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Bridge mixtures of random walks on an Abelian group

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In this paper, we characterize (mixtures of) bridges of a continuous time random walk with values in a countable Abelian group. Our main tool is a conditional version of Mecke's formula from the point process theory, which allows us to study, as transformation on the path space, the addition of random loops. Thanks to the lattice structure of the set of loops, we even obtain a sharp characterization. At the end, we discuss several examples to illustrate the richness of such random processes. We observe in particular how their structure depends on the algebraic properties of the underlying group.

Keywords: random walk on Abelian group; reciprocal class; stochastic bridge

Introduction

Given a reference Markov probability on the time interval [0, 1], define the set of all probability measures obtained as mixtures of its bridges. This set was called *reciprocal class* by Jamison since all these probability measures enjoy a time symmetry property called reciprocal property, which is a weaker version of the Markov property. For a detailed comparison between these properties, we refer to the recent survey [9]. These processes were first introduced by Schrödinger in [19] to study the dynamics of a Brownian particle with prescribed laws at the initial and the final times, see, for example, [8]. Jamison initiated later in a series of papers [5–7] a rigorous mathematical study of these processes. Krener underlined the importance of some quantities related to the bridges of diffusions, which he then called *reciprocal characteristics*. The problem of computing, interpreting and using them to characterize bridge mixtures has attracted the attention of many authors in the context of diffusions (see, e.g., [1,16,20,21]).

The study of mixtures of bridges of particular jump processes has been started more recently by Murr with the case of counting processes, see [12] and [4]. Then results concerning the characterization of random bridges of a compound Poisson process have been obtained in [2], in the particular case where (i) the state space is $\mathbb{R}^d$ and (ii) the support of the jump measure is a finite set of A different types of jumps. There, the approach is to study separately the jump-times of the paths and their type distribution.

In this paper, we propose to characterize bridge mixtures of random walks in the following more general framework: the state space is a *countable Abelian group* G. See, for example, [17] for a review on random walks on groups.

Our tool is, by working directly at the level of the path space, to exhibit a family of characterizing integral equations. The equations (8) we obtain can be viewed as a generalization of (the iterated) Mecke's formula, which characterizes Poisson random measures via transformations which consist in adding one point to the canonical process. Indeed, first we add several jumps to the canonical process, and secondly we work under the constraint that the added paths are loops, that is they should have as initial and final value the identity element.

However, our method is efficient only if one can assure that the set of loop paths is rich enough to allow to transform any given path of the random walk into any other one having the same initial and final value, only by adding and removing a finite number of well chosen elementary loops. This assumption on the support $G_\nu \subseteq G$ of the jump measure ν is formalized through (H1) and (H2), see Section 2.2.

As an interesting byproduct of our integral characterization of the bridge mixtures of a ν-random walk, we get the identification thanks to equation (8) of the associated family of reciprocal characteristics (9). These quantities remain unchanged on the whole set of random walks having the same bridges, see Corollary 6.

The paper is organized as follows. In Section 1, we set up the necessary definitions and notations regarding random walks on groups, and provide a formula on the path space satisfied by them. In Section 2, we state and prove our main result: the integral formula derived before on the path space is in fact an efficient way to characterize the whole set of bridge mixtures of a random walk. In the last section, we present several examples to illustrate the richness of the kind of processes we are dealing with.

1. Random walk on Abelian groups

1.1. The random walk as Poisson random measure

Let $(G, +)$ be a countable Abelian group with identity element e. We denote by $\mathbb{D}([0, 1], G)$ the space of càdlàg paths for the topology induced by the discrete metric in G. Note that, because of the existence of left and right limits, paths in $\mathbb{D}([0, 1], G)$ have finitely many jumps. $\mathbb{D}([0, 1], G)$ is equipped with its canonical sigma-algebra $\mathcal{F}$ and its canonical filtration $(\mathcal{F}_t)_{t\in[0,1]}$.

For any ν nonnegative finite measure on G, we call *ν-random walk* on G a Markov probability on $\mathbb{D}([0, 1], G)$ denoted by $\mathbb{P}_\nu$ whose infinitesimal generator $\mathscr{G}$ is given by

$$(\mathscr{G}\phi)(g) := \sum_{g'\in G} \nu(g')\big(\phi(g + g') - \phi(g)\big), \qquad g \in G,$$

for any ϕ bounded function. In the rest of the paper, $G_\nu \subseteq G$ denotes the support of ν, that is the set of allowed jumps of the ν-random walk. The path space $\Omega \subset \mathbb{D}([0, 1], G)$ is the set of paths with jumps in G_ν.

Mecke proved in [10] an integral characterization of Poisson point processes on general spaces which we recall in Proposition 1 in a form adapted to our framework. In the spirit of Murr [13] and Privault ([14], Section 6.4.4) who studied real-valued processes with independent increments, we turn Mecke's formula into a characterization of random walks on G in Proposition 2. Let us first introduce some notations.

For a measurable space $\mathcal{X}$, we denote by $\mathcal{P}(\mathcal{X})$ the set of probability measures on $\mathcal{X}$ and by $\mathcal{M}(\mathcal{X})$ the set of finite point measures, that is

$$\mathcal{M}(\mathcal{X}) := \left\{ \sum_{i=1}^{N} \delta_{x_i} : x_i \in \mathcal{X}, N \in \mathbb{N} \right\}.$$

$\mathcal{B}^+(\mathcal{X})$ denotes the set of nonnegative bounded measurable functions over $\mathcal{X}$. We will often choose for $\mathcal{X}$ the following product space Γ of time-space elements:

$$\Gamma := [0,1] \times G \qquad \ni \gamma = (t,g). \tag{1}$$

We identify trajectories in $\mathbb{D}([0,1], G)$ and point measures in $\mathcal{M}(\Gamma)$ via the following canonical bijective map M:

$$X \mapsto M_X := \sum_{0 \le t \le 1} \sum_{g \in G} \delta_{(t,g)} \mathbf{1}_{\{\Delta X_t = g\}}. \tag{2}$$

A useful observation is that the image measure of the ν-random walk under M is a Poisson random measure on Γ with intensity the finite measure $dt \otimes \nu$.

1.2. An integral characterization and its iteration

Mecke's original idea was to characterize any Poisson random measure by mean of an integral formula (see Satz 3.1 in [10]), via the change of measures which consists to add one (random) atom to the initial point measure, as in the right-hand side of equation (3). Adapted to our context it reads as follows.

Proposition 1. *For $\tilde{\mathbb{P}} \in \mathcal{P}(\mathcal{M}(\Gamma))$ the following assertions are equivalent*:

(i) *$\tilde{\mathbb{P}}$ is the Poisson random measure with intensity measure $\rho = dt \otimes \nu$ on Γ.*
(ii) *For all $\Phi \in \mathcal{B}^+(\mathcal{M}(\Gamma) \times \Gamma)$,*

$$\int \int_\Gamma \Phi(\mu, \gamma) \mu(d\gamma) \tilde{\mathbb{P}}(d\mu) = \int \int_\Gamma \Phi(\mu + \delta_\gamma, \gamma) \rho(d\gamma) \tilde{\mathbb{P}}(d\mu). \tag{3}$$

Remark that the left-hand side of (3) also reads $\int \sum_{\gamma \in \mu} \Phi(\mu, \gamma) \tilde{\mathbb{P}}(d\mu)$ where the notation $\gamma \in \mu$ means that the points $\gamma \in \Gamma$ build the support of the point measure μ: one integrates the function Φ under the Campbell measure associated with $\tilde{\mathbb{P}}$. Thus, (3) determines the Campbell measure of a Poisson random measure as the shifted product measure of itself with its intensity.

Let us adapt this tool to $\mathbb{D}([0,1], G)$. First, for $\gamma = (t,g) \in \Gamma$, we consider the simple step function $g\mathbf{1}_{[t,1]} \in \mathbb{D}([0,1], G)$ and then the transformation $X \mapsto X + g\mathbf{1}_{[t,1]}$ on the path space which consists in adding one jump g at time t. It corresponds to the addition of one atom to a random measure. Indeed, under any probability $\mathbb{P} \in \mathcal{P}(\mathbb{D}([0,1], G))$ satisfying $\mathbb{P}(X_t = X_{t^-}) = 1$ for all $t \in [0,1]$, one has:

$$M_{X + g\mathbf{1}_{[t,1]}} = M_X + \delta_\gamma \qquad \mathbb{P}\text{-a.s.} \tag{4}$$

This simple observation is all what is needed to rewrite Proposition 1 in the language of random walks.

Proposition 2. *For* $\mathbb{P} \in \mathcal{P}(\mathbb{D}([0,1], G))$ *the following assertions are equivalent*:

(i) $\mathbb{P}$ *is a* ν*-random walk on* G.
(ii) *For all* $F \in \mathcal{B}^+(\mathbb{D}([0,1], G) \times \Gamma)$,

$$E_{\mathbb{P}}\biggl(\int_{\Gamma} F(X,\gamma) M_X(d\gamma)\biggr) = E_{\mathbb{P}}\biggl(\int_{\Gamma} F(X + g\mathbf{1}_{[t,1]}, \gamma)\rho(d\gamma)\biggr), \tag{5}$$

where M_X *is defined through* (2).

Proof. (i) $\Rightarrow$ (ii). Since $\mathbb{P}_\nu$ is ν-random walk, M_X is a Poisson random measure with intensity $dt \otimes \nu$. Then Mecke's formula holds for $\tilde{\mathbb{P}} := \mathbb{P} \circ M^{-1}$. Since M is invertible and its inverse is measurable we can plug into (3) test functions Φ of the form $F(X,\gamma)$ and the conclusion follows.

(ii) $\Rightarrow$ (i). Let $\mathbb{P} \in \mathcal{P}(\mathbb{D}([0,1], G))$ satisfying (5). We define $\tilde{\mathbb{P}} := \mathbb{P} \circ M^{-1} \in \mathcal{P}(\mathcal{M}(\Gamma))$. Then, by considering test functions of the form $\Phi = F(M_X, \gamma)$ and using the fact that $M_{X+g\mathbf{1}_{[t,1]}} = M_X + \delta_\gamma \mathbb{P} \otimes \rho$-a.s., we deduce that $\tilde{\mathbb{P}}$ is a Poisson random measure with intensity $\rho = dt \otimes \nu$ by Proposition 1. Observing that

$$X_t = \sum_{g \in G} g M_X\bigl([0,t] \times \{g\}\bigr)$$

the conclusion follows using (4). $\square$

To prepare the characterization of bridges which we will prove in the next section, we now present an iteration of the formula (5). For this purpose, we define the set Δ_n, union of all diagonals of Γ^n:

$$\Delta_n := \bigl\{\bar{\gamma} \in \Gamma^n : \exists i \neq j, \gamma_i = \gamma_j\bigr\}.$$

In the above definition, and in all what follows, a typical element of Γ^n is denoted by $\bar{\gamma}$ (recall the definition of Γ in (1)). We are ready to state the multivariate Mecke formula satisfied under the ν-random walk.

Proposition 3. *Let* $\mathbb{P}_\nu$ *be a* ν*-random walk on* G. *Then, for any* $n \geq 1$ *and any test function* $F \in \mathcal{B}^+(\mathbb{D}([0,1], G) \times \Gamma^n)$,

$$\begin{aligned} &E_{\mathbb{P}_\nu}\biggl(\int_{\Gamma^n \setminus \Delta_n} F(X, \bar{\gamma}) M_X^{\otimes n}(d\bar{\gamma})\biggr) \\ &\quad = E_{\mathbb{P}_\nu}\biggl(\int_{\Gamma^n} F(X + g_1\mathbf{1}_{[t_1,1]} + \cdots + g_n\mathbf{1}_{[t_n,1]}, \bar{\gamma})\rho^{\otimes n}(d\bar{\gamma})\biggr). \end{aligned} \tag{6}$$

We do not prove this formula here. A proof can be found in Chapter 5 of Mecke's habilitation, see [11] for an english translation, or in the book [18], Corollary 3.2.3.

Remark 4. In general it is not true that $\int_{\Delta_n} F(X,\bar{\gamma})M_X^{\otimes n}(d\bar{\gamma})=0$. Indeed, if γ is an atom of M_X then $\underbrace{(\gamma,\dots,\gamma)}_{n \text{ times}}$ is an atom of $M_X^{\otimes n}$ which belongs to Δ_n.

2. Bridge mixtures and their characterization

2.1. Random bridges of a random walk

First, consider the set of pairs $(x,y)\in G^2$ for which the bridge of the ν-random walk is meaningful:

$$S(\nu) := \left\{(x,y)\in G^2 : \mathbb{P}_\nu(X_1=y|X_0=x)>0\right\}.$$

Then, for $(x,y)\in S(\nu)$ the *bridge* $\mathbb{P}_\nu^{xy}$ between x and y is defined by

$$\mathbb{P}_\nu^{xy}(\cdot) := \frac{\mathbb{P}_\nu(\cdot\cap\{X_0=x,X_1=y\})}{\mathbb{P}_\nu(X_0=x,X_1=y)}.$$

We now define our main object of study. It is the set of probability measures on Ω that can be written as a mixture of the bridges of $\mathbb{P}_\nu$:

$$\mathrm{Rec}(\nu)=\left\{\mathbb{Q}\in\mathcal{P}(\Omega):\mathbb{Q}=\int_{S(\nu)}\mathbb{P}_\nu^{xy}\mathbb{Q}_{01}(dx\,dy)\right\},$$

where $\mathbb{Q}_{01}$ denotes the joint marginal law of $\mathbb{Q}$ at times 0 and 1. Let us note that for (x,y) fixed in $S(\nu)$, the bridge $\mathbb{P}_\nu^{xy}$ itself belongs to the set $\mathrm{Rec}(\nu)$.

For a recent and short review on bridge mixtures via a stochastic analysis approach and the treatment of basic examples we refer the reader for example, to [15].

2.2. Loops and their skeletons

We call *loop* a path in $\mathbb{D}([0,1],G)$ that starts and ends at the identity element e. For each path $X\in\Omega$, we define its *skeleton* as the application $\varphi_X:G_\nu\to\mathbb{N}$ defined by:

$$\varphi_X(g) := M_X\big([0,1]\times\{g\}\big).$$

Thus, $\varphi_X(g)$ counts how many times the jump g occurs along the path X.

If X is a loop, we observe that

$$\sum_{g\in G_\nu}\varphi_X(g)g=e.$$

Therefore, as X varies in the set of all possible loops, φ_X varies in the set

$$\mathscr{L}^+ := \left\{\varphi \in \mathbb{N}^{G_\nu} : \sum_{g\in G_\nu} \varphi(g)g = e, \ell(\varphi) < +\infty\right\}, \tag{7}$$

where $\ell(\varphi) := \sum_{g\in G_\nu} |\varphi(g)|$ is the *length* of φ. Enlarging this set to the maps φ with negatives values by considering

$$\mathscr{L} := \left\{\varphi \in \mathbb{Z}^{G_\nu} : \sum_{g\in G_\nu} \varphi(g)g = e, \ell(\varphi) < +\infty\right\},$$

one recovers for $\mathscr{L}$ a lattice structure, which will be very useful. In particular, $\mathscr{L}$ admits a *basis* $\mathscr{B}$. Suppose now that one can choose $\mathscr{B} \subset \mathscr{L}^+$, which is the case if the following assumption (H1) is satisfied:

$$\mathrm{Span}(\mathscr{L}^+) = \mathscr{L}, \tag{H1}$$

where $\mathrm{Span}(\mathscr{L}^+)$ is, as usual, the set of all integer combinations of elements of $\mathscr{L}^+$. From now on, we fix such a basis $\mathscr{B}$.

To any $\varphi^* \in \mathscr{B}$, we can associate the – nonempty – set of loops whose skeleton is φ^*:

$$\Omega_{e,\varphi^*} := \{X \in \Omega : X_0 = X_1 = e \text{ and } \varphi_X = \varphi^*\}.$$

These paths have exactly $\varphi^*(g)$ jumps of type g, for all $g \in G_\nu$.

Furthermore, we have to assume that each jump in G_ν belongs to (at least) the skeleton of one loop, that is, the following assumption holds:

$$\forall g \in G_\nu \qquad \text{there exists } \varphi \in \mathscr{L} \text{ such that } \varphi(g) > 0. \tag{H2}$$

Note that w.l.o.g. we can assume that this skeleton φ belongs indeed to the basis $\mathscr{B}$.

As we shall see in Section 2.4, assumptions (H1) and (H2) allow a fruitful decomposition of the path space Ω. Heuristically, one can transform one path into any other one having the same initial and final values, by subsequently adding and removing loops whose skeleton belongs to $\mathscr{B}$. However, let us first state our main result.

2.3. Main result: An integral characterization of bridge mixtures

In the next theorem, we state that the identity (6) appeared in Proposition 3 is not only valid under any mixture of bridges of $\mathbb{P}_\nu$ but indeed characterizes them, if one restricts the set of test functions F to some well chosen subset.

For each skeleton φ^* in the basis $\mathscr{B}$, consider the following set of bounded measurable test functions on $\mathbb{D}([0,1], G) \times \Gamma^{\ell(\varphi^*)}$:

$$\mathscr{H}_{\varphi^*} := \{F : F(X, \bar{\gamma}) \equiv \mathbf{1}_{\{g_1 \mathbf{1}_{[t_1,1]} + \cdots + g_{\ell(\varphi^*)} \mathbf{1}_{[t_{\ell(\varphi^*)},1]} \in \Omega_{e,\varphi^*}\}} F(X, \bar{\gamma})\}.$$

Therefore, we will restrict our attention to perturbations of the sample paths consisting in adding a loop whose skeleton is equal to φ^*. Now we are ready for stating and proving the main result.

Theorem 5. *The probability measure $\mathbb{Q} \in \mathcal{P}(\Omega)$ belongs to the set* $\mathrm{Rec}(\nu)$ *if and only if for any skeleton φ^* in the basis $\mathscr{B}$ and for all test functions $F \in \mathscr{H}_{\varphi^*}$, we have*:

$$\begin{aligned} &E_{\mathbb{Q}}\left(\int_{\Gamma^n\setminus\Delta_n} F(X,\bar{\gamma})M_X^{\otimes n}(d\bar{\gamma})\right) \\ &\quad = \Phi^{\nu}_{\varphi^*} E_{\mathbb{Q}}\left(\int_{\Gamma^n} F(X+g_1\mathbf{1}_{[t_1,1]}+\cdots+g_n\mathbf{1}_{[t_n,1]},\bar{\gamma})(dt\otimes\Lambda)^{\otimes n}(d\bar{\gamma})\right), \end{aligned} \tag{8}$$

where $n=\ell(\varphi^*)$, $\Lambda := \sum_{g\in G}\delta_g$ is the counting measure on G and

$$\Phi^{\nu}_{\varphi^*} := \prod_{g\in G_\nu} \nu(g)^{\varphi^*(g)} \in \mathbb{R}^+. \tag{9}$$

In particular, if (8) *holds true under $\mathbb{Q}$ satisfying $\mathbb{Q}(X_0=x, X_1=y)=1$ for some $(x,y)\in S(\nu)$, then $\mathbb{Q}$ is nothing else but the bridge $\mathbb{P}^{xy}_\nu$.*

The result above carries two main messages. First, it shows that a conditional version of the multivariate Mecke formula characterizes bridges of random walks and their mixtures, generalizing the known fact that Mecke formula characterizes random walks. Second, it shows that the natural way to decompose paths of bridges is into loops, rather than into single step functions, as usual.

The positive coefficient $\Phi^{\nu}_{\varphi^*}$ appearing in (8), usually called *reciprocal characteristics*, only depends on the jump measure ν and on the skeleton φ^*. Letting vary φ^* they determine $\mathrm{Rec}(\nu)$ in the following sense.

Corollary 6. *Let ν and μ two nonnegative finite measures on G with the same support. The sets of bridge mixtures* $\mathrm{Rec}(\nu)$ *and* $\mathrm{Rec}(\mu)$ *coincide if and only if*

$$\Phi^{\mu}_{\varphi^*} = \Phi^{\nu}_{\varphi^*}, \qquad \forall \varphi^* \in \mathscr{B}. \tag{10}$$

In that case the bridges of both ν- and μ-random walk on G coincide too.

Remark 7. There is a remarkable probabilistic interpretation of the number $\Phi^{\nu}_{\varphi^*}$ as the leading factor, in the short-time expansion, of the probability that the ν-random walk follows a loop with skeleton φ^*. This is proven for Markov processes on graphs in [3].

2.4. Proof of the main theorem

($\Rightarrow$) We use, as main argument, the specific form of the density with respect to $\mathbb{P}_\nu$ of any probability measure in $\mathrm{Rec}(\nu)$ as it was proved in [2], Proposition 1.5:

$$\mathbb{Q}\in\mathrm{Rec}(\nu) \Rightarrow \mathbb{Q} \ll \mathbb{P}_\nu, \quad \text{and} \quad \frac{d\mathbb{Q}}{d\mathbb{P}_\nu} = h(X_{01}) \qquad \text{for some } h: G\times G\to\mathbb{R}^+,$$

where we write X_{01} for the vector (X_0, X_1). Take now any $F \in \mathscr{H}_{\varphi^*}$. Then, using Identity (6), the definition of $\mathscr{H}_{\varphi^*}$ and the fact that $(X + g_1\mathbf{1}_{[t_1,1]} + \cdots + g_n\mathbf{1}_{[t_n,1]})_{01} = X_{01}$, one gets

$$
\begin{aligned}
&E_{\mathbb{Q}}\left(\int_{\Gamma^n} F(X + g_1\mathbf{1}_{[t_1,1]} + \cdots + g_n\mathbf{1}_{[t_n,1]}, \bar{\gamma})(dt \otimes \nu)^{\otimes n}(d\bar{\gamma})\right) \\
&\overset{F\in\mathscr{H}_{\varphi^*}}{=} \Phi^{\nu}_{\varphi^*} E_{\mathbb{Q}}\left(\int_{\Gamma^n} F(X + g_1\mathbf{1}_{[t_1,1]} + \cdots + g_n\mathbf{1}_{[t_n,1]}, \bar{\gamma})(dt \otimes \Lambda)^{\otimes n}(d\bar{\gamma})\right) \\
&= \Phi^{\nu}_{\varphi^*} E_{\mathbb{P}_\nu}\left(h(X_{01})\int_{\Gamma^n} F(X + g_1\mathbf{1}_{[t_1,1]} + \cdots + g_n\mathbf{1}_{[t_n,1]}, \bar{\gamma})(dt \otimes \Lambda)^{\otimes n}(d\bar{\gamma})\right) \\
&= \Phi^{\nu}_{\varphi^*} E_{\mathbb{P}_\nu}\left(\int_{\Gamma^n} h\big((X + g_1\mathbf{1}_{[t_1,1]} + \cdots + g_n\mathbf{1}_{[t_n,1]})_{01}\big)\right. \\
&\qquad \left.\times F(X + g_1\mathbf{1}_{[t_1,1]} + \cdots + g_n\mathbf{1}_{[t_n,1]}, \bar{\gamma})(dt \otimes \Lambda)^{\otimes n}(d\bar{\gamma})\right) \\
&= E_{\mathbb{P}_\nu}\left(\int_{\Gamma^n\setminus\Delta_n} h(X_{01})F(X,\bar{\gamma})M_X^{\otimes n}(d\bar{\gamma})\right) \\
&= E_{\mathbb{Q}}\left(\int_{\Gamma^n\setminus\Delta_n} F(X,\bar{\gamma})M_X^{\otimes n}(d\bar{\gamma})\right)
\end{aligned}
$$

which completes the proof of the first implication.

($\Leftarrow$) The converse implication is more sophisticated and needs several steps.

Let us introduce the set of paths which correspond to the support of ey-bridges, $y \in G$:

$$\Omega_y := \{X \in \Omega : X_0 = e, X_1 = y\}.$$

Now we partition Ω_y according to the skeleton of its elements:

$$\Omega_y = \bigcup_{\varphi\in\mathscr{L}_y^+} \Omega_{y,\varphi}, \qquad \Omega_{y,\varphi} := \Omega_y \cap \{X \in \Omega : \varphi_X = \varphi\},$$

$$\text{where } \mathscr{L}_y^+ = \left\{\varphi \in \mathbb{N}^{G_\nu} : \sum_{g\in G_\nu} g\varphi(g) = y, \ell(\varphi) < +\infty\right\}.$$

In order to discretize the time, we introduce a mesh $h \in \mathbb{N}^*$ and partition $\Omega_{y,\varphi}$ by specifying the number of different jumps occurred in each h-dyadic interval. That is, we consider functions $\theta : \{0, \ldots, 2^h - 1\} \times G_\nu \longrightarrow \mathbb{N}$ and we look for paths which have $\theta(k, g)$ jumps of type g during the time interval $I_k^h := (2^{-h}k, 2^{-h}(k + 1)]$, for each k and each $g \in G_\nu$. For each skeleton φ, we define the set

$$\Theta_\varphi^h := \left\{\theta : \left\{0, \ldots, 2^h - 1\right\} \times G_\nu \longrightarrow \mathbb{N}, \sum_{0\le k\le 2^h-1} \theta(k, g) = \varphi(g), \forall g \in G_\nu\right\}$$

of all possible h-dyadic time repartition of the jumps, compatible with the skeleton φ. We thus obtain $\Omega_{y,\varphi} = \bigcup_{\theta \in \Theta^h_\varphi} \Omega^{h,\theta}_{y,\varphi}$ where

$$\Omega^{h,\theta}_{y,\varphi} := \{X \in \Omega_y : M_X(I^h_k \times \{g\}) = \theta(k,g), 0 \le k < 2^h, g \in G_\nu\}.$$

Consider the set

$$V := \{v = (\varphi, \theta) \text{ with } \varphi \in \mathscr{L}^+_y, \theta \in \Theta^h_\varphi\} \tag{11}$$

of pairs of skeletons connecting e to y and h-dyadic time repartition of their jumps. Elements of this set are discrete versions of paths of Ω: the spatial structure of the path is given by the skeleton φ, and the time structure is approximated by θ. One equips V with the following l^1-metric:

$$d(v, \tilde{v}) := \sum_{(k,g) \in \{0,\dots,2^h-1\} \times G_\nu} |\theta - \tilde{\theta}|(k,g) \in \mathbb{N}, \qquad v = (\varphi, \theta), \tilde{v} = (\tilde{\varphi}, \tilde{\theta}) \in V.$$

Take now two paths $X, X' \in \Omega_y$ and their trace v, v' on V. Our aim is to find a way to transform X into X' (resp., v into v') by adding or removing a finite number of loops whose skeletons belong to the basis $\mathscr{B}$. Let us introduce the following relation:

$$v_1 = (\varphi_1, \theta_1) \hookrightarrow v_2 = (\varphi_2, \theta_2) \qquad \text{if } \varphi_2 \in \varphi_1 + \mathscr{B} \quad \text{and} \quad \theta_2 - \theta_1 \in \Theta^h_{\varphi_2 - \varphi_1}.$$

We shall now use assumptions (H1) and (H2).

Lemma 8. *For each v and $\tilde{v} \neq v \in V$ on can construct a connecting finite sequence $v_1, \dots, v_N = \tilde{v}_{\tilde{N}}, \tilde{v}_{\tilde{N}-1}, \dots, \tilde{v}_1$ in V such that*

$$v \hookrightarrow v_1 \hookrightarrow \cdots \hookrightarrow v_N = \tilde{v}_{\tilde{N}} \hookleftarrow \tilde{v}_{\tilde{N}-1} \cdots \hookleftarrow \tilde{v}_1 \hookleftarrow \tilde{v}.$$

Proof. We distinguish two cases:

Case (i). The skeletons φ and $\tilde{\varphi}$ coincide.

In this case, it is sufficient to show that we can construct v_1 and $\tilde{v}_1$ in V such that $v \hookrightarrow v_1$, $\tilde{v} \hookrightarrow \tilde{v}_1$, $\varphi_1 = \tilde{\varphi}_1$ and $d(v_1, \tilde{v}_1) \le d(v, \tilde{v}) - 1$. The conclusion would then follow by iterating this procedure until $d(v_K, \tilde{v}_K) = 0$, that is, $v_K = \tilde{v}_K$. At this point, we have constructed a chain from v to v_K, and another one from $\tilde{v}$ to $\tilde{v}_K$. Joining them, we obtain a chain from v to $\tilde{v}$ and the conclusion follows.

Therefore, let us indicate how to construct v_1 and $\tilde{v}_1$. Since $\theta \neq \tilde{\theta}$ but $\varphi = \tilde{\varphi}$ there exists a jump $g \in G_\nu$ and two time intervals I^h_k and I^h_l such that $\theta(k,g) \ge \tilde{\theta}(k,g) + 1$ and $\theta(l,g) \le \tilde{\theta}(l,g) - 1$. Moreover, thanks to (H2) there exists at least one skeleton φ^* in the basis $\mathscr{B}$ containing the jump g: $\varphi^*(g) > 0$. Consider now any time repartition $\theta_1 \in \Theta^h_{\varphi^*}$ such that $\theta_1(l,g) \ge 1$. We then construct $\tilde{\theta}_1$ as follows:

$$\tilde{\theta}_1 = \theta_1 + \mathbf{1}_{\{(k,g)\}} - \mathbf{1}_{\{(l,g)\}}.$$

It is simple to check that $v_1 := (\varphi+\varphi^*, \theta+\theta_1)$, $\tilde{v}_1 := (\varphi+\varphi^*, \tilde{\theta}+\tilde{\theta}_1)$ fulfill the desired requirements. By construction, $v \hookrightarrow v_1$, $\tilde{v} \hookrightarrow \tilde{v}_1$ and $v_1, \tilde{v}_1$ have the same skeleton. Moreover,

$$\left|\theta+\theta_1-(\tilde{\theta}+\tilde{\theta}_1)\right| = |\theta-\tilde{\theta}| - \mathbf{1}_{\{(k,g),(l,g)\}}$$

so that $d(v_1, \tilde{v}_1) = d(v, \tilde{v}) - 2$.

Case (ii). The skeletons φ and $\tilde{\varphi}$ differ.

We first observe that, if $\varphi, \tilde{\varphi} \in \mathscr{L}_y^+$ thus $\varphi - \tilde{\varphi} \in \mathscr{L}$. Since $\mathscr{B}$ is a basis of the lattice $\mathscr{L}$ (see (H1)), there exist $(\varphi_j^*)_{j=1}^K, (\tilde{\varphi}_i^*)_{i=1}^{\tilde{K}} \subseteq \mathscr{B}$ such that

$$\varphi + \sum_{j=1}^{K} \varphi_j^* = \tilde{\varphi} + \sum_{i=1}^{\tilde{K}} \tilde{\varphi}_i^*.$$

Let us now choose for all j and i a time repartition $\theta_j \in \Theta^h_{\tilde{\varphi}_j^*}$ and $\tilde{\theta}_i \in \Theta^h_{\tilde{\varphi}_i}$. It is straightforward to verify that, if we define

$$v_0 = v, \qquad v_j := \left(\varphi + \sum_{j'=1}^{j} \varphi_{j'}^*, \theta + \sum_{j'=1}^{j} \theta_{j'}\right),$$

$$\tilde{v}_0 = \tilde{v}, \qquad \tilde{v}_i := \left(\tilde{\varphi} + \sum_{i'=1}^{i} \tilde{\varphi}_{i'}^*, \tilde{\theta}^* + \sum_{i'=1}^{i} \tilde{\theta}_{i'}\right)$$

then $(v_j)_{j=0}^K$, $(\tilde{v}_i)_{i=0}^{\tilde{K}}$ are two sequences connecting v to v_K and $\tilde{v}$ to $\tilde{v}_{\tilde{K}}$. By construction, $v_K, \tilde{v}_{\tilde{K}}$ have the same skeleton and one can use case (i) again. □

In the next lemma, we compare the probability of the paths in $\Omega^{h,\theta}_{y,\varphi}$ and those obtained by adding a loop with skeleton $\varphi^* \in \mathscr{B}$, under $\mathbb{Q}$ and under $\mathbb{P}_\nu$, see Figure 1.

Lemma 9. *Let $y \in G$, $h \in \mathbb{N}^*$, $\varphi \in \mathscr{L}_y^+$, $\theta \in \Theta^h_\varphi$ be fixed. Suppose* (8) *holds under $\mathbb{Q}$. Then, for any $\varphi^* \in \mathscr{B}$ and $\theta^* \in \Theta^h_{\varphi^*}$,*

$$\frac{\mathbb{Q}(\Omega^{h,\theta+\theta^*}_{y,\varphi+\varphi^*})}{\mathbb{P}_\nu(\Omega^{h,\theta+\theta^*}_{y,\varphi+\varphi^*})} = \frac{\mathbb{Q}(\Omega^{h,\theta}_{y,\varphi})}{\mathbb{P}_\nu(\Omega^{h,\theta}_{y,\varphi})}. \tag{12}$$

Proof. Take an arbitrary ordering of the support of θ^*: $(\tilde{k}_1, \tilde{g}_1), \ldots, (\tilde{k}_N, \tilde{g}_N)$. To simplify the notation, we write θ_j (resp., θ_j^*) for $\theta(\tilde{k}_j, \tilde{g}_j)$ (resp. $\theta^*(\tilde{k}_j, \tilde{g}_j)$). Consider the test function $F(X, \bar{\gamma}) = f(X)v(\bar{\gamma})$, where

$$f = \mathbf{1}_{\Omega^{h,\theta+\theta^*}_{y,\varphi+\varphi^*}} \quad \text{and} \quad v(\bar{\gamma}) = \mathbf{1}_{\Omega^{h,\theta^*}_{e,\varphi^*}}(g_1 \mathbf{1}_{[t_1,1]} + \cdots + g_n \mathbf{1}_{[t_n,1]}) \qquad \text{with } n = \ell(\varphi^*).$$

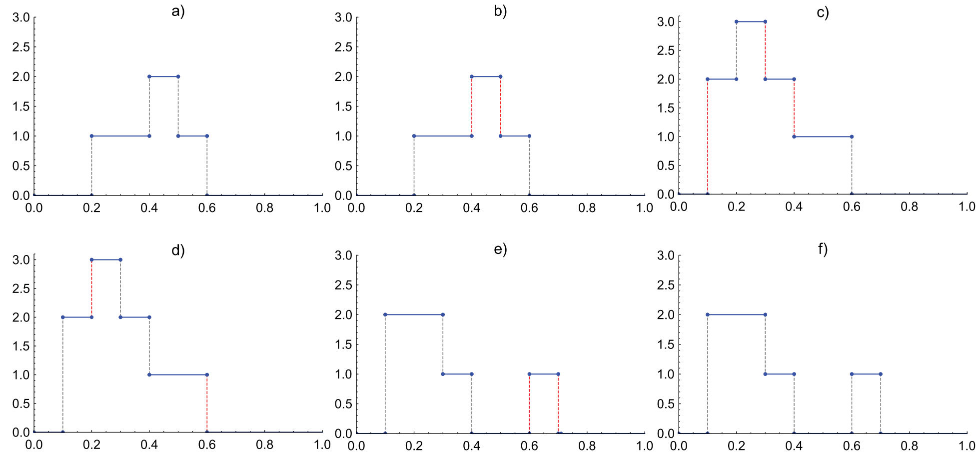

Figure 1. In this picture we illustrate by an example the proof of Lemma 9. Take $G = (\mathbb{Z}, +)$, and $G_\nu = \{-1, 1, 2\}$, situation which is treated in Section 3.1.1. $\mathscr{B} = \{\varphi_1, \varphi_2\}$, where $\varphi_1 := \mathbf{1}_1 + \mathbf{1}_{-1}$ and $\varphi_2 := \mathbf{1}_2 + 2\mathbf{1}_{-1}$, is a basis fulfilling (H1) and (H2). The picture shows how to transform the path (a) in the path (f) by mean of addition and cancellation of loops whose skeleton belongs to $\mathscr{B}$. All loops that are either added or removed are denoted by red dashed lines, which correspond to their jumps. At first, following case (ii), we have to modify the loop (a) to match its skeleton $(2, 2, 0)$ with that of (f), $(3, 1, 1)$. Therefore in (b) we remove a loop with skeleton φ_1, then in (c) add back a loop with skeleton φ_2. The skeleton is now the desired one. Now we follow case (i): we shift one jump of height -1 and one of height 1 further right. Since those two jumps form a loop with skeleton φ_1 we simply delete them in (d) and add a new loop with the same skeleton, but now with the desired jump times in (e).

It is straightforward that

$$f \circ (X + g_1 \mathbf{1}_{[t_1,1]} + \cdots + g_n \mathbf{1}_{[t_n,1]}) v(\bar{\gamma}) = \mathbf{1}_{\Omega^{h,\theta}_{y,\varphi}}(X) v(\bar{\gamma}) \qquad \mathbb{Q} \otimes \rho^n \text{ a.e.}$$

Therefore, since $F \in \mathscr{H}_{\varphi^*}$, (8) holds and its right-hand side rewrites as

$$\Phi^{\nu}_{\varphi^*}\left(\int_{\Gamma^n} v(\bar{\gamma})(dt \otimes \Lambda)^{\otimes n}(d\bar{\gamma})\right)\mathbb{Q}\left(\Omega^{h,\theta}_{y,\varphi}\right).$$

Concerning the left-hand side, let us first rewrite it as

$$E_{\mathbb{Q}}\left(f(X) \int_{\Gamma^n \setminus \Delta_n} v(\bar{\gamma}) M_X^{\otimes n}(d\bar{\gamma})\right).$$

Our aim is to show by a direct computation that the (discrete) stochastic integral $\int_{\Gamma^n \setminus \Delta_n} v(\bar{\gamma}) \times M_X^{\otimes n}(d\bar{\gamma})$ is actually constant for that choice of v if $X \in \Omega^{h,\theta+\theta^*}_{y,\varphi+\varphi^*}$.

First, we observe that an atom $\bar{\gamma} \in \Gamma^n \setminus \Delta_n$ of $M_X^{\otimes n}$ contributes (with the value 1) to the integral if and only if $g_1 \mathbf{1}_{[t_1,1]} + \cdots + g_n \mathbf{1}_{[t_n,1]} \in \Omega_{e,\varphi^*}^{h,\theta^*}$, that is if

$$\sharp\{i : \gamma_i \in I_{k_j}^h \times \{\tilde{g}_j\}\} = \theta_j^*, \qquad 1 \le j \le N. \tag{13}$$

We then need to count the atoms of $M_X^{\otimes n}$ satisfying (13). This is equivalent to count all ordered lists of $n = \ell(\varphi^*)$ atoms of M_X verifying that:

(1) the list contains no repetitions;
(2) for all $1 \le j \le N$, the number of elements in the list which belong to $I_{k_j}^h \times \{\tilde{g}_j\}$ is θ_j^*.

Therefore, for each j, we first choose a subset of cardinality θ_j^* among $\theta_j + \theta_j^*$ elements (recall that $X \in \Omega_{y,\varphi+\varphi^*}^{h,\theta+\theta^*}$). To do that, we have $\binom{\theta_j+\theta_j^*}{\theta_j^*}$ choices. Then we decide how to sort the list, and for this, there are $n!$ possibilities.

Therefore

$$\mathbf{1}_{\Omega_{y,\varphi+\varphi^*}^{h,\theta+\theta^*}}(X) \int_{\Gamma^n \setminus \Delta_n} v(\bar{\gamma}) M_X^{\otimes n}(d\bar{\gamma}) = \mathbf{1}_{\Omega_{y,\varphi+\varphi^*}^{h,\theta+\theta^*}}(X) n! \prod_{j=1}^N \binom{\theta_j + \theta_j^*}{\theta_j^*}$$

and (8) rewrites as

$$\Phi_{\varphi^*}^{\nu} \int_{\Gamma^n} v(\bar{\gamma})(dt \otimes \Lambda)^{\otimes n}(d\bar{\gamma}) \mathbb{Q}(\Omega_{y,\varphi}) = n! \prod_{j=1}^N \binom{\theta_j + \theta_j^*}{\theta_j^*} \mathbb{Q}\big(\Omega_{y,\varphi+\varphi^*}^{h,\theta+\theta^*}\big). \tag{14}$$

Since equation (8) holds under $\mathbb{P}_\nu$, equation (14) holds under $\mathbb{P}_\nu$ as well. Since $\mathbb{P}_\nu$ gives positive probability to both events $\Omega_{y,\varphi}$ and $\Omega_{y,\varphi+\varphi^*}^{h,\theta+\theta^*}$, the identity (12) follows.

Remark that, with the notation of the above lemma, if we define $v := (\varphi, \theta)$ and $w := (\varphi + \varphi^*, \theta + \theta^*)$, then $v \hookrightarrow w$.

Lemma 8 allows us to extend the conclusion of Lemma 9 to the whole set of skeletons, as we will prove now.

Lemma 10. *Let $y \in G, h \in \mathbb{N}^*, \varphi, \tilde{\varphi} \in \mathscr{L}_y^+, \theta \in \Theta_\varphi^h, \tilde{\theta} \in \Theta_{\tilde{\varphi}}^h$ be fixed. Suppose (8) holds under $\mathbb{Q}$. Then,*

$$\frac{\mathbb{Q}(\Omega_{y,\tilde{\varphi}}^{h,\tilde{\theta}})}{\mathbb{P}_\nu(\Omega_{y,\tilde{\varphi}}^{h,\tilde{\theta}})} = \frac{\mathbb{Q}(\Omega_{y,\varphi}^{h,\theta})}{\mathbb{P}_\nu(\Omega_{y,\varphi}^{h,\theta})}. \tag{15}$$

Proof. We observe that $v = (\varphi, \theta)$ and $\tilde{v} = (\tilde{\varphi}, \tilde{\theta})$ are elements of V. As proved above, there exists a connecting sequence $(v_i)_{i=0}^K := (\varphi_i, \theta_i)_{i=0}^K$, with $v_0 = v, v_K = \tilde{v}$, linking v to $\tilde{v}$, and such that either $v_i \hookleftarrow v_{i+1}$ or $v_i \hookrightarrow v_{i+1}$. This entitles us to apply recursively Lemma 9 to any pair v_i, v_{i+1} and obtain

$$\frac{\mathbb{Q}(\Omega^{h,\theta_{i+1}}_{y,\varphi_{i+1}})}{\mathbb{P}_\nu(\Omega^{h,\theta_{i+1}}_{y,\varphi_{i+1}})} = \frac{\mathbb{Q}(\Omega^{h,\theta_i}_{y,\varphi_i})}{\mathbb{P}_\nu(\Omega^{h,\theta_i}_{y,\varphi_i})} = \cdots = \frac{\mathbb{Q}(\Omega^{h,\theta}_{y,\varphi})}{\mathbb{P}_\nu(\Omega^{h,\theta}_{y,\varphi})}.$$

The conclusion follows with $i = N-1$. □

We can now complete the proof of the converse implication of the main theorem.

Fix $x, y \in G$ with $\mathbb{Q}(X_{01} = (x, y)) > 0$. W.l.o.g. we assume that $x = e$. Thanks to Lemma 10 we know that for any mesh h, there exists a positive constant c_h such that

$$\mathbb{Q}\big(\Omega^{h,\theta}_{y,\varphi}\big) = c_h \mathbb{P}_\nu\big(\Omega^{h,\theta}_{y,\varphi}\big), \qquad \forall \varphi \in \mathscr{L}^+_y, \theta \in \Theta^h_\varphi.$$

Now we show that the proportionality constant does not depend on the scale of the time discretisation: $c_h = c_{h+1}$. To this aim, let us observe that

$$\mathbb{Q}(\Omega_y) = \sum_{(\varphi,\theta)\in V} \mathbb{Q}\big(\Omega^{h,\theta}_{y,\varphi}\big) = \sum_{(\varphi,\theta)\in V} c_h \mathbb{P}_\nu\big(\Omega^{h,\theta}_{y,\varphi}\big) = c_h \mathbb{P}_\nu(\Omega_y).$$

In the same way one gets $\mathbb{Q}(\Omega_y) = c_{h+1}\mathbb{P}_\nu(\Omega_y)$ which implies that $c_h = c_{h+1}$. Therefore, there exists a constant $c > 0$ such that

$$\mathbb{Q}\big(\Omega^{h,\theta}_{y,\varphi}\big) = c\mathbb{P}_\nu\big(\Omega^{h,\theta}_{y,\varphi}\big), \qquad \forall h \in \mathbb{N}^*, \varphi \in \mathscr{L}^+_y, \theta \in \Theta^h_\varphi.$$

By standard approximation arguments this implies the equality between $\mathbb{Q}$ and $c\mathbb{P}_\nu$ on $\Omega_y \cap \mathcal{F}$ which then implies $\mathbb{Q}^{ey} = \mathbb{P}^{ey}_\nu$. The conclusion follows. □

Remark 11. Consider the identities (8) for $G = \mathbb{R}^d$ and compute them for particular test functions F which only depend on the skeleton of the paths. These equations, indexed by the skeletons in $\mathscr{B}$, then characterize the (marginal) distribution of the random vector defined as the number of jumps of any type occurred during the time interval [0, 1], as it was done in [2]. Note that for the unconstrained random walk the distribution of this random vector is a multivariate Poisson law, see, for example, [2] Section 2.2.1.

3. Examples

In this section, we present several examples of random walks defined on finite or infinite Abelian groups G.

For each example, we verify if assumptions (H1) and (H2) are satisfied by computing a basis $\mathscr{B}$ of skeleton of loops. We give explicitly the associated characteristics (9). In some cases, we also write down the integral formula (8), highlighting how it is influenced by the geometrical properties of the underlying group G.

Finally, for a fixed random walk $\mathbb{P}_\nu$ on G, we address the question of finding all random walks $\mathbb{P}_\mu$ which have the same bridges than $\mathbb{P}_\nu$, that is, using Corollary 6, we solve equation (10) and identify the set of probability measures:

$$\mathrm{Rec}(\nu) \cap \{\mathbb{P}_\mu : \mu \text{ finite measure on } G_\nu\}.$$

We will see that, in some cases, this set reduces to the singleton $\mathbb{P}_\nu$ and in other cases, this set is nontrivial.

3.1. The group $G = \mathbb{Z}$ is infinite

3.1.1. *The finite support G_ν of the jump measure ν contains* $\{-1, 1\}$

For any $z \in G_\nu \setminus \{1\}$ we define on G_ν the non negative map φ_z as follows:

$$\varphi_z = \mathbf{1}_z + |z|\mathbf{1}_{-\mathrm{sgn}(z)}.$$

It corresponds to the skeleton of paths with one jump of type z and $|z|$ jumps of type $-\mathrm{sgn}(z)$. As candidate for the lattice basis of $\mathscr{L}$, we propose

$$\mathscr{B} := \{\varphi_z\}_{z \in G_\nu \setminus \{1\}}.$$

Assumption (H2) is trivially satisfied and it is clear that the elements of $\mathscr{B}$ are linearly independent. Therefore, we only need to check if $\mathscr{B}$ spans $\mathscr{L}$, that is, if for each $\phi \in \mathscr{L}$, there exist integer coefficients $\alpha_z \in \mathbb{Z}$ such that

$$\forall \bar{z} \in G_\nu, \qquad \phi(\bar{z}) = \sum_{\substack{z \in G_\nu \\ z \neq 1}} \alpha_z \varphi_z(\bar{z}). \tag{16}$$

We now verify that the following choice is the right one:

$$\text{For } z \in G_\nu \setminus \{-1, +1\} \alpha_z = \phi(z) \quad \text{and} \quad \alpha_{-1} = \phi(-1) - \sum_{\substack{z \in G_\nu \\ z > 1}} z\phi(z).$$

- $\bar{z} \notin \{-1, +1\}$. Since $\varphi_{\bar{z}}$ is the only element of $\mathscr{B}$ whose support contains $\bar{z}$, we have

$$\phi(\bar{z}) = \alpha_{\bar{z}} \varphi_{\bar{z}}(\bar{z}) = \sum_{\substack{z \in G_\nu \\ z \neq 1}} \alpha_z \varphi_z(\bar{z}).$$

- $\bar{z} = -1$. Notice that -1 belongs to the support of any φ_z, as soon as $z > 1$. Therefore,

$$\phi(-1) = \sum_{\substack{z \in G_\nu \\ z > 1}} \phi(z) z + \alpha_{-1} = \sum_{\substack{z \in G_\nu \\ z > 1}} \alpha_z \varphi_z(-1) + \alpha_{-1}\varphi_{-1}(-1) = \Bigg(\sum_{\substack{z \in G_\nu \\ z \neq 1}} \alpha_z \varphi_z\Bigg)(-1).$$

- $\bar{z} = 1$. Notice that $+1$ belongs to the support of any φ_z, as soon as $z \le -1$. Recall that $\phi \in \mathscr{L}$. Therefore,

$$\begin{aligned}\phi(1) &= -\sum_{\substack{z\in G_\nu\\ z\neq 1}} \phi(z)z = \sum_{\substack{z\in G_\nu\\ z<1}} -\phi(z)z + \phi(-1)\\ &= \sum_{\substack{z\in G_\nu\\ z\le -1}} \alpha_z\varphi_z(1) = \Biggl(\sum_{\substack{z\in G_\nu\\ z\neq 1}} \alpha_z\varphi_z\Biggr)(1).\end{aligned}$$

Let us now compute the reciprocal characteristics associated to each skeleton in $\mathscr{B}$:

$$\Phi^\nu_{\varphi_z} = \nu\bigl(-\operatorname{sgn}(z)\bigr)^{|z|}\nu(z), \qquad z \in G_\nu \setminus \{1\}.$$

Finally, thanks to Corollary 6, we obtain

$$\begin{aligned}\mathbb{P}_\mu \in \operatorname{Rec}(\nu) \quad &\Leftrightarrow \quad \forall z \in G_\nu \setminus \{1\}, \qquad \mu\bigl(-\operatorname{sgn}(z)\bigr)^{|z|}\mu(z) = \nu\bigl(-\operatorname{sgn}(z)\bigr)^{|z|}\nu(z)\\ &\Leftrightarrow \quad \exists \alpha > 0 \qquad \text{such that } \frac{d\mu}{d\nu}(z) = \alpha^z.\end{aligned}$$

Example 12 (Simple random walks: $G_\nu = \{-1, 1\}$***).*** Due to the above computations, the basis $\mathscr{B}$ of the lattice $\mathscr{L}$ reduces to the singleton $\{\varphi_{-1}\}$ and the unique reciprocal characteristics is given by

$$\Phi^\nu_{\varphi_{-1}} = \nu(-1)\nu(1).$$

Therefore, the only loops which appear in the integral characterization (8) have length $n = \ell(\varphi_{-1}) = 2$. Test functions of the form

$$F\bigl(X, (\gamma_1, \gamma_2)\bigr) = f(X)\mathbf{1}_{\{g_1=1, g_2=-1\}}h(t_1, t_2)$$

belong to $\mathscr{H}_{\varphi_{-1}}$. Such functions are supported by pairs (γ_1, γ_2) building a path with one jump $+1$ at time t_1 and one jump -1 at time t_2. The identity (8) now reads as:

$$\begin{aligned}&E_{\mathbb{Q}}\Biggl(f(X) \sum_{\substack{(t_1,t_2):\Delta X_{t_1}=1\\ \Delta X_{t_2}=-1}} h(t_1, t_2)\Biggr)\\ &\quad = \nu(-1)\nu(1)\int_{[0,1]^2} E_{\mathbb{Q}}\bigl(f(X + g_1\mathbf{1}_{[t_1,1]} + g_2\mathbf{1}_{[t_2,1]})\bigr)h(t_1, t_2)\,dt_1\,dt_2.\end{aligned}$$

As in Remark 11, if we consider test functions f which only depend on the skeletons of the paths, $f(X) = v(\varphi_X)$, we obtain that the distribution $\chi(dn_{-1}, dn_1) \in \mathcal{P}(\mathbb{N}^2)$ of the number n_{-1} (resp., n_1) of negative (resp., positive) jumps is characterized by the system of equations: for all

$v \in \mathcal{B}^+(\mathbb{N}^2)$,

$$\int v(n_{-1}, n_1) n_{-1} n_1 \chi(dn_{-1}, dn_1) = \nu(-1)\nu(1) \int v(n_{-1}+1, n_1+1)\chi(dn_{-1}, dn_1),$$
$$\chi(n_1 = n_{-1}) = 1.$$

This result coincides with [2], Example 2.18.

3.1.2. $G_\nu = \{1, 2\}$

In that case, since -1 does not belong to the support of the jump measure, it leads to a case where (H2) is not satisfied. It is straightforward to prove that the lattice $\mathscr{L}$ is one-dimensional and is equal to $\{z\varphi^*, z \in \mathbb{Z}\}$ where

$$\varphi^*(1) = 2, \qquad \varphi^*(2) = -1.$$

Clearly $\mathscr{L}$ does not admit a non negative basis.

3.2. G is the cyclic group $\mathbb{Z}/N\mathbb{Z}$

We now consider the finite cyclic group $G := \mathbb{Z}/N\mathbb{Z} =: \{\mathbf{0}, \mathbf{1}, \mathbf{2}, \dots, \mathbf{N-1}\}$.

3.2.1. *The support G_ν of the jump measure reduces to* $\{\mathbf{-1}, \mathbf{1}\}$

This case corresponds to nearest neighbour random walks. The non negative basis $\mathscr{B} := \{\varphi_{N-1}, \varphi^*\}$ where

$$\varphi_{N-1} = \mathbf{1}_{\mathbf{1}} + \mathbf{1}_{\mathbf{N-1}} = \mathbf{1}_{\mathbf{1}} + \mathbf{1}_{\mathbf{-1}} \quad \text{and} \quad \varphi^* = N\mathbf{1}_{\mathbf{1}}$$

is suitable. The associated characteristics are

$$\Phi^\nu_{\varphi_{N-1}} = \nu(\mathbf{1})\nu(\mathbf{-1}) \quad \text{and} \quad \Phi^\nu_{\varphi^*} = \nu(\mathbf{1})^N.$$

The existence of the second invariant $\Phi^\nu_{\varphi^*}$ corresponding to the loop around the cycle $\{\mathbf{0}, \mathbf{1}, \mathbf{2}, \dots, \mathbf{N-1}\}$ implies that $\mathbb{P}_\nu$ is the unique nearest neighbour random walk of the set $\mathrm{Rec}(\nu)$. This differs from the nearest neighbour random walk on $\mathbb{Z}$, treated in Example 12. We proved there that any random walk $\mathbb{P}_\mu$, with μ satisfying $\mu(-1)\mu(1) = \nu(-1)\nu(1)$, induces the same set of bridges.

The distribution χ of the random vector $(n_{\mathbf{-1}}, n_{\mathbf{1}})$ under the $\mathbf{00}$-bridge is given by the following system of integral equations, satisfied for any test function v on $\mathbb{N}^2$:

$$\int v(n_{\mathbf{-1}}, n_{\mathbf{1}}) n_{\mathbf{-1}} n_{\mathbf{1}} \chi(dn_{\mathbf{-1}}, dn_{\mathbf{1}}) = \nu(\mathbf{-1})\nu(\mathbf{1}) \int v(n_{\mathbf{-1}}+1, n_{\mathbf{1}}+1)\chi(dn_{\mathbf{-1}}, dn_{\mathbf{1}}),$$
$$\int v(n_{\mathbf{-1}}, n_{\mathbf{1}}) n_{\mathbf{1}} \cdots \big(n_{\mathbf{1}} - (N-1)\big)\chi(dn_{\mathbf{-1}}, dn_{\mathbf{1}}) = \nu(\mathbf{1})^N \int v(n_{\mathbf{-1}}, n_{\mathbf{1}}+N)\chi(dn_{\mathbf{-1}}, dn_{\mathbf{1}}),$$
$$\chi(n_{\mathbf{1}} - n_{\mathbf{-1}} \in N\mathbb{Z}) = 1.$$

3.2.2. *The support G_ν of the jump measure covers $\mathbb{Z}/N\mathbb{Z} \setminus \{\mathbf{0}\}$*

We now consider a random walk on $\mathbb{Z}/N\mathbb{Z}$ which can jump anywhere: $G_\nu = \mathbb{Z}/N\mathbb{Z} \setminus \{\mathbf{0}\}$. Here, we focus for simplicity on the case $N = 4$, which is the first nontrivial example, and disintegrate the jump measure ν as follows:

$$\nu = \nu(\mathbf{1})\delta_{\mathbf{1}} + \nu(\mathbf{2})\delta_{\mathbf{2}} + \nu(\mathbf{3})\delta_{\mathbf{3}}, \qquad \nu(\mathbf{1})\nu(\mathbf{2})\nu(\mathbf{3}) > 0.$$

It can be proven along the same lines as in the previous examples, that a suitable nonnegative basis for the lattice $\mathscr{L}$ is given by $\mathscr{B} = \{\varphi^*, \eta^*, \xi^*\}$ where

$$\varphi^* = \mathbf{1}_{\mathbf{1}} + \mathbf{1}_{\mathbf{3}}, \qquad \eta^* = 4\mathbf{1}_{\mathbf{1}}, \qquad \xi^* = 2\mathbf{1}_{\mathbf{1}} + \mathbf{1}_{\mathbf{2}}.$$

Hence, the associated characteristics are:

$$\Phi^\nu_{\varphi^*} = \nu(\mathbf{1})\nu(\mathbf{3}), \qquad \Phi^{\eta^*}_\nu = \nu(\mathbf{1})^4, \qquad \Phi^\nu_{\xi^*} = \nu(\mathbf{1})^2\nu(\mathbf{2}).$$

We now turn our attention to the integral formula (8). Simple functions $F \in \mathscr{H}_{\xi^*}$ are of the form:

$$F\big(X, (\gamma_1, \gamma_2, \gamma_3)\big) = f(X)\mathbf{1}_{\{g_1=g_2=\mathbf{1}, g_3=\mathbf{2}\}} h(t_1, t_2, t_3).$$

$(\gamma_1, \gamma_2, \gamma_3)$ is in the support of F if two jumps of value $\mathbf{1}$ happen at times t_1, t_2 and one jump of value $\mathbf{2}$ at time t_3, leading to a global null displacement since $\mathbf{4} = \mathbf{0}$. The formula (8) reads:

$$\begin{aligned} &E_{\mathbb{Q}}\Bigg(f(X) \sum_{\substack{(t_1,t_2,t_3): t_1 \neq t_2, \\ \Delta X_{t_1} = \Delta X_{t_2} = \mathbf{1} \\ \Delta X_{t_3} = \mathbf{2}}} h(t_1, t_2, t_3)\Bigg) \\ &\quad = \nu(\mathbf{1})^2\nu(\mathbf{2}) E_{\mathbb{Q}}\bigg(\int_{[0,1]^3} f(X + g_1\mathbf{1}_{[t_1,1]} + \cdots + g_3\mathbf{1}_{[t_3,1]}) h(t_1, t_2, t_3)\bigg) dt_1\, dt_2\, dt_3. \end{aligned}$$

The distribution of the random vector (n_1, n_2, n_3) under the $\mathbf{00}$-bridge is given by the following identities, valid for any $v : \mathbb{N}^3 \to \mathbb{R}$:

$$\begin{aligned} &\int v(n_1, n_2, n_3) n_1 n_3 \chi(dn_1, dn_2, dn_3) = \nu(\mathbf{1})\nu(\mathbf{3}) \int (v(n_1 + 1, n_2, n_3 + 1)\chi(dn_1, dn_2, dn_3), \\ &\int v n_1(n_1 - 1)(n_1 - 2)(n_1 - 3)\chi(dn_1, dn_2, dn_3) \\ &\quad = \nu(\mathbf{1})^4 \int v(n_1 + 4, n_2, n_3)\chi(dn_1, dn_2, dn_3), \\ &\int v(n_1, n_2, n_3) n_1(n_1 - 1) n_2 \chi(dn_1, dn_2, dn_3) \\ &\quad = \nu(\mathbf{1})^2\nu(\mathbf{2}) \int v(n_1 + 2, n_2 + 1, n_3)\chi(dn_1, dn_2, dn_3), \\ &\chi(n_1\mathbf{1} + n_2\mathbf{2} + n_3\mathbf{3} = \mathbf{0}) = 1. \end{aligned}$$

In this situation, again $\mathbb{P}_\nu$ is the unique random walk of the set $\mathrm{Rec}(\nu)$.

3.3. The state space is a product group

Consider the product of two groups, say G and G', and two non negative finite measures on them, say ν and ν', such that in both cases (H1) and (H2) are satisfied. Then, the product group $G \times G'$ equipped with the product measure $\nu \otimes \nu'$ fulfills (H1) and (H2) too. The key idea is as follows: if $\mathscr{B}$ and $\mathscr{B}'$ are suitable basis of G and G' then we can define for all $\eta \in \mathscr{B}$,

$$\varphi_\eta : G_\nu \times G_{\nu'} \to \mathbb{N}, \qquad \varphi_\eta\big(g, g'\big) = \eta(g)$$

and for all $\eta' \in \mathscr{B}'$,

$$\varphi_{\eta'} : G_\nu \times G_{\nu'} \to \mathbb{N}, \qquad \varphi_{\eta'}\big(g, g'\big) = \eta'\big(g'\big).$$

The set $\mathscr{B}_\otimes = \{\varphi_\eta\}_{\eta \in \mathscr{B}} \cup \{\varphi_{\eta'}\}_{\eta' \in \mathscr{B}'}$ is an appropriate basis for the lattice of skeletons defined on the product group.

Example 13 (Random walk on the d-dimensional discrete hypercube $(\mathbb{Z}/2\mathbb{Z})^d$). The d-dimensional discrete hypercube is the d-product of the cyclic group with two elements. We denote by $(e_1, \dots, e_d)$ its canonical basis.

A random walk on the hypercube is defined uniquely through its jump measure $\nu = \sum_{i=1}^d \nu(i)\delta_{e_i}$. Since it can be realized as the product of d random walks on $\mathbb{Z}/2\mathbb{Z}$, the basis $\mathscr{B} := \{\varphi_i^*\}_{1 \le i \le d}, \varphi_i^* = 2\mathbf{1}_{e_i}$, is a suitable choice.

For the integral characterization it is enough to consider loops of length $\ell = 2$. However, we have here d different skeletons to consider. Test functions of the form

$$F(X, \gamma) = f(X)\mathbf{1}_{\{g_1 = g_2 = e_i\}} h(t_1, t_2), \qquad 1 \le i \le d,$$

belong to $\mathscr{H}_{\varphi_i^*}$. For any $i \in \{1, \dots, d\}$ fixed, (8) reads as:

$$E_\mathbb{Q}\Bigg(f(X) \sum_{\substack{(t_1,t_2): t_1 \neq t_2, \\ \Delta X_{t_1} = \Delta X_{t_2} = e_i}} h(t_1, t_2)\Bigg)$$
$$= \nu(i)^2 \int_{[0,1]^2} E_\mathbb{Q}\big(f\big(X + g_1\mathbf{1}_{[t_1,1]} + g_2\mathbf{1}_{[t_2,1]}\big)\big) h(t_1, t_2)\, dt_1\, dt_2.$$

Concerning the distribution of the random vector $(n_{e_1}, \dots, n_{e_d})$, it has independent marginals χ_i, $1 \le i \le d$, which are characterized through the system of equations: for all $v \in \mathcal{B}^+(\mathbb{N})$,

$$\int v(n)n(n-1)\chi_i(dn) = \nu(i)^2 \int v(n+1)\chi_i(dn),$$
$$\chi_i(n \in 2\mathbb{N}) = 1.$$

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