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Fractional Brownian motion satisfies two-way crossing

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We prove the following result: For $(Z_t)_{t\in\mathbf{R}}$ a fractional Brownian motion with arbitrary Hurst parameter, for any stopping time τ, there exist arbitrarily small $\varepsilon > 0$ such that $Z_{\tau+\varepsilon} < Z_\tau$, with asymptotic behaviour when $\varepsilon \searrow 0$ satisfying a bound of iterated logarithm type. As a consequence, fractional Brownian motion satisfies the "two-way crossing" property, which has important applications in financial mathematics.

Keywords: fractional Brownian motion; law of the iterated logarithm; stopping time; two-way crossing

1. Introduction

1.1. Context

In this article, we consider a filtered probability space $(\Omega, \mathbb{P}, \mathcal{B}, (\mathcal{B}_t)_{t\in\mathbf{R}})$. Notation '$\omega$' will implicitly refer to eventualities of Ω; we will use it from time to time when needing to make the dependency on the random phenomenon perfectly clear. We consider a (bilateral) Brownian motion $(W_t)_{t\in\mathbf{R}}$ whose increments are adapted to our filtered space, which means, for all $t \in \mathbf{R}$, for all $u \leq 0$, $(W_{t+u} - W_t)$ is $\mathcal{B}_t$-measurable, while for all $v \geq 0$, $(W_{t+v} - W_t)$ is independent from $\mathcal{B}_t$.

We fix once for all some arbitrary parameter $H \in (0, 1)$ (so that, in the sequel, "absolute" constants may actually depend on H) such that $H \neq 1/2$; moreover, in all this article, $(H - 1/2)$ may be referred to as η. Then we consider the fractional Brownian motion (fBm) $(Z_t)_{t\in\mathbf{R}}$ driven by W with Hurst parameter H, which means that

$$Z_t := C_1 \int_{\mathbf{R}} \bigl((t-s)_+^\eta - (-s)_+^\eta\bigr)\, dW_s$$

(with the convention that $0^r = 0\ \forall r \in \mathbf{R}$), where

$$C_1 := \left(\frac{1}{2H} + \int_0^\infty \bigl((1+s)^\eta - s^\eta\bigr)^2\, ds\right)^{-1/2}.$$

Then, the properties of Z are well known: it is a centred Gaussian process whose increments are adapted to $(\mathcal{B}_t)_t$, with $\mathrm{Var}(Z_t - Z_s) = |t-s|^{2H}$ and $Z_0 = 0$ a.s.; its trajectories are locally $(H-\varepsilon)$-Hölder with divergence in $O(|t|^{H+\varepsilon})$ at infinity, etc. (see, e.g., [10], Chapter 2).

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Remark 1.1. The integral in the r.h.s. of (1.1) should not be seen as an actual Itô integral, but rather as a compact writing for some complicated deterministic integral, which corresponds to the "Itô" formulation via (formal) integrating by parts. Namely: (here in the case $t > 0$),

$$\begin{aligned} C_1^{-1} Z_t &= \int_{\mathbf{R}} \big((t-s)_+^\eta - (-s)_+^\eta\big)\, dW_s \\ &= \int_{-\infty}^0 \big((t-s)^\eta - (-s)^\eta\big)\, dW_s + \int_0^t (t-s)^\eta\, dW_s \\ &= \big[\big((t-s)^\eta - (-s)^\eta\big) W_s\big]_{s=-\infty}^0 - \eta \int_{-\infty}^0 \big((t-s)^{\eta-1} - (-s)^{\eta-1}\big) W_s\, ds \\ &\quad + \big[(t-s)^\eta (W_s - W_t)\big]_{s=0}^t - \eta \int_0^t (t-s)^{\eta-1} (W_s - W_t)\, ds \\ &= t^\eta W_t - \eta \int_{-\infty}^t \big((t-s)^{\eta-1} - (-s)_+^{\eta-1}\big)(W_s - \mathbf{1}_{s>0} W_t)\, ds, \end{aligned} \tag{1.1}$$

where all the computations are licit (with absolutely converging integrals) because of the properties of regularity and slow divergence of the (ordinary) Brownian motion.

Remark 1.2. It has to be stressed that in all this article, actually we are not interested in the values themselves of the processes W and Z, but rather in their *increments*. This way, the fact that $W_0, Z_0 = 0$ should be considered as a mere convention, completely unessential though convenient.

1.2. Main result

The main result of this article is the following theorem.

Theorem 1.1. *In the context of Section 1.1, for τ any stopping time adapted to $(\mathcal{B}_t)_t$, one has almost-surely*:

$$\liminf_{v \searrow 0} \frac{Z_{\tau+v} - Z_\tau}{(\log|\log v|)^{1/2} v^H} \leq -C_1 H^{-1/2}.$$

Remark 1.3. What is the intuitive meaning of Theorem 1.1? Well, we know that fractional Brownian motion does not have Markov nor martingale property, which means that the future trajectory of the fBm after a stopping time may differ qualitatively from what it would be at a fixed time. Our theorem says however that future trajectory cannot differ too wildly from its ordinary behaviour, in the sense that some weak form of the law of the iterated logarithm (LIL) always has to remain true.

My motivation for proving Theorem 1.1 was its application to financial mathematics, which will be explained in Section 6; but the theorem may also be interesting even on its own right.

Remark 1.4. It is interesting to compare (1.1) with the standard LIL for the fBm, which states (cf. [1], Corollary 3.1) that in the case $\tau \equiv 0$:

$$\liminf_{v \searrow 0} \frac{Z_v - Z_0}{(\log|\log v|)^{1/2} v^H} = -\sqrt{2}.$$

Indeed, as soon as $H \neq 1/2$, one has $C_1 H^{-1/2} < \sqrt{2}$; therefore, provided the r.h.s. of (1.1) is sharp, Theorem 1.1 does allow for a deviation from the standard LIL behaviour of the fBm after a stopping time – though not a too wild one, as already said. I conjecture that the r.h.s. of (1.1) is sharp indeed, and that it would be attained for τ being a hitting time, from above for $H \geq 1/2$, resp. from below for $H \leq 1/2$ – for instance, for $H \geq 1/2$, I believe that the following stopping time would fit equality in equation (1.1):

$$\tau := \inf\{t \geq 0 | Z_t = Z_0 + 1\}.$$

Remark 1.5. As the increments of Z are adapted to the filtration $(\mathcal{B}_t)_t$, obviously in Theorem 1.1 we may replace that filtration by the filtration generated by the increments of Z.

Remark 1.6. In this article, we are only considering the case $H \neq 1/2$, but Theorem 1.1 is trivially valid for $H = 1/2$ too, since then the fBm Z is nothing but the ordinary Brownian motion W itself, for which the result follows immediately from the Markov property and the local properties of oBm.

1.3. Outline of the paper

Sections 2–5 will be devoted to proving Theorem 1.1. In Section 2, we will see how one can get rid of the notion of stopping time to get Theorem 1.1 back to a result on the trajectories of the fractional Brownian motion. In Section 3, we will make the needed result on fBm's trajectories more precise, by establishing a kind of law of the iterated logarithm for some variant of the fBm. Next, an issue will be that we have to control the probability of an event being a union over a continuous infinity of t's: that issue will be handled by Section 4, in which we will use regularity estimates on the fBm to get our continuous union back to a finite union. Finally, after all these simplifications it will only remain to prove some estimates on Gaussian vectors, which will be the work of Section 5.

Some technical results will be postponed to Appendices. In particular, in Appendix A we will compute the precise expression of the "drift operator" appearing in Lemma 2.1 describing what the law of the fBm becomes when you condition it by a stopping time: this formula, though not actually required to prove our main result, looks indeed intrinsically worthy to be written down to my eyes. Also, in Appendices B and C we will prove two lemmas on resp. the supremum of Gaussian processes and the inverse of nearly diagonal matrices.

Finally in Section 6 we will explain how Theorem 1.1 can be applied to yield some new results in financial mathematics.

2. Conditional future of the fractional Brownian motion

2.1. Preliminary definitions

To begin with, it will be convenient to set some notation for certain sets of trajectories:

Definition 2.1 (Sets $\mathcal{P}$ and $\mathcal{F}$).

- *We denote by $\mathcal{P}$ [like "past"] the set of the (deterministic) paths $(X_u)_{u\leq 0}$ such that:*
  1. $X_0 = 0$;
  2. *X is locally $(H-\varepsilon)$-Hölder for all $\varepsilon > 0$;*
  3. $X_u \overset{u\to-\infty}{=} O(|u|^{H+\varepsilon})$ *for all $\varepsilon > 0$.*
- *Similarly, we denote by $\mathcal{F}$ [like "future"] the space of the paths $(X_v)_{v\geq 0}$ satisfying the analogues of conditions 1–3 for non-negative times.*

Remark 2.1. With the notation of Definition 2.1, one has almost-surely that, for all $t \in \mathbf{R}$, $(Z(\omega)_{t+u} - Z(\omega)_t)_{u\leq 0} \in \mathcal{P}$ and $(Z(\omega)_{t+v} - Z(\omega)_t)_{v\geq 0} \in \mathcal{F}$ (cf. [10], Propositions 1.6 and 2.2.3).

We also define a certain "drift operator":

Definition 2.2 ("Drift operator" D). *Let $\mathbf{D}\colon \mathcal{P} \to \mathcal{F}$ be the linear operator such that, for $X \in \mathcal{P}$:*

$$(\mathbf{D}X)_v := \int_{-\infty}^{0} K(u,v) X_u \, du,$$

where

$$K(u,v) := \frac{\eta}{\Pi(\eta)\Pi(-\eta)} \bigg\{ \eta \int_{-\infty}^{0} \big(\mathbf{1}_{s>u} \xi_{\eta-1}(s-u, v) \xi_{-\eta-1}(-s, s-u) - \xi_{\eta-1}(-u, v) \xi_{-\eta-1}(-s, -u) \big) \, ds - v(v-u)^{\eta-1}(-u)^{-\eta-1} \bigg\}, \tag{2.1}$$

where $\Pi(\cdot)$ is Euler's pi function extrapolating the factorial, and where we denote, for $r \in \mathbf{R}$, $a, b > 0$:

$$\xi_r(a,b) := (a+b)^r - a^r.$$

Remark 2.2. Note that, since $H \in (0,1)$, the integrals in (2.1) and (2.2) do converge (absolutely) indeed, and $\mathbf{D}$ is well-defined on the whole $\mathcal{P}$ with values in $\mathcal{F}$.

Remark 2.3. The equations (2.1)–(2.2) defining $\mathbf{D}$, though interesting as such, shall not play an essential role in this article. What is really important to have in mind is the moral *meaning* of this operator: actually $\mathbf{D}$ was defined so that, informally,

$$(\mathbf{D}X)_v = \mathbb{E}\big(Z_v \,\big|\, (Z_u)_{u\leq 0} = (X_u)_{u\leq 0} \big).$$

The formal meaning of (2.3) will be made clear by Lemma 2.1 below.

Finally, we define a process called the "Lévy fractional Brownian motion", which is a kind of unilateral version of the "classic" fBm.

Definition 2.3 (Lévy fBm). *If* $(W_v)_{v\geq 0}$ *is a* (*unilateral*) *ordinary Brownian motion*, *then the process* $(Y_v)_{v\geq 0}$ *defined by*

$$Y_v := C_1 \int_0^v (v-s)^\eta \, dW_s$$

(*interpreted* via *the same integration by parts trick as in* (1.1)) (*and where we recall that* C_1 *is defined by* (1.1)) *is called a* Lévy fractional Brownian motion (*with Hurst parameter* H) *– or, more accurately, the law of this process* (*which* (2.3) *defines without ambiguity*) *is called "the law of the Lévy fBm"*.

Remark 2.4. From the regularity properties of the oBm, it is easy to check that the trajectories of the Lévy fBm lie in $\mathcal{F}$ a.s.

2.2. Conditioning lemma

Now we can state the key lemma of this section.

Lemma 2.1. *In the context of Section* 1.1, *for* τ *a stopping time*, $((Z_{\tau+v}-Z_\tau)_{v\geq 0} - \mathbf{D}((Z_{\tau+u} - Z_\tau)_{u\leq 0}))$ *is independent of* $\mathcal{B}_\tau$, *and its law is the Lévy fBm.*

Remark 2.5. In other words, Lemma 2.1 states that, conditionally to $\mathcal{B}_\tau$ (or, morally, knowing the past trajectory of Z until τ), the law of the future trajectory of Z is equal to a "deterministic" *drift term* $\mathbf{D}((Z_{\tau+u}-Z_\tau)_{u\leq 0})$ plus a "random" *noise term* being a Lévy fBm.

Proof of Lemma 2.1. As the increments of W are adapted to $(\mathcal{B}_t)_t$, conditionally to $\mathcal{B}_\tau$, the past trajectory $(W_{\tau+u}-W_\tau)_{u\leq 0}$ of (the increments of) W is deterministic, while its future trajectory $(W_{\tau+v}-W_\tau)_{v\geq 0}$ still has the unconditioned law of a standard oBm. Therefore, for $t\geq 0$, we split

$$\begin{aligned} Z_{\tau+t}-Z_\tau &\underset{(1.1)}{=} C_1 \int_{s\in\mathbf{R}} \big((\tau+t-s)_+^\eta - (\tau-s)_+^\eta\big)\, dW_s \\ &\underset{s\leftarrow s-\tau}{=} C_1 \int_{s\in\mathbf{R}} \big((t-s)_+^\eta - (-s)_+^\eta\big)\, dW_{\tau+s} \\ &= C_1 \int_{u=-\infty}^{0} \big((t-u)^\eta - (-u)^\eta\big)\, dW_{\tau+u} \\ &\quad + C_1 \int_{v=0}^{t} (t-v)^\eta \, dW_{\tau+v}, \end{aligned} \tag{2.2}$$

in which the first term is deterministic and given by some function of $(W_{\tau+u} - W_\tau)_{u\le 0}$, while the second term (seen as a trajectory indexed by t) has the law of the Lévy fBm indeed.

To end the proof, it remains to show that the aforementioned first term (seen as a trajectory indexed by t) is equal to $\mathbf{D}((Z_{\tau+u} - Z_\tau)_{u\le 0})$ indeed. Since this point is actually not needed to prove our main result, we will postpone it to Appendix A. □

2.3. Reformulation of the main theorem

Thanks to Lemma 2.1, we will be able to get a sufficient condition for Theorem 1.1 in which there are no stopping times any more. But first we observe that proving Theorem 1.1 is tantamount to proving that for any $\Lambda < C_1 H^{-1/2}$, for any stopping time τ:

$$\mathbb{P}\left(\liminf_{v\searrow 0} \frac{Z_{\tau+v} - Z_\tau}{(\log|\log v|)^{1/2} v^H} \ge -\Lambda\right) = 0.$$

When the event of the probability in equation (2.3) holds true, we will say that $\tau(\omega)$ is a Λ-*slow time* for the trajectory $(Z(\omega)_t)_{t\in\mathbf{R}}$.

☛ *From now on, we fix an arbitrary $\Lambda > -C_1 H^{-1/2}$ all along.*

Now we need an *ad hoc* definition:

Definition 2.4 (Nasty path). *We say that a deterministic path $(X_u)_{u\le 0} \in \mathcal{P}$ is* nasty, *and we denote "$X \in \mathcal{A}$", when, for Y a Lévy fBm*:

$$\mathbb{P}\big(0 \textit{ is a } \Lambda\textit{-slow time for } (\mathbf{D}X + Y(\omega))\big) > 0.$$

Remark 2.6. $\mathcal{A}$ is a measurable subset of $\mathcal{P}$,[1] as can be checked along the following lines:

- The drift operator $\mathbf{D}\colon \mathcal{P} \to \mathcal{F}$ is measurable. (This is obvious from its characterization (2.2) as an explicit kernel operator).
- Denoting $[\![n[\![:= \{0, \dots, n-1\}$ and $\vec{X}_I := (X_i)_{i\in I}$, for all $n \in \mathbf{N}$, for all $\vec{v}_{[\![n[\![} \in (\mathbf{R}_+)^{[\![n[\![}$, $\mathbb{P}(\vec{Y}_{\vec{v}_{[\![n[\![}} \in \prod_{i\in[\![n[\![} [a_i, b_i])$ is a measurable function of $(\vec{a}_{[\![n[\![}, \vec{b}_{[\![n[\![})$. (This is because $\vec{Y}_{\vec{v}_{[\![n[\![}}$ is a Gaussian random vector with known parameters.)
- Therefore, for all $n \in \mathbf{N}$, for all $\vec{v}_{[\![n[\![} \in (\mathbf{R}_+)^{[\![n[\![}$, for all $\vec{a}_{[\![n[\![}, \vec{b}_{[\![n[\![} \in \mathbf{R}^{[\![n[\![}$, the probabilities

  $$\mathbb{P}\Big(\forall i \in [\![n[\![\ \big(\mathbf{D}X + Y(\omega)\big)_{v_i} \in [a_i, b_i]\Big)$$

  are measurable functions of X; by Dynkin's π-λ theorem, it follows that for all measurable $A \subset \mathcal{F}$, $\mathbb{P}(\mathbf{D}X + Y(\omega) \in A)$ is a measurable function of X.

[1]The σ-algebra considered on $\mathcal{P}$ (resp. on $\mathcal{F}$) is obviously (the trace of) the product σ-algebra – a trajectory in $\mathcal{P}$ (resp. $\mathcal{F}$) being seen as a function from $\mathbf{R}_-$ (resp. $\mathbf{R}_+$) to $\mathbf{R}$.

- But "0's being a Λ-slow time" is a measurable subset of $\mathcal{F}$ (because the trajectories of $\mathcal{F}$ are continuous, so they can be described fully from a countable set of time indices), so the probability in the l.h.s. of (2.4) is a measurable function of X, which finally implies that $\mathcal{A}$ is measurable.

That vocabulary being set, equation (2.3), and hence Theorem 1.1, will be a consequence of the following.

Proposition 2.2. *In the context of Section* 1.1:

$$\mathbb{P}\big(\exists t \in \mathbf{R}\ \big(Z(\omega)_{t+u} - Z(\omega)_t\big)_{u\le 0} \in \mathcal{A}\big) = 0.$$

Proof of equation (2.3) **from Proposition 2.2.** Let τ be any stopping time. We introduce the notation $\mathbb{P}'$ for the law of the Lévy fBm Y, and the probability $\mathbb{P}'_{\tau,\omega}$ defined as the pushforward measure of $\mathbb{P}'$ by the map $(Y_v)_{v\ge 0} \mapsto (X_t)_{t\in\mathbf{R}}$ from $\mathcal{F}$ to $\mathcal{C}(\mathbf{R},\mathbf{R})$ defined by:

$$X_t = \begin{cases} Z_t, & t \le \tau(\omega); \\ Z_{\tau(\omega)} + \mathbf{D}\big(\big(Z(\omega)_{\tau(\omega)+u} - Z(\omega)_{\tau(\omega)}\big)_{u\le 0}\big) + Y_{t-\tau(\omega)}, & t > \tau(\omega), \end{cases}$$

so that, by Lemma 2.1, $(\mathbb{P}'_{\tau,\omega})_{\omega\in\Omega}$ is the conditional probability of $\mathbb{P}$ given $\mathcal{B}_\tau$. (That conditional probability is regular, by the same arguments as in Remark 2.6.) Then, writing the law of total probability w.r.t. $\mathcal{B}_\tau$:[2]

$$\begin{aligned} &\mathbb{P}(\tau \text{ is a } \Lambda\text{-slow time for } Z) \\ &\quad = \mathbb{E}\big(\mathbb{P}'_{\tau,\omega}\big(\tau(\omega) \text{ is } \Lambda\text{-slow time for } Z(\omega')\big)\big) \\ &\quad = \mathbb{E}\big(\mathbb{P}'\big(0 \text{ is a } \Lambda\text{-slow time for } \big(\mathbf{D}\big(Z(\omega)_{\tau(\omega)+u} - Z(\omega)_{\tau(\omega)}\big)_{u\le 0} + Y(\omega')\big)\big)\big) \\ &\quad = \mathbb{E}\big(0 \text{ whenever } (Z_{\tau+u} - Z_\tau)_{u\le 0} \notin \mathcal{A}\big), \end{aligned} \tag{2.3}$$

which is zero by Proposition 2.2. □

So, in the sequel, our new goal will be to prove Proposition 2.2.

3. Local behaviour of fBm's trajectories

3.1. A law of the iterated logarithm for the Lévy fBm

☛ *In all this article, we denote* $[\![n[\![:= \mathbf{N} \cap [0,n) = \{0,1,2,\dots,n-1\}$. *A subset* $\mathcal{I} \subset \mathbf{N}$ *will be said to be* thick *when it has positive upper asymptotic density:*

$$\limsup_{n\to\infty} \frac{|\mathcal{I} \cap [\![n[\![|}{n} > 0.$$

[2] In the following computation, to avoid confusions, I denoted by 'ω' the variable bound to operator $\mathbb{E}$, and by 'ω'' the variable bound to probabilities $\mathbb{P}'$ and $\mathbb{P}'_{\tau,\omega}$.

The first main result of this section is the following lemma.

Lemma 3.1. *Let $r \in (0,1)$ and let $\mathcal{I}$ be a thick subset of* $\mathbf{N}$; *then, for Y a Lévy fBm*:

$$\liminf_{\substack{i\in\mathcal{I}\\ i\to\infty}} \frac{Y_{r^i}}{(\log i)^{1/2} r^{Hi}} \leq -C_1 H^{-1/2} \qquad a.s.$$

Remark 3.1. From Lemma 3.1, one can deduce immediately that

$$\liminf_{v\searrow 0} \{Y_v \,/\, (\log|\log v|)^{1/2} v^H\} \leq -C_1 H^{-1/2},$$

which is the classic formulation of (one sense of) a law of the iterated logarithm, hence the title of this subsection. Note by the way that one could also prove that there is actually equality in (3.1) as well as in the LIL derived from it; but that will not be needed here.

Proof of Lemma 3.1. It will be convenient in this proof to assume that Y is driven by some oBm W according to (2.3). Then, for $v \geq 0$, let us define

$$\tilde{Y}_v := C_1 \int_{rv}^{v} (v-s)^\eta \, dW_s,$$

resp.

$$Y'_v := Y_v - \tilde{Y}_v = C_1 \int_0^{rv} (v-s)^\eta \, dW_s.$$

First let us study the $\tilde{Y}_{r^i}$'s. Obviously these random variables are independent, with $Y_{r^i} / r^{Hi} \sim \mathcal{N}(0, C_1^2(1-r)^{2H}/2H)$ $\forall i$. Now, using that $\mathbb{P}(\mathcal{N}(0,1) \leq -x) \geq e^{-x^2/2}/2\sqrt{2\pi}x$ for $x \geq 1$,[3] we get that for i large enough: (having fixed some arbitrary small $\varepsilon \in (0,1)$),

$$\begin{aligned} &\mathbb{P}\big(\tilde{Y}_{r^i} \,/\, (\log i)^{1/2} r^{Hi} \leq -(1-\varepsilon) C_1 (1-r)^H H^{-1/2}\big) \\ &\quad = \mathbb{P}\big(\mathcal{N}(0,1) \leq -(1-\varepsilon)\sqrt{2}(\log i)^{1/2}\big) \\ &\quad \geq i^{-(1-\varepsilon)^2} \,/\, (1-\varepsilon) 4\sqrt{\pi} (\log i)^{1/2} \overset{i\to\infty}{=} \Omega\big(i^{-1}\big), \end{aligned} \tag{3.1}$$

where "$f(i) = \Omega(g(i))$" means that $g(i) = O(f(i))$. As $\mathcal{I}$ is thick, the series $\sum_{i\in\mathcal{I}} i^{-1}$ is divergent, thus so is

$$\sum_{i\in\mathcal{I}} \mathbb{P}\left(\frac{\tilde{Y}_{r^i}}{(\log i)^{1/2} r^{Hi}} \leq -(1-\varepsilon) C_1 (1-r)^H H^{-1/2}\right).$$

[3]This is because of convexity of the density $y \mapsto \varphi(y) := e^{-y^2/2} / \sqrt{2\pi}$ on $(-\infty, -1]$: from this property you deduce that $\int_{-\infty}^{-x} \varphi(y)\,dy \geq \varphi(-x)^2/2\varphi'(-x) = \varphi(-x)/2x$.

Since the events concerning the different $\tilde{Y}_{r^i}$'s are independent, it follows by the (second) Borel–Cantelli lemma that almost-surely there are infinitely many $i \in \mathcal{I}$ for which $\tilde{Y}_{r^i} \,/\, (\log i)^{1/2} r^{Hi} \leq -(1-\varepsilon) C_1 (1-r)^H H^{-1/2}$, so that:

$$\liminf_{\substack{i \in \mathcal{I} \\ i \to \infty}} \frac{\tilde{Y}_{r^i}}{(\log i)^{1/2} r^{Hi}} \leq -(1-\varepsilon) C_1 (1-r)^H H^{-1/2} \qquad \text{a.s.,}$$

in which the factor $(1-\varepsilon)$ can be removed by letting $\varepsilon \to 0$.

Now let us handle the Y'_{r^i}'s. One has $Y'_{r^i} \,/\, r^{Hi} \sim \mathcal{N}(0, C_1^2(1-(1-r)^{2H})/2H)$; therefore, using that $\mathbb{P}(\mathcal{N}(0,1) \geq x) \leq e^{-x^2/2}$ for all x,[4] we get that: (having fixed some arbitrary small $\varepsilon > 0$),

$$\begin{aligned} &\mathbb{P}\bigl(Y'_{r^i} \,/\, (\log i)^{1/2} r^{Hi} \geq (1+\varepsilon) C_1 \bigl(1-(1-r)^{2H}\bigr)^{1/2} H^{-1/2}\bigr) \\ &\quad = \mathbb{P}\bigl(\mathcal{N}(0,1) \geq (1+\varepsilon)\sqrt{2}(\log i)^{1/2}\bigr) \leq i^{-(1+\varepsilon)^2}. \end{aligned} \tag{3.2}$$

The series $\sum_{i \in \mathcal{I}} i^{-(1+\varepsilon)^2}$ is convergent since $\sum_{i \in \mathbf{N}} i^{-(1+\varepsilon)^2}$ is, thus so is

$$\sum_{i \in \mathcal{I}} \mathbb{P}\biggl(\frac{Y'_{r^i}}{(\log i)^{1/2} r^{Hi}} \geq (1+\varepsilon) C_1 \bigl(1-(1-r)^{2H}\bigr)^{1/2} H^{-1/2}\biggr).$$

It follows by the (first) Borel–Cantelli lemma that almost-surely there are only finitely many i's for which $\tilde{Y}_{r^i} \,/\, (\log i)^{1/2} r^{Hi} \geq (1+\varepsilon) C_1 (1-(1-r)^{2H})^{1/2} H^{-1/2}$, so that:

$$\limsup_{\substack{i \in \mathcal{I} \\ i \to \infty}} \frac{Y'_{r^i}}{(\log i)^{1/2} r^{Hi}} \leq (1+\varepsilon) C_1 \bigl(1-(1-r)^{2H}\bigr)^{1/2} H^{-1/2} \qquad \text{a.s.,}$$

in which the factor $(1+\varepsilon)$ can be removed by letting $\varepsilon \to 0$.

Summing (3.1) and (3.1), we get an intermediate result.

Proposition 3.2. *Under the assumptions of Lemma* 3.1, *almost-surely*:

$$\liminf_{i \in \mathcal{I}} \frac{Y_{r^i}}{(\log i)^{1/2} r^{Hi}} \leq -\lambda(r),$$

where

$$\lambda(r) := C_1 \bigl((1-r)^H - \bigl(1-(1-r)^{2H}\bigr)^{1/2}\bigr) H^{-1/2}.$$

So, now it remains to improve the constant $\lambda(r)$ in (3.2) into $C_1 H^{-1/2}$. For this, we begin with observing that the Lévy fBm is scale-invariant with exponent H (by which I mean that for $a \in \mathbf{R}_+^*$, $(Y_{av} \,/\, a^H)_{v \geq 0}$ is also a Lévy fBm); therefore, Proposition 3.2 has the following corollary.

[4]This is because $\mathbb{E}(e^{x\mathcal{N}(0,1)}) = e^{x^2/2}$, from which the claimed formula follows by Markov's inequality.

Corollary 3.3. *Under the assumptions of Lemma 3.1, for $a \in \mathbf{R}_+^*$, one has almost-surely*:

$$\liminf_{i\in\mathcal{I}} \frac{Y_{ar^i}}{a^H(\log i)^{1/2}r^{Hi}} \le -\lambda(r).$$

Now let $k > 1$ be an arbitrary large integer; and take $l \in [\![k[\![$ such that $\mathcal{J} := \{j \in \mathbf{N} | kj + l \in \mathcal{I}\}$ is thick – such an l exists since $\mathcal{I}$ itself is thick. Then one has:

$$\begin{aligned}\liminf_{i\in\mathcal{I}} \frac{Y_{r^i}}{(\log i)^{1/2}r^{Hi}} &\le \liminf_{j\in\mathcal{J}} \frac{Y_{r^{kj+l}}}{(\log(kj+l))^{1/2}r^{H(kj+l)}} \\ &= \liminf_{j\in\mathcal{J}} \frac{Y_{r^l(r^k)^j}}{(r^l)^H(\log j)^{1/2}(r^k)^{Hj}},\end{aligned} \tag{3.3}$$

where in the last equality we used that $(\log(kj+l))^{1/2} \overset{j\to\infty}{\sim} (\log j)^{1/2}$. But, applying Corollary 3.3 with 'r' $= r^k$, 'a' $= r^l$ and '$\mathcal{I}$' $= \mathcal{J}$, the r.h.s. of (3.3) is bounded above by $-\lambda(r^k)$; so,

$$\liminf_{i\in\mathcal{I}} \frac{Y_{r^i}}{(\log i)^{1/2}r^{Hi}} \le -\lambda\bigl(r^k\bigr).$$

Letting k tend to infinity, $\lambda(r^k)$ tends to $C_1H^{-1/2}$, which finally proves (3.1). □

3.2. Nastiness condition as a limit

For all the sequel of this article, we fix some $r \in (0,1)$ small enough (in a sense to be made precise later); we also fix arbitrarily two parameters $\alpha \in (0, C_1H^{-1/2} - \Lambda)$ and $p \in (0,1)$. Then we define, for all $n > 0$:

$$\mathcal{A}_n := \Bigl\{(X_u)_{u\le 0} \in \mathcal{P} \Bigm| \operatorname{card}\Bigl\{i \in [\![n[\![\Bigm| (\mathbf{D}X)_{r^i} \ge \alpha(\log i)_+^{1/2} r^{Hi}\Bigr\} \ge pn\Bigr\}.$$

Then we have the following connection between $\mathcal{A}$ and the $\mathcal{A}_n$'s:

Lemma 3.4.

$$\mathcal{A} \subset \liminf_{n\in\mathbf{N}} \mathcal{A}_n.$$

Proof. We prove the contrapositive inclusion. Let $(X_u)_{u\le 0} \in \mathcal{P}$ be such that $X \notin \liminf\{\mathcal{A}_n\}$, that is, the set $\{n | X \notin \mathcal{A}_n\}$ is unbounded; and set

$$\mathcal{I} := \Bigl\{i \in \mathbf{N} \Bigm| (\mathbf{D}X)_{r^i} < \alpha(\log i)_+^{1/2} r^{Hi}\Bigr\}.$$

One has by definition that $|\mathcal{I} \cap [\![n[\![| / n \ge 1 - p$ for all n such that $X \notin \mathcal{A}_n$; as these n are unbounded and $1 - p > 0$, it follows that $\mathcal{I}$ is thick. Therefore, Lemma 3.1 gives that for almost-all Lévy fBm $Y(\omega)$, one has that

$$\liminf_{i\in\mathcal{I}} \frac{Y(\omega)_{r^i}}{(\log i)^{1/2}r^{Hi}} \le -C_1H^{-1/2}.$$

On the other hand, the definition of $\mathcal{I}$ obviously implies that

$$\liminf_{i\in\mathcal{I}} \frac{(\mathbf{D}X)_{r^i}}{r^{Hi}(\log i)^{1/2}} \leq \alpha.$$

Summing (3.2) and (3.2), it follows that almost-surely:

$$\liminf_{i\in\mathcal{I}} \frac{(\mathbf{D}X + Y(\omega))_{r^i}}{(\log i)^{1/2} r^{Hi}} \leq \alpha - C_1 H^{-1/2} < -\Lambda,$$

so that $X \notin \mathcal{A}$. □

3.3. Second reformulation of the main theorem

Thanks to the work of this section, we are now able to show that the following result will be a sufficient condition for Proposition 2.2.

Proposition 3.5. *In the context of Section* 1.1:

$$\mathbb{P}\big(\exists t \in [0,1]\ \big(Z(\omega)_{t+u} - Z(\omega)_t\big)_{u\leq 0} \in \mathcal{A}_n\big) \overset{n\to\infty}{\to} 0.$$

Proof of Proposition 2.2 from Proposition 3.5. Proposition 3.4 implies that

$$\begin{aligned} &\big\{\omega \in \Omega \big| \exists t \in [0,1]\ \big(Z(\omega)_{t+u} - Z(\omega)_t\big)_{u\leq 0} \in \mathcal{A}\big\} \\ &\quad \subset \liminf_{n\to\infty} \big\{\omega \big| \exists t \in [0,1]\ \big(Z(\omega)_{t+u} - Z(\omega)_t\big)_{u\leq 0} \in \mathcal{A}_n\big\}; \end{aligned} \tag{3.4}$$

therefore, by the (first) Borel–Cantelli lemma, Proposition 3.5 yields that

$$\mathbb{P}\big(\exists t \in [0,1]\ \big(Z(\omega)_{t+u} - Z(\omega)_t\big)_{u\leq 0} \in \mathcal{A}\big) = 0.$$

But, since the increments of the fractional Brownian motion are stationary, in (3.3) we may replace $[0,1]$ by $[n, n+1]$ for all $n \in \mathbf{Z}$; and then, by countable union, we get the wished result (2.2). □

So, in the sequel, our new goal will be to prove Proposition 3.5.

4. Pathwise control via pointwise control

4.1. A regularity result

One of our issues to prove Proposition 3.5 is that we have to bound the probability of an event defined as a union for uncountably infinitely many t's. To overcome this issue, we will need a tool to "get rid of the trajectorial aspects" of the problem: this is the work of this section.

First, we need a little notation.

Definition 4.1 (Processes $\hat{\Gamma}_i$ and variables Γ_i). *Within the context of Section 1.1, for $i \in \mathbf{N}$, we define the following random process (indexed by $t \in \mathbf{R}$):*

$$\hat{\Gamma}_i(\omega)_t := \frac{\mathbf{D}((Z(\omega)_{t+u} - Z(\omega)_t)_{u \le 0})_{r^i}}{r^{Hi}}.$$

We also define the following random variable:

$$\Gamma_i(\omega) := \hat{\Gamma}_i(\omega)_0 = \frac{\mathbf{D}((Z(\omega)_u)_{u \le 0})_{r^i}}{r^{Hi}}.$$

Then, the main result of this section is the following.

Lemma 4.1. *In the context of this section, there exist absolute*[5] *constants $C_{\mathrm{a}} > 0, C_{\mathrm{b}} < \infty$ (whose exact expressions do not matter) such that for all $i \in \mathbf{N}$, for all $T > 0$:*

$$\mathbb{P}\big(\exists t \in [0, T]\ \big|(\hat{\Gamma}_i)_t - \Gamma_i\big| \ge 1\big) \le C_{\mathrm{b}} \exp\big(-C_{\mathrm{a}}\big(r^i / T\big)^{2H \wedge 1}\big).$$

Proof. First, since Z is scale-invariant with exponent H (and operator $\mathbf{D}$ preserves that scale invariance), it will be enough to prove Lemma 4.1 for $i = 0$; so we will only handle that case. Then the subscript i becomes useless, so we remove it in our notation.

Because of the characterization (2.3) of $\mathbf{D}$, $\hat{\Gamma}_t$ may be written as a function of W:

$$\hat{\Gamma}_t = C_1 \int_{-\infty}^{t} \big((t + 1 - s)^\eta - (t - s)^\eta\big)\, dW_s.$$

That shows that $\hat{\Gamma}$ is a stationary centred Gaussian random process, with

$$\begin{aligned} \operatorname{Var}(\hat{\Gamma}_t - \hat{\Gamma}_0) &= C_1^2 \int_{\mathbf{R}} \big(\mathbf{1}_{s \le t}(t + 1 - s)^\eta - \mathbf{1}_{s \le 0}(1 - s)^\eta - (t - s)_+^\eta + (-s)_+^\eta\big)^2\, ds \\ &\overset{t \to 0}{=} O\big(t^{2H \wedge 1}\big). \end{aligned} \tag{4.1}$$

To go further, we need the following lemma, whose proof is postponed to Appendix B:

Lemma 4.2. *Let $(X_t)_{t \in [0,1]}$ be a centred Gaussian process such that $X_0 = 0$ a.s. and*

$$\forall t, s \in [0, 1] \qquad \operatorname{Var}(X_t - X_s) \le |t - s|^{2\theta}$$

for some $\theta \in (0, 1]$. In such a case it is known that X has a continuous version (by the Kolmogorov theorem) and that, for this continuous version, the random variable $\|X\|(\omega) := \sup_{t \in [0,1]} |X(\omega)_t|$ is sub-Gaussian (by the Fernique theorem), that is, there exist constants $C_{\mathrm{c}} > 0, C_{\mathrm{d}} < \infty$ such that

$$\forall x \ge 0 \qquad \mathbb{P}\big(\|X\| \ge x\big) \le C_{\mathrm{d}} \exp\big(-C_{\mathrm{c}} x^2\big).$$

[5] Remember that in this article, "absolute" constants may actually depend on H.

The present lemma states that, moreover, the constants C_c and C_d can be made explicit, only depending on θ.

We apply Lemma 4.2 in the following way. From (4.1), one has $\mathrm{Var}(\hat{\Gamma}_t - \hat{\Gamma}_s) \leq C_\mathrm{e}|t-s|^{2H\wedge 1}$ for all $t, s \in [0,1]$, for some $C_\mathrm{e} < \infty$. Therefore, for $T \leq 1$, the random process

$$X(\omega)_t := \big(C_\mathrm{e} T^{2H\wedge 1}\big)^{-1/2}\big(\hat{\Gamma}(\omega)_{tT} - \Gamma(\omega)\big)$$

satisfies the assumptions of Lemma 4.2 with 'θ' $= H \wedge 1/2$, so that (4.2) yields:

$$\mathbb{P}\big(\exists t \in [0,T]\ \big|(\hat{\Gamma}_i)_t - \Gamma_i\big| \geq x\big) \leq C_\mathrm{d} \exp\big(-C_\mathrm{c} C_\mathrm{e}^{-1} x^2 \,/\, T^{2H\wedge 1}\big).$$

This implies (4.1) for $T \leq 1$, with constants not depending on T. On the other hand, up to replacing C_b by $(e^{C_\mathrm{a}} \vee C_\mathrm{b})$, (4.1) is automatically true for $T > 1$; so the proof of Lemma 4.1 is completed. □

4.2. Third reformulation of the main result

Now, we will see how Proposition 4.1 allows one to find an easier sufficient condition for Proposition 3.5. First of all, we have to introduce a little notation: in this section, we fix some arbitrary $\alpha' \in (0,\alpha)$, $p' \in (0,p)$, and we define $\mathcal{A}'_n$ by the variant of equation (3.2) in which α and p are replaced by resp. α' and p'; also, we fix some arbitrary $\tilde{r} \in (0,r)$, and we set

$$T_n := \tilde{r}^n.$$

Now, we introduce the following events of Ω:

Definition 4.2 (Events** A_n, A'_n, $\bar{A}_n$, $\bar{A}_n^k$ **and** $\bar{A}_n^*$**).

$$A_n := \big\{\omega \big| \big(Z(\omega)_u\big)_{u\leq 0} \in \mathcal{A}_n\big\}; \tag{4.2}$$

$$A'_n := \big\{\omega \big| \big(Z(\omega)_u\big)_{u\leq 0} \in \mathcal{A}'_n\big\}; \tag{4.3}$$

$$\bar{A}_n := \big\{\omega \big| \exists t \in [0,T_n]\ \big(Z(\omega)_{t+u} - Z(\omega)_t\big)_{u\leq 0} \in \mathcal{A}_n\big\}; \tag{4.4}$$

$$\bar{A}_n^k := \big\{\omega \big| \exists t \in \big[kT_n, (k+1)T_n\big]\ \big(Z(\omega)_{t+u} - Z(\omega)_t\big)_{u\leq 0} \in \mathcal{A}_n\big\}; \tag{4.5}$$

$$\bar{A}_n^* := \big\{\omega \big| \exists t \in [0,1]\ \big(Z(\omega)_{t+u} - Z(\omega)_t\big)_{u\leq 0} \in \mathcal{A}_n\big\}. \tag{4.6}$$

Then we claim the following lemma.

Lemma 4.3. *For n large enough,*

$$\omega \in \bar{A}_n \setminus A'_n \quad \Rightarrow \quad \exists i \in [\![n [\![, \exists t \in [0,T_n] \qquad \big|\hat{\Gamma}_i(\omega)_t - \Gamma_i(\omega)\big| \geq 1.$$

Proof. Assume that $\omega \in \bar{A}_n \setminus A'_n$. Then the fact that $\omega \in \bar{A}_n$ means that there exists some $t \in [0, T_n]$ such that $(Z_{t+u} - Z_t)_{u \le 0} \in \mathcal{A}_n$. For such a t, going back to the definitions (3.2) and (4.1) of $\mathcal{A}_n$ and $\hat{\Gamma}_i$, this means that

$$\operatorname{card}\{i \in [\![n[\![\mid (\hat{\Gamma}_i)_t \ge \alpha(\log i)_+^{1/2}\} \ge pn.$$

Similarly, the fact that $\omega \notin A'_n$ means that

$$\operatorname{card}\{i \in [\![n[\![\mid \Gamma_i \ge \alpha'(\log i)_+^{1/2}\} < p'n.$$

Therefore, there exist at least $(p - p')n$ indices 'i' such that $(\hat{\Gamma}_i)_t \ge \alpha(\log i)_+^{1/2}$ while (for the same i) $\Gamma_i < \alpha'(\log i)_+^{1/2}$. Necessarily one these indices is $\ge (p - p')n - 1$; thus, for such an i, one has:

$$(\hat{\Gamma}_i)_t - \Gamma_i \ge (\alpha - \alpha')\big(\log\big((p - p')n - 1\big)\big)^{1/2}.$$

But, provided $n \ge (e^{(\alpha-\alpha')^{-2}} + 1) / (p - p')$, the r.h.s. of (4.2) is ≥ 1; so in the end we have found $i \in [\![n[\![$, $t \in [0, T_n]$ such that $|\hat{\Gamma}_i(\omega)_t - \Gamma_i(\omega)| \ge 1$, proving the lemma. □

Combining Lemma 4.3 with Lemma 4.1, we get that

$$\mathbb{P}(\bar{A}_n \setminus A'_n) \le \sum_{i=0}^{n-1} C_\mathrm{b} \exp\big(-C_\mathrm{a}\big(r^i / T_n\big)^{2H\wedge 1}\big),$$

in which the right-hand side is obviously bounded by

$$nC_\mathrm{b} \exp\big(-C_\mathrm{a}(r / \tilde{r})^{(2H\wedge 1)n}\big),$$

which shows that $\mathbb{P}(\bar{A}_n \setminus A'_n)$ decreases superexponentially in n (that is, faster than any exponential).

Now $\bar{A}^*_n \subset \bigcup_{k \in [\![\lceil 1/T_n \rceil [\![} \bar{A}^k_n$, where $\mathbb{P}(\bar{A}^k_n) = \mathbb{P}(\bar{A}_n)\ \forall k$ by translation invariance, so it follows that

$$\mathbb{P}(\bar{A}^*_n) \le \lceil 1 / T_n \rceil \mathbb{P}(\bar{A}_n) \le \lceil \tilde{r}^{-n} \rceil \big(\mathbb{P}(A'_n) + \mathbb{P}(\bar{A}_n \setminus A'_n)\big) = \lceil \tilde{r}^{-n} \rceil \mathbb{P}(A'_n) + o(1).$$

Our goal being to prove that $\mathbb{P}(\bar{A}^*_n) \to 0$ as $n \to \infty$ (that is just rewriting Proposition 3.5 with the notation of this section), it will be sufficient for that to prove the following proposition.

Proposition 4.4. $\mathbb{P}(A'_n)$ *decreases superexponentially in* n.

So, as A'_n corresponds to a condition on a finite-dimensional Gaussian vector, we have managed to get completely rid of the trajectorial aspects of the problem! Now our ultimate goal will be to prove Proposition 4.4.

Remark 4.1. As the "prime" symbols would be somehow cumbersome, we will drop them in the sequel, thus actually proving the superexponential decrease of $\mathbb{P}(A_n)$. Nevertheless this should not be confusing, as the constraints on α and p (and therefore on A_n) are the same as on α' and p' (and therefore on A'_n).

5. Final computations: Controlling a Gaussian vector

5.1. Covariance structure

☛ *In this section, for 'X' a symbol and I a discrete set, "$\vec{X}_I$" will be a shorthand for "$(X_i)_{i\in I}$".*

So, our goal is to prove the superexponential decay of $\mathbb{P}(A_n)$, which can be rewritten as

$$\mathbb{P}(A_n) = \mathbb{P}\big(\operatorname{card}\big\{i \in [\![n[\![\,\big|\, \Gamma_i \geq \alpha(\log i)_+^{1/2}\big\} \geq pn\big)$$

(where Γ_i was defined by (4.1)). (5.1) obviously implies that

$$\mathbb{P}(A_n) \leq \sum_{\substack{I\subset[\![n[\![\\ |I|\geq pn}} \mathbb{P}\big(\forall i \in I\ \Gamma_i \geq \alpha(\log i)_+^{1/2}\big).$$

As there are only 2^n subsets of $[\![n[\![$, to prove that $\mathbb{P}(A_n)$ decreases superexponentially it is therefore sufficient to prove that

$$\sup_{\substack{I\subset[\![n[\![\\ |I|\geq pn}} \mathbb{P}\big(\forall i \in I\ \Gamma_i \geq \alpha(\log i)_+^{1/2}\big)$$

decreases superexponentially.

Now, by (2.3) one has

$$\Gamma_i(\omega) = C_1 r^{-Hi} \int_{-\infty}^{0} \big(\big(r^i - s\big)^\eta - (-s)^\eta\big)\, dW(\omega)_s;$$

therefore $\vec{\Gamma}_{\mathbf{N}}$ is a centred Gaussian vector, with:

$$\begin{aligned}
\operatorname{Cov}(\Gamma_i, \Gamma_j) &= C_1^2 r^{-(i+j)H} \int_{-\infty}^{0} \big(\big(r^i - s\big)^\eta - (-s)^\eta\big)\big(\big(r^j - s\big)^\eta - (-s)^\eta\big)\, ds \\
&= C_1^2 r^{-|i-j|H} \int_{-\infty}^{0} \big(\big(r^{|i-j|} - s\big)^\eta - (-s)^\eta\big)\big((1-s)^\eta - (-s)^\eta\big)\, ds \qquad (5.1)\\
&\leq C_{\mathrm{f}} r^{(1/2-|\eta|)|i-j|},
\end{aligned}$$

for some absolute constant $C_{\mathrm{f}} < \infty$. Therefore, provided r was chosen small enough, we have the following control on the covariance matrix of $\vec{\Gamma}_{\mathbf{N}}$:

$$\operatorname{Cov}(\Gamma_i, \Gamma_j) = \sigma^2 \qquad \text{for } i = j; \tag{5.2}$$

$$\left|\operatorname{Cov}(\Gamma_i, \Gamma_j)\right| \le \sigma^2 \varepsilon^{|i-j|} \qquad \text{for } i \neq j, \tag{5.3}$$

where $\varepsilon > 0$ is some small parameter which will be fixed later, and where $\sigma := \operatorname{Var}(\Gamma)^{1/2} > 0$ (since $H \neq 1/2$).

5.2. Density estimates

To exploit (5.2)–(5.3), we need the following lemma (whose proof is postponed to Appendix C):

Lemma 5.1. *For $n \in \mathbf{N}$, $\varepsilon > 0$, let $\mathbf{A} =: ((a_{ij}))_{i,j \in [\![n[\![}$ be a square matrix such that $a_{ii} = 1$ $\forall i$ and $|a_{ij}| \le \varepsilon^{|i-j|}$ $\forall i \neq j$. Then*:

$$\det \mathbf{A} \ge \exp\bigl(-n \Phi_{\mathrm{g}}(\varepsilon) \varepsilon^2\bigr),$$

where $\Phi_{\mathrm{g}} \colon (0, \infty) \to [1, \infty]$ is some absolute function (in particular, not depending on n) such that $\Phi_{\mathrm{g}}(\varepsilon) \overset{\varepsilon \to 0}{\to} 1$ – we will call such a function a quasi-one function. *In particular, provided ε is small enough, $\mathbf{A}$ is invertible. The present lemma asserts moreover that, then, denoting $\mathbf{A}^{-1} =: ((b_{ij}))_{i,j}$:*

$$|b_{ij}| \le 2^{|i-j|-1} \bigl(\Phi_{\mathrm{h}}(\varepsilon) \varepsilon\bigr)^{|i-j|} \qquad \forall i \neq j; \tag{5.4}$$

$$|b_{ii} - 1| \le 2 \Phi_{\mathrm{i}}(\varepsilon) \varepsilon^2 \qquad \forall i, \tag{5.5}$$

for Φ_{i} and Φ_{h} some other absolute "quasi-one functions".

We apply Lemma 5.1 to the covariance matrix of $\vec{\Gamma}_{[\![n[\![}$ (assuming ε was chosen small enough so that Φ_{g} is finite); then, the formula for the density of Gaussian vectors gives that:

$$\begin{aligned} &\frac{d\mathbb{P}(\vec{\Gamma}_{[\![n[\![} = \vec{\gamma}_{[\![n[\![})}{d\gamma} \\ &\quad \le \left(\frac{\exp(\Phi_{\mathrm{g}}(\varepsilon)\varepsilon^2)}{2\pi\sigma^2}\right)^{n/2} \exp\Biggl(\frac{1}{2\sigma^2}\Biggl(\bigl(-1 + 2\Phi_{\mathrm{i}}(\varepsilon)\varepsilon^2\bigr) \sum_{i \in [\![n[\![} \gamma_i^2 \\ &\qquad + \frac{1}{2} \sum_{\substack{i,j \in [\![n[\![\\ i \neq j}} |\gamma_i||\gamma_j| \times \bigl(2\Phi_{\mathrm{h}}(\varepsilon)\varepsilon\bigr)^{|i-j|}\Biggr)\Biggr). \end{aligned} \tag{5.6}$$

Bounding above $|\gamma_i||\gamma_j|$ by $\frac{1}{2}(\gamma_i^2+\gamma_j^2)$, that is bounded again by

$$\big(\Phi_{\mathrm{j}}(\varepsilon)/2\pi\sigma^2\big)^{n/2}\exp\bigg(-\frac{1}{2\Phi_{\mathrm{k}}(\varepsilon)\sigma^2}\sum_{i\in[\![n[\![}\gamma_i^2\bigg),$$

where

$$\Phi_{\mathrm{j}}(\varepsilon):=\exp\big(\Phi_{\mathrm{g}}(\varepsilon)\varepsilon^2\big)$$

and

$$\Phi_{\mathrm{k}}(\varepsilon):=\bigg(1-2\Phi_{\mathrm{i}}(\varepsilon)\varepsilon^2-\frac{1}{2}\sum_{z\in\mathbf{Z}^*}\big(2\Phi_{\mathrm{h}}(\varepsilon)\varepsilon\big)^{|z|}\bigg)_+^{-1}$$

are "quasi-one functions" again.

In the sequel, we assume that ε was chosen small enough so that $\Phi_{\mathrm{j}}(\varepsilon), \Phi_{\mathrm{k}}(\varepsilon)<\infty$; and we define the following vectorial random variable (which we are actually only interested in through its law).

Definition 5.1 (Variable $\vec{\Pi}_{\mathbf{N}}$). *$\vec{\Pi}_{\mathbf{N}}$ is a random vector on $\mathbf{R}^{\mathbf{N}}$ whose entries are i.i.d. $\mathcal{N}(0, \Phi_{\mathrm{k}}(\varepsilon)\sigma^2)$.*

Then, equation (5.2) can be rephrased into:

$$\frac{d\mathbb{P}(\vec{\Gamma}_{[\![n[\![}=\vec{\gamma}_{[\![n[\![})}{d\mathbb{P}(\vec{\Pi}_{[\![n[\![}=\vec{\gamma}_{[\![n[\![})}\le\big(\Phi_{\mathrm{j}}(\varepsilon)\Phi_{\mathrm{k}}(\varepsilon)\big)^{n/2}\qquad\text{uniformly in }\vec{\gamma}_{[\![n[\![}.$$

Therefore, for $I\subset[\![n[\![$ with $|I|\ge pn$:

$$\mathbb{P}\big(\forall i\in I\ \Gamma_i\ge\alpha(\log i)_+^{1/2}\big)\le\big(\Phi_{\mathrm{j}}(\varepsilon)\Phi_{\mathrm{k}}(\varepsilon)\big)^{n/2}\mathbb{P}\big(\forall i\in I\ \Pi_i\ge\alpha(\log i)_+^{1/2}\big).$$

But

$$\begin{aligned}&\mathbb{P}\big(\forall i\in I\ \Pi_i\ge\alpha(\log i)_+^{1/2}\big)\\&\quad=\prod_{i\in I}\mathbb{P}\big(\Pi_i\ge\alpha(\log i)_+^{1/2}\big)\\&\quad=\prod_{i\in I}\mathbb{P}\big(\mathcal{N}(0,1)\ge C_{\mathrm{l}}(\log i)_+^{1/2}\big)\le\prod_{i\in I}\exp\big(-C_{\mathrm{l}}^2(\log i)_+/2\big)\\&\quad=\prod_{i\in I}(i\vee 1)^{-C_{\mathrm{l}}^2/2}\le\big(|I|-1\big)_+!^{-C_{\mathrm{l}}^2/2}\le\big(\lceil pn\rceil-1\big)_+!^{-C_{\mathrm{l}}^2/2}\end{aligned}\tag{5.7}$$

(with $C_{\mathrm{l}}:=\alpha\ /\ \Phi_{\mathrm{k}}(\varepsilon)^{1/2}\sigma$), where the penultimate inequality comes from ordering $I=:\{i_0,i_1,\dots,i_{|I|-1}\}$ with $i_0<i_1<\cdots$, and observing that then $i_j\ge j$ for all j, so that $(i_j\vee 1)^{-C_{\mathrm{l}}^2/2}\le(j\vee 1)^{-C_{\mathrm{l}}^2/2}$.

Combining (5.2) with (5.7) shows that $\mathbb{P}(\Gamma_i \geq \alpha(\log i)_+^{1/2}\ \forall i \in I)$ decreases superexponentially in n uniformly in I, which finally proves Proposition 4.4 and hence Theorem 1.1.

6. Application to financial mathematics

6.1. Two-way crossing property

Theorem 1.1 implies in particular that a stopping time τ can almost-never be a local minimum at right for the trajectory of Z, that is, that for almost-all ω, there exist arbitrarily small $\varepsilon > 0$ such that $Z(\omega)_{\tau(\omega)+\varepsilon} < Z(\omega)_{\tau(\omega)}$. Symmetrically (since $-Z$ has the same law as Z), τ can almost-never be a local maximum for Z. The conjunction of these two facts implies that fBm has the "two-way crossing property" introduced by Bender in [2] (whose definition is recalled below for the sake of self-containment).

Definition 6.1 (Two-way crossing property). *A process X (whose increments[6] are) adapted to a filtration $(\mathcal{B}_t)_t$ is said to satisfy the* two-way crossing (TWC) property *when, for any stopping time τ adapted to $(\mathcal{B}_t)_t$, one has almost-surely*:

$$\inf\{t > \tau | X_t > X_\tau\} = \inf\{t > \tau | X_t < X_\tau\}.$$

Corollary 6.1. *For any $H \neq 1/2$, fractional Brownian motion (hence also geometric fBm) satisfies the two-way crossing property.*

6.2. Financial context

The motivation for this work comes from the study of (geometric) fractional Brownian motion as a model for price processes in financial mathematics, which is a natural choice for modelling correlations and scale invariance, as was first suggested by [9]. FBm is however not a semimartingale, which in the classic setting leads to arbitrage opportunities, as was shown by Delbaen and Schachermayer [6]: so, an efficient market should not allow for fBm price processes. Rogers [12] showed however that arbitrage strategies for a fBm price process shall necessarily rely on making numerous transactions on extremely short scales of time. That suggested to look whether relaxing slightly some assumptions of the classic financial theory would nevertheless lead to a consistent theory allowing for fBm price processes: many work on this issue was carried out over the last two decades.

A first idea is to impose restrictions on the authorized trading strategies. In [3], it was suggested to restrict oneself to the so-called *Cheridito class* of strategies, in which a minimal waiting time is imposed between two transactions. In such a context, it can be shown rather easily that fBm is arbitrage-free indeed. An intermediate class of trading strategies between the Cheridito class and the full class of predictable strategies, called *simple strategies*, was investigated by Bender

[6] Or logarithmic increments in the case of geometric fBm.

[2]: a simple strategy allows only for a fixed, finite number of transactions, but transaction times may be arbitrarily close to each other. In his paper Bender introduced the (TWC) property, and showed that this property (together with conditional full support, see below) was a necessary and sufficient condition for simple arbitrage to be ruled out. At that time it remained an open question however whether fBm did have this (TWC) property. My work answers positively to this question, and therefore proves that geometric fBm is proof against simple arbitrage. This was actually my initial motivation for this article.

Another approach for limiting arbitrage opportunities is to allow for any trading strategy, but to introduce transaction costs, generally proportional, of arbitrarily small rate. In this context, Guasoni [7] proved that fBm is arbitrage-free indeed. The proof of that property already relied on studying the properties of fBm after a stopping time: a sufficient condition for arbitrage-freeness is indeed the *stickiness* property, which says that the future trajectory of the price process after any stopping time may remain arbitrarily close to its initial value for an arbitrarily large amount of time, with positive probability. The stickiness property itself is a particular case of the *conditional full support* property, which says that future trajectories of the process after any stopping time have full support among the space of continuous trajectories (for the topology of locally uniform convergence). Conditional full support was shown to hold for fBm by Guasoni, Rásonyi and Schachermayer [8].

Still in presence of transaction costs, more recently Czichowsky, Schachermayer *et al.* investigated on existence of a so-called *shadow price process* for an asset whose price is modelled by a non-semimartingale process (like geometric fBm): a shadow price process is a semimartingale process taking values in the bid-ask spread, whose optimal trading strategy in absence of transaction costs coincides with the optimal strategy for the actual price process in presence of transaction costs. The notion of optimality considered here is to maximise the expectation value of some (reasonable) utility function. Using the stickiness property, Czichowsky and Schachermayer [5] proved first that geometric fBm does admit a shadow price indeed in the case the utility function is bounded from above and defined on the whole $\mathbf{R}$ (e.g., $U(x) = 1 - e^{-x}$). Then they realised that a process satisfying Bender's (TWC) condition (together with some boundedness estimate) would actually have a shadow price for a much larger class of utility functions, including in particular logarithmic and concave power utilities. Hence, they used my results in [4] to show that geometric fBm admits a shadow price for such utilities.

Appendix A: Conditional expectation of the fBm

This appendix is devoted to ending the proof of Lemma 2.1 initiated in Section 2.2. At the point we have got to, what remains to do is showing that

$$C_1 \int_{s=-\infty}^{0} \big((v-s)^\eta - (-s)^\eta\big)\, dW_{\tau+s}$$

(seen as a trajectory indexed by $v \in \mathbf{R}_+$) is actually equal to $\mathbf{D}((Z_{\tau+u} - Z_\tau)_{u\leq 0})$ with $\mathbf{D}$ defined by (2.1)–(2.2), where W is the ordinary Brownian motion driving the fBm Z. To alleviate notation, actually we will only prove this result for $\tau \equiv 0$, the original case being the same up to time translation of the increments (hence the informal definition (2.3) of $\mathbf{D}$).

The starting point for our computation is the *Pipiras–Taqqu formula*, which says that equation (1.1) defining the past increments of Z as a function of the past increments of W has an "inverse" giving back the past increments of W from the past increments of Z:

Proposition A.1 ([11], Corollary 1.1). *In the context of Section 1.1, one has almost-surely, for all t:*

$$W_t = \frac{C_1^{-1}}{\Pi(\eta)\Pi(-\eta)} \int_{\mathbf{R}} \big((t-s)_+^{-\eta} - (-s)_+^{-\eta}\big)\, dZ_s.$$

(*Recall that $\Pi(\bullet)$ is Euler's pi function extrapolating the factorial.*)

☛ *From now on in this appendix, it will be convenient to shorthand "$1 \,/\, \Pi(\eta)\Pi(-\eta)$" into "$C_H$".*

So, let us use (A.1) to get (2.2)–(2.1). First, like (1.1), equations (A) and (A.1) have to be interpreted by integrating by parts: for $v \geq 0$, $s \leq 0$, that means resp. that:

$$(\mathrm{A}) = \eta C_1 \int_{-\infty}^{0} \xi_{\eta-1}(-s,v) W_s\, ds;$$

$$\frac{W_t}{C_1^{-1} C_H} = \eta \int_{-\infty}^{t} \xi_{-\eta-1}(t-s,-t)(Z_s - Z_t)\, ds + \eta \int_{t}^{0} (-s)^{-\eta-1} Z_s ds + (-t)^{-\eta} Z_t, \tag{A.1}$$

where we recall that $\xi_r(a,b) := (a+b)^r - a^r$. Hence, (A) is equal to:

$$\eta^2 C_H \int_{s=-\infty}^{0} \int_{u=-\infty}^{s} \xi_{\eta-1}(-s,v)\xi_{-\eta-1}(s-u,-s)(Z_u - Z_s)\, ds\, du \tag{A.2}$$

$$+ \eta^2 C_H \int_{s=-\infty}^{0} \int_{u=s}^{0} \xi_{\eta-1}(-s,v)(-u)^{-\eta-1} Z_u\, ds\, du \tag{A.3}$$

$$+ \eta C_H \int_{-\infty}^{0} \xi_{\eta-1}(-s,v)(-s)^{-\eta} Z_s\, ds. \tag{A.4}$$

Now we are going to rewrite each of the terms (A.2)–(A.4) as an integral against $Z_u\, du$, in order to get (2.2)–(2.1). First, Term (A.4) is already of the wanted form, up to renaming 's' into 'u'. Next, Term (A.3) simplifies into:

$$(\mathrm{A.3}) = \eta^2 C_H \int_{u=-\infty}^{0} \left(\int_{s=-\infty}^{u} \xi_{\eta-1}(-s,v)\, ds \right) (-u)^{-\eta-1} Z_u\, du$$

$$= -\eta C_H \int_{-\infty}^{0} \big[\xi_\eta(-s,v)\big]_{s=-\infty}^{u} (-u)^{-\eta-1} Z_u du \tag{A.5}$$

$$= -\eta C_H \int_{-\infty}^{0} \xi_\eta(-u, v)(-u)^{-\eta-1} Z_u \, du.$$

Term (A.2) is the hardest to get into the wanted form, because splitting naively the factor $(Z_u - Z_s)$ would yield divergent integrals. To bypass that problem, we first make a truncation: for ε a small positive number,

$$\begin{aligned}
(\text{A.2}) &\approx \eta^2 C_H \int_{s=-\infty}^{0} \int_{u=-\infty}^{(1+\varepsilon)s} \xi_{\eta-1}(-s, v)\xi_{-\eta-1}(s-u, -s)(Z_u - Z_s)\, ds\, du \\
&= \eta^2 C_H \iint_{\substack{s<0 \\ u<(1+\varepsilon)s}} \xi_{\eta-1}(-s, v)\xi_{-\eta-1}(s-u, -s) Z_u \, ds\, du \qquad \text{(A.6)} \\
&\quad - \eta^2 C_H \iint_{\substack{s<0 \\ u<(1+\varepsilon)s}} \xi_{\eta-1}(-s, v)\xi_{-\eta-1}(s-u, -s) Z_s \, ds\, du. \qquad \text{(A.7)}
\end{aligned}$$

By the change of variables $(s, u) \leftarrow (u - s, u)$,

$$\begin{aligned}
(\text{A.6}) &= \eta^2 C_H \int_{u=-\infty}^{0} \left(\int_{s=-\infty}^{\varepsilon u/(1+\varepsilon)} \mathbf{1}_{s>u} \xi_{\eta-1}(s-u, v)\xi_{-\eta-1}(-s, s-u)\, ds \right) Z_u \, du \\
&\approx \eta^2 C_H \int_{u=-\infty}^{0} \left(\int_{s=-\infty}^{\varepsilon u} \mathbf{1}_{s>u} \xi_{\eta-1}(s-u, v)\xi_{-\eta-1}(-s, s-u)\, ds \right) Z_u \, du
\end{aligned} \qquad \text{(A.8)}$$

(where by "≈" we mean that, for all v, the difference between the two members from either side of the '≈' sign tends to 0 as $\varepsilon \to 0$, as one can check by simple estimates); and by the change of variables $(s, u) \leftarrow (u - s, s)$,

$$(\text{A.7}) = -\eta^2 C_H \int_{u=-\infty}^{0} \left(\int_{s=-\infty}^{\varepsilon u} \xi_{\eta-1}(-u, v)\xi_{-\eta-1}(-s, -u)\, ds \right) Z_u \, du.$$

So,

$$(\text{A.2}) \approx \eta^2 C_H \int_{-\infty}^{0} \left(\int_{-\infty}^{\varepsilon u} J(v, u, s)\, ds \right) Z_u \, du,$$

with

$$J(v, u, s) := \mathbf{1}_{s>u} \xi_{\eta-1}(s-u, v)\xi_{-\eta-1}(-s, s-u) - \xi_{\eta-1}(-u, v)\xi_{-\eta-1}(-s, -u).$$

But $\int^0 J(v, u, s)\, ds$ does converge, so, letting ε tend to 0, we get in the end:

$$(\text{A.2}) = \eta^2 C_H \int_{-\infty}^{0} \left(\int_{-\infty}^{0} J(v, u, s)\, ds \right) Z_u \, du.$$

Summing (A.4), (A.5) and (A), and observing that $\xi_{\eta-1}(-u, v)(-u)^{-\eta} - \xi_\eta(-u, v) \times (-u)^{-\eta-1} = -v(v-u)^{\eta-1}(-u)^{-\eta-1}$, finally yields equations (2.2)–(2.1).

Appendix B: Explicit estimate for the supremum of Gaussian processes

Proof of Lemma 4.2. Let $\theta \in (0,1]$ and let X satisfying the assumptions of the lemma. Then obviously, for the continuous version of X:

$$\|X\|(\omega) \leq \sum_{i=0}^{\infty} \sup_{a \in [\![2^i [\![} \left| X(\omega)_{a2^{-i}} - X(\omega)_{(a+1)2^{-i}} \right|.$$

Therefore, for $(\gamma_i)_i$ a sequence of positive real numbers such that $\sum_i \gamma_i = 1$, one has that, for all $x \geq 0$:

$$\begin{aligned} \mathbb{P}\big(\|X\| \geq x\big) &\leq \sum_{i=0}^{\infty} \mathbb{P}\Big(\sup_{a \in [\![2^i [\![} |X_{a2^{-i}} - X_{(a+1)2^{-i}}| \geq \gamma_i x \Big) \\ &\leq \sum_i 2^i \sup_a \mathbb{P}\big(|X_{a2^{-i}} - X_{(a+1)2^{-i}}| \geq \gamma_i x\big). \end{aligned} \tag{B.1}$$

But, uniformly in a,

$$\begin{aligned} \mathbb{P}\big(|X_{a2^{-i}} - X_{(a+1)2^{-i}}| \geq \gamma_i x\big) &= \mathbb{P}\big(\mathcal{N}(0,1) \geq \gamma_i x \,/\, \mathrm{Var}^{1/2}(X_{a2^{-i}} - X_{(a+1)2^{-i}})\big) \\ &\leq \mathbb{P}\big(\mathcal{N}(0,1) \geq 2^{i\theta} \gamma_i x\big) \leq 2 \exp\big(-2^{2i\theta-1} (\gamma_i x)^2\big); \end{aligned} \tag{B.2}$$

so, taking $\gamma_i := (1 - 2^{-\theta/2}) 2^{-i\theta/2}$:

$$\mathbb{P}\big(\|X\| \geq x\big) \leq \sum_{i=0}^{\infty} 2^{i+1} \exp\big(-\big(1 - 2^{-\theta/2}\big)^2 \times 2^{i\theta-1} x^2\big).$$

Provided $x \geq 2 \,/\, (1 - 2^{-\theta/2}) \theta^{1/2} =: C_{\mathrm{m}}(\theta)$, one has (bounding $2^{i\theta}$ below by $(1 + i\theta \log 2)$)

$$\exp\big(-2^{i\theta} x^2/2\big(1 - 2^{-\theta/2}\big)^2\big) \leq \exp\big(-x^2/2\big(1 - 2^{-\theta/2}\big)^2 - \log(4) i\big),$$

so that, for $x \geq C_{\mathrm{m}}$:

$$\mathbb{P}\big(\|X\| \geq x\big) \leq \left(\sum_{i=0}^{\infty} 2^{i+1} 4^{-i} \right) \exp\big(-x^2/2\big(1 - 2^{-\theta/2}\big)^2\big) =: 4 e^{-C_{\mathrm{c}}(\theta) x^2}.$$

On the other hand, for $x < C_{\mathrm{m}}$ one has obviously $\mathbb{P}(\|X\| \geq x) \leq 1$; so equation (4.2) follows with $C_{\mathrm{d}} := 4 \vee e^{C_{\mathrm{c}} C_{\mathrm{m}}^2}$. □

Appendix C: Almost diagonal matrices

Proof of Lemma 5.1. Consider $\mathbf{A}$ satisfying the assumptions of the lemma, and denote $\mathbf{I}_n - \mathbf{A} =: \mathbf{H}$. The first part of this proof will consist in deriving estimates on the entries of $\mathbf{H}$ and its powers. Denote respectively,

$$\mathbf{H} =: \big((h_{ij})\big)_{i,j\in[\![n[\![};\tag{C.1}$$

$$\mathbf{H}^k =: \big((h_{ij}^{(k)})\big)_{i,j\in[\![n[\![} \qquad \forall k \geq 0.\tag{C.2}$$

Then the assumptions of the lemma ensure that one has $|h_{ii}| = 0\ \forall i$, resp. $|h_{ij}| \leq \varepsilon^{|i-j|}\ \forall i \neq j$, and hence

$$\big|h_{ij}^{(k+1)}\big| \leq \sum_{i'\in[\![n[\![} |h_{ii'}|\big|h_{i'j}^{(k)}\big| = \sum_{i'\neq i} \varepsilon^{|i-i'|}\big|h_{i'j}^{(k)}\big| \qquad \forall i, j, \forall k.$$

That suggests to define by induction:

$$\begin{cases} \mathfrak{h}_z^{(0)} := \mathbf{1}_{z=0}, & \forall z \in \mathbf{Z}, \\ \mathfrak{h}_z^{(k+1)} := \displaystyle\sum_{z'\neq z} \varepsilon^{|z-z'|}\mathfrak{h}_{z'}^{(k)}, & \forall z \in \mathbf{Z}, \forall k \geq 0, \end{cases}$$

so that one has

$$\big|h_{ij}^{(k)}\big| \leq \mathfrak{h}_{i-j}^{(k)} \qquad \forall i, j, \forall k.$$

The interest of having introduced the $\mathfrak{h}_z^{(k)}$'s is that these are easier to bound than the $h_{ij}^{(k)}$'s themselves. To bound the $\mathfrak{h}_z^{(k)}$'s, we begin with observing that one has obviously by induction:

$$\begin{aligned} \mathfrak{h}_z^{(k)} &= \sum_{\substack{(0=:s_0,s_1,s_2,\ldots,s_{k-1},z=:s_k)\\ 0\neq s_1,s_1\neq s_2,\ldots,s_{k-1}\neq z}} \varepsilon^{\sum_{i\in[\![k[\![}|s_i-s_{i+1}|} \\ &= \sum_{n\geq 0} \operatorname{card}\Big\{(0=: s_0, s_1, \ldots, s_{k-1}, z =: s_k)\Big| s_{i+1}\neq s_i\ \forall i, \sum_i |s_i - s_{i+1}| = n\Big\}\varepsilon^n. \end{aligned}\tag{C.3}$$

To bound the cardinality appearing in (C.3), we observe that a $(k+1)$-tuple $(0, s_1, s_2, \ldots, s_{k-1}, z)$ such that $s_{i+1} \neq s_i\ \forall i$ and $\sum_i |s_i - s_{i+1}| = n$ (we will call such a $(k+1)$-tuple as *valid*) can be *coded* by a word of n symbols from $\{+, +_|, -, -_|\}$, in the following way: successively, for each i we write $(|s_{i+1} - s_i| - 1)$ symbols "$\operatorname{sgn}(s_{i+1} - s_i)$" followed by a symbol "$\operatorname{sgn}(s_{i+1} - s_i)_|$" – for instance, for $k = 5$, $z = 2$, $n = 8$, one would have

$$(0, 1, 4, 2, 1, 2) \mapsto \text{“}+_|{+}{+}{+}_|{-}{-}_|{-}_|{+}_|\text{”}.$$

Obviously such a coding in injective. Moreover, for given k, z, n, an n-character word may be the image of a valid $(k+1)$-tuple only if z and n have the same parity, that the word contains

$(n+z)/2$ symbols from $\{+, +_|\}$ vs. $(n-z)/2$ symbols from $\{-, -_|\}$, and that k exactly of the n symbols, necessarily including the last one, are from $\{+_|, -_|\}$. Henceforth:

$$\begin{aligned} &\operatorname{card}\left\{(0, s_1, s_2, \ldots, s_{k-1}, z) \middle| s_{i+1} \neq s_i \ \forall i \text{ and } \sum_i |s_i - s_{i+1}| = n\right\} \\ &\quad \leq \mathbf{1}_{2|n-z} \binom{n}{(n-|z|)/2} \binom{n-1}{k-1}. \end{aligned} \tag{C.4}$$

In the end, combining (C), (C.3) and (C.4):

$$\left|h_{ij}^{(k)}\right| \leq \sum_{m \geq 0} \binom{|i-j|+2m}{m} \binom{|i-j|+2m-1}{k-1} \varepsilon^{|i-j|+2m} \qquad \forall i, j, \forall k.$$

After these preliminary estimates, let us turn to proving the lemma itself. We begin with the first part, namely, bounding $\det \mathbf{A}$ from below. For $\mathbf{X} =: ((x_{ij}))_{i,j}$ an $n \times n$ matrix, denote

$$\|\mathbf{X}\| := \sup_{j \in [\![n[\![} \sum_{i \in [\![n[\![} |x_{ij}| :$$

$\|\bullet\|$ is the operator norm of $\mathbf{X}$ when seen as an operator from $\ell^1([\![n[\![)$ into itself, so it is sub-multiplicative. Then, the formula

$$\log(\mathbf{I}_n - \mathbf{H}) = \sum_{k=1}^{\infty} k^{-1} \mathbf{H}^k$$

converges as soon as $\|\mathbf{H}\| < 1$, then yielding:

$$\begin{aligned} \left|\operatorname{tr} \log(\mathbf{I}_n - \mathbf{H})\right| &\leq \sum_{k=1}^{\infty} k^{-1} \left|\operatorname{tr} \mathbf{H}^k\right| \\ &\leq 0 + \frac{1}{2}\left|\operatorname{tr} \mathbf{H}^2\right| + n \sum_{k \geq 3} k^{-1} \left\|\mathbf{H}^k\right\| \\ &\leq \frac{1}{2}\left|\operatorname{tr} \mathbf{H}^2\right| + n \sum_{k \geq 3} k^{-1} \|\mathbf{H}\|^k. \end{aligned} \tag{C.5}$$

But the assumptions on the entries of $\mathbf{H}$ imply that

$$\|\mathbf{H}\| \leq \sum_{z \in \mathbf{Z}^*} \varepsilon^{|z|} = \frac{2\varepsilon}{1-\varepsilon},$$

which is < 1 as soon as $\varepsilon < 1/3$; and on the other hand, we get from (C) that, for all i,

$$\left|h_{ii}^{(2)}\right| \leq \sum_{m \geq 1} \binom{2m}{m} (2m-1) \varepsilon^{2m} \leq 2\varepsilon^2 + \sum_{m \geq 2} 2^{2m} (2m-1) \varepsilon^{2m} =: 2\Phi_{\mathrm{n}}(\varepsilon) \varepsilon^2,$$

so that $|\operatorname{tr}\mathbf{H}^2| \le 2n\Phi_{\mathrm{n}}(\varepsilon)\varepsilon^2$. In the end:

$$\begin{aligned}\det\mathbf{A} &= \det\exp\log(\mathbf{I}_n - \mathbf{H}) = \exp\operatorname{tr}\log(\mathbf{I}_n - \mathbf{H}) \\ &\ge \exp\biggl(-n\Phi_{\mathrm{n}}(\varepsilon)\varepsilon^2 - n\sum_{k\ge 3} k^{-1}\Bigl(\frac{2\varepsilon}{1-\varepsilon}\Bigr)^k\biggr) =: \exp\bigl(-n\Phi_{\mathrm{g}}(\varepsilon)\varepsilon^2\bigr),\end{aligned} \tag{C.6}$$

which is equation (5.1).

Now let us handle the second part of the lemma, namely, bounding the entries of $(\mathbf{A}^{-1} - \mathbf{I}_n)$. Provided $\|\mathbf{H}\| < 1$, one has

$$\mathbf{A}^{-1} - \mathbf{I}_n = \sum_{k=1}^{\infty} \mathbf{H}^k,$$

so that

$$|b_{ij} - \mathbf{1}_{i=j}| \le \sum_{k=1}^{\infty} \bigl|h_{ij}^{(k)}\bigr| \qquad \forall i, j.$$

To bound the r.h.s. of (C), we write that, starting from (C):

$$\begin{aligned}\sum_{k\ge 1}\bigl|h_{ij}^{(k)}\bigr| &\le \sum_{m\ge 0}\binom{|i-j|+2m}{m}\biggl(\sum_{k=1}^{|i-j|+2m}\binom{|i-j|+2m-1}{k-1}\biggr)\varepsilon^{|i-j|+2m} \\ &= \sum_{m\ge 0}\binom{|i-j|+2m}{m}\mathbf{1}_{|i-j|+2m\ge 1}2^{|i-j|+2m-1}\varepsilon^{|i-j|+2m} \\ &= \biggl(\sum_{m\ge 0}\mathbf{1}_{|i-j|+2m\ge 1}\binom{|i-j|+2m}{m}\bigl(4\varepsilon^2\bigr)^m\biggr)2^{|i-j|-1}\varepsilon^{|i-j|}.\end{aligned} \tag{C.7}$$

But we observe that, for $z \ge 1$, $x > 0$,

$$\begin{aligned}\sum_{m\ge 0}\binom{z+2m}{m}x^m &= 1 + \sum_{m\ge 1}\binom{z+2m}{m}x^m \le 1 + \sum_{m\ge 1}\frac{(z+2m)^m}{m!}x^m \\ &\le 1 + \sum_{m\ge 1}\frac{(2z)^m + (4m)^m}{m!}x^m \\ &\le 1 + \sum_{m\ge 1}\frac{(2zx)^m}{m!} + \sum_{m\ge 1}(4ex)^m = e^{2zx} + \frac{4ex}{1-4ex} \\ &\le \Bigl(e^{2x} + \frac{4ex}{1-4ex}\Bigr)^z =: \Phi_{\mathrm{o}}(x)^z,\end{aligned} \tag{C.8}$$

so that here, for $i \neq j$:

$$\sum_{k=1}^{\infty} \left|h_{i,j}^{(k)}\right| \le \Phi_{\mathrm{o}}\left(4\varepsilon^2\right)^{|i-j|} 2^{|i-j|-1} \varepsilon^{|i-j|} =: 2^{|i-j|-1}\left(\Phi_{\mathrm{h}}(\varepsilon)\varepsilon\right)^{|i-j|},$$

which is equation (5.4).

Equation (5.5) for the case $i = j$ is derived in the same way as (5.4), with just a few minor differences at the beginning of the computation: namely, in the l.h.s. of (C.7), we treat apart the cases "$k = 1$" (which yields zero here since $h_{ii} = 0$ by assumption) and "$k = 2$" (which has already been handled by (C)); then all the sequel is the same. □

Remark C.1. The bounds (5.1)–(5.5) of Lemma 5.1 are optimal at first order. Actually this is much more than needed to prove Theorem 1.1, and we could have got a sufficient result with a shorter proof; but it seemed interesting to me to state the sharp version of the lemma.

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