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Irreducibility of stochastic real Ginzburg–Landau equation driven by α-stable noises and applications

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We establish the irreducibility of stochastic real Ginzburg–Landau equation with α-stable noises by a maximal inequality and solving a control problem. As applications, we prove that the system converges to its equilibrium measure with exponential rate under a topology stronger than total variation and obeys the moderate deviation principle by constructing some Lyapunov test functions.

Keywords: α-stable noises; exponential ergodicity; irreducibility; moderate deviation principle; stochastic real Ginzburg–Landau equation

1. Introduction

Consider the stochastic real Ginzburg–Landau equation driven by α-stable noises on torus $\mathbb{T} := \mathbb{R}/\mathbb{Z}$ as follows:

$$\mathrm{d}X - \partial_\xi^2 X\,\mathrm{d}t - \left(X - X^3\right)\mathrm{d}t = \mathrm{d}L_t, \tag{1.1}$$

where $X : [0, +\infty) \times \mathbb{T} \times \Omega \to \mathbb{R}$ and L_t is an α-stable noise. It is known [22] that equation (1.1) admits a unique mild solution X in the càdlàg space almost surely. As $\alpha \in (3/2, 2)$, X is a Markov process with a unique invariant measure π (see Section 2 below for details). By the uniqueness (see [4]), π is ergodic in the sense that

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \Psi(X_t)\,\mathrm{d}t = \int \Psi\,\mathrm{d}\pi, \qquad \mathbb{P}\text{-a.s.}$$

for all initial state x_0 and all continuous and bounded functions Ψ.

The irreducibility is a fundamental concept in stochastic dynamic system, and plays a crucial role in the research of ergodic theory. See, for instance, the classical work [10] and the books for stochastic infinite dimensional systems [4,15].

It is well known that one usually solves a control problem to prove the irreducibility for stochastic partial differential equations (SPDEs) driven by Wiener noises, see [3,4]. For SPDEs driven by α-stable noises, when the system is linear or Lipschitz, Priola and Zabczyk proved the

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irreducibility in the same line [19]. However, due to the discontinuity of trajectories and the lack of second moment, the control problem in [19] is much harder than those in the Wiener noises case. To our knowledge, there seem no other literatures about the irreducibility of stochastic systems with α-stable noises.

In this paper, we prove that the system (1.1) is irreducible by following the spirit in [3] and [19]. Due to the non-Lipschitz nonlinearity, the control problem in our setting is much harder and a maximal inequality is needed.

The ergodicity of the system (1.1) has been proved in [22] in the sense that X converges to a unique invariant measure under the weak topology, but the convergence speed is not addressed. In this paper, thanks to the irreducibility and the strong Feller property (established in [22]), we prove that the system (1.1) converges to the invariant measure exponentially fast under a topology stronger than total variation by constructing a Lyapunov test function.

Another application of our irreducibility result is to establish moderate deviation principle (MDP) of (1.1). Thanks to [21], the MDP is obtained by verifying the same Lyapunov condition as above.

Finally, we recall some literatures on the study of invariant measures and the long time behavior of stochastic systems driven by α-stable type noises. [17,18] studied the exponential mixing for a family of semi-linear SPDEs with Lipschitz nonlinearity, while [9] obtained the existence of invariant measures for 2D stochastic Navier–Stokes equations forced by α-stable noises with $\alpha \in (1,2)$. [23] proved the exponential mixing for a family of 2D SDEs forced by degenerate α-stable noises. For the long term behaviour about stochastic system drive by Lévy noises, we refer to [6,7,9,12–14] and the literatures therein.

The paper is organized as follows. In Section 2, we first give a brief review of some known results about the existence and uniqueness of solutions and invariant probability measures for stochastic Ginzburg–Landau equations. We will also present the main theorems in this section. In Section 3, we prove that the system X is irreducible. In the last section, we first recall some results about moderate deviations and exponential convergence for general strong Feller Markov processes, and then we prove moderate deviations and exponential convergence for X by constructing appropriate Lyapunov test functions.

2. Stochastic real Ginzburg–Landau equations

Let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ be equipped with the usual Riemannian metric, and let $d\xi$ denote the Lebesgue measure on $\mathbb{T}$. For any $p \geq 1$, let

$$L^p(\mathbb{T};\mathbb{R}) := \left\{ x : \mathbb{T} \to \mathbb{R}; \|x\|_{L^p} := \left(\int_{\mathbb{T}} \left| x(\xi) \right|^4 \mathrm{d}\xi \right)^{1/4} < \infty \right\}.$$

Then

$$H := \left\{ x \in L^2(\mathbb{T};\mathbb{R}); \int_{\mathbb{T}} x(\xi)\,\mathrm{d}\xi = 0 \right\}$$

is a separable real Hilbert space with inner product

$$\langle x, y \rangle_H := \int_{\mathbb{T}} x(\xi) y(\xi)\,\mathrm{d}\xi \qquad \forall x, y \in H.$$

For any $x \in H$, let

$$\|x\|_H := \|x\|_{L^2} = \big(\langle x, x\rangle_H\big)^{1/2}.$$

Let $\mathbb{Z}_* := \mathbb{Z} \setminus \{0\}$. It is well known that

$$\big\{e_k; e_k = e^{i2\pi k\xi}, k \in \mathbb{Z}_*\big\}$$

is an orthonormal basis of H. For each $x \in H$, it can be represented by Fourier series

$$x = \sum_{k\in\mathbb{Z}_*} x_k e_k \qquad \text{with } x_k \in \mathbb{C}, x_{-k} = \overline{x_k}.$$

Let Δ be the Laplace operator on H. It is well known that $D(\Delta) = H^{2,2}(\mathbb{T}) \cap H$. In our setting, Δ can be determined by the following relations: for all $k \in \mathbb{Z}_*$,

$$\Delta e_k = -\gamma_k e_k \qquad \text{with } \gamma_k = 4\pi^2|k|^2,$$

with

$$H^{2,2}(\mathbb{T}) \cap H = \left\{x \in H; x = \sum_{k\in\mathbb{Z}_*} x_k e_k, \sum_{k\in\mathbb{Z}_*} |\gamma_k|^2|x_k|^2 < \infty\right\}.$$

Denote

$$A = -\Delta, \qquad D(A) = H^{2,2}(\mathbb{T}) \cap H.$$

Define the operator A^σ with $\sigma \geq 0$ by

$$A^\sigma x = \sum_{k\in\mathbb{Z}_*} \gamma_k^\sigma x_k e_k, \qquad x \in D\big(A^\sigma\big),$$

where $\{x_k\}_{k\in\mathbb{Z}_*}$ are the Fourier coefficients of x, and

$$D\big(A^\sigma\big) := \left\{x \in H : x = \sum_{k\in\mathbb{Z}_*} x_k e_k, \sum_{k\in\mathbb{Z}_*} |\gamma_k|^{2\sigma}|x_k|^2 < \infty\right\}.$$

Given $x \in D(A^\sigma)$, its norm is

$$\big\|A^\sigma x\big\|_H := \left(\sum_{k\in\mathbb{Z}_*} |\gamma_k|^{2\sigma}|x_k|^2\right)^{1/2}.$$

Moreover, let

$$V := D\big(A^{1/2}\big) \quad \text{and} \quad \|x\|_V := \big\|A^{1/2}x\big\|_H. \tag{2.1}$$

Notice that V is densely and compactly embedded in H.

We shall study 1D stochastic Ginzburg–Landau equation on $\mathbb{T}$ as the following

$$\begin{cases} \mathrm{d}X_t + AX_t\,\mathrm{d}t = N(X_t)\,\mathrm{d}t + \mathrm{d}L_t, \\ X_0 = x_0, \end{cases} \tag{2.2}$$

where

(i) the nonlinear term N is defined by

$$N(u) = u - u^3, \qquad u \in H.$$

(ii) $L_t = \sum_{k\in\mathbb{Z}_*} \beta_k l_k(t) e_k$ is an α-stable process on H with $\{l_k(t)\}_{k\in\mathbb{Z}_*}$ being i.i.d. 1-dimensional symmetric α-stable process sequence with $\alpha > 1$, see [20]. Moreover, we assume that there exist some $C_1, C_2 > 0$ so that $C_1\gamma_k^{-\beta} \le |\beta_k| \le C_2\gamma_k^{-\beta}$ with $\beta > \frac{1}{2} + \frac{1}{2\alpha}$.

Let $C > 0$ be a constant and let $C_p > 0$ be a constant depending on the parameter p. We shall often use the following inequalities [22]:

$$\|A^\sigma e^{-At}\|_H \le C_\sigma t^{-\sigma} \qquad \forall \sigma > 0\ \forall t > 0; \tag{2.3}$$

$$\|N(x)\|_V \le C\big(\|x\|_V + \|x\|_V^3\big) \qquad \forall x \in V; \tag{2.4}$$

$$\|AN(x)\|_H \le C\big(1 + \|x\|_V^2\big)\big(1 + \|Ax\|_H^2\big); \tag{2.5}$$

$$\|x\|_{L^4}^4 \le \|x\|_V^2 \|x\|_H^2 \qquad \forall x \in V. \tag{2.6}$$

Here we consider a general E-valued càdlàg Markov process,

$$\big(\Omega, \{\mathcal{F}_t^0\}_{t\ge 0}, \mathcal{F}, \{X_t^x\}_{t\ge 0, x\in E}, (\mathbb{P}_x)_{x\in E}\big)$$

whose transition probability is denoted by $\{P_t(x, dy)\}_{t\ge 0}$, where $\Omega := D([0,+\infty); E)$ is the space of the càdlàg functions from $[0,+\infty)$ to E equipped with the Skorokhod topology, $\mathcal{F}_t^0 = \sigma\{X_s, 0 \le s \le t\}$ is the natural filtration.

For all $f \in b\mathcal{B}(E)$ (the space of all bounded measurable functions), define

$$P_t f(x) = \int_E P_t(x, \mathrm{d}y) f(y) \qquad \text{for all } t \ge 0, x \in E.$$

For any $t > 0$, P_t is said to be *strong Feller* if $P_t\varphi \in C_b(E)$ for any $\varphi \in b\mathcal{B}(E)$; P_t is *irreducible* in E if $P_t 1_O(x) > 0$ for any $x \in E$ and any non-empty open subset O of E.

Definition 2.1. *We say that a predictable H-valued stochastic process $X = (X_t^x)$ is a mild solution to equation (2.2) if, for any $t \ge 0$, $x \in H$, it holds ($\mathbb{P}$-a.s.):*

$$X_t^x(\omega) = e^{-At}x + \int_0^t e^{-A(t-s)} N\big(X_s^x(\omega)\big)\,\mathrm{d}s + \int_0^t e^{-A(t-s)}\,\mathrm{d}L_s(\omega). \tag{2.7}$$

The following existence and uniqueness results for the solutions and the invariant measure can be found in [22].

Theorem 2.2 ([22]). *The following statements hold*:

(1) *If $\alpha \in (1,2)$ and $\beta > \frac{1}{2} + \frac{1}{2\alpha}$, for every $x \in H$ and $\omega \in \Omega$ a.s., equation* (2.2) *admits a unique mild solution $X_\cdot^x(\omega) \in D([0,\infty); H) \cap D((0,\infty); V)$.*

(2) *$X = (X_t^x)_{t \geq 0, x \in H}$ is a Markov process. If $\alpha \in (3/2, 2)$ and $\frac{1}{2} + \frac{1}{2\alpha} < \beta < \frac{3}{2} - \frac{1}{\alpha}$ are further assumed, the transition probability P_t of X is strong Feller in H for any $t > 0$.*

(3) *If $\alpha \in (3/2, 2)$ and $\frac{1}{2} + \frac{1}{2\alpha} < \beta < \frac{3}{2} - \frac{1}{\alpha}$, X admits a unique invariant measure, and the invariant measure is supported on V.*

Our first main result is the following theorem about the irreducibility.

Theorem 2.3. *Assume that $\alpha \in (1,2)$ and $\beta > \frac{1}{2} + \frac{1}{2\alpha}$. For any initial value $x \in H$, the Markov process $X = \{X_t^x\}_{t \geq 0, x \in H}$ to the equation* (2.2) *is irreducible in H.*

Remark 2.4. By the well-known Doob's theorem (see [4]), the strong Feller property and the irreducibility imply that X admits at most one unique invariant probability measure. This gives another proof to the uniqueness of invariant measure.

As an application of our irreducibility result (together with strong Feller property), we have the following exponential ergodicity under a topology stronger than total variation. Recall that in [22] by ergodicity we mean that X has a unique invariant measure under the weak topology. Theorem 2.5 below gives not only an ergodic theorem in stronger sense but also exponential convergence speed.

Theorem 2.5. *Assume that $\alpha \in (3/2, 2)$ and $\frac{1}{2} + \frac{1}{2\alpha} < \beta < \frac{3}{2} - \frac{1}{\alpha}$. Let π be the unique invariant probability measure of X. Then there exist some positive constants $M > 1, \rho \in (0,1), \theta > 0$ satisfying that $\int \Psi \, d\pi < +\infty$, where $\Psi(x) := (M + \|x\|_H^2)^{1/2}$, and π is exponentially ergodic in the sense that*

$$\sup_{|f| \leq \Psi} \left| P_t f(x) - \int f \, d\pi \right| \leq \theta \Psi(x) \cdot \rho^t \qquad \forall x \in H, t \geq 0. \tag{2.8}$$

Remark 2.6. Let $(B_\Psi, \|\cdot\|_\Psi)$ be the Banach space of all real measurable functions f on H such that

$$\|f\|_\Psi := \sup_{x \in H} \frac{|f(x)|}{\Psi(x)} < +\infty.$$

The exponential convergence (2.8) means that

$$\left\| (P_t - \pi)(f) \right\|_\Psi \leq \theta \|f\|_\Psi \cdot \rho^t,$$

that is, P_t has a spectral gap near its largest eigenvalue 1 in B_Ψ.

Let $\mathcal{M}_b(H)$ be the space of signed σ-additive measures of bounded variation on H equipped with the Borel σ-field $\mathcal{B}(H)$. On $\mathcal{M}_b(H)$, we consider the topology $\sigma(\mathcal{M}_b(H), b\mathcal{B}(H))$, the so called τ-topology of convergence against all bounded Borel functions which is stronger than the usual weak convergence topology $\sigma(\mathcal{M}_b(H), C_b(H))$, see [5], Section 6.2.

Let

$$\mathfrak{L}_t(A) := \frac{1}{t}\int_0^t \delta_{X_s}(A)\,\mathrm{d}s \qquad \text{for any measurable set } A,$$

where δ_a is the Dirac measure at a. According to Corollary 2.5 in [21], the system X has the following exponential ergodicity.

Corollary 2.7. *Under the conditions of Theorem 2.5, the following results hold*:

(a) $\mathfrak{L}_t$ *converges to* π *with an exponential rate w.r.t. the* τ*-topology. More precisely for any neighborhood* $\mathcal{N}(\pi)$ *of* π *in* $(\mathcal{M}_b(H), \tau)$,

$$\sup_{K \subset\subset H} \limsup_{t\to+\infty} \frac{1}{t} \log \sup_{x\in K} \mathbb{P}_x\big(\mathfrak{L}_t \notin \mathcal{N}(\pi)\big) < 0.$$

Here $K \subset\subset H$ *means that* K *is a compact set in* H.

(b) *The process* X *is exponentially recurrent in the sense below*: *for any compact* K *in* H *with* $\pi(K) > 0$, *there exists some* $\lambda_0 > 0$ *such that for any compact* K' *in* H,

$$\sup_{x\in K'} \mathbb{E}_x \exp\big(\lambda_0 \tau_K(T)\big) < +\infty,$$

where $\tau_K(T) = \inf\{t \ge T; X_t \in K\}$ *for any* $T > 0$.

Another application of our irreducibility result (together with strong Feller property) is to establish MDP for the system (1.1). To this end, let us first briefly recall MDP as follows.

Let $b(t) : \mathbb{R}^+ \to (0, +\infty)$ be an increasing function verifying

$$\lim_{t\to\infty} b(t) = +\infty, \qquad \lim_{t\to\infty} \frac{b(t)}{\sqrt{t}} = 0, \tag{2.9}$$

define

$$\mathfrak{M}_t := \frac{1}{b(t)\sqrt{t}} \int_0^t (\delta_{X_s} - \pi)\,\mathrm{d}s. \tag{2.10}$$

Then *moderate deviations* of $\mathfrak{L}_t$ from its *asymptotic limit* π is to estimate

$$\mathbb{P}_\mu(\mathfrak{M}_t \in A), \tag{2.11}$$

where A is some measurable set in $(\mathcal{M}_b(H), \tau)$, a given domain of deviation. Here $\mathbb{P}_\mu$ is the probability measure of the system X with initial measure μ. When $b(t) = 1$, this becomes an estimation of the central limit theorem; and when $b(t) = \sqrt{t}$, it is exactly the large deviations. $b(t)$ satisfying (2.9) is between those two scalings, called *scaling of moderate deviations*, see [5].

Now we are at the position to state our MDP result.

Theorem 2.8. *In the context of Theorem 2.5, for any initial measure μ verifying $\mu(\Psi) < +\infty$, the measure $\mathbb{P}_\mu(\mathfrak{M}_t \in \cdot)$ satisfies the large deviation principle w.r.t. the τ-topology with speed $b^2(t)$ and the rate function*

$$I(\nu) := \sup\left\{\int f\,\mathrm{d}\nu - \frac{1}{2}\sigma^2(f);\ f \in b\mathcal{B}(H)\right\} \qquad \forall \nu \in M_b(H), \tag{2.12}$$

where

$$\sigma^2(f) = \lim_{t\to\infty} \frac{1}{t}\mathbb{E}^\pi\left(\int_0^t \big(f(X_s) - \pi(f)\big)\,\mathrm{d}s\right)^2 \tag{2.13}$$

exists in $\mathbb{R}$ for every $f \in B_\Psi \supset b\mathcal{B}(H)$. More precisely, the following three properties hold:

(a1) *for any $a \geq 0$, $\{\nu \in \mathcal{M}_b(H); I(\nu) \leq a\}$ is compact in $(\mathcal{M}_b(H), \tau)$*;

(a2) (*the upper bound*) *for any closed set F in $(\mathcal{M}_b(H), \tau)$*,

$$\limsup_{T\to\infty} \frac{1}{b^2(T)} \log \mathbb{P}_\mu(\mathfrak{M}_T \in F) \leq -\inf_F I;$$

(a3) (*the lower bound*) *for any open set G in $(\mathcal{M}_b(H), \tau)$*,

$$\liminf_{T\to\infty} \frac{1}{b^2(T)} \log \mathbb{P}_\mu(\mathfrak{M}_T \in G) \geq -\inf_G I.$$

3. Irreducibility in *H*

In this section, we shall prove that $X = \{X_t^x\}_{t\geq 0, x\in H}$ in the system (2.2) is irreducible in H. Together with the strong Feller property established in [22], Theorem 6.1, this gives another proof to the existence of at most one invariant measure by classical Doob's theorem.

3.1. Irreducibility of stochastic convolution

Let us first consider the following Ornstein–Uhlenbeck process:

$$\mathrm{d}Z_t + AZ_t\,\mathrm{d}t = \mathrm{d}L_t, \qquad Z_0 = 0, \tag{3.1}$$

where $L_t = \sum_{k\in\mathbb{Z}_*} \beta_k l_k(t) e_k$ is an α-stable process on H. It is well known that

$$Z_t = \int_0^t e^{-A(t-s)}\,\mathrm{d}L_s = \sum_{k\in\mathbb{Z}_*} z_k(t) e_k,$$

where

$$z_k(t) = \int_0^t e^{-\gamma_k(t-s)} \beta_k\,\mathrm{d}l_k(s).$$

The following maximal inequality can be found in [22], Lemma 3.1.

Lemma 3.1. *For any $T > 0, 0 \le \theta < \beta - \frac{1}{2\alpha}$ and all $0 < p < \alpha$, we have*

$$\mathbb{E} \sup_{0 \le t \le T} \left\| A^\theta Z_t \right\|_H^p \le C T^{p/\alpha},$$

where C depends on α, θ, β, p.

The following lemma is concerned with the support of the distribution of $(\{Z_t\}_{0 \le t \le T}, Z_T)$.

Lemma 3.2. *For any $T > 0, 0 < p < \infty$, the random variable $(\{Z_t\}_{0 \le t \le T}, Z_T)$ has a full support in $L^p([0,T]; V) \times V$. More precisely, for any $\phi \in L^p([0,T]; V), a \in V, \varepsilon > 0$,*

$$\mathbb{P}\left(\int_0^T \|Z_t - \phi_t\|_V^p \, \mathrm{d}t + \|Z_T - a\|_V < \varepsilon \right) > 0.$$

Proof. First, by Lemma 3.1, we have $Z \in L^\infty([0,T]; V)$, a.s. For any $N \in \mathbb{N}$, let H_N be the Hilbert space spanned by $\{e_k\}_{1 \le k \le N}$, and let $\pi_N : H \to H_N$ be the orthogonal projection. Notice that π_N is also an orthogonal projection in V. Define

$$\pi^N = I - \pi_N, \qquad H^N = \pi^N H.$$

By the independence of $\pi_N Z$ and $\pi^N Z$, for any $\phi_t \in L^p([0,T]; V), a \in V$, we have

$$\begin{aligned}
&\mathbb{P}\left(\int_0^T \|Z_t - \phi_t\|_V^p \, \mathrm{d}t + \|Z_T - a\|_V < \varepsilon \right) \\
&\quad \ge \mathbb{P}\left(\int_0^T \left\|\pi_N (Z_t - \phi_t)\right\|_V^p \, \mathrm{d}t + \left\|\pi_N (Z_T - a)\right\|_V < \frac{\varepsilon}{2^{p+1}}, \right. \\
&\qquad \left. \int_0^T \left\|\pi^N (Z_t - \phi_t)\right\|_V^p \, \mathrm{d}t + \left\|\pi^N (Z_T - a)\right\|_V < \frac{\varepsilon}{2^{p+1}} \right) \\
&\quad = \mathbb{P}\left(\int_0^T \left\|\pi_N (Z_t - \phi_t)\right\|_V^p \, \mathrm{d}t + \left\|\pi_N (Z_T - a)\right\|_V < \frac{\varepsilon}{2^{p+1}} \right) \\
&\qquad \times \mathbb{P}\left(\int_0^T \left\|\pi^N (Z_t - \phi_t)\right\|_V^p \, \mathrm{d}t + \left\|\pi^N (Z_T - a)\right\|_V < \frac{\varepsilon}{2^{p+1}} \right).
\end{aligned}$$

By the same argument as in the Section 4.2 of [19], we obtain

$$\mathbb{P}\left(\int_0^T \left\|\pi_N (Z_t - \phi_t)\right\|_V^p \, \mathrm{d}t + \left\|\pi_N (Z_T - a)\right\|_V < \frac{\varepsilon}{2^{p+1}} \right) > 0.$$

To finish the proof, it suffices to show

$$\mathbb{P}\left(\int_0^T \left\|\pi^N (Z_t - \phi_t)\right\|_V^p \, \mathrm{d}t + \left\|\pi^N (Z_T - a)\right\|_V < \frac{\varepsilon}{2^{p+1}} \right) > 0.$$

For any $\theta \in (\frac{1}{2}, \beta - \frac{1}{2\alpha})$, by Lemma 3.1 (with $p = 1$ therein), the spectral gap inequality and Chebyshev inequality, we have for any η

$$\begin{aligned}
\mathbb{P}\Big(\sup_{0\le t\le T}\|\pi^N Z_t\|_V \le \eta\Big) &= 1 - \mathbb{P}\Big(\sup_{0\le t\le T}\|\pi^N Z_t\|_V > \eta\Big) \\
&\ge 1 - \mathbb{P}\Big(\sup_{0\le t\le T}\|\pi^N A^\theta Z_t\|_H > \eta\gamma_N^{\theta-1/2}\Big) \\
&\ge 1 - \mathbb{P}\Big(\sup_{0\le t\le T}\|A^\theta Z_t\|_H > \eta\gamma_N^{\theta-1/2}\Big) \\
&\ge 1 - C_{\alpha,\beta,T}\,\eta^{-1}\gamma_N^{1/2-\theta},
\end{aligned}$$

where $C_{\alpha,\beta,T}$ depends on α, β, T. By the previous inequality, as long as N (depending on ε, p, ϕ) is sufficiently large, we have

$$\mathbb{P}\bigg(\sup_{0\le t\le T}\|\pi^N Z_t\|_V \le \frac{\varepsilon}{2^{2p+2}}\bigg) > 0$$

and

$$\int_0^T \|\pi^N \phi_t\|_V^p \,\mathrm{d}t + \|\pi^N a\|_V < \frac{\varepsilon}{2^{2p+2}}.$$

Hence,

$$\begin{aligned}
&\mathbb{P}\bigg(\int_0^T \|\pi^N(Z_t - \phi_t)\|_V^p \,\mathrm{d}t + \|\pi^N(Z_T - a)\|_V < \frac{\varepsilon}{2^{p+1}}\bigg) \\
&\quad\ge \mathbb{P}\bigg(\int_0^T \|\pi^N Z_t\|_V^p \,\mathrm{d}t + \|\pi^N Z_T\|_V < \frac{\varepsilon}{2^{2p+2}}, \int_0^T \|\pi^N \phi_t\|_V^p \,\mathrm{d}t + \|\pi^N a\|_V < \frac{\varepsilon}{2^{2p+2}}\bigg) \\
&\quad= \mathbb{P}\bigg(\int_0^T \|\pi^N Z_t\|_V^p \,\mathrm{d}t + \|\pi^N Z_T\|_V < \frac{\varepsilon}{2^{2p+2}}\bigg) \\
&\quad> 0.
\end{aligned}$$

The proof is complete. □

3.2. A control problem for the deterministic system

Consider the deterministic system in H,

$$\partial_t x(t) + Ax(t) = N\big(x(t)\big) + u(t), \qquad x(0) = x_0, \tag{3.2}$$

where $u \in L^2([0,T]; V)$. By using the similar argument in the proof of Lemma 4.2 in [22], for every $x(0) = x_0 \in H, u \in L^2([0,T]; V)$, the system (3.2) admits a unique solution $x(\cdot) \in$

$C([0,T];H)\cap C((0,T];V)$. Moreover, $\{x(t)\}_{t\in[0,T]}$ has the following form:

$$x(t)=e^{-At}x_0+\int_0^t e^{-A(t-s)}N\big(x(s)\big)\,\mathrm{d}s+\int_0^t e^{-A(t-s)}u(s)\,\mathrm{d}s \qquad \forall t\in[0,T]. \tag{3.3}$$

Next, we shall prove that the deterministic system is approximately controllable in time $T>0$.

Lemma 3.3. *For any $T>0,\varepsilon>0,a\in V$, there exists some $u\in L^\infty([0,T];V)$ such that the system (3.2) satisfies that*

$$\|x(T)-a\|_V<\varepsilon.$$

Proof. We shall prove the lemma by the following three steps.

Step 1. *Regularization.* For any $t_0\in(0,T]$, let $u(t)=0$ for all $t\in[0,t_0]$. Then the system (3.2) admits a unique solution $x(\cdot)\in C([0,t_0];H)\cap C((0,t_0];V)$ with the following form:

$$x(t)=e^{-At}x_0+\int_0^t e^{-A(t-s)}N\big(x(s)\big)\,\mathrm{d}s \qquad \forall 0<t\le t_0.$$

Step 2. *Approximation at time T and linear interpolation.* For any $a\in V,\varepsilon>0$, there exists a constant $\theta>0$ such that

$$\|e^{-\theta A}a-a\|_V\le\varepsilon.$$

Setting $x(t)=\frac{t-t_0}{T-t_0}e^{-\theta A}a+\frac{T-t}{T-t_0}x(t_0)$ for all $t\in[t_0,T]$. Then $x(\cdot)\in C((0,T];V)$. By (3.2), we have

$$u(t)=\frac{e^{-\theta A}a-x(t_0)}{T-t_0}-Ax(t)-N\big(x(t)\big) \qquad \forall t\in[t_0,T].$$

Step 3. It remains to show that $u\in L^\infty([0,T];V)$. By (2.3), (2.4) and the constructions of $\{x(t)\}_{t\in[0,T]}$ and $\{u(t)\}_{t\in[0,T]}$ above, it is sufficient to show that $Ax(t_0)\in V$. For any $t\in[t_0/2,t_0]$,

$$\begin{aligned}x(t)&=e^{-(t-t_0/3)A}x(t_0/3)+\int_{t_0/3}^t e^{-(t-s)A}N\big(x(s)\big)\,\mathrm{d}s\\&=e^{-(t-t_0/2)A}x(t_0/2)+\int_{t_0/2}^t e^{-(t-s)A}N\big(x(s)\big)\,\mathrm{d}s.\end{aligned}$$

By (2.3), we have for $t>\frac{t_0}{2}$,

$$\begin{aligned}\|Ax(t)\|_H&=\|Ae^{-(t-t_0/3)A}x(t_0/3)\|_H+\left\|A\int_{t_0/3}^t e^{-(t-s)A}N\big(x(s)\big)\,\mathrm{d}s\right\|_H\\&\le\|Ae^{-(t-t_0/3)A}x(t_0/3)\|_H+\int_{t_0/3}^t\|A^{1/2}e^{-(t-s)A}\|\cdot\|A^{1/2}N\big(x(s)\big)\|_H\,\mathrm{d}s \qquad (3.4)\\&\le C_1(t-t_0/3)^{-1}\|x(t_0/3)\|_H+\int_{t_0/3}^t C_{1/2}(t-s)^{-1/2}\|N\big(x(s)\big)\|_V\,\mathrm{d}s.\end{aligned}$$

Since $x(\cdot) \in C((0, t_0]; V)$, $\sup_{t\in[t_0/3,t_0]} \|x(s)\|_V < \infty$. Together with (2.4) and (3.4), we obtain that

$$\sup_{t\in[t_0/2,t_0]} \|Ax(t)\|_H < \infty.$$

By the previous inequality and (2.5), we have

$$\begin{aligned}
\|A^{3/2}x(t_0)\|_H &= \left\| A^{3/2}e^{(t_0/2)A}x(t_0/2) + A^{3/2}\int_{t_0/2}^{t_0} e^{-(t-s)A}N\big(x(s)\big)\,\mathrm{d}s \right\|_H \\
&\le \|A^{3/2}e^{-(t_0/2)A}x(t_0/2)\|_H + \int_{t_0/2}^{t_0} \|A^{1/2}e^{-(t_0-s)A}\| \cdot \|AN\big(x(s)\big)\|_H\,\mathrm{d}s \\
&\le C_{3/2}(t_0/2)^{-3/2}\|x(t_0/2)\|_H + \int_{t_0/2}^{t_0} C_{1/2}(t_0-s)^{-1/2}\|AN\big(x(s)\big)\|_H\,\mathrm{d}s \\
&\le C_{3/2}(t_0/2)^{-3/2}\|x(t_0/2)\|_H + C\sup_{s\in[t_0/2,t_0]}\big(1+\|x(s)\|_V^2\big)\cdot\big(1+\|Ax(s)\|_H^2\big) \\
&< \infty,
\end{aligned}$$

which means that $Ax(t_0) \in V$. The proof is complete. □

3.3. Irreducibility in *H*

Now we prove Theorem 2.3 by following the idea in [19], Theorem 5.4.

Proof of Theorem 2.3. For any $x_0 \in H$, $t > 0$, we have $X_t^{x_0} \in V$ a.s. by Theorem 2.2. Since X is Markov in H, for any $a \in H$, $T > 0$, $\varepsilon > 0$,

$$\begin{aligned}
\mathbb{P}\big(\|X_T^{x_0} - a\|_H < \varepsilon\big) &= \int_V \mathbb{P}\big(\|X_T^{x_0} - a\|_H < \varepsilon | X_t^{x_0} = v\big)\mathbb{P}\big(X_t^{x_0} \in \mathrm{d}v\big) \\
&= \int_V \mathbb{P}\big(\|X_{T-t}^{v} - a\|_H < \varepsilon\big)\mathbb{P}\big(X_t^{x_0} \in \mathrm{d}v\big).
\end{aligned}$$

To prove that

$$\mathbb{P}\big(\|X_T^{x_0} - a\|_H < \varepsilon\big) > 0,$$

it is sufficient to prove that for any $T > 0$,

$$\mathbb{P}\big(\|X_T^{x_0} - a\|_H < \varepsilon\big) > 0 \qquad \text{for all } x_0 \in V.$$

Next, we prove the theorem under the assumption of the initial value $x_0 \in V$ in the following two steps.

Step 1. For any $a \in H, \varepsilon > 0$, there exists some $\theta > 0$ such that $e^{-\theta A} a \in V$ and

$$\left\| a - e^{-\theta A} a \right\|_H \leq \frac{\varepsilon}{4}. \tag{3.5}$$

For any $T > 0$, by Lemma 3.3 and the spectral gap inequality, there exists some $u \in L^\infty([0, T]; V)$ such that the system

$$\dot{x} + Ax = N(x) + u, \qquad x(0) = x_0,$$

satisfies that

$$\left\| x(T) - e^{-\theta A} a \right\|_H \leq \left\| x(T) - e^{-\theta A} a \right\|_V < \frac{\varepsilon}{4}. \tag{3.6}$$

Putting (3.5) and (3.6) together, we have

$$\left\| x(T) - a \right\|_H < \frac{\varepsilon}{2}. \tag{3.7}$$

Step 2. We shall consider the systems (3.8) and (3.9) as follows:

$$\begin{cases} \dot{z} + Az = u, & z(0) = 0, \\ \dot{y} + Ay = N(y + z), & y(0) = x_0 \in V, \end{cases} \tag{3.8}$$

and

$$\begin{cases} \mathrm{d}Z_t + AZ_t \, \mathrm{d}t = \mathrm{d}L_t, & Z_0 = 0; \\ \mathrm{d}Y_t + AY_t \, \mathrm{d}t = N(Y_t + Z_t) \, \mathrm{d}t, & Y_0 = x_0 \in V. \end{cases} \tag{3.9}$$

By the arguments in the proof of Lemma 4.2 in [22], for any $x_0 \in V, u \in L^2([0, T]; V)$, the systems (3.8) and (3.9) admit the unique solutions $(y(\cdot), z(\cdot)) \in C([0, T]; V)^2$ and $(Y_\cdot, Z_\cdot) \in C([0, T]; V)^2$, a.s. Furthermore, denote

$$x(t) = y(t) + z(t), \qquad X_t = Y_t + Z_t \qquad \forall t \geq 0.$$

For any $0 \leq t \leq T$,

$$\begin{aligned} &\left\| Y_t - y(t) \right\|_H^2 + 2 \int_0^t \left\| Y_s - y(s) \right\|_V^2 \, \mathrm{d}s \\ &\quad = 2 \int_0^t \left\langle Y_s - y(s), N(X_s) - N\big(x(s)\big) \right\rangle_H \mathrm{d}s \\ &\quad = 2 \int_0^t \left\| Y_s - y(s) \right\|_H^2 \mathrm{d}s + 2 \int_0^t \left\langle Y_s - y(s), Z_s - z(s) \right\rangle_H \mathrm{d}s \\ &\qquad - 2 \int_0^t \left\langle Y_s - y(s), X_s^3 - x^3(s) \right\rangle_H \mathrm{d}s. \end{aligned}$$

Let us estimate the third term of the right-hand side. Denoting $\Delta Y_s = Y_s - y(s)$ and $\Delta Z_s = Z_s - z(s)$, we have

$$\begin{aligned}
&\int_0^t \langle Y_s - y(s), X_s^3 - x^3(s)\rangle_H \,\mathrm{d}s \\
&\quad= \int_0^t \langle \Delta Y_s, \big[\Delta Y_s + \Delta Z_s + x(s)\big]^3 - x^3(s)\rangle_H \,\mathrm{d}s \\
&\quad= \int_0^t \langle \Delta Y_s, [\Delta Y_s + \Delta Z_s]^3 + 3[\Delta Y_s + \Delta Z_s]^2 x(s) \\
&\qquad + 3[\Delta Y_s + \Delta Z_s]x^2(s)\rangle_H \,\mathrm{d}s \\
&\quad= \int_0^t \langle \Delta Y_s, (\Delta Y_s)^3 + 3(\Delta Y_s)^2\Delta Z_s + 3\Delta Y_s(\Delta Z_s)^2 + (\Delta Z_s)^3\rangle_H \,\mathrm{d}s \\
&\qquad + 3\int_0^t \langle \Delta Y_s, \big[(\Delta Y_s)^2 + 2\Delta Y_s\Delta Z_s + (\Delta Z_s)^2\big]x(s)\rangle_H \,\mathrm{d}s \\
&\qquad + 3\int_0^t \langle \Delta Y_s, [\Delta Y_s + \Delta Z_s]x^2(s)\rangle_H \,\mathrm{d}s.
\end{aligned}$$

Since $\frac{3}{4}(\Delta Y_s)^4 + 3(\Delta Y_s)^3 x(s) + 3(\Delta Y_s)^2 x^2(s) \geq 0$, from the above relation we have

$$\begin{aligned}
&\int_0^t \langle Y_s - y(s), X_s^3 - x^3(s)\rangle_H \,\mathrm{d}s \\
&\quad\geq \int_0^t \langle \Delta Y_s, 3(\Delta Y_s)^2\Delta Z_s + 3\Delta Y_s(\Delta Z_s)^2 + (\Delta Z_s)^3\rangle_H \,\mathrm{d}s \\
&\qquad + 3\int_0^t \langle \Delta Y_s, \big[2\Delta Y_s\Delta Z_s + (\Delta Z_s)^2\big]x(s)\rangle_H \,\mathrm{d}s \\
&\qquad + 3\int_0^t \langle \Delta Y_s, \Delta Z_s x^2(s)\rangle_H \,\mathrm{d}s + \frac{1}{4}\int_0^t \|\Delta Y_s\|_{L^4}^4 \,\mathrm{d}s.
\end{aligned}$$

Using the following Young inequalities: for all $y, z \in L^4(\mathbb{T};\mathbb{R})$,

$$\begin{aligned}
\big|\langle y, z\rangle_H\big| &= \left|\int_{\mathbb{T}} y(\xi)z(\xi)\,\mathrm{d}\xi\right| \leq \frac{\int_{\mathbb{T}} y^4(\xi)\,\mathrm{d}\xi}{80} + C\int_{\mathbb{T}} z^{4/3}(\xi)\,\mathrm{d}\xi, \\
\big|\langle y^2, z\rangle_H\big| &= \left|\int_{\mathbb{T}} y^2(\xi)z(\xi)\,\mathrm{d}\xi\right| \leq \frac{\int_{\mathbb{T}} y^4(\xi)\,\mathrm{d}\xi}{80} + C\int_{\mathbb{T}} z^2(\xi)\,\mathrm{d}\xi, \\
\big|\langle y^3, z\rangle_H\big| &= \left|\int_{\mathbb{T}} y^3(\xi)z(\xi)\,\mathrm{d}\xi\right| \leq \frac{\int_{\mathbb{T}} y^4(\xi)\,\mathrm{d}\xi}{80} + C\int_{\mathbb{T}} z^4(\xi)\,\mathrm{d}\xi,
\end{aligned}$$

and the Hölder inequality, we further get

$$\begin{aligned}
&\int_0^t \langle Y_s - y(s), X_s^3 - x^3(s)\rangle_H \, \mathrm{d}s \\
&\quad \geq \frac{1}{80}\int_0^t \|\Delta Y_s\|_{L^4}^4 \, \mathrm{d}s - 7C\int_0^t \|\Delta Z_s\|_{L^4}^4 \, \mathrm{d}s \\
&\qquad - 6C\int_0^t \|\Delta Z_s x(s)\|_{L^2}^2 \, \mathrm{d}s - 3C\int_0^t \|(\Delta Z_s)^2 x(s)\|_{L^{4/3}}^{4/3} \, \mathrm{d}s \\
&\qquad - 3C\int_0^t \|\Delta Z_s x^2(s)\|_{L^{4/3}}^{4/3} \, \mathrm{d}s \\
&\quad \geq \frac{1}{80}\int_0^t \|\Delta Y_s\|_{L^4}^4 \, \mathrm{d}s - 7C\int_0^t \|\Delta Z_s\|_{L^4}^4 \, \mathrm{d}s \\
&\qquad - 6C\int_0^t \|\Delta Z_s\|_{L^4}^2 \|x(s)\|_{L^4}^2 \, \mathrm{d}s - 3C\int_0^t \|\Delta Z_s\|_{L^4}^{8/3} \|x(s)\|_{L^4}^{4/3} \, \mathrm{d}s \\
&\qquad - 3C\int_0^t \|\Delta Z_s\|_{L^4}^{4/3} \|x(s)\|_{L^4}^{8/3} \, \mathrm{d}s.
\end{aligned}$$

Since $x(t) = y(t) + z(t) \in C([0, T]; V)$, by (2.6), there exists a constant C_T such that

$$\sup_{s\in[0,T]} \|y(s) + z(s)\|_{L^4} \leq \sup_{s\in[0,T]} \|y(s) + z(s)\|_H^{1/2} \cdot \|y(s) + z(s)\|_V^{1/2} \leq C_T.$$

Consequently, there is some constant $C_T > 0$ satisfying that

$$\begin{aligned}
&\|Y_t - y(t)\|_H^2 + 2\int_0^t \|Y_s - y(s)\|_V^2 \, \mathrm{d}s \\
&\quad \leq 3\int_0^t \|Y_s - y(s)\|_H^2 \, \mathrm{d}s + \int_0^t \|Z_s - z(s)\|_H^2 \, \mathrm{d}s \\
&\qquad + C_T\int_0^t \big(\|Z_s - z(s)\|_{L^4}^4 + \|Z_s - z(s)\|_{L^4}^2 + \|Z_s - z(s)\|_{L^4}^{8/3} + \|Z_s - z(s)\|_{L^4}^{4/3} \, \mathrm{d}s\big) \, \mathrm{d}s.
\end{aligned}$$

Therefore, by the spectral gap inequality and Gronwall's inequality, we have

$$\|Y_T - y(T)\|_H^2 \leq C_T \sum_{i\in\Lambda} \int_0^T \|Z_s - z(s)\|_V^i \, \mathrm{d}s, \tag{3.10}$$

where $\Lambda := \{4/3, 2, 8/3, 4\}$. This inequality, together with Lemma 3.2, (3.7), implies

$$\begin{aligned}
&\mathbb{P}\big(\|X_T - a\|_H < \varepsilon\big) \\
&\quad = \mathbb{P}\big(\|Y_T - y(T) + Z_T - z(T) + x(T) - a\|_H < \varepsilon\big)
\end{aligned}$$

$$
\begin{aligned}
&\geq \mathbb{P}\big(\|Y_T - y(T)\|_H \leq \varepsilon/4, \|Z_T - z(T)\|_H \leq \varepsilon/4, \|x(T) - a\|_H < \varepsilon/2\big) \\
&= \mathbb{P}\big(\|Y_T - y(T)\|_H \leq \varepsilon/4, \|Z_T - z(T)\|_H \leq \varepsilon/4\big) \\
&\geq \mathbb{P}\bigg(\sum_{i\in\Lambda}\int_0^T \|Z_s - z(s)\|_V^i \,\mathrm{d}s + \|Z_T - z(T)\|_V \leq C_{T,\varepsilon}\bigg) \\
&> 0.
\end{aligned}
$$

The proof is complete. □

4. The proofs of Theorems 2.5 and 2.8

4.1. Several general results for strong Feller Markov processes

In this subsection, we recall some general results about moderate deviations and exponential convergence for general strong Feller Markov processes, borrowed from [21].

We say that a measurable function $f : H \to \mathbb{R}$ belongs to the extended domain $\mathbf{D}_e(\mathcal{L})$ of the generator $\mathcal{L}$ of (P_t), if there is a measurable function $g : H \to \mathbb{R}$ so that $\int_0^t |g|(X_s)\,\mathrm{d}s < +\infty, \forall t > 0, \mathbb{P}_x$-a.s. and

$$
f(X_t) - f(X_0) - \int_0^t g(X_s)\,\mathrm{d}s, \qquad t \geq 0
$$

is a càdlàg $\mathbb{P}_x$-local martingale for all $x \in H$. In that case, $g := \mathcal{L}f$.

Theorem 4.1 ([11], Theorem 5.2c, or [21], Theorem 2.4). *Assume that the process (X_t) is strong Feller, irreducible and aperiodic (see [11] for definition, that is the case if $P_T(\cdot, K) > 0$ over H for some compact K verifying $\pi(K) > 0$). If there are some continuous function $1 \leq \Psi \in \mathbf{D}_e(\mathcal{L})$, compact subset $K \subset H$ and constants $\varepsilon, C > 0$ such that*

$$
-\frac{\mathcal{L}\Psi}{\Psi} \geq \varepsilon \mathbb{1}_{K^c} - C\mathbb{1}_K, \tag{4.1}
$$

then there is a unique invariant probability measure π satisfying

$$
\int \Psi\,\mathrm{d}\pi < +\infty,
$$

and there are some constants $\theta > 0$ and $0 < \rho < 1$ such that for all $t \geq 0$,

$$
\sup_{|f|\leq\Psi}\left|P_t f(z) - \int f\,\mathrm{d}\pi\right| \leq \theta\Psi(z)\cdot\rho^t, \qquad z \in H. \tag{4.2}
$$

For the measure-valued process $\mathfrak{M}_t$ defined in (2.10), we have the following large deviations result.

Theorem 4.2 ([21], Theorem 2.6). *Assume that the process (X_t) is strong Feller, irreducible, aperiodic and satisfies (4.1). For any initial measure μ verifying $\mu(\Psi) < +\infty$, the measure $\mathbb{P}_\mu(\mathfrak{M}_t \in \cdot)$ satisfies the large deviation principle w.r.t. the τ-topology with the speed $b^2(t)$ and the rate function*

$$I(\nu) := \sup\left\{\int f\,\mathrm{d}\nu - \frac{1}{2}\sigma^2(f);\, f \in b\mathcal{B}(H)\right\} \qquad \forall \nu \in \mathcal{M}_b(H), \tag{4.3}$$

where

$$\sigma^2(f) = \lim_{t\to\infty} \frac{1}{t}\mathbb{E}_\pi\left(\int_0^t \big(f(X_s) - \pi(f)\big)\,\mathrm{d}s\right)^2 \tag{4.4}$$

exists in $\mathbb{R}$ for every $f \in B_\Psi \supset b\mathcal{B}(H)$.

4.2. The proofs of main results

In this subsection, we shall prove Theorems 2.5 and 2.8 based on the above theorems. The main technique is to construct a suitable Lyapunov test function.

Proofs of Theorems 2.5 and 2.8. By Theorems 2.2 and 2.3, the system (2.2) is strong Feller, irreducible and aperiodic in H. Indeed, as the invariant measure π is supported on V, there exists a bounded closed ball $F \subset V$ satisfying $\pi(F) > 0$. Notice that F is compact in H. Since the system X is strong Feller and irreducible in H, by [4], Theorem 4.2.1, the invariant measure π is equivalent to all measures $P_t(x,\cdot)$, for all $x \in H, t > 0$. Consequently, $P_t(x,F) > 0$ for all $x \in H, t > 0$, which implies that the system is aperiodic.

By Theorems 4.1 and 4.2, we now construct a suitable Lyapunov function Ψ satisfying (4.1). Take

$$\Psi(x) := \big(M + \|x\|_H^2\big)^{1/2}, \tag{4.5}$$

where M is a large constant to be determined later. By Lemma 4.3 below, we have $\Psi \in \mathbf{D}_e(\mathcal{L})$.

Recall $x^m = \pi_m x$, we have

$$\begin{aligned}
\mathcal{L}\Psi\big(x^m\big) &= \big\langle -Ax^m, D\Psi\big(x^m\big)\big\rangle + \big\langle N\big(x^m\big), D\Psi\big(x^m\big)\big\rangle \\
&\quad + \sum_{|i|\le m}\int_{|y_i|\le 1}\big[\Psi\big(x^m + \beta_i y_i e_i\big) - \Psi\big(x^m\big) - \beta_i y_i D_{e_i}\Psi\big(x^m\big)\big]\nu(\mathrm{d}y_i) \\
&\quad + \sum_{|i|\le m}\int_{|y_i|> 1}\big[\Psi\big(x^m + \beta_i y_i e_i\big) - \Psi\big(x^m\big)\big]\nu(\mathrm{d}y_i) \\
&=: J_1^m + J_2^m + J_3^m + J_4^m.
\end{aligned} \tag{4.6}$$

Here

$$D_{e_i}\Psi\big(x^m\big) := \frac{x_i}{\Psi(x^m)}, \qquad D\Psi\big(x^m\big) := \sum_{|i|\le m}\big(D_{e_i}\Psi\big(x^m\big)\big)e_i = \frac{x^m}{\Psi(x^m)}.$$

For the first term, using the integration by parts formula, we have

$$\big\langle -Ax^m, D\Psi\big(x^m\big)\big\rangle = \left\langle -Ax^m, \frac{x^m}{\Psi(x^m)}\right\rangle = -\frac{\|x^m\|_V^2}{\Psi(x^m)}. \tag{4.7}$$

For the second term, by (2.5) in [22] (note that $N(x)$ here equals $-N(x)$ in [22]), we have

$$\big\langle N\big(x^m\big), D\Psi\big(x^m\big)\big\rangle = \left\langle N\big(x^m\big), \frac{x^m}{\Psi(x^m)}\right\rangle \le \frac{1}{4\Psi(x^m)}. \tag{4.8}$$

For any $h \in H$,

$$\big|\big\langle h, D^2\Psi\big(x^m\big)h\big\rangle\big| = \frac{\|h^m\|_H^2}{\sqrt{M+\|x^m\|^2}} - \frac{|\langle h^m, x^m\rangle|^2}{(M+\|x^m\|_H^2)^{3/2}} \le \frac{\|h\|_H^2}{\sqrt{M+\|x^m\|_H^2}}.$$

This inequality, together with Taylor's formula, implies that

$$\big|\Psi\big(x^m+\beta_i y_i e_i\big) - \Psi\big(x^m\big) - \beta_i y_i D_{e_i}\Psi\big(x^m\big)\big| \le \frac{\beta_i^2 y_i^2}{\sqrt{M+\|x^m\|_H^2}}.$$

Thus, for the third term, we have

$$\begin{aligned}
&\sum_{|i|\le m}\int_{|y_i|\le 1}\big|\Psi\big(x^m+\beta_i y_i e_i\big) - \Psi\big(x^m\big) - \beta_i y_i D_{e_i}\Psi\big(x^m\big)\big|\nu(\mathrm{d}y_i)\\
&\le \sum_{|i|\le m}\int_{|y_i|\le 1}\frac{\beta_i^2 y_i^2}{\sqrt{M+\|x^m\|_H^2}}\nu(\mathrm{d}y_i)\\
&= \frac{\sum_{|i|\le m}\beta_i^2}{\sqrt{M+\|x^m\|_H^2}}\int_{|y|\le 1}\frac{|y|^{1-\alpha}}{C_\alpha}\,\mathrm{d}y\\
&= \frac{2\sum_{|i|\le m}\beta_i^2}{C_\alpha(2-\alpha)\sqrt{M+\|x^m\|_H^2}} < \frac{2\sum_{i\in\mathbb{Z}_*}\beta_i^2}{C_\alpha(2-\alpha)\sqrt{M+\|x^m\|_H^2}} < +\infty,
\end{aligned} \tag{4.9}$$

where ν is the Lévy measure of 1-dimensional α-stable process and we have used the assumption of β_i in (ii) in the last inequality.

By Taylor's formula again, there exists $\tilde{x}^m \in H$ satisfying that

$$\big|\Psi\big(x^m+\beta_i y_i e_i\big) - \Psi\big(x^m\big)\big| = \big|\big\langle D_{e_i}\Psi\big(\tilde{x}^m\big), \beta_i y_i e_i\big\rangle\big| = \frac{|\tilde{x}_i\beta_i y_i|}{\sqrt{M+\|\tilde{x}^m\|_H^2}} \le |\beta_i y_i|.$$

For the fourth term, we have

$$\begin{aligned}
&\left|\sum_{|i|\le m}\int_{|y_i|>1}\big[\Psi\big(x^m+\beta_i y_i e_i\big)-\Psi\big(x^m\big)\big]\nu(\mathrm{d}y_i)\right| \\
&\quad\le \sum_{|i|\le m}\int_{|y_i|>1}|\beta_i y_i|\nu(\mathrm{d}y_i) \\
&\quad= \sum_{|i|\le m}\int_{|y_i|>1}\frac{|\beta_i y_i|}{C_\alpha|y_i|^{1+\alpha}}\,\mathrm{d}y_i=\frac{2\sum_{|i|\le m}|\beta_i|}{C_\alpha(\alpha-1)}<\frac{2\sum_{i\in\mathbb{Z}_*}|\beta_i|}{C_\alpha(\alpha-1)}<+\infty,
\end{aligned} \tag{4.10}$$

where we have used the assumption of β_i in (ii) again.

Putting (4.6)–(4.10) together, we obtain that for any $x\in H$,

$$\begin{aligned}
-\frac{\mathcal{L}\Psi(x^m)}{\Psi(x^m)}&=-\frac{J_1^m+J_2^m+J_3^m+J_4^m}{\Psi(x^m)} \\
&\ge\frac{\|x^m\|_V^2}{M+\|x^m\|_H^2}-\frac{1}{4(M+\|x^m\|_H^2)}-\frac{2\sum_{i\in\mathbb{Z}_*}\beta_i^2}{C_\alpha(2-\alpha)(M+\|x^m\|_H^2)} \\
&\quad-\frac{2\sum_{i\in\mathbb{Z}_*}|\beta_i|}{C_\alpha(\alpha-1)\sqrt{M+\|x^m\|_H^2}}.
\end{aligned} \tag{4.11}$$

Let

$$K:=\big\{x\in V;\ \|x\|_V^2\le M\big\}.$$

Then K is compact in H. By (3) in Theorem 2.2, choose M large enough such that $\pi(K)>0$ and

$$\frac{1}{4M}+\frac{2\sum_{i\in\mathbb{Z}_*}\beta_i^2}{C_\alpha(2-\alpha)M}+\frac{2\sum_{i\in\mathbb{Z}_*}|\beta_i|}{C_\alpha(\alpha-1)\sqrt{M}}\le\frac{1}{4}\qquad\forall x\in H. \tag{4.12}$$

Since $\lim_{m\to\infty}\Psi(x^m)=\Psi(x)$ and $\lim_{m\to\infty}\mathcal{L}\Psi(x^m)$ has limit for $x\in V$, by the closable property of $\mathcal{L}$, we immediately get $\mathcal{L}\Psi(x)=\lim_{m\to\infty}\mathcal{L}\Psi(x^m)$ and thus

$$-\frac{\mathcal{L}\Psi(x)}{\Psi(x)}\ge-\frac{1}{4},\qquad x\in K. \tag{4.13}$$

For any $x\in K^c$, by (4.11) and (4.12), we have

$$-\frac{\mathcal{L}\Psi(x^m)}{\Psi(x^m)}\ge\frac{\|x^m\|_V^2}{M+\|x^m\|_H^2}-\frac{1}{4}\ge\frac{1}{4}.$$

This implies

$$-\frac{\mathcal{L}\Psi(x)}{\Psi(x)} \geq \frac{\|x^m\|_V^2}{M+\|x^m\|_H^2} - \frac{1}{4} \geq \frac{1}{4} \qquad \forall x \in V \cap K^c \tag{4.14}$$

and

$$-\frac{\mathcal{L}\Psi(x)}{\Psi(x)} = \infty \qquad \forall x \in H \setminus \bigl(V \cap K^c\bigr). \tag{4.15}$$

Putting (4.13)–(4.15) together, we immediately obtain

$$-\frac{\mathcal{L}\Psi(x)}{\Psi(x)} \geq \frac{1}{4}\mathbb{1}_{K^c} - \frac{1}{4}\mathbb{1}_K.$$

The proof is complete. □

Lemma 4.3. *For Ψ defined in (4.5), we have $\Psi \in \mathbf{D}_e(\mathcal{L})$.*

Before proving $\Psi \in \mathbf{D}_e(\mathcal{L})$, let us first briefly review some well known facts about α-stable process for using Itô formula. Let $\{l_j(t)\}_{j\geq 1}$ be a sequence of i.i.d. 1-dimensional α-stable processes. The Poisson random measure associated with $l_j(t)$ is defined by

$$N^{(j)}(t,\Gamma) := \sum_{s\in(0,t]} 1_\Gamma\bigl(l_j(s)-l_j(s-)\bigr) \qquad \forall t>0, \forall \Gamma \in \mathcal{B}\bigl(\mathbb{R}\setminus\{0\}\bigr).$$

By Lévy–Itô's decomposition (cf. [1], page 126, Theorem 2.4.16), one has

$$l_j(t) = \int_{|x|\leq 1} x\tilde{N}^{(j)}(t,\mathrm{d}x) + \int_{|x|>1} xN^{(j)}(t,\mathrm{d}x),$$

where $\tilde{N}^{(j)}$ is the compensated Poisson random measure defined by

$$\tilde{N}^{(j)}(t,\Gamma) = N^{(j)}(t,\Gamma) - t\nu(\Gamma).$$

Proof of Lemma 4.3. The proof follows from [8], Section 3, in spirit. Let $T>0$ be an arbitrary but finite number, we shall consider the stochastic system in $[0,T]$. Consider the Galerkin approximation of (2.2):

$$\mathrm{d}X_t^m + AX_t^m\,\mathrm{d}t = N^m\bigl(X_t^m\bigr)\,\mathrm{d}t + \mathrm{d}L_t^m, \qquad X_0^m = x^m, \tag{4.16}$$

where $X_t^m = \pi_m X_t$, $N^m(X_t^m) = \pi_m[N(X_t^m)]$, $L_t^m = \sum_{|k|\leq m}\beta_k l_k(t)e_k$, π_m is the orthogonal projection defined in the proof of Lemma 3.2. By a standard argument [2,16], for all $x\in W$ with $W=H$ or $W=V$ we have

$$\mathbb{E}\Bigl[\sup_{0\leq t\leq T}\bigl\|X_t^m\bigl(x^m\bigr)\bigr\|_W\Bigr] \leq C_W(x,T), \tag{4.17}$$

$$\lim_{m\to\infty}\mathbb{E}\Big[\sup_{0\le t\le T}\big\|X_t^m\big(x^m\big)-X_t(x)\big\|_W\Big]=0, \tag{4.18}$$

where $C_W(x,T)>0$ is finite.

Write $\Psi(u):=(M+\|u\|_H^2)^{1/2}$ for all $u\in H$, it follows from Itô formula [1] that

$$\Psi\big(X_t^m\big)-\Psi\big(x^m\big)+I_1^m(t)-I_2^m(t)=I_3^m(t)+I_4^m(t), \tag{4.19}$$

where

$$\begin{aligned}
I_1^m(t)&:=\int_0^t\frac{\|X_s^m\|_V^2}{(M+\|X_s^m\|_H^2)^{1/2}}\,\mathrm{d}s-\int_0^t\frac{\langle X_s^m,N(X_s^m)\rangle_H}{(M+\|X_s^m\|_H^2)^{1/2}}\,\mathrm{d}s,\\
I_2^m(t)&:=\sum_{|j|\le m}\int_0^t\int_{\mathbb{R}}\Big[\Psi\big(X_s^m+y\beta_je_j\big)-\Psi\big(X_s^m\big)-\frac{\langle X_s^m,y\beta_je_j\rangle}{(M+\|X_s^m\|_H^2)^{1/2}}1_{\{|y|\le1\}}\Big]\nu(\mathrm{d}y)\,\mathrm{d}s,\\
I_3^m(t)&:=\sum_{|j|\le m}\int_0^t\int_{|y|\le1}\big[\Psi\big(X_s^m+y\beta_je_j\big)-\Psi\big(X_s^m\big)\big]\tilde N^{(j)}(\mathrm{d}s,\mathrm{d}y),\\
I_4^m(t)&:=\sum_{|j|\le m}\int_0^t\int_{|y|>1}\big[\Psi\big(X_s^m+y\beta_je_j\big)-\Psi\big(X_s^m\big)\big]\tilde N^{(j)}(\mathrm{d}s,\mathrm{d}y).
\end{aligned}$$

By a Taylor expansion argument similar to (4.9) and (4.10) below, for all $T>0$ we have

$$\mathbb{E}\Big[\sup_{0\le t\le T}\big|I_2^m(t)\big|\Big]\le CT,$$

$$\mathbb{E}\Big[\sup_{0\le t\le T}\big|I_3^m(t)\big|^2\Big]\le CT,$$

$$\mathbb{E}\Big[\sup_{0\le t\le T}\big|I_4^m(t)\big|\Big]\le CT.$$

Moreover, by a Taylor expansion argument similar to (4.9) and (4.10) below again, we get that as $m_1\to\infty, m_2\to\infty$,

$$\mathbb{E}\Big[\sup_{0\le t\le T}\big|I_2^{m_1}(t)-I_2^{m_2}(t)\big|\Big]\to0,$$

$$\mathbb{E}\Big[\sup_{0\le t\le T}\big|I_3^{m_1}(t)-I_3^{m_2}(t)\big|^2\Big]\to0,$$

$$\mathbb{E}\Big[\sup_{0\le t\le T}\big|I_4^{m_1}(t)-I_4^{m_2}(t)\big|^p\Big]\to0,$$

for all $1\le p<\alpha$. Hence, there exist I_2, I_3 and I_4 such that

$$\lim_{m\to\infty}\mathbb{E}\Big[\sup_{0\le t\le T}\big|I_2^m(t)-I_2(t)\big|\Big]=0, \tag{4.20}$$

$$\lim_{m\to\infty}\mathbb{E}\Big[\sup_{0\le t\le T}\big|I_3^m(t)-I_3(t)\big|^2\Big]=0, \tag{4.21}$$

$$\lim_{m\to\infty}\mathbb{E}\Big[\sup_{0\le t\le T}\big|I_4^m(t)-I_4(t)\big|^p\Big]=0, \tag{4.22}$$

where I_2, I_3 and I_4 have the same forms as I_2^m, I_3^m and I_4^m but with $\sum_{|i|\le m}$ replaced by $\sum_{i\in\mathbb{Z}_*}$ and X^m replaced by X. It is also easy to verify that I_3 is an L^2 martingale and that I_4 is an L^1 martingale.

Next we shall show below, taking a subsequence if necessary, that

$$\lim_{m\to\infty} I_1^m(t)=I_1(t)\qquad \forall 0\le t\le T, \text{a.s.}, \tag{4.23}$$

where $I_1(t)$ has the same form as $I_1^m(t)$ but with X^m replaced by X. Collecting (4.20)–(4.23), taking a subsequence if necessary and letting $m\to\infty$ in (4.19), we obtain

$$\Psi(X_t)-\Psi(x)+I_1(t)-I_2(t)=I_3(t)+I_4(t). \tag{4.24}$$

Since I_3 and I_4 are L_2 and L_1 martingales respectively, taking

$$\begin{aligned} g(t)=&-\int_0^t\frac{\|X_s\|_V^2}{(M+\|X_s\|_H^2)^{1/2}}\,\mathrm{d}s+\int_0^t\frac{\langle X_s,N(X_s)\rangle_H}{(M+\|X_s\|_H^2)^{1/2}}\,\mathrm{d}s\\ &+\sum_{j\in\mathbb{Z}_*}\int_0^t\int_{\mathbb{R}}\Big[\Psi(X_s+y\beta_je_j)-\Psi(X_s)-\frac{\langle X_s,y\beta_je_j\rangle_H}{(M+\|X_s\|_H^2)^{1/2}}1_{\{|y|\le1\}}\Big]\nu(\mathrm{d}y)\,\mathrm{d}s, \end{aligned}$$

we immediately verify that $\Psi\in\mathbf{D}_e(\mathcal{L})$ for $t\in[0,T]$. Since $T>0$ is arbitrary, $\Psi\in\mathbf{D}_e(\mathcal{L})$ for $t\in[0,\infty)$.

It remains to prove (4.23). Taking a subsequence if necessary and letting $m\to\infty$ in (4.19), by Fatou lemma and the fact $\langle x,N(x)\rangle\le\frac14$ from [22] we have

$$\mathbb{E}\Big[\sup_{t\in[0,T]}\big(M+\|X_t\|_H^2\big)^{1/2}\Big]+\mathbb{E}\Big[\int_0^T\frac{\|X_s\|_V^2}{(M+\|X_s\|_H^2)^{1/2}}\,\mathrm{d}s\Big]\le\big(M+\|x\|_H^2\big)^{1/2}+CT+CT^{1/2}.$$

This implies

$$\int_0^t\frac{\|X_s\|_V^2}{(M+\|X_s\|_H^2)^{1/2}}\,\mathrm{d}s<\infty\qquad\forall t\in[0,T], \text{a.s.} \tag{4.25}$$

It is easy to check $\frac{\|X_s^m\|_V^2}{(M+\|X_s^m\|_H^2)^{1/2}}$ is increasing in m for every $s>0$ and

$$\frac{\|X_s^m\|_V^2}{(M+\|X_s^m\|_H^2)^{1/2}}\le\frac{\|X_s\|_V^2}{(M+\|X_s\|_H^2)^{1/2}},\qquad s>0.$$

Hence, by (4.18) and the Lesbegue dominated convergence theorem, taking a subsequence if necessary, we get

$$\lim_{m\to\infty}\int_0^t \frac{\|X_s^m\|_V^2}{(M+\|X_s^m\|_H^2)^{1/2}}\,\mathrm{d}s = \int_0^t \frac{\|X_s\|_V^2}{(M+\|X_s\|_H^2)^{1/2}}\,\mathrm{d}s \qquad \text{a.s.} \tag{4.26}$$

Furthermore, observe that

$$\begin{aligned}\left|\langle x, N(x)\rangle_H\right| &\le \int_{\mathbb{T}} \left|x(\xi)\right|^2 \mathrm{d}\xi + \int_{\mathbb{T}} \left|x(\xi)\right|^4 \mathrm{d}\xi \\ &\le \|x\|_H^2 + \|x\|_\infty^2 \|x\|_H^2 \le \|x\|_H^2 + \tilde{C}\|x\|_V^2 \|x\|_H^2,\end{aligned}$$

where the last inequality is by Sobolev embedding. Note that for all $s \in (0, T]$

$$\begin{aligned}\left|\frac{\langle X_s^m, N(X_s^m)\rangle_H}{(M+\|X_s^m\|_H^2)^{1/2}}\right| &\le \frac{(\sup_{0\le t\le T}\|X_t^m\|_H^2)(1+\tilde{C}\|X_s^m\|_V^2)}{(M+\|X_s^m\|_H^2)^{1/2}} \\ &\le \frac{(\sup_{0\le t\le T}\|X_t^m\|_H^2)(1+\tilde{C}\|X_s\|_V^2)}{(M+\|X_s\|_H^2)^{1/2}} \qquad \text{a.s.}\end{aligned}$$

Hence, taking a subsequence if necessary, by (4.17), (4.18) and (4.25) with the Lesbegue dominated convergence theorem, we obtain

$$\lim_{m\to\infty}\int_0^t \frac{\langle X_s^m, N(X_s^m)\rangle_H}{(M+\|X_s^m\|_H^2)^{1/2}}\,\mathrm{d}s = \int_0^t \frac{\langle X_s, N(X_s)\rangle_H}{(M+\|X_s^m\|_H^2)^{1/2}}\,\mathrm{d}s \qquad \text{a.s.} \tag{4.27}$$

Combining (4.26) and (4.27), we immediately get the desired equation (4.23). □

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