"Building" exact confidence nets

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Confidence nets, that is, collections of confidence intervals that fill out the parameter space and whose exact parameter coverage can be computed, are familiar in nonparametric statistics. Here, the distributional assumptions are based on invariance under the action of a finite reflection group. Exact confidence nets are exhibited for a single parameter, based on the root system of the group. The main result is a formula for the generating function of the coverage interval probabilities. The proof makes use of the theory of "buildings" and the Chevalley factorization theorem for the length distribution on Cayley graphs of finite reflection groups.

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1. Introduction

It is well known, and usually attributed to Wilks [31], that the order statistics from a random sample provide nonparametric confidence intervals for percentiles from a distribution: every interval formed by the order statistics covers a given percentile with a computable probability. For the median the probabilities are of binomial form. We shall refer to the situation in which the set of coverage intervals cover the real line and the coverage probability of each interval is computable as a *confidence net*.

An interesting example is given by Hartigan [16,17] for the median, given an independent sample from a distribution symmetric about the median. There, the net is based on all sub-sample means: for a sample $\{y_i | i \in \mathbf{n} = \{1, \ldots, n\}\}$ and $S \subset \mathbf{n}$, a subsample mean is $\frac{1}{|S|} \sum_{i \in S} y_i$, in which each of the 2^n intervals has coverage probability $\frac{1}{2^n}$. Hartigan's typical value theorem [16] is the basis for random subsampling, namely a resampling plan to construct confidence intervals for the centre of a symmetric distribution on a real line. Atkins and Sherman [4] derived a group-theoretic condition on a set of subsamples of a random sample from a continuous random variable symmetric about zero to be sufficient to provide typical values for zero. With the current interest in very large data sets, subsampling from complex data can be viewed as a natural solution to the computational issues. While many methods have been devised to provide unbiased and efficient estimation of average quantiles, to our knowledge no such method exploits invariance under the action of a finite reflection group. Knowledge of invariance provides an omnibus method for constructing covering nets, which cannot be obtained by inverting selected nonpara-

metric multivariate rank tests (see [21]). Hartigan's work has also had impact in the theory of the bootstrap and resampling (see Efron [8], Efron and Tibshirani [9]).

Another example is the set of intervals formed by pairwise means, $\{\frac{y_i+y_j}{2}\}$, sometimes called Walsh averages. These are the basis for one version of the Hodges–Lehmann estimator for a mean [18], which is the empirical median of all pairwise means (including the single observation). There are strong connections to the Wilcoxon signed rank (sum) test where the same generating function as derived for the group of type B_n in this paper is used in the computation of critical values [24,29]. Indeed, the current paper could be represented as a group theoretic generalisation of the generating function approach of these papers, or, given the duality between testing and confidence intervals, as a way to invert certain permutation tests (see, for example, Tritchler [28]).

We first give an account of a general construction of nonparametric confidence interval nets and then specialise to the case of finite reflection groups, showing the relation to the root systems of the groups. Finite reflection groups have been classified completely up to isomorphism, and via this classification are also known as finite Coxeter groups, which also have a purely algebraic definition based on their presentations. We will not elaborate on this classification here but refer the reader to [20] or [1]. Sections 3.1 and 3.2 cover in some detail the case of the finite Coxeter groups of type B_n (the hyperoctahedral groups) and type D_n. It turns out that in the case of type B_n the interval boundaries are the pairwise means, mentioned above, together with the single observations. In the case of type D_n they are the pairwise means, but excluding the single observations. The generating functions turn out to be familiar from the theory of partitions in number theory.

In Section 4, the main result of the paper is given, namely a generating function for the interval probabilities for a general finite Coxeter group (with one exception). Specifically, we show (Theorem 4.2) that the frequency distribution for the intervals of the confidence net based on a (almost any) finite irreducible Coxeter group is given by the generating function

$$G(q) = \frac{\prod_{i=1}^{m}(1-q^{d_i})}{\prod_{i=1}^{n}(1-q^i)},$$

where $d_1, \ldots, d_m$ are the basic invariant degrees of the group. As an example, when the group is the Coxeter group of type B_2, the generating function is $G(q) = 1 + 2q + 2q^2 + 2q^3 + q^4$. This indicates that over the five intervals, the relative (coverage) probability of the parameter θ being in one of the middle three intervals is twice that of it being in one of the extremal intervals. Details of this example and others appear in Sections 3 and 4.

The proof of this result is given in Section 6 and relies on showing that the probabilities are derived from the Coxeter length function for the quotient of the Coxeter group by the symmetric (permutation) group (the finite Coxeter group of type A_n). To translate the geometry of the confidence net into group theory requires the theory of buildings (given in Section 5) and specifically the mapping of intervals into "chambers" and the full collection of intervals, nets, into "galleries" formed by chambers. Because of the strong links with group theory, we also put this paper forward as a contribution to the rapidly developing area of "algebraic statistics", in which there has been renewed interest in permutation tests; see, for example, Morton *et al.* [25].

There is a long tradition of the study of "statistics" (also called indices) such as the length function, on groups. For example, Reiner [27] studied the extension of such statistics from the

symmetric groups to type B_n. Adin and Roichman [2] defined a new index called the flag major index whose length was equidistributed in the type B case. They used this to study group actions on polynomial rings. Geometric distance problems in genomic rearrangements can be reduced to Coxeter length problems [10,11]. In statistics, Diaconis [7], Chapter 4C, makes the connection between length distributions and non-parametric tests.

Our general formula (Theorem 4.2) agrees with that for B_n and D_n already derived in Sections 3.1 and 3.2, using a counting argument on the raw inequalities describing the cones of the groups. The generating functions for the exceptional groups E_6, E_7, E_8 and for the groups of type A_n are given as examples after the main proof. The net in the case of E_8 has a remarkable 93 cells. While these E_n cases can only be used for sample sizes 6, 7 and 8 respectively, they are nonetheless of independent interest. The paper concludes with short sections on an example not in the group class, and some simple asymptotics.

2. Confidence nets

Let Y be a random n-vector with probability density function $f(y, \theta)$, where θ is an unknown k-dimensional parameter. For most of this paper, we will study the case $k = 1$, but begin in this section with the general set-up. We assume that Y can be transformed by a measurable transformation $T(y, \theta)$, typically θ-dependent, to a random variable Z:

$$Z = T(Y, \theta),$$

which is also n-dimensional and has a distribution some of whose properties are known, independently of θ.

Assume there exists a finite collection of sets $\{C_i, i = 1, \ldots, m\}$, such that:

1. $\bigcup C_i = \mathbb{R}^n$.
2. The measure with respect to Z of any intersection $C_i \cap C_j, i \neq j$, is zero.
3. $\text{prob}\{Z \in C_i\} = \alpha_i, i = 1, \ldots, m$.
4. The α_i are positive, do not depend on θ, and $\sum_{i=1}^m \alpha_i = 1$.

Define, for fixed y

$$S_i(y) = \{\theta : Z \in C_i\}.$$

Thus, $S_i(y)$ is the inverse of the function $T(y, \theta)$ for fixed y and

$$\text{prob}\{S_i(y) \ni \theta\} = \alpha_i.$$

We should note that typically k (the dimension of θ) is very much smaller than n.

A confidence net is based on the following coarsening in the description of the coverage sets S_i using geometric considerations. Suppose that there are N random sets $\{U_j(Y), j = 0, \ldots, N-1\}$ in θ-space whose intersections cover θ with zero probability, $\bigcup_{j=0}^{N-1} U_j = \mathbb{R}^k$, and such that for any $i = 1, \ldots, m$, there is a mapping $j = u(i)$ such that

$$S_i(y) = U_j(y),$$

and moreover that every U_j can be obtained in this way. The mapping $u(\cdot)$ is typically a many-to-one mapping, and given any j we can define the pre-image:

$$u^{-1}(j) = \{i : j = u(i)\}.$$

This implies that the $U_j(Y)$ are themselves (random) coverage sets with coverage probabilities

$$p_j = \text{prob}\{U_j(Y) \ni \theta\} = \sum_{i \in u^{-1}(j)} \alpha_i$$

for $j = 0, \ldots, N - 1$. Note also that since $\sum_{j=0}^{N-1} \alpha_i = 1$, we have $\sum_{i=0}^{N-1} p_i = 1$. We refer to the set $\{U_j(Y)\}$ as an *exact confidence net*. We summarise this in the following definition.

Definition 2.1. *For a parametric statistical model with random variable Y (possibly multivariate), an* (exact) confidence net *is a collection of data dependent sets $U_j(Y)$, $j = 0, \ldots, N - 1$ whose union is the whole parameter space, and such that the probability that $U_j(Y)$ covers the parameter θ is a known quantity α_j, $j = 0, \ldots, N - 1$ and such that any intersection of the U_j covers θ with probability zero.*

Here we take a classical statistical approach to coverage nets. Thus, the notion is that the user *declares* the sets U_j. Theories of inference based on collections of coverage sets can be thought of as part of a well developed theory of belief functions based on upper and lower probabilities and the theory of random sets based on Choquet capacities (see [6,30]). In terms of the former, a coverage net is essentially a theory of random sets in which the upper and lower probabilities coincide, and in which the Choquet capacity functional is additive over the σ-algebra of unions of sets. That is to say, for the sets $\{U_j\}$ we have for $i \neq j$,

$$\text{prob}\{U_i \cup U_j \ni \theta\} = \text{prob}\{U_i \ni \theta\} + \text{prob}\{U_j \ni \theta\},$$

and so on.

3. Reflection groups and cones

Let Y be an n-dimensional random vector and θ be a univariate parameter ($k = 1$). Define

$$Z(Y, \theta) = (Y_1 - \theta, Y_2 - \theta, \ldots, Y_n - \theta)^T.$$

Let G be a finite reflection group acting on $\mathbb{R}^n$ and let $\{C_i, i = 1, \ldots, m\}$, where $m = |G|$, be the collection of cones in $\mathbb{R}^n$ that are the transformations under G of the fundamental cone C_1. Our key condition, corresponding to condition (3) in Section 2, is that every such cone has the same probability content with respect to the distribution of Z:

$$\text{prob}\{Z \in C_i\} = \frac{1}{m}, \qquad i = 1, \ldots, |G|.$$

Each statement $\{y \in C_i\}$ yields a statement $\theta \in U_j$. To find $\{U_j\}$ we need to provide the mapping $i \mapsto j = u(i)$. The U_j are intervals and it is enough to give their endpoints. Then, for each j we can define the count $N_j = |u^{-1}(j)|$, namely the number of cones giving $U_j(y)$. Then under our assumptions

$$p_j = \text{prob}\{U_j \ni \theta\} = \frac{N_j}{m}, \qquad j = 0, \ldots, N-1.$$

Fortunately, although the orders of the groups can be very large, the geometry of finite reflection groups can be understood in terms of their root systems, and the number of roots is orders of magnitude smaller. In all that follows the counts N_j will have a factor of $n!$, the order of the symmetric group, and it is somewhat neater, therefore, to work with $n_j = \frac{N_j}{n!}$. The main objective of this paper is to find, for different groups, the generating function for the $\{n_j\}$:

$$G(q) = \sum_{j=0}^{N-1} n_j q^j.$$

Every finite reflection group is defined by its *roots*. These are vectors $\{a_j\}$ that define the perpendiculars to the defining hyperplanes

$$H_j = \left\{ x : a_j^T x = 0 \right\}$$

forming the walls of the cones C_i. Roots are identified with half-spaces and therefore come in pairs: $\pm a_j$, which are important in the classification of these groups (for more details on root systems and the classification of finite reflection groups, see for instance [20]).

Before we proceed to a general approach, in the next two subsections we will use an elementary discussion of inequalities and a counting argument to derive the generating functions $G(q)$, in two cases.

3.1. The hyperoctahedral groups: Type B_n

The group of type B_n (we will refer to the group simply as "B_n") operates on points $z \in \mathbb{R}^n$ by permutation and sign change of the coordinates. It has order $2^n n!$. Its fundamental cone C_1 is (by convention) given by

$$z_1 \geq z_2 \geq \cdots \geq z_n \geq 0,$$

and it has fundamental roots given by each of the inequalities above. In standard notation the roots are

$$\{e_1 - e_2, e_2 - e_3, \ldots, e_{n-1} - e_n, e_n\},$$

where the e_i are unit vectors. The fundamental roots are thus

$$(1, -1, 0, \ldots, 0)^T, (0, 1, -1, 0, \ldots, 0)^T, \ldots, (0, \ldots, 0, 1)^T.$$

It is important to repeat that all other roots come from transformation of these roots under the group. Each cone C_i is obtained by transformation of C_1 under a suitable group element. We can describe the cones compactly by inequalities:

$$\pm z_{\pi(1)} \geq \pm z_{\pi(2)} \geq \cdots \geq \pm z_{\pi(n)} \geq 0,$$

where $\pi = (\pi(1), \ldots, \pi(n))$ ranges over all $n!$ permutations of $\{1, \ldots, n\}$.

Substituting $z_i = y_i - \theta$ for $i = 1, \ldots, n$, we have

$$\pm(y_{\pi(1)} - \theta) \geq \pm(y_{\pi(2)} - \theta) \geq \cdots \geq \pm(y_{\pi(n)} - \theta) \geq 0.$$

Thus, every cone C_i is defined by a set of inequalities for θ, each of which yields a U_j interval for θ.

Consider the case B_3. First fix the order z_1, z_2, z_3. The set of inequalities which describes the cones for this order is

$$+(y_1 - \theta) \geq +(y_2 - \theta) \geq +(y_3 - \theta) \geq 0,$$
$$+(y_1 - \theta) \geq +(y_2 - \theta) \geq -(y_3 - \theta) \geq 0,$$
$$+(y_1 - \theta) \geq -(y_2 - \theta) \geq +(y_3 - \theta) \geq 0,$$
$$-(y_1 - \theta) \geq +(y_2 - \theta) \geq +(y_3 - \theta) \geq 0,$$
$$+(y_1 - \theta) \geq -(y_2 - \theta) \geq -(y_3 - \theta) \geq 0,$$
$$-(y_1 - \theta) \geq +(y_2 - \theta) \geq -(y_3 - \theta) \geq 0,$$
$$-(y_1 - \theta) \geq -(y_2 - \theta) \geq +(y_3 - \theta) \geq 0,$$
$$-(y_1 - \theta) \geq -(y_2 - \theta) \geq -(y_3 - \theta) \geq 0.$$

We are interested in the index of the interval that covers θ, that is, its *position* among the subset means, because this determines the interval U_j for θ. There are $\binom{3}{2} + 3 = 6$ subset means, namely the pairwise means $\frac{y_i + y_j}{2}$ ($i \neq j$) and the individual y_i's. For instance, the first row of inequalities above yields $\theta \leq y_1, y_2, y_3$, and consequently θ is also less than each of the pairwise means, placing it in the zero-th position in interval U_0. A less trivial example is the fourth row of inequalities, whose manipulations yield

$$y_1, \frac{y_1 + y_2}{2}, \frac{y_1 + y_3}{2} \leq \theta \leq y_2, y_3, \frac{y_2 + y_3}{2},$$

so that θ is covered by U_3. Note that other inequalities, such as $y_3 \leq y_2$, follow from the same row of the list of inequalities above, but these do not affect the coverage interval for θ. The fifth row of inequalities also places θ in third position. Repeating this for each row, we obtain θ, respectively, in intervals: $U_0, U_1, U_2, U_3, U_3, U_4, U_5, U_6$ (noting the double representation of the middle interval). We obtain the same distribution for all permutations π. This gives the n_i count as $1, 1, 1, 2, 1, 1, 1$, with generating function

$$G(q) = 1 + q + q^2 + 2q^3 + q^4 + q^5 + q^6.$$

For B_4 a similar calculation yields $4!2^4 = 384$ sets of inequalities, in blocks of $2^4 = 16$, one block of inequalities for each permutation, as for B_3. There are 11 intervals formed by the 10 values $\{y_i, \frac{y_i+y_j}{2}, i, j = 1, \ldots, 4, i \neq j\}$. The n_i count is 1, 1, 1, 2, 2, 2, 2, 2, 1, 1, 1 (summing to 16) with generating function

$$G(q) = 1 + q + q^2 + 2q^3 + 2q^4 + 2q^5 + 2q^6 + 2q^7 + q^8 + q^9 + q^{10}.$$

From these examples, we see how to evaluate the vector $(n_0, \ldots, n_{N-1})$ for B_n. For any line of inequalities (cone) the kth interval covers θ if and only if there are exactly k elements from the set of possible boundaries $\{y_i, \frac{y_i+y_j}{2}, i < j\}$ less than or equal to θ. Thus, for our initial permutation and a particular cone:

$$n_k = \left|\{i : y_i \leq \theta\}\right| + \left|\left\{(i, j) : i \leq j; \frac{y_i + y_j}{2} \leq \theta\right\}\right|.$$

Now consider which sign combinations on the $(y_i - \theta)$ lead to a contribution to n_k. The possibilities are:

1. $y_i \leq \theta$: a single $-$ at position i.
2. $\frac{y_i+y_j}{2} \leq \theta$: a pair $-, +$ in positions $i < j$, respectively.
3. $\frac{y_i+y_j}{2} \leq \theta$: a pair $-, -$ in positions $i < j$, respectively.

Define indicator functions which capture the sign combination: $x_i = 1, 0$ for $-, +$ in the ith position, respectively. We can then set up a counting function to capture n_i:

$$\psi(x_1, \ldots, x_n) = \sum_{i=1}^{n} x_i + \sum_{i<j}^{n} x_i(1 - x_j) + \sum_{i<j}^{n} x_i x_j = \sum_{i=1}^{n} i x_i.$$

Now, considering the x_i as independent Bernoulli random variables, each $\{i x_i\}$ is independent with support $\{0, i\}$ and probability generating function $\frac{1}{2}(1 + q^i)$, by convolution, and multiplying by 2^n, we see that

$$G_n(q) = (1 + q)(1 + q^2) \cdots (1 + q^n),$$

which we confirm in the cases $n = 3, n = 4$, above.

3.2. The groups of type D_n

The group of type D_n (which we again will call simply the group D_n) is the group of permutations with an even number of sign changes, and has order $2^{n-1}n!$. It has fundamental cone

$$z_1 \geq z_2 \geq \cdots \geq z_n, \qquad z_{n-1} + z_n \geq 0.$$

The roots are

$$\{e_1 - e_2, e_2 - e_3, \ldots, e_{n-1} - e_n, e_{n-1} + e_n\},$$

giving

$$(1, -1, 0, \ldots, 0)^T, (0, 1, -1, 0, \ldots, 0)^T, \ldots, (0, \ldots, 0, 1, -1)^T, (0, \ldots, 0, 1, 1)^T.$$

In the case $n = 4$, the inequalities are

$$z_1 \geq z_2 \geq z_3 \geq z_4; \qquad z_3 \geq -z_4,$$

giving

$$y_1 - \theta \geq y_2 - \theta \geq y_3 - \theta \geq y_4 - \theta; \qquad y_3 - \theta \geq -(y_4 - \theta).$$

Now D_4 allows permutations of the coordinates as well as *even* numbers of sign changes. Therefore, the possible signs are as follows (the vertical and horizontal lines will be explained shortly):

$$
\begin{array}{ccc|c}
+ & + & + & + \\
- & - & + & + \\
- & + & - & + \\
+ & - & - & + \\
\hline
- & + & + & - \\
+ & - & + & - \\
+ & + & - & - \\
- & - & - & - \\
\end{array}
$$

The second line, for example, gives

$$-(y_1 - \theta) \geq -(y_2 - \theta) \geq y_3 - \theta \geq y_4 - \theta; \qquad y_3 - \theta \geq -(y_4 - \theta),$$

from which we deduce

$$\theta \geq \frac{y_1 + y_3}{2}, \frac{y_1 + y_4}{2}, \frac{y_2 + y_3}{2}, \frac{y_2 + y_4}{2}, \frac{y_1 + y_2}{2};$$
$$\theta \leq \frac{y_3 + y_4}{2}.$$

The inequality $\theta \geq \frac{y_1 + y_2}{2}$ is found by first noting that $\theta \geq y_2 \geq y_1$. It is tempting to include the singletons y_i in the set of boundary points, but not all y_j can be determined in this way which means that intervals using the y_i are not fully computable.

Following the last remark, we determine coverage of θ given by all pair means $\frac{y_i + y_j}{2}, i < j$.

We shall need to account for the following possibilities using slightly more complicated rules than for B_n:

1. $y_i \leq \theta$: a single $-$ at position i, for $i = 1, \ldots, n - 1$. This is used to help place the pair-means.
2. $\frac{y_i + y_n}{2} \leq \theta$: $-$ in position i and $-$ in position n for $i = 1, \ldots, n - 1$.
3. $\frac{y_i + y_j}{2} \leq \theta$ for $1 \leq i < j \leq n$: a pair $-, +$ in positions i, j respectively,

4. $\frac{y_i+y_j}{2} \le \theta$ for $1 \le i < j \le n-1$: a pair $-,-$ in positions i, j respectively. This follows by noting that $y_i, y_j \le \theta$, as in rule (1), above.

We can split rule (3), above, into two cases: when the $+$ is in position $1, \ldots, n-1$; and when the $+$ is in position n (hence the vertical line in the preceding figure). The latter can be combined with rule 2 to give rule 1. Again we take the indicator with $x_i = 1$ for $-$ and $x_i = 0$ for $+$ and our counter is

$$\psi(x) = \sum_{i=1}^{n-1} x_i + \sum_{i<j}^{n-1} x_i(1-x_j) + \sum_{i<j}^{n-1} x_i x_j = \sum_{i=1}^{n-1} i x^i.$$

Thus, the generating function for the n_i, again using a convolution argument, is

$$G_{n-1}(q) = (1+q^2)(1+q^3)\cdots(1+q^{n-1}).$$

In the following section, we re-derive these generating functions in a general framework, using the theory of buildings.

4. The main result

As mentioned in the Introduction we will use some theory developed around the concept of indices (sometimes call "statistics") attached to an element g of a group G. MacMahon [23] discussed, for the symmetric group, *descent, excedance, length* and the *major index*. Authors are often interested in the frequency of the distinct values of an index as g ranges over the whole group, and there are strong combinatorial results, going back to MacMahon, showing that one index has the same distribution as another, even though the actual indices (as mappings) are different. This work is relevant for us because (i) we have a special index which is the value j of our interval U_j of the net construction; (ii) generating functions play an important role; and (iii) the study of such indices is being extended to finite reflection groups such as B_n and D_n.

A starting point for the construction of these indices is the Cayley graph of a group. If S is the set of generators of our group, then the Cayley graph is a graph $(E, V)_G$ where each vertex $v_g \in V$ is labelled by a group element $g \in G$ and each edge $e_{g,h}$ by a single right multiplication by a generator $s \in S$: $h = gs$; only generators may be used. The length, $l(g)$ of a group element $g \in G$ is the length of the minimal path on the graph from the identity e to g, when each edge counts unity:

$$l(g) = \min\{k \ge 0 : g = s_{i_1} s_{i_2} \cdots s_{i_k}, \text{ for } s_{i_1} \in S\}.$$

The Cayley graph for the group B_2 has two generators which we may take (on the (z_1, z_2) plane) as (i) s_1 the reflection in the line $z_1 = z_2$ and (ii) s_2 the reflection in the $z_1 = 0$ axis. The Cayley graph and corresponding lengths are given in Figure 1.

The length frequency distribution is $\{f_0, f_1, \ldots, f_m\}$ where $f_j = \#\{g : l(g) = j, g \in G\}$ and m is the diameter of the group. The generating function for the length frequencies is
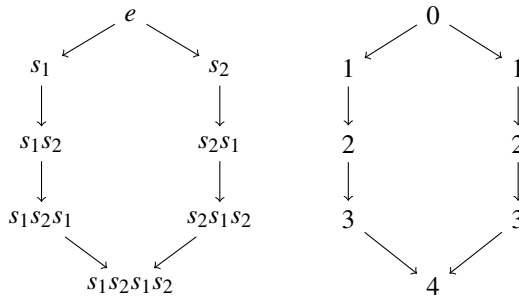
$$G(q) = \sum_{j=0}^{m} f_j q^j.$$

Figure 1. The Cayley graph for B_2 with its elements' lengths.

We can compute $G(q)$ using the Chevalley factorization theorem (see for instance [20], Section 3.15):

Theorem 4.1. *Let W be an irreducible Coxeter group. Then the length generating function is*

$$G_W(q) = \prod_{j=1}^{m} \frac{1 - q^{d_j}}{1 - q},$$

where $d_1, \ldots, d_m$ are the basic invariant degrees of the group.

Table 1, taken from [20], Section 3.7, lists the degrees for the crystallographic Coxeter groups. The polynomial $G_W(q)$ is known as the *Poincaré polynomial* of W.

With our running B_2 example,

$$G_W(q) = \frac{(1 - q^2)(1 - q^4)}{(1 - q)^2} = 1 + 2q + 2q^2 + 2q^3 + q^4,$$

giving the frequencies $(1, 2, 2, 2, 1)$, as expected from the graph in Figure 1.

Table 1. Degrees for the crystallographic Coxeter groups

Type	$d_1, d_2, \ldots$
A_n	$2, 3, \ldots, n + 1$
B_n, C_n	$2, 4, \ldots, 2n - 2, 2n$
D_n	$2, 4, 6, \ldots, 2n - 2, n$
E_6	$2, 5, 6, 8, 9, 12$
E_7	$2, 6, 8, 10, 12, 14, 18$
E_8	$2, 8, 12, 14, 18, 20, 24, 30$
F_4	$2, 6, 8, 12$
G_2	$2, 6$

The length distribution for the symmetric group is

$$G_{S_n}(q) = \prod_{i=1}^{n} \frac{1 - q^i}{1 - q}.$$

In the main theorem, which follows, the formula is obtained by dividing the generating function for the length distribution of our group given in Theorem 4.1, by that for the symmetric group.

Theorem 4.2. *The generating function for the frequency distribution for the intervals of the confidence net based on a finite irreducible Coxeter group G of any type except F_4 is given by*

$$G(q) = \frac{\prod_{j=1}^{m}(1 - q^{d_j})}{\prod_{i=1}^{n}(1 - q^j)},$$

where $d_1, \ldots, d_m$ are the basic invariant degrees of the group.

The exclusion of F_4 in the theorem statement is necessary because the result depends on the symmetric group being a maximal parabolic subgroup of G, which holds in all cases except F_4 (see for instance [12], Appendix A). We leave the calculation of the generating function for F_4 as an exercise along the lines of the examples in Sections 3.1 and 3.2.

Before we prove this theorem (in Section 6), we demonstrate with two examples that its results agree with those calculated in Sections 3.1 and 3.2.

For B_n the formula in Theorem 4.2 gives

$$\frac{G_W(q)}{G_n(q)} = \frac{\prod_{i=1}^{n}(1 - q^{2i})}{\prod_{i=1}^{n}(1 - q^j)} = \prod_{i=1}^{n}(1 + q^i),$$

as expected. Note that we have two ways of counting the number of intervals: the number of live roots, following Lemma 5.1, and the degree of $G(s)$:

$$n + \binom{n}{2} = \sum_{j=1}^{n} j.$$

For D_n the formula is

$$\frac{G_W(q)}{G_n(q)} = \frac{\prod_{j=1}^{n-1}(1 - q^{2j})(1 - q^n)}{\prod_{i=1}^{n}(1 - q^j)} = \prod_{j=1}^{n-1}(1 + q^j),$$

again, as expected.

Before we proceed to the proof of Theorem 4.2, we need to introduce some more of the tools of the theory of buildings.

5. Rays, chambers and Cayley graphs

Returning to our construction from Section 3, in vector notation we have

$$Z(y, \theta) = y - \theta \mathbf{j}, \tag{5.1}$$

where $\mathbf{j} = (1, 1, \dots, 1)^T$. For fixed y, we shall refer to the one dimensional affine subspace defined by (5.1), as θ varies, as the *ray* from y, denoted E_y:

$$E_y = \{z : z = y - \theta \mathbf{j}, \theta \in \mathbb{R}\}.$$

We have:

Lemma 5.1. *Let* $\{C_i\}$ *be the collection of all cones generated in the standard way by a finite reflection group, and let* y *be a non-zero vector.*

1. *If* y *is in general position* (*not lying in any defining hyperplane* H_i), *then the ray* E_y *intersects the faces of a fixed number* N *of the cones* C_i *at values* $\theta_1(y) < \theta_2(y) < \cdots < \theta_{N-1}(y)$.
2. *Any* $\theta_j(y)$ *is given by*

$$\theta_j(y) = \frac{a^T y}{a^T a},$$

   for some positive root a *which is not orthogonal to* $\mathbf{j}$.
3. $N - 1$ *is the number of roots not orthogonal to* $\mathbf{j}$.

Proof. By elementary geometry, when y is in general position the ray E_y intersects every defining hyperplane H_i exactly once except when $\mathbf{j}$ lies in an H_i, in which case it does not intersect that H_i. Because for any hyperplane H_i, its root a_i, by definition defines the orthogonal subspace to H_i, the latter condition is equivalent to being orthogonal to a_i. Each cone has two intersection points except for the end cones when the intersection are at $\theta_1(y)$ and $\theta_{N-1}(y)$. Part (2) follows since the intersection points satisfy: $a_j^T(y - \theta \mathbf{j}) = 0$. $\qquad\square$

We refer to hyperplanes H_i as being *live* if their roots are not orthogonal to $\mathbf{j}$. The intervals we require are

$$U_0 = (-\infty, \theta_1(y)], \qquad U_1 = [\theta_1(y), \theta_2(y)], \qquad \dots,$$
$$U_{N-2} = [\theta_{N-2}(y), \theta_{N-1}(y)], \qquad U_{N-1} = [\theta_{N-1}(y), \infty).$$

To prove Theorem 4.2, we need to introduce some of the group-theoretic geometry behind it. An excellent reference for further reading on this topic is [1], Chapter 1.

The *chamber graph* of a finite reflection group has cones (called chambers in this context) as vertices, with two cones having an edge if they share a common face. A path in the chamber graph is called a *gallery*: imagine a walk through chambers with doors in the common wall (facet). With each edge given length unity, distance between chambers is defined (as for a Cayley graph)

by the shortest distance between the chambers, and we call the corresponding gallery *minimal*. A gallery is minimal if it does not cross any wall more than once ([1], Proposition 1.56). Since a straight line in general position (in an obvious sense) cannot cut any wall of a chamber more than once, it defines a minimal gallery. For both the Cayley graph and the chamber graph, C_e is the cone corresponding to the identity element e of the group, and we call this the fundamental cone.

Following Lemma 5.1, the ray E_y defines a gallery that we denote G_y. This gallery starts in the identity chamber C_e and has length N (there are N chambers along it). The index j of a chamber C_g yielding the interval

$$U_j = \big[\theta_j(y), \theta_{j+1}(y)\big]$$

is the distance in the gallery G_y from C_e to C_g. Consequently, the number of cones $|u^{-1}(j)|$ that map into a given index j is the number of group elements of distance j from C_e along the gallery G_y.

Now consider reflections in the walls of a chamber. Suppose this chamber is a translation by w of the fundamental chamber C_e, so that its faces are translations of the fundamental hyperplanes that are the faces of C_e. If H_s is a face of C_e (for a generator s of G), then it is translated by w to wH_s, and reflection in this hyperplane corresponds to action by the reflection wsw^{-1} (in general this is not a fundamental reflection). Thus, reflection in the face wH_s of wC_e gives the chamber given by the left multiplication of w by wsw^{-1}, namely $wsw^{-1}w = ws$, or wsC_e. In other words, we move from the chamber wC_e to the chamber wsC_e. Thus, movement along a gallery corresponds to *right* multiplication by a generator.

As an aside, it is worth noting that the movement along the gallery by right multiplication provides a correspondence between the chamber graph and the Cayley graph, in which the movement along edges is given by left multiplication $w \to sw$. The chamber graph, however, is the natural place for our results because it has a very direct link with the geometry.

6. Proof of Theorem 4.2

Theorem 4.2 states, in effect, that the distribution of distances of group elements along galleries defined by the rays E_y is the same as the distribution of lengths of minimal coset representatives when the quotient of G is taken by the symmetric group S_n. This is because the numerator of this generating function is the Poincaré polynomial of the group, and the denominator is that of the symmetric group (see [20], Section 1.11 for more details). For background reading on the theory of reflection groups and buildings there are many good sources, but we recommend in particular Abramenko and Brown [1], Humphreys [20], and Kane [22].

To prove Theorem 4.2, it suffices to show that the set of group elements along the galleries defined by the rays E_y is precisely the set of minimal length coset representatives of S_n in G, for G of the types given in the theorem. We prove this in Proposition 6.3, below, but first a short lemma.

Lemma 6.1. *The roots from S_n are all orthogonal to $\mathbf{j} = (1, 1, \ldots, 1)^T$.*

Proof. Action by any element of S_n fixes j; that is, reflection in any root from S_n fixes **j**, which means that the root must be orthogonal to **j**.   □

There are some well-known facts about Coxeter groups that we refer to in what follows, gathered in the lemma below.

Lemma 6.2. *Let W be a finite Coxeter group, and W' a parabolic subgroup of W.*

1. *The minimal length elements of the cosets of W' in W are unique.*
2. *The minimal length elements of the cosets of W' in W add in length when multiplied by any element of W'.*
3. *If W' is a parabolic subgroup of W, then the length distribution of distinguished coset representatives of W' in W is given by $G_W(t)/G_{W'}(t)$, where $G_W(t)$ and $G_{W'}(t)$ are the respective Poincaré polynomials.*
4. *The length of the longest word w_0 in S_n is the number of positive roots in the root system of S_n.*

Proof. These statements are all given in various texts, but in particular all are in [20]: for (1) and (2) see [20], Section 1.10; for (3) see [20], Section 1.11; and for (4) see [20], Section 1.8. □

Proposition 6.3. *For chambers defined by the action of a finite Coxeter group on $\mathbb{R}^n$, let the gallery G_y be the series of adjacent chambers beginning with C_e, defined by the ray E_y where y is some point in C_e.*

(i) *The group elements labelling chambers in G_y are all minimal length S_n-coset representatives.*
(ii) *Every S_n-coset has its minimal length element appearing on a gallery G_y for some y.*

Proof. First, note that S_n is a parabolic subgroup of every finite Coxeter group G except F_4, and we consider G acting on $\mathbb{R}^n$ ([12], Appendix A). Explicitly: S_n is a parabolic subgroup of the groups of types A_n, B_n and D_n that we consider acting on $\mathbb{R}^n$, and of the groups of types E_n for $n = 6, 7, 8$ (acting on $\mathbb{R}^n$); S_3 and S_4 are parabolic subgroups of the groups H_3 and H_4 acting on $\mathbb{R}^3$ and $\mathbb{R}^4$, respectively; and S_2 is (rather trivially) a parabolic subgroup of each of the dihedral groups $I_2(m)$ acting on $\mathbb{R}^2$.

Part (i). We begin by showing that the element corresponding to the *last* chamber in the gallery is in the same coset as the longest word in the group G. The length of the longest word w_0 in S_n is the number of positive roots in the root system of S_n (Lemma 6.2(4)), which is the number of roots orthogonal to **j** (Lemma 6.1). Let $\mathcal{C}$ be the S_n-coset containing w_0. Because minimal coset elements add in length with any element of S_n (Lemma 6.2(2)), the minimal coset representative in $\mathcal{C}$ must have length the number of positive roots *not* orthogonal to **j**.

An element of $\mathcal{C}$ appears in some gallery G_y, for some y, by Lemma 6.3. The number of chambers in each gallery is the number of roots not orthogonal to **j**. Since crossing each hyperplane from one chamber to the next along the gallery adds at most 1 in length, the longest element on any gallery is at most length the number of roots not orthogonal to **j**. Therefore the longest word

of the group must be in a coset whose minimal length element is precisely the number of roots not orthogonal to $\mathbf{j}$. This can only be the last chamber in the gallery.

We now show that all other group elements on the gallery are minimal right coset representatives.

Take a minimal right coset representative $s_{i_1} \cdots s_{i_m}$ on the gallery G_y. We claim that the preceding element on the gallery, $s_{i_2} \cdots s_{i_m}$, is also a minimal length coset representative. If not, then there is a $w \in S_n$ satisfying $\ell(s_{i_2} \cdots s_{i_m} w) < \ell(w) + (m-1)$ (length is additive for minimal coset representatives in Coxeter groups; see Lemma 6.2(2)). But then $\ell(s_{i_1} s_{i_2} \cdots s_{i_m} w) < \ell(w) + (m-1) + 1$ since multiplying by a generator can add at most 1 to the minimal length. That is, $\ell(s_{i_2} \cdots s_{i_m} w) < \ell(w) + m$, contradicting the minimality of $s_{i_1} \cdots s_{i_m}$. It follows that all elements corresponding to cones along a gallery G_y are minimal coset representatives.

Part (ii). It suffices to show that each S_n coset contains an element in the gallery G_y. Consider a point $Z = (Z_1, \ldots, Z_n)$ in the fundamental cone that is also on the ray $E_y = \{y - \theta \mathbf{j} | \theta \in \mathbb{R}\}$. The inequality its coordinates must satisfy is

$$Z_n \geq \cdots \geq Z_1 \geq 0 \tag{6.1}$$

for $Z = (Z_1, \ldots, Z_n)$. Because the entries in $\mathbf{j}$ are all equal, this means $y_n \geq \cdots \geq y_1 \geq 0$ for $y = (y_1, \ldots, y_n)$.

Moving along the ray E_y in the positive direction (decreasing θ) does not change the inequalities in equation (6.1) (all components stay positive) and hence the ray stays in the fundamental cone. Increasing θ moves the ray through the gallery into different cones as first $Z_n \geq \cdots \geq Z_2 \geq 0 \geq Z_1$, then $Z_n \geq \cdots \geq Z_3 \geq 0 \geq Z_2 \geq Z_1$ and so on.

On the other hand, acting by S_n on a point on the ray permutes the entries, but this fixes the entries of $\mathbf{j}$ and simply permutes the entries of y. So a chamber on the gallery corresponds to an ordering of form $Z_n \geq \cdots \geq Z_{i+1} \geq 0 \geq Z_i \geq \cdots \geq Z_1$, and the other chambers in its S_n coset are obtained by permuting these entries.

Now consider an arbitrary point v in $\mathbb{R}^n$ and denote the cone it is contained within by $C_{w(v)}$. The action of S_n permutes the entries of v, and there is a permutation that puts the entries in increasing order. Every point that is in increasing order is in a cone that is on a ray-gallery, so we are done. $\qquad \square$

We are now in a position to prove our main result. Recall that this gives a generating function for the frequency distribution of intervals in the confidence net based on an irreducible finite Coxeter group.

Proof of Theorem 4.2. The set of all rays E_y from the identity chamber to the last chamber (labelled by g) gives the set of all possible galleries from 1 to g. Each group element in each gallery is a minimum length coset representative of S_n in G, and all S_n-cosets have their minimal length representative occurring in such a gallery (Proposition 6.3).

Recall that $u^{-1}(j)$ is the set of chambers that are in the jth position along a ray E_y. When we start with the identity chamber, this is simply the set of group elements of length j that appear in galleries. From Proposition 6.3, this is the set of minimal coset representatives of length j. So the number n_j is the number of minimal coset representatives of length j, and this is given by our formula, by Lemma 6.2(3). $\qquad \square$

7. Further examples

In Sections 3.1 and 3.2, we gave examples of confidence nets from our theory for types B and D respectively, and recalculated them using Theorem 4.2 after its statement. Here we add types E_6, E_7, E_8 and A_n. The remaining types of finite Coxeter group (omitted) are types H_3 and H_4, the dihedral groups $I_2(m)$, and the group F_4 (to which Theorem 4.2 does not apply).

7.1. Type E

Inserting the d_j values for E_6, E_7 and E_8 from Table 1 and obtaining help in factorization from Maple we have the following formulae:

$$E_6 :\ (q+1)(q^2+1)(q^2-q+1)(q^4-q^2+1)(q^2+q+1)(q^6+q^3+1)(q^4+1)$$

$$
\begin{aligned}
E_7 :\ & (q+1)^4(q^2-q+1)^2(q^6+q^3+1)(q^6-q^3+1) \\
& (q^6-q^5+q^4-q^3+q^2-q+1) \\
& (q^2+1)(q^2+q+1)(q^4-q^2+1)(q^4-q^3+q^2-q+1)(q^4+1)
\end{aligned}
$$

$$
\begin{aligned}
E_8 :\ & (q^4+q^3+q^2+q+1)(q^6+q^3+1)(q^6-q^3+1)(q^4+1) \\
& (q^6-q^5+q^4-q^3+q^2-q+1)(q^8-q^7+q^5-q^4+q^3-q+1) \\
& (q^8+q^7-q^5-q^4-q^3+q+1)(q^8-q^6+q^4-q^2+1)(q^8-q^4+1) \\
& (q^2+q+1)^2(q^4-q^3+q^2-q+1)^2(q^2+1)^2 \\
& (q^4-q^2+1)^2(q^2-q+1)^3(q+1)^4.
\end{aligned}
$$

Let us consider E_8 in a little more detail. It has order $2^{14}3^5 5^2 7 = 696\,729\,600$, meaning that $\mathbb{R}^8$ is split into this many cones. The root system is described in the standard way as:

$$\{\pm e_i \pm e_j : 1 \le i < j\}, \qquad \left\{ \frac{1}{2}\sum_{i=1}^{8}\lambda_i e_i : \lambda_i = \pm 1, \prod_{i=1}^{8}\lambda_i = 1 \right\}.$$

Again we have two ways of counting. The number of live roots are those not orthogonal to $\mathbf{j} = (1, 1, 1, 1, 1, 1, 1, 1)^T$. From the first set above, we have those of the form $(1, 1, 0, \ldots)$, namely $\binom{8}{2}$. From the second set we have all those for which the number of ones and zeros is different and even, being careful not to double count. This gives

$$\binom{8}{2} + 1 + \binom{8}{2} + \binom{8}{6} = 92.$$

On the other hand $G(s) = a_0 + a_1 q + \cdots$ is a polynomial of degree 92 whose $N = 93$ coefficients are laid out below to show the symmetry.

1	1	1	2	3	6	6	8	10	13	17
21	26	32	38	46	55	64	74	86	98	112
127	142	157	175	193	211	230	249	267	287	307
325	343	361	377	393	409	421	432	443	452	458
464	466	466	466	464						
458	452	443	432	421	409	393	377	361	343	325
307	287	267	249	230	211	193	175	157	142	127
112	98	86	74	64	55	46	38	32	26	21
17	13	10	8	6	4	3	2	1	1	1

7.2. Type A_n

The usual interpretation of the action of type A_n is as the restriction of the symmetric group S_{n+1} to the hyperplane: $H : \sum_{i=1}^{n=1} x_i = 0$. When $n = 2$ this yields a figure in 2-dimensions with cones with apex angle $\frac{1}{3}\pi$. The role of S_n in the above examples is now played by A_{n-1}. Referring to the first entry in Table 1 this gives the generating function

$$G(q) = \frac{\prod_{i=2}^{n+1}(2 - q^i)}{\prod_{i=2}^{n+1}(2 - q^i)} = \sum_{i=0}^{n+1} q^i,$$

giving a discrete uniform distribution on the net chambers.

The statistical interpretation takes a little care. There are different choices one can make for the representation of A_{n-1} as a subgroup of A_n. A simple choice is for A_{n-1} to be the restriction to the hyperplane H of the group that permutes the first n coordinates. Let us require that the "data" Y also lies in H and that the model is given by

$$Y = \theta \mathbf{k} + Z,$$

where $\text{prob}\{Z \in C_i\} = \frac{1}{(n+1)!}$ and the C_i are the cones of A_n in H (with similar assumptions as in the Introduction). The key is to make the vector $\mathbf{k}$, which is the analogue of the previous $\mathbf{j}$, to be invariant under A_{n-1}. Thus, we can take

$$\mathbf{k} = (1, 1, \ldots, 1, -n)^T.$$

Following Lemma 5.1, we find the boundary of the net chamber by taking the intersection of the ray $Y - \theta \mathbf{k}$ with the live root of A_n that is all those not orthogonal to $\mathbf{k}$. These are

$$(1, 0, \ldots, 0, -1)^T, (0, 1, \ldots, 0, -1)^T, \ldots, (0, 0, \ldots, 0, 1, -1)^T.$$

Taking the jth member of this list first we see that the boundary is given by

$$y_j - \theta - (y_{n+1} + n\theta) = y_j - y_{n+1} - (n + 1)\theta.$$

But since $Y \in H$, we have $y_{n+1} = -\sum_{i=1}^{n} y_i$. This means that the boundaries are the n sample quantities

$$\frac{1}{n+1}\left(2y_j + \sum_{i \neq j}^{n} x_i\right), \qquad j = 1, \ldots, n,$$

giving $n+1$ chambers, as expected.

8. A non-group cone example

Exact coverage nets also arise for the situation in which $\mathbb{R}^n$ is divided into cones that are congruent, but not arising as the fundamental cones of a reflection group. Consider the partition of the positive orthant into n cones generate by a "long diagonal" and $n-1$ principal axes. Leaving out the first principal axis, we obtain generators:

$$(1, 1, \ldots, 1)^T, (0, 1, 0, \ldots, 0)^T, (0, 0, 1, \ldots, 0)^T, \ldots, (0, 0, \ldots, 1)^T.$$

The other cones are generated by successively omitting principal axes. Now take all sign changes to reach all other quadrants. This divides $\mathbb{R}^n$ into $n2^n$ congruent cones.

Assume the Z-probability content of each cone is equal and apply the method used for the other examples. We first check how many, and which, walls are cut by a typical ray, and group together the cones which lead to the same "index", as above. The number of planes is $2n+1$. After a little work, it turns out that the intervals formed by the order statistics $y_{(1)} < y_{(2)} < \cdots < y_{(n)}$ and all neighbour pairs $\frac{y_{(i)}+y_{(j)}}{2}$ form a net of $2n$ intervals.

The successive net vectors (ignoring commas) are the rows below for $n = 2, \ldots, 6$.

			1	1	1	1			
		1	1	2	2	1	1		
	1	2	3	3	3	3	1	1	
1	1	4	4	6	6	4	4	1	1
1 1	5	5	10	10	10	10	5	5	1	1

Note how each row is constructed by repeating the integer of the previous row of the Pascal triangle eg the row 1, 6, 15, 20, 15, 6, 1 is split $6 \to (1, 5), 15 \to (5, 10), 20 \to (10, 10)$, giving the last row of the tableau above. The generating function is

$$(1+q)(1+q^2)^{n-1}.$$

9. Some asymptotics

The generating function for B_n is well-known in the theory of partitions. It is the generating function for the number partition of an integer into at most n distinct parts. The infinite version $G(q) = \prod(1+q^i)_{i=1}^{\infty}$ gives the number of partitions into j distinct parts, with no other

restrictions, and the two generating functions are identical up to q^n. The general $G(q)$ has a long history. Following their celebrated work on partitions [15], Hardy and Ramanujan also studied this case, giving an asymptotic formula, see [14,19]. For an extensive review, see [3].

Noting the convergence of the Binomial distribution to the Normal (and following computer experimentation), it is natural to conjecture that for $G_n(s)$ the $\{a_n\}$ follow an asymptotic distribution, and indeed this is the case. The associated probability distribution is that of the random variable

$$U = \sum_{j=1}^{n} j V_j,$$

where the V_i are i.i.d. Bernoulli random variables with probability $\frac{1}{2}$. Then, a theorem of Hájek and Šidák [13] for sums of independent random variables with unequal means and variance gives $U \sim N(\mu, \sigma)$ where $\mu = \frac{1}{2}\sum_{j=1}^{n} = \frac{1}{4}n(n+1)$ and $\sigma^2 = \frac{1}{4}\sum_{j=1}^{n} j^2 = \frac{1}{24}n(n+1)(2n+1)$.

An Edgeworth-type expansion shows that the standardized random variable $\frac{U-\mu}{s}$ can be approximated by

$$\phi(u)\left(1 + \frac{\kappa_4}{24}\mathcal{H}_4(u)\right),$$

where ϕ is the standard Normal density, κ_4 is the fourth cumulant of the standardized variable, and $\mathcal{H}_4 = u^4 - 6u^2 + 3$ is the order 4 standard Hermite polynomial.

After a little work we derive, for B_n,

$$\kappa_4 = -\frac{12}{5}\frac{3n^2 + 3n - 1}{n(n+1)(2n+1)}.$$

Keeping the $\mathrm{O}(\frac{1}{n})$ terms, we have the approximation

$$\phi(u)\left(1 - \frac{3}{20}\mathcal{H}_4(u)\frac{1}{n} + \mathrm{O}\left(\frac{1}{n^2}\right)\right).$$

There are similar result for D_n.

For E_8 the distribution mean and variance are $(\mu, \sigma^2) = (46, \frac{6811}{3})$ and probabilities roughly follow a normal distribution with this mean and variance. The approximation using H_4 is surprisingly good. For the standardized distribution

$$\frac{\kappa_4}{24} = -\frac{365\,311}{28\,896\,080} = -0.01264\ldots.$$

Converting the approximation back to the original cell probabilities the maximum absolute deviation and the root mean squared error are approximately 1.5×10^{-4} and 7.3×10^{-5}, respectively. For the Edgeworth-type approximation, the integral (or in this case the sum) of the approximate probability will typically not be unity. In this case, the sum of the approximands is $1.0001534\ldots$ so that the error is of the same order as the maximum deviation.

10. Conclusions and further work

This paper is a contribution to coverage problems in which, essentially, there are only group symmetry conditions on the underlying distribution. The most obvious limitation of the present paper is that it only covers a single parameter, although much classical non-parametrics is of this type. The long term aim is to use the ideas of this paper to develop coverage nets based on chambers in more than one dimension. Roughly, the requirements are (i) a large group that houses the distributional assumptions, (ii) a smaller sub-group under which the (linear) model is invariant, (iii) a valid quotient or coset operation, (iv) the use of the Chevalley factorization formula to perform the counting, and (v) more extensive use of the theory of buildings. Another challenge is to apply the theory to infinite groups such as affine Weyl groups which already have applications in physics, material science and genomics [5,10,26]. Finally, there may be theory with upper and lower probabilities where exactness is hard to find which would lead to some kind of group-based belief functions.

Acknowledgments

References

[1] Abramenko, P. and Brown, K.S. (2008). *Buildings*: *Theory and Applications*. *Graduate Texts in Mathematics* **248**. New York: Springer. MR2439729

[2] Adin, R.M. and Roichman, Y. (2001). The flag major index and group actions on polynomial rings. *European J. Combin*. **22** 431–446. MR1829737

[3] Andrews, G.E. (1998). *The Theory of Partitions*. Cambridge: Cambridge Univ. Press. MR1634067

[4] Atkins, J.E. and Sherman, G.J. (1992). Sets of typical subsamples. *Statist*. *Probab*. *Lett*. **14** 115–117. MR1173408

[5] Bodner, M., Patera, J. and Peterson, M. (2012). Affine reflection groups for tiling applications: Knot theory and DNA. *J. Math*. *Phys*. **53** 013516, 21. MR2919555

[6] Dempster, A.P. (1967). Upper and lower probabilities induced by a multivalued mapping. *Ann*. *Math*. *Stat*. **38** 325–339. MR0207001

[7] Diaconis, P. (1988). *Group Representations in Probability and Statistics*. *Institute of Mathematical Statistics Lecture Notes—Monograph Series* **11**. Hayward, CA: IMS. MR0964069

[8] Efron, B. (1982). *The Jackknife*, *the Bootstrap and Other Resampling Plans*. *CBMS-NSF Regional Conference Series in Applied Mathematics* **38**. Philadelphia, PA: SIAM. MR0659849

[9] Efron, B. and Tibshirani, R. (1986). Bootstrap methods for standard errors, confidence intervals, and other measures of statistical accuracy. *Statist*. *Sci*. **1** 54–77. MR0833275

[10] Egri-Nagy, A., Gebhardt, V., Tanaka, M.M. and Francis, A.R. (2014). Group-theoretic models of the inversion process in bacterial genomes. *J. Math*. *Biol*. **69** 243–265. MR3215080

[11] Francis, A.R. (2014). An algebraic view of bacterial genome evolution. *J. Math*. *Biol*. **69** 1693–1718. MR3275210

[12] Geck, M. and Pfeiffer, G. (2000). *Characters of Finite Coxeter Groups and Iwahori–Hecke Algebras. London Mathematical Society Monographs. New Series* **21**. New York: The Clarendon Press. MR1778802

[13] Hájek, J. and Šidák, Z. (1967). *Theory of Rank Tests*. Prague: Academic Press. MR0229351

[14] Hardy, G.H., Littlewood, J.E. and Pólya, G. (1988). *Inequalities. Cambridge Mathematical Library*. Cambridge: Cambridge Univ. Press. MR0944909

[15] Hardy, G.H. and Ramanujan, S. (1918). Asymptotic Formule in Combinatory Analysis. *Proc. Lond. Math. Soc*. **S2-17** 75. MR1575586

[16] Hartigan, J.A. (1969). Using subsample values as typical values. *J. Amer. Statist. Assoc*. **64** 1303–1317. MR0261737

[17] Hartigan, J.A. (1975). Necessary and sufficient conditions for asymptotic joint normality of a statistic and its subsample values. *Ann. Statist*. **3** 573–580. MR0391346

[18] Hodges, J.L. Jr. and Lehmann, E.L. (1963). Estimates of location based on rank tests. *Ann. Math. Stat*. **34** 598–611. MR0152070

[19] Hua, L. (1942). On the number of partitions of a number into unequal parts. *Trans. Amer. Math. Soc*. **51** 194–201. MR0006195

[20] Humphreys, J.E. (1990). *Reflection Groups and Coxeter Groups. Cambridge Studies in Advanced Mathematics* **29**. Cambridge: Cambridge Univ. Press. MR1066460

[21] Jurečková, J. and Kalina, J. (2012). Nonparametric multivariate rank tests and their unbiasedness. *Bernoulli* **18** 229–251. MR2888705

[22] Kane, R. (2001). *Reflection Groups and Invariant Theory*. New York: Springer. MR1838580

[23] MacMahon, P.A. (1915–1916). *Combinatorial Analysis*, *Vols. 1 and* 2. Cambridge: Cambridge University Press. Reprinted by Chelsea, New York, 1960.

[24] Mitic, P. (1996). Critical values for the wilcoxon signed rank statistic. *Mathematica Journal* **6** 73–77.

[25] Morton, J., Pachter, L., Shiu, A., Sturmfels, B. and Wienand, O. (2009). Convex rank tests and semigraphoids. *SIAM J. Discrete Math*. **23** 1117–1134. MR2538642

[26] Prince, E. (1994). *Mathematical Techniques in Crystallography and Material Science*, 2nd ed. Berlin: Springer. MR1313726

[27] Reiner, V. (1993). Signed permutation statistics. *European J. Combin*. **14** 553–567. MR1248063

[28] Tritchler, D. (1984). On inverting permutation tests. *J. Amer. Statist. Assoc*. **79** 200–207. MR0742870

[29] Van de Wiel, M.A., Di Bucchianico, A. and Van der Laan, P. (1999). Symbolic computation and exact distributions of nonparametric test statistics. *J. R. Stat. Soc*., *Ser. D Stat*. **48** 507–516.

[30] Wasserman, L.A. and Kadane, J.B. (1990). Bayes' theorem for Choquet capacities. *Ann. Statist*. **18** 1328–1339. MR1062711

[31] Wilks, S.S. (1948). Order statistics. *Bull. Amer. Math. Soc*. **54** 6–50. MR0025122