sigma_k}] | \mathcal{H}_{t \wedge \sigma_k}]}{E[E[dQ(k)/dP | \mathcal{H}_{T \wedge \sigma_k}] | \mathcal{H}_{t \wedge \sigma_k}]} \\ &= \frac{E[E[dQ(k)/dP | \mathcal{H}_{T \wedge \sigma_k}](S_{t \wedge \sigma_k} - S_{T \wedge \sigma_k}) | \mathcal{H}_{t \wedge \sigma_k}]}{E[E[dQ(k)/dP | \mathcal{H}_{T \wedge \sigma_k}] | \mathcal{H}_{t \wedge \sigma_k}]} \\ &= 0, \end{aligned}$$

which follows from $\tilde{S}_{T \wedge \sigma_k} = S_{T \wedge \sigma_k}$, $E[dQ(k)/dP | \mathcal{H}_{T \wedge \sigma_k}] = 1$, and the martingale property of S up to σ_k under P . Now note that $P(dQ(k)/dP = 1) \geq P(\sigma_k \wedge T = T) \rightarrow 1, k \rightarrow \infty$, which implies that $dQ(k)/dP$ tends to 1 in probability, hence almost surely along a subsequence. Using Scheffé's theorem, we can find k with $\|Q(k) - P\|_{\text{tv}} < \chi$. The last statement is clear since \mathcal{G}_T is independent of \mathcal{F}_T . \square

Corollary 5.2. *Let Assumption 2.5 be in force. Let $p \geq 1$ be arbitrary and let S be a sticky local martingale. Then for all $\chi > 0$ there exists $Q \sim P$ with $\|Q - P\|_{\text{tv}} < \chi$ and a Q -martingale \tilde{S} with respect to the enlarged filtration \mathbb{H} such that*

$$E_Q \sup_{t \in [0, T]} |S_t - \tilde{S}_t|^p < \chi$$

is satisfied. \square

Remark 5.3. A *strict local martingale* is a local martingale which is not a martingale. It is not difficult to construct sticky strict local martingales by using Proposition 3.8 (also Corollary 3.9) of Elworthy et al. [8]. We can choose any continuous and nonincreasing $m : \mathbb{R}^+ \rightarrow (0, 1]$ with $m(0) = 1$ and let $M_t = 1/R_{r^{-1}(m(t))}$, where R_t is a 3-dimensional Bessel process starting from 1 and $r(t) = E(1/R_t)$. Then from Proposition 3.8 of Elworthy et al. [8], M_t is a strict local martingale with $m(t) = EM_t$. (M_t is strict local martingale as long as $m(t)$ is not a constant). M_t is sticky as it is obtained from a sticky process R by transformation under continuous function $\frac{1}{x}$, $x > 0$ and by bounded time change.

Remark 5.4. Strict local martingales (which are not martingales) have been suggested as models for financial bubbles, see Protter [22]. In this context, P is the pricing measure, S is the price process of risky assets. Theorem 5.1 and Corollary 5.2 reiterate the word of caution already pronounced in Guasoni and Rásonyi [11]: an arbitrarily small misspecification of option and asset prices (that is, mistaking Q for P and \tilde{S} for S) may destroy the “bubble phenomenon” generated by S under P since \tilde{S} is a martingale under Q , admitting no bubbles.

6. Application to mathematical finance

A central concept of mathematical finance is arbitrage, that is, riskless profit. Such opportunities should not exist in an efficient market. Arbitrage theory is well-understood in idealized models of financial markets where the presence of frictions (transaction fees, liquidity effects) is disregarded, see, for example, [6]. There is also a fairly clear picture in the case of proportional transaction costs, where, roughly speaking, trading costs are linear functions of the trading speed, see [17]. Illiquid markets, however, show new phenomena due to a *superlinear* dependence of trading costs on the trading speed. In such market models, a characterization for the absence of arbitrage in terms of dual variables has been provided in the paper [12], see Theorem 6.2 below.

We will apply the results of the present paper to show that a very large class of candidate price processes (namely the sticky ones) enjoy an absence of arbitrage property in markets with superlinear liquidity effects. We now briefly sketch (a slightly simplified version of) the model in Guasoni and Rásonyi [12], see [12] for further details.

Staying on the stochastic basis $(\Omega, \mathcal{F}, P, \mathbb{F})$, let S describe the price of d risky assets in a financial market when trading is (infinitely) slow. Liquidity effects will be described by a cost function $G : \Omega \times [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}_+$ which is assumed $\mathcal{O} \otimes \mathcal{B}(\mathbb{R}^d)$ -measurable where \mathcal{O} is the optional sigma-field. We furthermore assume that $G(\omega, t, \cdot)$ is convex with $G(\omega, t, x) \geq G(\omega, t, 0) := 0$ for all ω, t, x . Henceforth, set $G_t(x) := G(\omega, t, x)$, i.e. the dependence on ω is omitted, and t is used as a subscript. There is also a riskless asset S_t^0 of price constant 1, $t \in [0, T]$.

A *feasible strategy* is a process ϕ in the class

$$\mathcal{A} := \left\{ \phi : \phi \text{ is a } \mathbb{R}^d\text{-valued, optional process, } \int_0^T |\phi_u| du < \infty \text{ a.s.} \right\}, \tag{20}$$

that is, the speed of trading ϕ_t at t is assumed to be finite and the traded quantity of stocks over $[0, T]$ as well.

With this definition, for a given strategy $\phi \in \mathcal{A}$ and an initial asset position $z = (z^0, \dots, z^d) \in \mathbb{R}^{d+1}$, the resulting positions at time $t \in [0, T]$ in the risky and safe assets are defined as:

$$X_t^i(z, \phi) := z^i + \int_0^t \phi_u^i du, \quad 1 \leq i \leq d, \tag{21}$$

$$X_t^0(z, \phi) := z^0 - \int_0^t \phi_u S_u du - \int_0^t G_u(\phi_u) du. \tag{22}$$

The main item in the following assumption is the superlinearity condition (23): it expresses that fast trading has an effect which is stronger than linear as a function of the trading speed.

Assumption 6.1. There is $\alpha > 1$ and $H > 0$ such that

$$G_t(x) \geq H|x|^\alpha, \quad \text{for all } \omega, t, x, \tag{23}$$

$$\int_0^T \left(\sup_{|x| \leq N} G_t(x) \right) dt < \infty \quad \text{a.s. for all } N > 0. \tag{24}$$

Define also

$$G_t^*(y) := \sup_{x \in \mathbb{R}^d} (xy - G_t(x)) \geq 0, \quad y \in \mathbb{R}^d, t \in [0, T].$$

We will apply results of Section 4 to investigate under which conditions such market models are free of arbitrage. An *arbitrage of the second kind* is a strategy $\phi \in \mathcal{A}$, such that $X_T^i(z, \phi) \geq 0$, $i = 0, 1, \dots, d$ with $z = (c, 0, \dots, 0)$ for some $c < 0$. Absence of arbitrage of the second kind (NA2) holds if no such opportunity exists.

We reproduce Theorem 4.2 of Guasoni and Rásonyi [12] below¹ which characterizes (NA2). The notation $L^p(Q)$ for $p \geq 1$ refers to the usual Banach space of d -dimensional random variables with finite p th (absolute) moment under the probability $Q \sim P$.

Theorem 6.2. Let \mathcal{F}_0 be trivial, let Assumption 6.1 hold, fix $1 < \beta < \alpha$ and let $1/\beta + 1/\gamma = 1$. (NA2) holds if and only if, for all $\chi > 0$, there exists $Q \sim P$ with

$$E_Q \int_0^T (1 + |S_t|)^{\beta\alpha/(\alpha-\beta)} dt < \infty$$

and an \mathbb{R}_+^{d+1} -valued Q -martingale Z with $Z_T \in L^\gamma(Q)$ such that $\{Z_t^i = 0, i = 1, \dots, T\} \subset \{Z_t^0 = 0\}$ a.s. for all t , $Z_0^0 = 1$ and $E_Q \int_0^T Z_t^0 G_t^*(\bar{Z}_t - S_t) dt < \chi$ where $\bar{Z}_t^i = (Z_t^i / Z_t^0) 1_{\{Z_t^0 \neq 0\}}$, $i = 1, \dots, d$. □

Theorem 4.1 ensures that a plethora of models satisfy (NA2).

¹Unfortunately, the conditions “ $\{Z_t^i = 0, i = 1, \dots, T\} \subset \{Z_t^0 = 0\}$ a.s. for all t , $Z_0^0 = 1$ ” are missing from the statement of that theorem in Guasoni and Rásonyi [12] (but they are apparently needed in view of the preceding results there). Here in Theorem 6.2 we state the corrected version.

Proposition 6.3. *Let \mathcal{F}_0 be trivial, let Assumption 6.1 hold. If S is sticky, then it satisfies (NA2).*

Proof. We enlarge the probability space so that Assumption 2.5 holds. Lemma 3.2 of Guasoni and Rásonyi [12] shows that there is a constant C such that, for all t , $G_t^*(y) \leq C|y|^{\alpha/(\alpha-1)}$. Set $\delta := \max\{\gamma, \beta\alpha/(\alpha - \beta)\}$, $g(x) := x^\delta$, $x \geq 0$ and notice that $\alpha/(\alpha - 1) \leq \gamma$ hence, for any random variable X ,

$$E_Q G_t^*(X) \leq C E_Q |X|^{\alpha/(\alpha-1)} \leq C E_Q^{\alpha/[(\alpha-1)\delta]} [|X|^\delta]. \tag{25}$$

Fix $\chi > 0$. Theorem 4.1 provides $Q \sim P$ and a Q -martingale \tilde{S} with respect to \mathbb{H} such that

$$E_Q g\left(\sup_{t \in [0, T]} |S_t - \tilde{S}_t|\right) < \chi. \tag{26}$$

A closer look at the details of those arguments shows that Lemma 4.5 is used with the choice $w(x) = 4^{2\delta}|x|^{2\delta} + 2|x|$. Remark 4.6 then yields

$$E_Q \sup_{k \in \mathbb{N}} |M_k|^{2\delta} < \infty$$

hence also

$$E_Q g\left(\sup_{k \in \mathbb{N}} |M_k|\right) < \infty.$$

We thus get

$$\begin{aligned} E_Q g\left(\sup_{t \in [0, T]} |Y_t|\right) &\leq E_Q g\left(\sup_{k \in \mathbb{N}} |Y_{\tau_k}| + \varepsilon\right) \\ &= E_Q g\left(\sup_{k \in \mathbb{N}} |M_k| + \varepsilon\right) < \infty, \end{aligned} \tag{27}$$

noting convexity of g . Since $|Y_t - S_t| < \varepsilon$, this implies

$$E_Q g\left(\sup_{t \in [0, T]} |S_t|\right) < \infty, \tag{28}$$

and hence also

$$E_Q g\left(\sup_{t \in [0, T]} |\tilde{S}_t|\right) < \infty, \tag{29}$$

by (26). Noting $\beta\alpha/(\alpha - \beta) \leq \delta$ and (28),

$$E_Q \int_0^T (1 + |S_t|)^{\beta\alpha/(\alpha-\beta)} dt < \infty.$$

Define the \mathcal{Q} -martingale $Z_t^0 := 1$, $Z_t^i := \tilde{S}_t^i$, $i = 1, \dots, d$. Clearly, $Z_T \in L^\gamma(\mathcal{Q})$ by (29) and $\delta \geq \gamma$. We deduce from (25) and (26) that

$$\begin{aligned} E_{\mathcal{Q}} \int_0^T Z_t^0 G_t^*(\bar{Z}_t - S_t) dt &= \int_0^T E_{\mathcal{Q}} G_t^*(\tilde{S}_t - S_t) dt \\ &\leq T C E_{\mathcal{Q}}^{\alpha/[(\alpha-1)\delta]} \left[\sup_{t \in [0, T]} g(|\tilde{S}_t - S_t|) \right] \\ &\leq T C \chi^{\alpha/[(\alpha-1)\delta]}, \end{aligned}$$

which goes to 0 as $\chi \rightarrow 0$. This implies (NA2) for the class \mathcal{A} defined with \mathbb{H} -optional processes. As \mathbb{F} is a subfiltration of \mathbb{H} , the result follows for \mathcal{A} defined with \mathbb{F} -optional processes. This finishes the proof. \square

Remark 6.4. Property (NA2) was established in Guasoni and Rásonyi [12] for the class of continuous processes S satisfying the CFS- O property, see Remark 2.3. Using arguments of Bender et al. [2], one could establish (NA2) for continuous sticky processes S . The essential novelty of Proposition 6.3 thus lies in allowing jumps for S .

7. Auxiliary results

For the proof of Theorem 4.1, we need the two Lemmas presented below. We fix some notations first. Scalar products in \mathbb{R}^d will be denoted by $\langle \cdot, \cdot \rangle$. $\overline{\mathbb{R}^d}$ denotes the one-point compactification of \mathbb{R}^d and $C(\overline{\mathbb{R}^d})$ denotes the set of \mathbb{R} -valued continuous functions on $\overline{\mathbb{R}^d}$. We let $C_+(\overline{\mathbb{R}^d}) := \{g \in C(\overline{\mathbb{R}^d}) : g(x) > 0, x \in \mathbb{R}^d\}$. We denote by $C_0(\overline{\mathbb{R}^d})$ the family of continuous functions with compact support on \mathbb{R}^d . As $\overline{\mathbb{R}^d}$ is compact, $C(\overline{\mathbb{R}^d})$ (equipped with the supremum norm) is a separable Banach space, a fortiori a Polish space. As $C_+(\overline{\mathbb{R}^d})$ is clearly a Borel subspace of $C(\overline{\mathbb{R}^d})$, the measurable selection theorem (see, e.g., III. 44–45. in [7]) applies to multifunctions with values in $C_+(\overline{\mathbb{R}^d})$. Fix a continuous function $w : \mathbb{R}^d \rightarrow \mathbb{R}_+$ with $w(0) = 0$.

The next lemma will provide a positive function f that is used for a measure change with density $f(Y)$ in the arguments of Lemma 4.5. The idea here is that, due to (30) below (which will be a consequence of stickiness in our applications of Lemma 7.1), one can guarantee that the “mean” $EYf(Y)$ is 0 while the “norm” $Ef(Y)w(Y)$ stays small, together with the “mass” $Ef(Y)1_{\{|Y| \geq \eta\}}$ allocated outside a small ball.

Lemma 7.1. *Let Y be an \mathbb{R}^d -valued random variable with $0 \in \mathcal{S}(Y)$. Assume that*

$$P(Y \in B(0, \epsilon)) > 0 \tag{30}$$

for all $\epsilon > 0$. Then for each $\eta > 0$ there exists $f \in C_+(\overline{\mathbb{R}^d})$ such that $Ef(Y) = 1$, $Ef(Y)w(Y) < \eta$, $Ef(Y)1_{\{|Y| \geq \eta\}} < \eta$, and $Ef(Y)Y = 0$.

Proof. Define $\tilde{w}(x) := w(x) + |x|$ and note that it suffices to show the result for \tilde{w} instead of w . Since $\mathcal{S}(Y) \subset \mathcal{D}(Y)$, $\mathcal{D}(Y)$ is a nonempty linear subspace of \mathbb{R}^d . If $\mathcal{D}(Y) = \{0\}$ then we set $f(y) := 1$ for all $y \in \mathbb{R}^d$. From now on, we assume that $\mathcal{D}(Y)$ has dimension at least 1. Define

$$\mathcal{A} := \{r \in C_+(\overline{\mathbb{R}^d}) : Er(Y)\tilde{w}(Y) < \eta/2, Er(Y) < \eta\}.$$

Now set $A := \{Er(Y)Y : r \in \mathcal{A}\}$. Clearly, $A \subset \mathcal{D}(Y)$ is a convex and nonempty set. To see this, observe that for any $h \in C_0(\mathbb{R}^d)$, $h \geq 0$,

$$r(y) = \kappa_1 h(y) + \kappa_2 e^{-\tilde{w}(y)} \tag{31}$$

lies in \mathcal{A} for $\kappa_1, \kappa_2 > 0$ small enough.

Denoting by $ri_D(A)$ the interior of A in the relative topology of $\mathcal{D}(Y)$ we claim that $0 \in ri_D(A)$. If this were not true, then there would exist a non-zero $l \in \mathcal{D}(Y)$ such that $\langle l, a \rangle \geq 0$ for all $a \in A$. This implies

$$E\langle l, Y \rangle r(Y) \geq 0 \tag{32}$$

for all r of the form (31) (with κ_1, κ_2 small enough). We can let $\kappa_2 \rightarrow 0$ and obtain that (32) also holds for all $r(y) = \kappa_1 h(y)$ with $h \in C_0(\mathbb{R}^d)$, $h \geq 0$. This clearly implies that $\langle l, Y \rangle \geq 0$ a.s. Therefore from $0 \in \mathcal{S}(Y)$, we obtain that $\langle l, Y \rangle = 0$ a.s., by Theorem 3 in [16] so $\langle l, z \rangle = 0$ for all $z \in \mathcal{D}(Y)$. But $\langle l, l \rangle > 0$ and we arrive at a contradiction.

It follows that $B(0, \delta) \cap \mathcal{D}(Y) \subset A$ for some $0 < \delta < \eta$. We choose $\delta > 0$ small enough such that $\sup_{|y| \leq \delta} \tilde{w}(y) \leq \eta/2$. Let us now take $m \in C_0(\mathbb{R}^d)$ that is positive in the interior of $B(0, \delta)$, vanishes elsewhere, and satisfies $Em(Y) = 1$. Such a function exists since $P(Y \in B(0, \delta/2)) > 0$.

Set $c := Em(Y)Y \in B(0, \delta) \cap \mathcal{D}(Y)$. There exists $r \in \mathcal{A}$ with $Er(Y)Y = -c$, so setting $f(y) := (r(y) + m(y))/E[r(Y) + m(Y)]$ we have $Ef(Y)Y = 0$. Obviously, $Em(Y)\tilde{w}(Y) < \eta/2$ and $E[r(Y) + m(Y)] > 1$ hence $Ef(Y)\tilde{w}(Y) < \eta$, using the definition of \mathcal{A} . It remains to check that

$$Ef(Y)1_{\{|Y| \geq \eta\}} = Er(Y)1_{\{|Y| \geq \eta\}}/E[r(Y) + m(Y)] \leq Er(Y)1_{\{|Y| \geq \eta\}} < \eta,$$

which follows from $\delta < \eta$ and the definition of \mathcal{A} . □

Now consider a sigma-algebra $\mathcal{K} \subset \mathcal{F}$ and an \mathbb{R}^d -valued random variable X . Let $Q : \mathcal{B}(\mathbb{R}^d) \times \Omega \rightarrow [0, 1]$ be the conditional law of X with respect to \mathcal{K} . We denote by δ_0 the Dirac measure at the origin. The following lemma is a “kernel version” of Lemma 7.1 above.

Lemma 7.2. *Let $0 \in \mathcal{S}(Q(\cdot, \omega))$ for a.s. ω and let $Q(B(0, \epsilon), \omega) > 0$ hold a.s. for each $\epsilon > 0$. Then for each $\eta > 0$, there is a $\mathcal{B}(\mathbb{R}^d) \otimes \mathcal{K}$ -measurable $j : \mathbb{R}^d \times \Omega \rightarrow (0, \infty)$ such that for almost all ω the following holds:*

$$\int_{\mathbb{R}^d} j(z, \omega) Q(dz, \omega) = 1,$$

$$\int_{\mathbb{R}^d} j(z, \omega) z Q(dz, \omega) = 0,$$

$$\int_{\mathbb{R}^d} j(z, \omega) w(z) Q(dz, \omega) < \eta,$$

$$\int_{\mathbb{R}^d} j(z, \omega) 1_{\{|z| \geq \eta\}} Q(dz, \omega) < \eta.$$

Furthermore, we may choose $j(z, \omega) := 1, z \in \mathbb{R}^d$ on $\{\omega : Q(\omega, \cdot) = \delta_0(\cdot)\}$.

Proof. For a.e. ω , one can apply Lemma 7.1 to a random variable Y that has law $Q(\cdot, \omega)$ to get a function $f_\omega \in C_+(\overline{\mathbb{R}^d})$. We may apply the measurable selection theorem on (Ω, \mathcal{K}, P) to get a mapping $\omega \rightarrow f_\omega$ that is $\mathcal{K}/\mathcal{B}(C(\overline{\mathbb{R}^d}))$ -measurable. Let $j(z, \omega) := f_\omega(z)$ which is $\mathcal{B}(\mathbb{R}^d) \otimes \mathcal{K}$ -measurable (since each f_ω is continuous). This clearly satisfies the conclusions of the present lemma. The last statement is clear from the proof of Lemma 7.1. \square

Remark 7.3. The technology used in Lemmas 7.1 and 7.2 was initiated in Dalang et al. [5]. It has been further developed by Y. Kabanov in a continuous-time context and found several applications in mathematical finance. Here we only refer to Kabanov and Stricker [18] as a representative example.

Finally, we present a simple Lemma that was useful for the proof of Lemma 4.9 above.

Lemma 7.4. *Let \mathcal{A}, \mathcal{B} be sigma-fields and let U, V be nonnegative random variables such that $A \vee \sigma(U)$ is independent of $\mathcal{B} \vee \sigma(V)$. Then*

$$E[UV|\mathcal{A} \vee \mathcal{B}] = E[U|\mathcal{A}]E[V|\mathcal{B}].$$

Proof. Let A (resp. B, C, D) be $\sigma(U)$ (resp. $\sigma(V), \mathcal{A}, \mathcal{B}$) measurable sets. By the monotone class theorem and by the definition of conditional expectations, it suffices to prove that

$$E[1_A 1_B 1_C 1_D] = E[E[1_A|\mathcal{A}]E[1_B|\mathcal{B}]1_C 1_D]. \tag{33}$$

By independence and by the \mathcal{A} (resp. \mathcal{B}) measurability of C (resp. D), the right-hand side of (33) is

$$E[E[1_A 1_C|\mathcal{A}]E[E[1_B 1_D|\mathcal{B}]]] = E[1_A 1_C]E[1_B 1_D],$$

which equals the left-hand side of (33) by independence of $1_A 1_C$ from $1_B 1_D$. \square

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