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Perimeters, uniform enlargement and high dimensions

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We study the isoperimetric problem in product spaces equipped with the uniform distance. Our main result is a characterization of isoperimetric inequalities which, when satisfied on a space, are still valid for the product spaces, up a to a constant which does not depend on the number of factors. Such dimension free bounds have applications to the study of influences of variables.

Keywords: influences; isoperimetry

1. Introduction

Let (X, d, μ) denote a metric probability space, where X is separable and μ is a Borel probability measure on (X, d). For a Borel subset A of X, we define, for $r > 0$, the open r-neighbourhood of A by $A_r = \{x \in X | d(x, A) < r\}$, and its outer and inner boundary measures (also called Minkowski contents) by

$$\mu^+(A) = \liminf_{r\to 0^+} \frac{\mu(A_r) - \mu(A)}{r}, \qquad \mu^-(A) = \mu^+(X \setminus A).$$

The isoperimetric problem consists in obtaining sharp lower bounds on the above quantities in terms of the measure $\mu(A)$. The isoperimetric function of (X, d, μ), denoted by $I_{(X,d,\mu)}$ (or simply I_μ when there is no ambiguity on the underlying metric space), is defined for $p \in [0, 1]$ as follows:

$$I_\mu(p) = \inf_{A\subseteq X;\mu(A)=p} \min\bigl(\mu^+(A), \mu^-(A)\bigr) \tag{1}$$

$$= \inf_{A\subseteq X;\mu(A)\in\{p,1-p\}} \mu^+(A), \tag{2}$$

where the infimum is taken over all Borel subsets A of X. As we can see from the definition, I_μ is the largest function such that, for every $A \subseteq X$, $\mu^+(A) \ge I_\mu(\mu(A))$ and for every $t \in [0, 1]$, $I_\mu(t) = I_\mu(1 - t)$. Notice also that $I_\mu(0) = I_\mu(1) = 0$.

Given metric probability spaces (X_i, d_i, μ_i), $i = 1, \dots, n$, several metric structures can be considered on the product probability space $(X_1 \times \cdots \times X_n, \mu_1 \otimes \cdots \otimes \mu_n)$. Throughout this paper, we equip this product with the supremum distance $d = d_\infty^{(n)}$ defined by

$$d_\infty^{(n)}\bigl((x_1, \dots, x_n), (y_1, \dots, y_n)\bigr) := \max_i d_i(x_i, y_i).$$

We shall also say that $d_\infty^{(n)}$ is the ℓ_∞-combination of the distances d_i, $1 \le i \le n$. The isoperimetric problem has been intensively studied in the Riemannian setting, where the geodesic distance on a product manifold is the ℓ_2-combination of the geodesic distance on the factors. Hence, from a geometric viewpoint, the choice of the ℓ_∞-combination is less natural than the one of the ℓ_2-combination $d_2^{(n)}((x_i)_{i=1}^n, (y_i)_{i=1}^n) = (\sum_i d(x_i, y_i)^2)^{1/2}$. Nevertheless, the study of the uniform enlargement has various motivations. We briefly explain some of them.

Firstly the isoperimetric problem for the uniform enlargement is technically easier to deal with in the setting of product spaces, due to the product structure of metric balls. This often allows to work by comparisons. For instance, Bollobás and Leader [8] study this problem for the uniform measure on the cube in order to solve the discrete isoperimetric problem on the grid. Since $d_\infty^{(n)} \le d_2^{(n)} \le \sqrt{n} d_\infty^{(n)}$, it easily follows that

$$\frac{1}{\sqrt{n}} I_{(X^n, d_\infty^{(n)}, \mu^n)} \le I_{(X^n, d_2^{(n)}, \mu^n)} \le I_{(X^n, d_\infty^{(n)}, \mu^n)}.$$

This approach was used for example, by Morgan [16] for products of two Riemannian manifolds.

Another motivation for studying the isoperimetric problem for the uniform enlargement is that it amounts to the study of the usual isoperimetric problem for a special class of sets. Let us explain this briefly in the setting of $\mathbb{R}^n$ equipped with a probability measure $d\mu(x) = \rho(x)\,dx$ and the ℓ_∞ distance. If ρ is continuous and $A \subset \mathbb{R}^n$ is a domain with Lipschitz boundary, its outer Minkowski content is

$$\mu^+(A) = \int_{\partial A} \left\| n_A(x) \right\|_1 \rho(x)\, d\mathcal{H}^{n-1}(x),$$

where $n_A(x)$ is a unit outer normal to A at x (unit for the Euclidean length), and $\mathcal{H}^{n-1}$ is the $n-1$-dimensional Hausdorff measure. Consequently, the boundary measure for the uniform enlargement coincides with the usual one $\int_{\partial A} \rho(x)\, d\mathcal{H}^{n-1}(x)$, for sets A such that almost surely on ∂A the outer normal is equal to a vector of the canonical basis of $\mathbb{R}^n$ (or its opposite). These so-called rectilinear sets comprise cartesian products of intervals $I_1 \times \cdots \times I_n$, their finite unions and their complements. Hence, the isoperimetric problem for the uniform enlargement is closely connected to the usual isoperimetric problem restricted to the class of rectilinear sets (actually, a smooth domain A can be approximated by rectilinear sets in such a way that their boundary measures approach the one of A for the uniform enlargement). Note that rectilinear sets naturally appear when studying the supremum of random variables, as $\{x \in \mathbb{R}^n \,|\, \max_i x_i \in [a, b]\}$ is rectilinear. This was one of the original motivations of Bobkov [5], Bobkov and Houdré [7] for studying isoperimetry for the uniform enlargement.

Eventually, let us mention that isoperimetric inequalities for the uniform enlargement naturally appear in the recent extension by Keller, Mossel and Sen [12] of the theory of influences of variables to the continuous setting.

Computing exactly the isoperimetric profile is a hard task, even in simple product spaces (see, e.g., the survey article by Ros [18]). However, various probabilistic questions involve sequences of independent random variables and require lower estimates on the isoperimetric profile of n-fold product spaces, which actually do not depend on the value of n. First, observe that for all integers $n \geq 1$,

$$I_{(X^{n+1},d_\infty^{(n+1)},\mu^{n+1})} \leq I_{(X^n,d_\infty^{(n)},\mu^n)},$$

which holds because for every set $A \subset X^n$, $\mu^{n+1}(A \times X) = \mu^n(A)$ and $(\mu^{n+1})^+(A \times X) = (\mu^n)^+(A)$. Therefore, one may define the so-called infinite dimensional isoperimetric profile of (X, d, μ) as follows: for $t \in [0, 1]$,

$$I_{\mu^\infty}(t) := \inf_{n\geq 1} I_{(X^n,d_\infty^{(n)},\mu^n)} \leq I_{(X,d,\mu)}.$$

This quantity has been investigated by Bobkov [5], Bobkov and Houdré [7] and Barthe [3]. In particular, Bobkov has put forward a sufficient condition for the equality $I_{\mu^\infty} = I_\mu$ to hold. This condition depends only on the function I_μ but it is rather restrictive. However, it allowed to get a natural family of isoperimetric inequalities for which there exists $K > 1$ such that $I_\mu \geq I_{\mu^\infty} \geq \frac{1}{K} I_\mu$. We shall say in this case that the isoperimetric inequality with profile I_μ tensorizes, up to a factor K.

The goal of this article is to provide a workable necessary and sufficient condition for the latter property to hold. We were inspired by a sufficient condition for tensorization, given by Milman [15] in the setting of ℓ_2 distances on products. We now describe the plan of the paper.

- In the next section, we recall the known sufficient condition for $I_{\mu^\infty} = I_\mu$ and propose a new one (in Theorem 2).
- Section 3 is devoted to approximate isoperimetric inequalities. Building on the results of Section 2, we provide a sufficient condition for tensorization up to a factor (see Theorem 3). By a careful study of product sets, we actually show that this condition is also necessary (Theorem 4). Combining the latter two theorems allows to describe exactly the isoperimetric profiles enjoying the approximate tensorization property. We state the result informally here, omitting the precise hypotheses on the underlying metric measure space (given in Theorem 5): there exists a constant $K > 1$ such that $I_\mu \geq I_{\mu^\infty} \geq \frac{1}{K} I_\mu$ if and only if there exists a constant $D > 1$ such that for all $0 < s < t < 1$,

$$\frac{I_\mu(s)}{s\log(1/s)} \leq D\frac{I_\mu(t)}{t\log(1/t)}.$$

- The final section draws consequences of our isoperimetric inequalities to the theory of influences of variables: following the argument of Keller, Mossel and Sen [12], we obtain an extension of the Kahn–Kalai–Linial theorem about the existence of a coordinate with a large influence (see Theorem 7).

Let us conclude this introduction with some useful notation. If (Y, ρ) is a metric space we define the modulus of gradient of a locally Lipschitz function $f : Y \to \mathbb{R}$ by:

$$|\nabla f|(x) = \limsup_{\rho(x,y)\to 0^+} \frac{|f(x) - f(y)|}{\rho(x, y)},$$

this quantity being zero at isolated points. Note that when the distance is given by a norm on a vector space, that is $\rho(x, y) = \|x - y\|$, and when f is differentiable, then the modulus of gradient coincides with $\|Df(x)\|_*$. We shall work under the following Hypothesis $(\mathcal{H})$: for every $m, n \in \mathbb{N}^*$ and for every locally Lipschitz function $f : X^{m+n} \to \mathbb{R}$, for μ^{m+n}-almost every point $(x, y) \in X^m \times X^n$:

$$|\nabla f|(x, y) = |\nabla_x f|(x, y) + |\nabla_y f|(x, y).$$

This assumption holds in various cases: when (X, d) is an open metric subset of a Minkowski space $(\mathbb{R}^n, \|\cdot\|)$ and when μ is absolutely continuous with respect to Lebesgue's measure, or for Riemannian manifolds when the measure is absolutely continuous with respect to the volume form (as a consequence of Rademacher's theorem of almost everywhere differentiability of Lipschitz functions). On the contrary, this hypothesis often fails in discrete settings.

2. Sharp isoperimetric inequalities

We start by recalling a couple of important results about extremal half-spaces for the isoperimetric problem. The first one below is due to Bobkov and Houdré [6] and deals with the real line. Before stating it, we need to introduce some notations. Let $\mathcal{M}$ be the set of Borel probability measures on $\mathbb{R}$ which are concentrated on a possibly unbounded interval (a, b) and have a density f which is positive and continuous on (a, b). For $\mu \in \mathcal{M}$, the distribution function $F_\mu(x) := \mu((-\infty, x])$ is one-to-one from (a, b) to $(0, 1)$ and one may define

$$J_\mu(t) = f\big(F_\mu^{-1}(t)\big), \qquad t \in (0, 1).$$

We may as well consider J_μ as a function on $[0, 1]$ by setting $J_\mu(0) = J_\mu(1) = 0$. The value of $J_\mu(t)$ represents the boundary measure of the half-line of measure t starting at $-\infty$. Let $\mathcal{L} \subset \mathcal{M}$ denote the set of (non-Dirac) log-concave probability measures on $\mathbb{R}$ (the density f is of the form e^{-c} for some convex function c).

Proposition 1 (Bobkov and Houdré [6]). *The map $\mu \mapsto J_\mu$ is one-to-one between the set $\mathcal{M}$ and the set of positive continuous functions on* $(0, 1)$. *It is also one-to-one between the subset $\mathcal{L}$ of log-concave probability measures and the set of positive concave functions on* $(0, 1)$. *Moreover for $\mu \in \mathcal{M}$, the following properties are equivalent*:

(i) $I_\mu = J_\mu$ (*meaning for any* $p \in (0, 1)$, *the infimum in* (1) *is attained for the set* $(-\infty, F_\mu^{-1}(p)]$),

(ii) *the measure μ is symmetric around its median, that is, J_μ is symmetric around $\frac{1}{2}$, and for all $p, q > 0$ such that $p + q < 1$,*

$$J_\mu(p + q) \le J_\mu(p) + J_\mu(q).$$

The next basic lemma allows to compare the various conditions on isoperimetric profiles that appear in the rest of the article. In particular, it shows that the above result encompasses a classical theorem of Borell, asserting that for even log-concave probability measures on $\mathbb{R}$, half-lines are solutions to the isoperimetric problem.

Lemma 1. *Let $T \in (0,+\infty]$ and $K : [0,T) \to \mathbb{R}^+$ be a non-negative function. Consider the following properties that K may verify*:

(i) *K is concave*,
(ii) *$t \mapsto K(t)/t$ is non-increasing*,
(iii) *for all $a, b \in [0,T)$ with $a+b<T$, it holds $K(a+b) \le K(a)+K(b)$.*

Then (i) $\Longrightarrow$ (ii) *and* (ii) $\Longrightarrow$ (iii).

Proof. If K is concave then $t \mapsto (K(t)-K(0))/t$ is non-increasing. Since $t \mapsto K(0)/t$ is non-increasing as well, the first implication follows. Assuming (ii) and without loss of generality $a \le b$,

$$K(a+b) \le (a+b)\frac{K(b)}{b} = a\frac{K(b)}{b} + K(b) \le K(a)+K(b). \qquad \square$$

The next result provides sharp isoperimetric inequalities in high dimensions. It goes back to the dissertation thesis of Bobkov.

Theorem 1 (Bobkov [5]). *Let $J : [0,1] \to \mathbb{R}^+$ be a concave function, with $J(t) = J(1-t)$ for all $t \in [0,1]$. Assume that for all $a, b \in [0,1]$,*

$$J(ab) \le aJ(b) + bJ(a). \tag{3}$$

Then for every space (X,d,μ) verifying Hypothesis $(\mathcal{H})$,

$$I_\mu \ge J \quad \Longrightarrow \quad I_{\mu^\infty} \ge J.$$

Moreover, there exists an even log-concave probability measure ν on $\mathbb{R}$ such that $I_\nu = I_{\nu^\infty} = J$ and for every n, coordinate half-spaces are solutions of the isoperimetric problem for ν^n.

Condition (3) may be verified in a few instances as $J(t) = t(1-t)$. However, it is not so easy to deal with, in particular in conjunction with the symmetry assumption. For these reasons, stronger conditions of more local nature are useful. In Barthe [3], it is shown that (3) is verified when J is concave, twice differentiable and $-1/J''$ is concave. Observe that condition (3) amounts to the subadditivity of the function $u \mapsto e^u J(e^{-u})$ on $\mathbb{R}^+$. Hence, using the second part of Lemma 1, we obtain that the condition "$t \mapsto J(t)/(t\log(1/t))$ is non-decreasing" implies (3) as well. By a tedious but straightforward calculation, this yields a neat variant of one of the main results of Barthe [3]:

Corollary 1. *For $\beta \in [0,1]$, the function K_β defined for $t \in [0,1]$ by*

$$K_\beta(t) := t(1-t)\log^\beta\left(\frac{3}{t(1-t)}\right),$$

satisfies that for every space (X,d,μ) verifying Hypothesis $(\mathcal{H})$ and all $c \ge 0$,

$$I_\mu \ge cK_\beta \quad \Longrightarrow \quad I_{\mu^\infty} \ge cK_\beta.$$

Let us point out that (3) is not the best sufficient condition for the conclusion of the above theorem to hold. The optimal condition given by Bobkov's approach is the following: for every Borel probability measure N on $[0,1]$,

$$J\left(\int t\,dN(t)\right) \le \int J(t)\,dN(t) + \int_0^1 J\bigl(N\bigl([0,t]\bigr)\bigr)\,dt.$$

Actually when $\mu \in \mathcal{F}$ is a probability measure on $\mathbb{R}$ and $J = J_\mu = I_\mu$, it is not hard to check, considering subgraphs, that the above condition is necessary and sufficient for having $I_\mu = I_{\mu^\infty}$. However this condition is hard to verify in practice, and most of the work in Bobkov's proof consists in showing that when J is concave, it boils down to (3).

Next, we develop a different approach to dimension free isoperimetric inequalities. We use classical methods to make a link between isoperimetric inequalities, and some Beckner-type functional inequalities, which nicely tensorize.

Lemma 2. *Let $a \in (0,1]$ and (X,d,μ) be a metric probability space. Let $c > 0$, then the following assertions are equivalent*:

(i) *For all $p \in [0,1]$, $cI_\mu(p) \ge p - p^{1/a}$*,
(ii) *For all locally Lipschitz functions $f : X \to [0,1]$, $c\int |\nabla f|\,d\mu \ge \int f\,d\mu - (\int f^a\,d\mu)^{1/a}$.*

Proof. Assuming (i), we apply the co-area inequality to an arbitrary locally Lipschitz function f (see, e.g., Bobkov and Houdré [6]); next we take advantage of the isoperimetric inequality for μ:

$$\begin{aligned} c\int |\nabla f|\,d\mu &\ge c\int_0^1 \mu^+\bigl(\{f \ge t\}\bigr)\,dt \\ &\ge \int_0^1 \bigl(\mu\bigl(\{f \ge t\}\bigr) - \mu\bigl(\{f \ge t\}\bigr)^{1/a}\bigr)\,dt \\ &= \int f\,d\mu - \int_0^1 \mu\bigl(\{f \ge t\}\bigr)^{1/a}\,dt. \end{aligned}$$

In order to conclude that the second assertion is valid, we apply the Minkowski inequality with exponent $1/a \ge 1$:

$$\begin{aligned} \left(\int_0^1 \mu\bigl(\{f \ge t\}\bigr)^{1/a}\,dt\right)^a &= \left(\int_0^1 \left(\int \mathbf{1}_{f(s)\ge t}\,d\mu(s)\right)^{1/a} dt\right)^a \\ &\le \int \left(\int_0^1 (\mathbf{1}_{f(s)\ge t})^{1/a}\,dt\right)^a d\mu(s) = \int f^a\,d\mu. \end{aligned}$$

The fact that the second assertion implies the first one is rather standard: one applies the functional inequalities to Lipschitz approximations of the characteristic function of an arbitrary Borel set $A \subset X$ (see Lemma 3.7 in Bobkov and Houdré [6]). This yields $c\mu^+(A) \ge \mu(A) -$

$\mu(A)^{1/a}$. Applying the inequality to $1-f$ instead of f and using $|\nabla f| = |\nabla(1-f)|$ and then taking approximations of $\mathbf{1}_A$ gives $c\mu^+(A) \geq 1-\mu(A)-(1-\mu(A))^{1/a}$ for all A, which is equivalent to $c\mu^-(A) \geq \mu(A) - \mu(A)^{1/a}$ for all Borel sets A. $\square$

The following extension of the classical subadditivity property of the variance is due to Latała and Oleszkiewicz [14]. It allowed them to devise functional inequalities with the tensorization property. Actually, they focused on Sobolev inequalities involving L_2-norms of gradients, with applications to concentration inequalities. Here we aim at functional inequalities involving L_1-norms of gradients and provide information about isoperimetric inequalities.

Lemma 3. *Let (Ω_1, μ_1) and (Ω_2, μ_2) be probability spaces and consider their product probability space $(\Omega, \mu) := (\Omega_1 \times \Omega_2, \mu_1 \otimes \mu_2)$. For any non-negative random variable Z defined on (Ω, μ) and having finite first moment and for any strictly convex function ϕ on $[0, +\infty)$ such that $\frac{1}{\phi''}$ is a concave function, the following inequality holds true*:

$$\mathbb{E}_\mu \phi(Z) - \phi(\mathbb{E}_\mu Z) \leq \mathbb{E}_\mu\big(\big[\mathbb{E}_{\mu_1}\phi(Z) - \phi(\mathbb{E}_{\mu_1} Z)\big] + \big[\mathbb{E}_{\mu_2}\phi(Z) - \phi(\mathbb{E}_{\mu_2} Z)\big]\big).$$

Proposition 2. *Let (X, d, μ) be a metric probability space verifying hypothesis $(\mathcal{H})$. Let $a \in [\frac{1}{2}, 1]$ and $c > 0$. If for all $p \in (0, 1)$, $I_\mu \geq c(p - p^{1/a})$, then for all $p \in (0, 1)$,*

$$I_{\mu^\infty}(p) \geq c\big(p - p^{1/a}\big).$$

Proof. By Lemma 2, we know that for every locally Lipschitz function $f : X \to [0, 1]$

$$\frac{1}{c}\int |\nabla f|\, d\mu \geq \int f\, d\mu - \left(\int f^a\, d\mu\right)^{1/a}. \tag{4}$$

We shall prove that this functional inequality tensorizes, meaning that for all n the same property is verified by μ^n. Applying Lemma 2 again will give the claimed dimension-free isoperimetric inequality.

Checking the tensorization property is done along the same lines as in Latała and Oleszkiewicz [14]. Assume that (X_1, ν_1, d_1) and (X_2, ν_2, d_2) satisfy (4). Since $a \in [\frac{1}{2}, 1]$, Lemma 3 applies to $\Phi(t) = t^{1/a}$ and gives

$$\begin{aligned}
&\int f\, d\nu_1\, d\nu_2 - \left(\int f^a\, d\nu_1\, d\nu_2\right)^{1/a} \\
&\quad = \int \Phi\big(f^a\big)\, d\nu_1\, d\nu_2 - \Phi\left(\int f^a\, d\nu_1\, d\nu_2\right) \\
&\quad \leq \int \left(\int \Phi\big(f^a\big)\, d\nu_1 - \Phi\left(\int f^a\, d\nu_1\right)\right) d\nu_2 + \int \left(\int \Phi\big(f^a\big)\, d\nu_2 - \Phi\left(\int f^a\, d\nu_2\right)\right) d\nu_1 \\
&\quad \leq \frac{1}{c}\int \big(|\nabla_1 f| + |\nabla_2 f|\big)\, d\nu_1\, d\nu_2,
\end{aligned}$$

where $|\nabla_i f|$ is the norm of the gradient of f taken with respect to the ith variable. When $\nu_1 = \mu^m$ and $\nu_2 = \mu^n$, we may apply Hypothesis $(\mathcal{H})$ to replace the function in the latter integral by the norm of the full gradient $|\nabla f|$. This allows to show by induction that for all n, μ^n verifies the claimed functional inequality. □

The later result readily extends as follows.

Theorem 2. *Let $c : [\frac{1}{2}, 1] \to \mathbb{R}^+$, and consider for $p \in [0, 1]$,*

$$L(p) := \sup_{a\in[1/2,1]} c(a) \max\bigl\{p - p^{1/a}, 1 - p - (1 - p)^{1/a}\bigr\}.$$

If (X, d, μ) satisfies $(\mathcal{H})$ and $I_\mu \geq L$ then

$$I_{\mu^\infty} \geq L.$$

Moreover, there exists an even probability measure ν on $\mathbb{R}$ such that $I_\nu = I_{\nu^\infty} = L$ and such that for all n, coordinate half-spaces are solutions to the isoperimetric problem for ν^n.

Proof. Observe that since, by definition, isoperimetric functions of probability measures are symmetric with respect to $\frac{1}{2}$, the property for all $p \in [0, 1]$, $I_\mu \geq c(p - p^{1/a})$ is equivalent to $I_\mu(p) \geq cM_a(p)$, for all p, where

$$M_a(p) := \max\bigl\{p - p^{1/a}, 1 - p - (1 - p)^{1/a}\bigr\}.$$

Hence, the fact that $I_\mu \geq L$ implies $I_{\mu^\infty} \geq L$ is a direct consequence of Proposition 2, applied for all values of a.

Next, it is not hard to check that for $a \in [\frac{1}{2}, 1]$, M_a is subadditive, being a supremum of two concave functions defined on $[0, 1]$. And, since the property "$J(x + y) \leq J(x) + J(y)$ for all x, y" is stable under supremum, it follows that L is also subadditive.

Hence, by Proposition 1, there exists an even probability measure ν on $\mathbb{R}$ such that $I_\nu = L$ and half-lines solve the isoperimetric problem for ν. As we just proved, $I_\nu \geq L$ ensures that $I_{\nu^\infty} \geq L$. Combining this with $L = I_\nu \geq I_{\nu^\infty}$ yields $I_{\nu^\infty} = L$. The coordinate half-space $\{x \in \mathbb{R}^n | x_1 \leq t\}$ has same measure and boundary measure (for ν^n) as the set $(-\infty, t]$ (for ν). It is then clear that it solves the isoperimetric problem. □

Remark that for $a \in (\frac{1}{2}, 1)$, the function $M_a(p) = \max\{p - p^{1/a}, 1 - p - (1 - p)^{1/a}\}$ is not concave, hence the measure ν_a is not log-concave. Actually, M_a does not even have its maximum at $\frac{1}{2}$. Hence, it cannot be obtained as a supremum of concave functions which are in addition symmetric around 1/2. Therefore, it gives a genuinely new example of a measure for which coordinate half-spaces solve the isoperimetric problem in any dimension (that could not be deduced from Theorem 1).

3. Approximate inequalities

Let us start with some notations. Given two non-negative functions f, g defined on a set $S \subset \mathbb{R}$ and $D \geq 1$, we write $f \approx_D g$ and say that f and g are equivalent up to a factor D if there exists $a > 0$ such that for all $x \in S$, $ag(x) \leq f(x) \leq Dag(x)$. We write $f \approx g$ when there exists D such that $f \approx_D g$.

We say that a non-negative function f defined on a set $S \subset \mathbb{R}$ is essentially non-decreasing (with constant $D \geq 1$) when there exists a non-decreasing function g on S such that $f \approx_D g$. In the same way, we may define the notion of essentially non-increasing functions.

Also, a non-negative function f defined on an interval is said to be essentially concave (or pseudoconcave) if it is equivalent to a concave function.

The next proposition provides workable formulations of the above definitions. The part about essentially concave functions is due to Peetre [17].

Lemma 4. *Let f be a non-negative function defined on $S \subset \mathbb{R}$. Then f is essentially non-decreasing (resp. essentially non-increasing) with constant $D \geq 1$ if and only if for every $s \leq t$ in S,*

$$f(s) \leq Df(t) \qquad \big(\text{resp. } f(t) \geq Df(s)\big).$$

When f is defined on $(0, +\infty)$, the following assertions are equivalent:

(i) *f is essentially concave with some constant C_1,*
(ii) *There exists $C_2 \geq 1$ such that for all $s, t \in \mathbb{R}_+^*$, $f(s) \leq C_2 \max(1, \frac{s}{t}) f(t)$.*
(iii) *There exists $C_3 \geq 1$ such that on $\mathbb{R}_+^*$, f is essentially non-decreasing and $t \mapsto \frac{f(t)}{t}$ is essentially non-increasing, both with constant C_3.*

Moreover, the smallest possible constants verify $C_1/2 \leq C_2 = C_3 \leq C_1$.

Proof. The argument for essentially non-decreasing functions is very simple and we skip it. Let us just point out that it involves the least non-decreasing function above f, which is given by $\acute{f}(t) := \sup\{f(x) \mid x \in S \cap (-\infty, t]\}$.

Next, let us focus on concavity issues. The equivalence of the last two statements is obvious.

Assume f is essentially concave on $\mathbb{R}_+^*$. Then there exists a concave function h on $\mathbb{R}_+^*$ which is equivalent to f. And as f is positive, h is positive, therefore, being concave, h is necessarily non-decreasing on $\mathbb{R}_+^*$. Moreover, $t \mapsto \frac{h(t)}{t}$ is non-increasing on $\mathbb{R}_+^*$. So f satisfies the third condition.

Eventually, let us assume the second condition and show that f is equivalent to a concave function. The natural guess is the least concave majorant of f, which is explicitly given for $t > 0$ by

$$\widehat{f}(t) := \sup\left\{\sum_{i=1}^n \lambda_i f(t_i) \middle| n \in \mathbb{N}^*, \lambda_i \geq 0, t_i > 0, \sum_{i=1}^n \lambda_i = 1 \text{ and } \sum_{i=1}^n \lambda_i t_i = t\right\}.$$

By definition $f \leq \widehat{f}$. Let $n \in \mathbb{N}^*$, $t \in \mathbb{R}_+^*$, $(\lambda_i)_{1 \leq i \leq n}$ and $(t_i)_{1 \leq i \leq n}$ such that, for all i, $\lambda_i \geq 0$, $\sum_{i=1}^n \lambda_i = 1$ and $\sum_{i=1}^n \lambda_i t_i = t$. Using the hypothesis, we obtain

$$\sum_{i=1}^n \lambda_i f(t_i) \leq C_2 \sum_{i=1}^n \lambda_i \max\left(1, \frac{t_i}{t}\right) f(t) \leq C_2 \left(\sum_{i=1}^n \lambda_i + \sum_{i=1}^n \frac{\lambda_i t_i}{t}\right) f(t) = 2C_2 f(t).$$

Therefore $f \leq \widehat{f} \leq 2C_2 f$ and we have shown that f is essentially concave. □

We are now ready to state our main results.

Theorem 3. *Let J be a non-negative function defined on $[0, 1]$ with $J(0) = 0$. Assume that it is symmetric around $\frac{1}{2}$ (i.e., for every $t \in [0, 1]$, $J(t) = J(1-t)$) and that the function*

$$t \in (0, 1) \mapsto \frac{J(t)}{t \log(1/t)}$$

is essentially non-decreasing with constant D. Then for every metric probability space (X, d, μ) satisfying Hypothesis $(\mathcal{H})$:

$$I_\mu \geq J \quad \Longrightarrow \quad I_{\mu^\infty} \geq \frac{1}{c_D} J,$$

with $c_D = 2(D/\log 2)^2 \leq 5D^2$. Moreover, there exists a symmetric log-concave measure ν on the real line such that, on $[0, 1]$, $J \approx I_\nu \approx I_{\nu^\infty}$.

If in addition J is concave, one can take $c_D = 2D$ for $D > 1$ and $c_1 = 1$.

Remark 1. This result should be compared to a theorem of Milman [15], where a similar condition is given for dimension-free isoperimetric inequalities for the ℓ_2-combination of distances on products (in other words for the Euclidean enlargement). His condition involves an essential monotonicity property of J/I_γ where γ is the one-dimensional standard Gaussian measure. On $(0, 1/2]$ it is known that $I_\gamma(t) \approx t\sqrt{\log(1/t)}$.

In order to formulate a converse statement, we introduce the following hypothesis: we say that (X, d, μ) enjoys the regularity property $(\mathcal{R})$ if for all $t \in (0, 1)$, $I_\mu(t) < +\infty$ and for all $n \in \mathbb{N}^*$, $t \in (0, 1)$ and $\varepsilon > 0$ there exists a Borel set $A \subset X^n$ with $\mu^n(A) = t$, $(\mu^n)^+(A) \leq I_{\mu^n}(t) + \varepsilon$ and

$$\left(\mu^n\right)^+(A) = \lim_{h \to 0^+} \frac{\mu^n(A_h \setminus A)}{h},$$

where the products X^n are equipped with the uniform distance. This hypothesis means that there are almost solutions of the isoperimetric problems for which the lim inf in the definition of the Minkowski content is actually a real limit. Thanks to Theorem 15 in Barthe [2] it is not hard to check this property for log-concave measures on the real line. We will give more comments on this hypothesis in Remark 3 below.

Theorem 4. *Let (X, d, μ) satisfy hypothesis $(\mathcal{R})$. Then the map*

$$t \in (0, 1) \mapsto \frac{I_{\mu^\infty}(t)}{t \log(1/t)}$$

is continuous and essentially non-decreasing.

We introduce two functions, both defined on $(0, 1)$ by: $J_0(t) = t$ and $J_1(t) = t \log \frac{1}{t}$. Combining Theorems 3 and 4, we can formulate our results as an equivalence.

Theorem 5. *Let (X, d, μ) denote a metric space equipped with a Borel probability measure μ and satisfying hypothesis $(\mathcal{R})$ and $(\mathcal{H})$. Then the following assertions are equivalent:*

(i) *There exists a constant C such that $\frac{I_\mu}{J_1}$ is essentially non-decreasing on $(0, 1)$ with constant C,*

(ii) *There exists a constant $K \geq 1$ such that, on $[0, 1]$, $\frac{1}{K} I_\mu \leq I_{\mu^\infty} \leq I_\mu$.*

The next lemma gives a different formulation of the main condition appearing in the previous theorems.

Lemma 5. *Let $K : [0, 1] \to \mathbb{R}_+$ be a non-negative function such that K is symmetric with respect to $\frac{1}{2}$ (i.e. for $t \in [0, 1]$, $K(t) = K(1 - t)$). Then the following assertions are equivalent:*

(i) *There is a constant C such that $\frac{K}{J_1}$ is essentially non-decreasing on $(0, 1)$ with constant C.*

(ii) *There exist constants C_0 and C_1 such that $\frac{K}{J_0}$ is essentially non-increasing on $(0, \frac{1}{2}]$ with constant C_0 and $\frac{K}{J_1}$ is essentially non-decreasing on $(0, \frac{1}{2}]$ with constant C_1.*

Moreover, the smallest possible constants verify $C \leq \frac{C_0 C_1}{\log 2}$ and $C_0 \leq \frac{C}{\log 2}$, $C_1 \leq C$.

Proof. We use the concavity of the map $t \mapsto (1 - t) \log \frac{1}{1-t}$, which yields, for every $t \in [0, \frac{1}{2}]$, $t \log 2 \leq (1 - t) \log \frac{1}{1-t} \leq t$. Assuming (i), $\frac{K}{J_1}$ is essentially non-decreasing on $(0, \frac{1}{2}]$ with constant C. For the second part of the assertion, let $0 < s \leq t \leq \frac{1}{2}$. Then,

$$\frac{K(t)}{t} \leq \frac{K(1-t)}{(1-t)\log(1/(1-t))} \leq C \frac{K(1-s)}{(1-s)\log(1/(1-s))} \leq \frac{C}{\log 2} \frac{K(s)}{s}. \tag{5}$$

For the converse implication: assuming (ii), we first check that $\frac{K}{J_1}$ is essentially non-decreasing on $[\frac{1}{2}, 1)$. Let $\frac{1}{2} \leq s \leq t < 1$, then

$$\frac{K(s)}{s \log(1/s)} \leq \frac{1}{\log 2} \frac{K(1-s)}{1-s} \leq \frac{C_0}{\log 2} \frac{K(1-t)}{1-t} \leq \frac{C_0}{\log 2} \frac{K(t)}{t \log(1/t)}.$$

To get the property on the whole interval $(0, 1)$, it suffices to use $\frac{1}{2}$ as an intermediate point. $\square$

The next corollary describes the possible size of an infinite dimensional isoperimetric profile:

Corollary 2. *Let (X, d, μ) denote a metric space equipped with a Borel probability measure μ and satisfying hypotheses $(\mathcal{R})$ and $(\mathcal{H})$.*

If $\inf_{t\in(0,\frac{1}{2}]} \frac{I_\mu(t)}{t} = 0$ then I_{μ^∞} is identically 0, else there exist $\alpha, \beta > 0$ such that for all $t \in [0, 1]$,

$$\alpha \min(t, 1-t) \le I_{\mu^\infty}(t) \le \beta \min\left(t \log \frac{1}{t}, (1-t) \log \frac{1}{1-t}\right).$$

Remark 2. The function defined on $[0, 1]$ by $t \mapsto \min(t, 1-t)$ is the isoperimetric function of the double-sided exponential measure on $\mathbb{R}$, $e^{-|x|}\, dx/2$. Using the notation and results of Corollary 1, we observe that it is equivalent to the function $K_0(t) = t(1-t)$. Moreover there is a log-concave probability measure ℓ_0 on the real line for which $K_0 = I_{\ell_0} = I_{\ell_0^\infty}$ (actually, ℓ_0 is the standard logistic measure ℓ with density $\frac{e^{-x}}{(1+e^{-x})^2}$ with respect to Lebesgue's measure). Hence, the lower bound is optimal up to the multiplicative factor.

The upper bound of I_{μ^∞} given in the above corollary is due to Bobkov and Houdré [6]. A similar remark applies to it: the quantity in the upper estimate is equivalent to the function K_1 of Corollary 1, which is also an infinite dimensional isoperimetric profile (of a measure which is reminiscent of Gumble laws, as its distribution function is of the order of $e^{-\beta e^{-y}}$ when $y \to -\infty$, for some $\beta > 0$).

The fact that the infinite dimensional isoperimetric profile is either trivial, or at least as big as the one of the exponential measure was already discovered, in slightly different forms, by Talagrand [19] and by Bobkov and Houdré [7].

Proof of Corollary 2. By Theorem 4, there exists $C \ge 1$ such that for all $t \in (0, 1/2]$,

$$I_{\mu^\infty}(t) \le Ct \log\left(\frac{1}{t}\right) \times \frac{2}{\log 2} I_{\mu^\infty}\left(\frac{1}{2}\right). \tag{6}$$

Applying Theorem 4 again, together with Lemma 5, we get that there exists $D \ge 1$ such that for all $t \in (0, 1/2]$,

$$D \frac{I_{\mu^\infty}(t)}{t} \ge 2 I_{\mu^\infty}\left(\frac{1}{2}\right).$$

Therefore, assuming $\inf_{t\in(0,1/2]} \frac{I_\mu(t)}{t} = 0$, and using that $I_\mu \ge I_{\mu^\infty}$, we can deduce that $I_{\mu^\infty}(\frac{1}{2}) = 0$. Then (6) and the symmetry of isoperimetric functions yield $I_{\mu^\infty} = 0$ pointwise.

Next, assume that there exists $\kappa > 0$ such that $I_\mu(t) \ge \kappa t$ for all $t \in (0, 1/2]$. Then Theorem 3 applies to $J(t) := \kappa \min(t, 1-t)$ (Lemma 5 gives a quick way to check the hypothesis) and gives $I_{\mu^\infty} \ge cJ$ for some $c > 0$. □

Remark 3. Our results are stated for general metric spaces, but are devised for continuous settings (e.g., for which the values taken by the measure cover all $[0, 1]$). This is why additional hypotheses appear in our statements. One may find Hypothesis $(\mathcal{H})$ quite natural (it is related to a.e. differentiability of Lipschitz functions). On the other hand, Hypothesis $(\mathcal{R})$ is more demanding, as it seems to require approximation theorems by smooth sets.

Let us point out a possible variant of Theorem 4 where all the hypotheses are incorporated in the structure of the ambient space: assume that X is a finite dimensional vector space of dimension p, that the distance d is induced by a norm N on X and that μ has a positive C^1 density h with respect to Lebesgue's measure, $\mu = h.\mathcal{L}^p$. We equip the product spaces X^n with d_∞, the ℓ_∞-combination of N, that is, for $x, y \in X^n$, $d_\infty(x, y) = \max_{1\leq i\leq n} N(x_i, y_i)$. Then, instead of using the Minkowski content as a definition of the boundary measure, let us choose the notion of generalized perimeter instead: if $A \subseteq X^n$ is measurable, then

$$P_{\mu^n,\infty} = \sup\left\{ \int_A \sum_{i=1}^n \left|\nabla_i(\varphi h)\right| d\mathcal{L}^{np} | \varphi \in \mathcal{C}_c^1\left(X^n\right) \text{ and } \sup_{x\in X^n} d_\infty\left(\varphi(x), 0\right) \leq 1 \right\},$$

where $|\nabla f|$ is the modulus of gradient of f.

Since the perimeter is defined as a supremum (recall that the Minkowski content is an inferior limit), the proof of Lemma 8 below does not require any regularity assumption. Hence the proof of Theorem 4 applies without any changes and does not require $(\mathcal{R})$. The proof of Theorem 3 also applies to this new setting, without assuming $(\mathcal{H})$, with the following main modification: instead of using functional inequalities for locally Lipschitz functions, we work in the class of functions of bounded variations. We refer the reader to the book of Ambrosio, Fusco and Pallara [1] for an exhaustive study of this approach in the Euclidean case. This requires to use various results about these functions: co-area inequality (Theorem 3.40), approximation by smooth functions (Theorem 3.9), approximate differentiability (Theorem 3.83 and Proposition 3.92 among others).

3.1. Proof of Theorem 3

We start with a few preliminary statements.

Lemma 6. *Consider a function $K : [0, 1] \to \mathbb{R}^+$ with $K(0) = 0$. Assume that K is symmetric with respect to $\frac{1}{2}$ and that $\frac{K}{J_1}$ is essentially non-decreasing on $(0, 1)$ with constant D. Then*

(i) *K is essentially non-decreasing on $[0, \frac{1}{2}]$ with constant $\frac{2D}{e\log 2}$,*

(ii) *K is essentially concave. More precisely, there exists a concave function $I : [0, 1] \to \mathbb{R}^+$, which is symmetric with respect to $\frac{1}{2}$, and is equivalent to K up to a factor $2D/\log 2$.*

Proof. Observe that the function $J_1(t) = t\log(1/t)$ is increasing on $(0, 1/e]$ and decreasing on $[1/e, 1)$. Its maximum is therefore $J(1/e) = 1/e$.

Assume that $0 \leq s \leq t \leq \frac{1}{2}$. Then, by hypothesis $K(s) \leq D\frac{J_1(s)}{J_1(t)}K(t)$. If $t \leq \frac{1}{e}$, we can conclude that $K(s) \leq DK(t)$. If $t \in (1/e, 1/2]$, we argue differently

$$K(s) \leq D\frac{J_1(s)}{J_1(t)}K(t) \leq D\frac{J_1(1/e)}{J_1(1/2)}K(t) = \frac{2D}{e\log 2}K(t).$$

This concludes the proof of (i).

Next, let us prove (ii). Consider the map $\widetilde{K}$ defined on $\mathbb{R}_+$ by:

$$\widetilde{K}(t) = \begin{cases} K(t), & \text{if } t \in [0, \frac{1}{2}), \\ K(\frac{1}{2}), & \text{if } t \geq \frac{1}{2}. \end{cases}$$

Combining (i) and the second part of (ii) in Lemma 5, one readily checks that $\widetilde{K}$ satisfies the hypothesis of Assertion (iii) in Lemma 4 with constant $\frac{D}{\log 2}$ ($\geq \frac{2D}{e\log 2}$). Hence there exists a concave function H which is equivalent to $\widetilde{K}$ on $(0, +\infty)$, up to a factor $2D/\log 2$. Define I to be the restriction of H to $(0, \frac{1}{2}]$, extended at 0 by $I(0) = 0$ and to $[0, 1]$ by symmetry with respect to $\frac{1}{2}$. Since H is concave and non-negative on $(0, +\infty)$, it is also non-decreasing. Therefore, the function I is concave as well. As $\widetilde{K} \approx H$ on $(0, +\infty)$, we obtain by restriction that $K \approx I$ on $(0, 1/2]$, up to the same constant. Since $K(0) = I(0) = 0$, and both I and K are symmetric with respect to $1/2$, we can conclude that $I \approx K$ on $[0, 1]$, up to a factor $2D/\log 2$. □

The following result shows how we exploit the essentially monotonicity properties of J/J_0 and J/J_1 where $J_0(t) = t$ and $J_1(t) = t\log(1/t)$.

Proposition 3. *Let $J : (0, 1) \to \mathbb{R}^+$ such that for all t, $J(t) = J(1-t)$. Assume that on $(0, 1/2]$, J/J_0 is essentially non-increasing with constant D_0 and J/J_1 is essentially non-decreasing with constant D_1. Then there exists a function $c : [1/2, 1) \to \mathbb{R}^+$ such that for all $t \in (0, 1)$,*

$$J(t) \geq \sup_{a\in[1/2,1)} c(a)\big(t - t^{1/a}\big),$$

and for all $t \in (0, \frac{1}{2}]$,

$$J(t) \leq 2D_0 \max(D_0, D_1) \sup_{a\in[1/2,1)} c(a)\big(t - t^{1/a}\big).$$

The proof of this proposition relies on the following statement, which is related to Barthe, Cattiaux and Roberto [4], Lemma 19.

Lemma 7. *Let $\Phi : (0, \frac{1}{\log 2}] \to \mathbb{R}^+$. If Φ is essentially non-increasing with constant C_0, then for all $y \in (0, \frac{1}{2}]$,*

$$\sup_{\alpha\in(0,1]} \Phi(\alpha)\big(1 - y^\alpha\big) \geq \frac{1}{2C_0}\Phi\left(\frac{1}{\log(1/y)}\right).$$

If, in addition, $s \mapsto s\Phi(s)$ is essentially non-decreasing with constant C_1, then for all $y \in (0, 1/2]$,

$$\sup_{\alpha\in(0,1]} \Phi(\alpha)\big(1 - y^\alpha\big) \leq \max(C_0, C_1)\Phi\left(\frac{1}{\log(1/y)}\right).$$

Proof. In order to bound the supremum from below, we just select an appropriate value for α: if $y \in (0, 1/e]$, choosing $\alpha = 1/\log(1/y) \in (0, 1]$, we get

$$\sup_{\alpha \in (0,1]} \Phi(\alpha)\big(1 - y^\alpha\big) \geq \big(1 - e^{-1}\big)\Phi\left(\frac{1}{\log(1/y)}\right).$$

If $y \in [1/e, 1/2]$, we choose $\alpha = 1$ and get

$$\sup_{\alpha \in (0,1]} \Phi(\alpha)\big(1 - y^\alpha\big) \geq (1 - y)\Phi(1) \geq \frac{1}{2}\Phi(1).$$

However, $y \geq 1/e$ ensures $1/\log(1/y) \geq 1$. Since Φ is essentially non-increasing, $C_0\Phi(1) \geq \Phi(1/\log(1/y))$. Hence the first claim is proved, with a constant $\min(1 - e^{-1}, 1/(2C_0)) = 1/(2C_0)$.

To prove the converse inequality, we change variables as follows: setting $c = \alpha \log(1/y)$,

$$\sup_{\alpha \in (0,1]} \Phi(\alpha)\big(1 - y^\alpha\big) = \sup_{c \in (0, \log(1/y)]} \big(1 - e^{-c}\big)\Phi\left(\frac{c}{\log(1/y)}\right).$$

For $c \geq 1$, we know that $\Phi(\frac{c}{\log(1/y)}) \leq C_0\Phi(\frac{1}{\log(1/y)})$ and we bound $1 - e^{-c}$ from above by 1. For $c \in (0, 1]$, we take advantage of the hypothesis on $x \mapsto x\Phi(x)$, in the form $c\Phi(cx) \leq C_1\Phi(x)$ for $x > 0$:

$$\big(1 - e^{-c}\big)\Phi\left(\frac{c}{\log(1/y)}\right) \leq C_1 \frac{1 - e^{-c}}{c}\Phi\left(\frac{1}{\log(1/y)}\right) \leq C_1\Phi\left(\frac{1}{\log(1/y)}\right).$$

These two estimates readily give the claim. □

Proof of Proposition 3. For $\alpha \in (0, 1/\log 2]$, we define

$$\Phi(\alpha) := \frac{J(e^{-1/\alpha})}{e^{-1/\alpha}} = \frac{J}{J_0}\big(e^{-1/\alpha}\big).$$

Since $e^{-1/\alpha} \in (0, 1/2]$, our hypothesis ensures that Φ is essentially non-increasing with constant D_0. Notice that

$$\alpha\Phi(\alpha) := \frac{J(e^{-1/\alpha})}{e^{-1/\alpha}\log(1/e^{-1/\alpha})} = \frac{J}{J_1}\big(e^{-1/\alpha}\big).$$

Hence by hypothesis, it is essentially non-decreasing with constant D_1. Therefore, we may apply the previous lemma to Φ. Since by definition, $\Phi(1/\log(1/y)) = J(y)/y$, it gives that for all $y \in (0, 1/2]$,

$$\frac{J(y)}{y} \geq \frac{1}{\max(D_0, D_1)} \sup_{\alpha \in (0,1]} \Phi(\alpha)\big(1 - y^\alpha\big),$$

and for all $y \in (0, 1/2]$,

$$\frac{J(y)}{y} \le 2D_0 \sup_{\alpha \in (0,1]} \Phi(\alpha)\big(1 - y^\alpha\big).$$

Multiplying these inequalities by y, and setting $a := 1/(1+\alpha)$, the former estimate gives for $y \in (0, 1/2]$

$$\begin{aligned} J(y) &\ge \frac{1}{\max(D_0, D_1)} \sup_{\alpha \in (0,1]} \Phi(\alpha)\big(y - y^{1+\alpha}\big) \\ &= \frac{1}{\max(D_0, D_1)} \sup_{a \in [1/2,1)} \Phi\left(\frac{1}{a} - 1\right)\big(y - y^{1/a}\big). \end{aligned} \tag{7}$$

Hence, we have prove the claimed lower bound on J with $c(a) = \Phi(a^{-1} - 1)/\max(D_0, D_1)$. We proceed in the same way with the upper bound on J. The ratio of the upper bound to the lower bound is $2D_0 \max(D_0, D_1)$.

It remains to extend the lower bound (7) to values $y \in (1/2, 1)$. To do this we use the symmetry of J and the fact that for all $a \in [1/2, 1]$ and all $s \in [1/2, 1)$, $1 - s - (1-s)^{1/a} \ge s - s^{1/a}$ (this follows from the comparison of second derivatives, observing that equality holds at $1/2$ and 1): for $y \in (1/2, 1)$,

$$J(y) = J(1-y) \ge c(a)\big(1 - y - (1-y)^{1/a}\big) \ge c(a)\big(y - y^{1/a}\big). \qquad \square$$

Proof of Theorem 3. Let us denote by $D_0 \ge 1$ the smallest constant such that J/J_0 is essentially non-increasing on $(0, 1/2]$ with constant D_0. Similarly, let $D_1 \ge 1$ be the smallest constant such that J/J_1 is essentially non-decreasing on $(0, 1/2]$ with constant D_1.

First, we apply Proposition 3. With the notation of the proposition, it follows that for all $a \in [1/2, 1)$ and all $t \in [0, 1]$,

$$I_\mu(t) \ge J(t) \ge c(a)\big(t - t^{1/a}\big).$$

Note that for $t = 0$ or $t = 1$ all quantities vanish. Next, Proposition 2 tell us that $I_{\mu^\infty}(t) \ge c(a)(t - t^{1/a})$. This is true for all a and all t, hence applying the second part of Proposition 3, we deduce that for all $t \in [0, 1/2]$,

$$I_{\mu^\infty}(t) \ge \sup_{a \in [1/2,1)} c(a)\big(t - t^{1/a}\big) \ge \frac{1}{2D_0 \max(D_0, D_1)} J(t).$$

Since both I_{μ^∞} and J are symmetric with respect to $1/2$, we can conclude that for all $t \in [0, 1]$, it holds $I_{\mu^\infty}(t) \ge J(t)/(2D_0 \max(D_0, D_1))$.

In the general case, we know by Lemma 5 that $D_0 \le D/\log 2$ and $D_1 \le D$. Therefore we get that $I_{\mu^\infty} \ge J/c_D$ with $c_D = 2D^2/(\log 2)^2$.

In the particular case where J is concave, we know that $J(t)/t$ is non-increasing, so that $D_0 = 1$ and $I_{\mu^\infty} \ge J/(2D_1) \ge J/(2D)$. The paragraph after Theorem 1 explains that when J is concave, symmetric and such that J/J_1 is non-decreasing, the inequality $I_\mu \ge J$ implies $I_{\mu^\infty} \ge J$. Hence, in this case the conclusion of Theorem 3 is valid with $c_1 = 1$.

Our final task is to find a log-concave measure ν on the real line such that $J \approx I_\nu \approx I_{\nu^\infty}$. By the second point in Lemma 6, there exists a function $I : [0,1] \to \mathbb{R}^+$ such that $I \approx J$ and I is concave and symmetric with respect to $\frac{1}{2}$. By a well known result of Bobkov and Houdré, any such function I is the isoperimetric profile of an even log-concave measure on the real line (this follows from Proposition 1 and Lemma 1). Let ν be an even log-concave measure on $\mathbb{R}$ such that $I_\nu = I \approx J$. Obviously $\frac{I_\nu}{J_1}$ is also essentially non-decreasing. Hence I_ν fulfils the assumption of the first part of the theorem. Therefore,

$$I_\nu \geq I_{\nu^\infty} \geq \frac{1}{c_D} I_\nu.$$

Putting everything together, we conclude that $J \approx I_\nu \approx I_{\nu^\infty}$. □

3.2. Proof of Theorem 4

First, we recall a classical property of infinite dimensional profiles, which comes from testing isoperimetric inequalities on product sets. It was put forward by Bobkov [5].

Lemma 8. *Let (X,d,μ) be a metric space equipped with a Borel probability measure μ and satisfying the regularity property $(\mathcal{R})$. Then the infinite-dimensional isoperimetric profile of (X,d,μ) satisfies, for every $a,b \in [0,1]$:*

$$I_{\mu^\infty}(ab) \leq a I_{\mu^\infty}(b) + b I_{\mu^\infty}(a). \tag{8}$$

Proof. The inequality is obvious if a or b is equal to 1, or to 0 since $I_{\mu^\infty}(0) = 0$. Let $a,b \in (0,1)$ and $\varepsilon > 0$. Let $m,n \in \mathbb{N}^*$. Let $A \subset X^m$ be any set with $\mu^m(A) = a$. Let $B \subset X^n$ be any set with $\mu^n(B) = b$, $(\mu^n)^+(B) \leq I_{\mu^n}(b) + \varepsilon$ and

$$\left(\mu^n\right)^+(B) = \lim_{h\to 0} \frac{\mu^n(B_h \setminus B)}{h},$$

which is possible thanks to Hypothesis $(\mathcal{R})$.

Then consider $A \times B \subseteq \mathbb{R}^{m+n}$. Obviously, $\mu^{m+n}(A \times B) = ab$. The uniform enlargement of a product set is still a product: $(A \times B)_h = A_h \times B_h$ for any $h > 0$. Therefore

$$\begin{aligned}\frac{1}{h}\mu^{m+n}\big((A \times B)_h \setminus (A \times B)\big) &= \frac{\mu^m(A_h)\mu^n(B_h) - \mu^m(A)\mu^n(B)}{h} \\ &= \frac{\mu^m(A_h \setminus A)}{h}\mu^n(B_h) + \mu^m(A)\frac{\mu^n(B_h \setminus B)}{h}.\end{aligned}$$

Since by hypothesis $\lim_{h\to 0}(\mu^n(B_h) - \mu^n(B))/h < +\infty$, we know that $\lim_{h\to 0}\mu^n(B_h) = \mu^n(B)$ (note the convergence holds by monotonicity). Taking upper limits in $h \to 0$, and observing that two of the three terms have limits, we deduce from the latter inequality that

$$\left(\mu^{m+n}\right)^+(A \times B) \leq \left(\mu^m\right)^+(A)\mu^n(B) + \mu^m(A)\left(\mu^n\right)^+(B). \tag{9}$$

Since $I_{\mu^{m+n}}(ab) \le (\mu^{m+n})^+(A \times B)$, we obtain after optimizing on sets A of measure a and using the hypothesis on the boundary measure of B:

$$I_{\mu^{m+n}}(ab) \le I_{\mu^m}(a)b + a\big(I_{\mu^n}(b) + \varepsilon\big).$$

Letting ε tend to 0, and m, n tend to $+\infty$ gives the claim (8). □

The symmetry property ($I_{\mu^\infty}(t) = I_{\mu^\infty}(1-t)$ for all $t \in 0, 1]$) and the two-points inequality (8) are enough to deduce Theorem 4, as the next statement shows:

Proposition 4. *Let* $I : [0, 1] \to [0, +\infty]$ *be an application satisfying that for all* $a, b \in [0, 1]$

$$I(a) = I(1-a) \quad \text{and} \quad I(ab) \le aI(b) + bI(a), \tag{10}$$

with the convention that $+\infty \times 0 = 0$.

If there exists $x_0 \in [0, 1]$ *such that* $\limsup_{x\to x_0} I(x) < +\infty$, *then* I *is continuous and* $t \mapsto I(t)/(t\log(1/t))$ *is essentially non-decreasing on* $(0, 1)$.

The condition of local boundedness around some point cannot be removed as shown by the following example: $I(t) = 0$ if $t \in \mathbb{Q}$ and $I(t) = +\infty$ otherwise.

The proof of the proposition uses the next two easy lemmas.

Lemma 9. *Let* $S \subset (0, 1)$ *be a set with the following stability property*:

$$(x \in S \text{ and } y \in S) \quad \Longrightarrow \quad (1 - x \in S \text{ and } xy \in S).$$

If S is not empty, then it is dense in $(0, 1)$. *Moreover, if S has non-empty interior then* $S = (0, 1)$.

In other words, if S is neither $\varnothing$ nor $(0, 1)$ then S and $(0, 1) \setminus S$ are dense in $(0, 1)$. This is the case for instance of $S = \mathbb{Q} \cap (0, 1)$.

Proof of Lemma 9. Let $t \in (0, 1)$ be an element of S, then for all $n \in \mathbb{N}^*$, $x_n := 1 - t^n$ belongs to S and the sequence (x_n) tends to 1. Given $0 < a < b < 1$, let us show that there is a point of S between a and b. Choose k large enough such that $x_k > \max(b, a/b)$. Then for all $n \ge 1$, $(x_k)^n \in S$. Obviously $x_k \ge b$ and $\lim_n (x_k)^n = 0$. Let n_0 be maximal with $(x_k)^{n_0} \ge b$. Then

$$b > x_k^{n_0+1} = x_k^{n_0} x_k > b \times \frac{a}{b} = a.$$

Hence, $x_k^{n_0+1} \in S \cap (a, b)$. This completes the proof of the density of S.

Assume now that $(a, b) \subset S$ for some $0 < a < b < 1$. Consider an arbitrary $x \in (0, 1)$ and let us show that $x \in S$. If $x \in (a, b)$, we have nothing to prove. If $x \in (0, a]$, we use the fact that S being non-empty is dense: there exists $y \in S \cap (x/b, x/a)$. Since S contains y and (a, b), the stability by product ensures that S also contains (ya, yb). Hence $x \in (ya, yb) \subset S$. Eventually, if $x \in [b, 1)$, we consider $1 - x \in (0, 1 - b]$. By the symmetry assumption $(1 - b, 1 - a) \subset S$, so the latter argument yields $1 - x \in S$. Using symmetry again, we may conclude that $x \in S$. □

The next lemma is a classical result about subadditive functions on $\mathbb{R}_+$ (see, e.g., Kuczma [13]).

Lemma 10. *Let* $K:\mathbb{R}_+\to\mathbb{R}$ *be a subadditive function with* $\lim_0 K=0$. *Then*

$$\lim_{h\to 0^+}\frac{K(h)}{h}=\sup_{t>0}\frac{K(t)}{t}.$$

Proof. Denote $S=\sup_{t>0}\frac{K(t)}{t}$. Given any $u<S$, there exists $x_0\in\mathbb{R}_+^*$ such that $K(x_0)>ux_0$. For any $h\in(0,x_0)$, write $x_0=nh+\delta$ with $n=\lfloor\frac{x_0}{h}\rfloor$ and $\delta\in[0,h)$. By subadditivity of K,

$$ux_0<K(x_0)\le nK(h)+K(\delta)=(x_0-\delta)\frac{K(h)}{h}+K(\delta).$$

Next, we let h tend to 0^+. In this case $\delta\to 0^+$ and $K(\delta)\to 0$, hence (for any $u<S$)

$$u\le\liminf_{h\to 0^+}\frac{K(h)}{h}.$$

Therefore, $S\le\liminf_{h\to 0^+}\frac{K(h)}{h}$. On the other hand, $\limsup_{h\to 0^+}\frac{K(h)}{h}\le S$ holds by definition. $\square$

Proof of Proposition 4. There is nothing to prove if I is identically 0, so we assume that I does not vanish everywhere. Observe that the two-points inequality in (10), applied for $a=b=0$, yields $I(0)=0$.

Consider the subset S_1 of $(0,1)$ of points x such that the function I is bounded on a neighbourhood of x. Our hypothesis $\limsup_{x\to x_0}I(x)<+\infty$ ensures that S_1 has non-empty interior. Thanks to (10), one readily checks that S_1 is stable by product and by symmetry with respect to $1/2$. Hence, Lemma 9 applies to S_1 and shows that $S_1=(0,1)$. This means that I is locally bounded at every point of $(0,1)$. By compactness, we deduce that I is bounded on any segment $[a,b]\subset(0,1)$.

The next step of the proof is an argument of Bobkov and Houdré, that we include for completeness. The two-points inequality implies by induction that for all $a\in[0,1]$ and any integer $k\ge 1$, $I(a^k)\le ka^{k-1}I(a)$. Let $t\in(0,1/e]$. Choosing $k=\lfloor\log(1/t)\rfloor\ge 1$ and $a=t^{1/k}$ in the latter inequality leads to

$$I(t)\le kt^{1-1/k}I\big(t^{1/k}\big)=t\big\lfloor\log(1/t)\big\rfloor\frac{I(t^{1/k})}{t^{1/k}}.$$

Using that for $x\ge 1$, $x/\lfloor x\rfloor\in[1,2]$, we obtain that $t^{1/k}=\exp(-\log(1/t)/\lfloor\log(1/t)\rfloor)\in[e^{-2},e^{-1}]$. Hence for $t\in(0,e^{-1}]$, $I(t)\le Ct\log(1/t)$ where $C=e^2\sup\{I(s);s\in[e^{-2},e^{-1}]\}$ is finite (by the previous point). In particular, this estimate implies that $I(t)$ tends to 0 when $t\neq 0$ tends to 0. Since $I(0)=0$, the function I is continuous at 0. By symmetry, it is also continuous at 1, with $I(1)=0$.

Consider the map $K:\mathbb{R}^+\to\mathbb{R}^+$ defined by $K(x)=e^xI(e^{-x})$. Then for all $x,y\in\mathbb{R}_+$, by (10)

$$K(x+y)=\frac{I(e^{-x}e^{-y})}{e^{-x}e^{-y}}\le\frac{e^{-x}I(e^{-y})+e^{-x}I(e^{-y})}{e^{-x}e^{-y}}=K(y)+K(x),$$

which means that K is subadditive. Moreover, since I is continuous at 1, K is continuous at 0, with $K(0) = I(1) = 0$.

For all $a > \varepsilon > 0$, we have by subadditivity $K(a+\varepsilon) \le K(a) + K(\varepsilon)$ and $K(a) \le K(a-\varepsilon) + K(\varepsilon)$. Letting ε tend to zero, we obtain for all $a > 0$

$$\limsup_{x\to a^+} K(x) \le K(a) \le \liminf_{x\to a^-} K(x).$$

In words, on $(0, +\infty)$, the function K is right-upper-semicontinuous and left-lower-semicontinuous. Since for all $t \in (0,1)$, $I(t) = tK(\log(1/t))$, if follows that on $(0,1)$ the function I is left-upper-semicontinuous and right-lower-semicontinuous. Note that "left" and "right" were exchanged, since $t \mapsto \log(1/t)$ is continuous and decreasing. The symmetry assumption $I(t) = I(1-t)$ allows to exchange once more: so I is also right-upper-semicontinuous and left-lower-semicontinuous on $(0,1)$. Thus I is continuous on $(0,1)$, and actually on $[0,1]$. Indeed, the continuity at the endpoints has already been established.

Next, let us draw another consequence of the above properties of K. Lemma 10 directly applies and gives that

$$\lim_{h\to 0^+} \frac{K(h)}{h} = \sup_{x>0} \frac{K(x)}{x} = \sup_{x>0} \frac{e^x I(e^{-x})}{x}.$$

Since we assume that I is not identically 0, the above limit, denoted by L, belongs to $(0, +\infty]$. We now translate this convergence in terms of I: using symmetry, for $t \in (0,1)$,

$$\frac{I(t)}{t} = \frac{I(1-t)}{t} = \frac{1-t}{t} K\left(\log\left(\frac{1}{1-t}\right)\right) = \frac{(1-t)\log(1/(1-t))}{t} \times \frac{K(\log(1/(1-t)))}{\log(1/(1-t))}.$$

When $t > 0$ tends to 0, the first ratio tends to 1, and the second to $L = \lim_{h\to 0^+} K(h)/h$. Therefore, we can deduce that $\lim_{t\to 0^+} I(t)/t = L \in (0, +\infty]$.

In order to turn this limit into a lower bound on $I(t)/t$ for $t \in (0, 1/2]$, we need to check that I does not vanish in $(0,1)$. To do this, let us consider the set $S_0 = \{x \in (0,1);\, I(x) = 0\}$. By (10), it is stable by product and symmetry around $1/2$. If it were non-empty, the first part of Lemma 9 would imply that S_0 is dense in $(0,1)$. By continuity of I, we would conclude that I is identically 0. Since we assumed that I does not vanish everywhere, it follows that $S_0 = \varnothing$. As a conclusion, the function I vanishes only at 0 and 1.

On $(0, 1/2]$ the map $t \mapsto I(t)/t$ is continuous, with positive values. Moreover, it has a positive (maybe infinite) limit at 0^+. As a consequence, there exists $c > 0$ such that $I(t) \ge ct$ for all $t \le 1/2$.

Let us deduce that I is essentially non-decreasing on $[0, 1/2]$. Let $0 \le s < t \le \frac{1}{2}$. Using the two-points inequality (10)

$$I(s) = I\left(t \times \frac{s}{t}\right) \le \frac{s}{t} I(t) + tI\left(\frac{s}{t}\right) \le I(t) + t \max I \le \left(1 + \frac{\max I}{c}\right) I(t),$$

where we have used that I is continuous on $[0,1]$ and $I(t) \ge ct$.

Eventually, let us prove that $\frac{I}{J_1}$ is essentially non-decreasing on $(0, 1)$. Let $0 < s < t < 1$. Then one can write $s = t^{k+\alpha}$ with $k = \lfloor \frac{\log 1/s}{\log 1/t} \rfloor \in \mathbb{N}^*$ and $\alpha \in [0, 1)$. By the two-points inequality for I:

$$\frac{I(s)}{s} = \frac{I(t^{k+\alpha})}{t^{k+\alpha}} \le k\frac{I(t)}{t} + \frac{I(t^\alpha)}{t^\alpha}. \tag{11}$$

Assume first that $t \ge \frac{1}{2}$. Then $t^\alpha \ge t \ge \frac{1}{2}$. We have shown that I is essentially non-decreasing on $[0, \frac{1}{2}]$ (with a constant denoted by D). By symmetry, it follows that I is essentially non-increasing on $[\frac{1}{2}, 1]$ with constant D. Hence,

$$\frac{I(t^\alpha)}{t^\alpha} \le D\frac{I(t)}{t^\alpha} \le D\frac{I(t)}{t}.$$

Combining this estimate with (11) gives

$$\frac{I(s)}{s} \le (k + D)\frac{I(t)}{t} \le (1 + D)k\frac{I(t)}{t} \le (1 + D)\frac{\log 1/s}{\log 1/t}\frac{I(t)}{t},$$

that is $\frac{I(s)}{J_1(s)} \le (1 + D)\frac{I(t)}{J_1(t)}$. In particular, we have shown that I/J_1 is essentially non-decreasing on $[1/2, 1]$. Using the symmetry of I, this implies that on $[0, 1/2]$ the function $I(t)/J_0(t) = I(t)/t$ is essentially non-increasing. This is actually explained in the first part of the proof of Lemma 5, see Equation (5). We have already shown that I is essentially non-decreasing on $(0, 1/2]$. Thus by symmetry, I is essentially non-increasing on $[1/2, 1]$, and so is the map $t \mapsto I(t)/t = I(t)/J_0(t)$. Therefore, I/J_0 is essentially non-increasing on the whole interval $(0, 1]$. Let us denote by D_0 the corresponding constant.

The latter fact allows to conclude: Let $0 < s < t < 1$. Since $t^\alpha \ge t$, we know that $\frac{I(t^\alpha)}{t^\alpha} \le D_0\frac{I(t)}{t}$. Combining this estimate with (11) gives,

$$\frac{I(s)}{s} \le k\frac{I(t)}{t} + \frac{I(t^\alpha)}{t^\alpha} \le (k + D_0)\frac{I(t)}{t} \le (1 + D_0)k\frac{I(t)}{t} \le (1 + D_0)\frac{\log 1/s}{\log 1/t}\frac{I(t)}{t}.$$

The proof is now complete. □

4. An application to geometric influences

This section is devoted to an application of Theorem 3 to geometric influences. The notion of influence of a variable on a boolean function plays an important role in discrete harmonic analysis, with applications to various fields (see, e.g., the survey article by Kalai and Safra [10] on threshold phenomena). Let us recall the definition: for a function $f : \{0, 1\}^n \to \{0, 1\}$, which can be viewed as a subset $A = \{x; f(x) = 1\}$ of $\{0, 1\}^n$, the influence of the ith variable with respect to a probability measure ν on the discrete cube $\{0, 1\}^n$ is

$$I_i(f) = I_i(A) := \mathbb{P}_{x\sim\nu}\big(f(x) \ne f\big(\tau_i(x)\big)\big) = \mathbb{P}_{x\sim\nu}\big(x \in A \text{ xor } \tau_i(x) \in A\big),$$

where $\tau_i(x)$ is the neighbour of x having different ith coordinate, $(\tau_i(x))_i = 1 - x_i$. Geometrically speaking, $I_i(A)$ measures the size of the edge boundary of A in the ith direction. A seminal result in the theory of influences is the KKL theorem (by Kahn, Kalai and Linial [9]). Based on the hypercontractivity inequality, it ensures the existence of a coordinate with a large influence for non-constant boolean functions.

Recent papers have developped the theory of influences in the case of a continuous space. They propose two different definitions: the h-influences of Keller [11] involve the measures of the intersections of a given set with all lines in the ith canonical direction, while the geometric influences of Keller, Mossel and Sen [12] involve the boundary measures of the intersections with lines in the ith direction.

Definition. *Let $n \in \mathbb{N}^*$, $i \in \{1, \dots, n\}$, $x \in \mathbb{R}^n$ and A a Borel subset of $\mathbb{R}^n$. For $z \in \mathbb{R}^{n-1}$, we set*

$$A_i^z = \left\{y \in \mathbb{R} | (z_1, \dots, z_{i-1}, y, z_i, \dots, z_{n-1}) \in A\right\}.$$

Let $\nu = \nu_1 \otimes \cdots \otimes \nu_n$ be a product probability measure on $\mathbb{R}^n$.

If $h : [0, 1] \to \mathbb{R}_+$ is a measurable function, the h-influence of the ith coordinate on A with respect to ν is defined by

$$\mathcal{I}_{\nu,i}^h(A) = \int_{\mathbb{R}^{n-1}} h\left(\nu_i\left(A_i^z\right)\right) d\widehat{\nu}^i(z),$$

where $\widehat{\nu}^i = \nu_1 \otimes \cdots \otimes \nu_{i-1} \otimes \nu_{i+1} \otimes \cdots \otimes \nu_n$.

The geometric influence of the ith coordinate on A with respect to the measure ν is given by

$$\mathcal{I}_{\nu,i}^{\mathcal{G}}(A) = \int_{\mathbb{R}^{n-1}} (\nu_i)^+\left(A_i^z\right) d\widehat{\nu}^i(z).$$

When the choice of the underlying measure is obvious, we simply write $\mathcal{I}_i^h(A)$ and $\mathcal{I}_i^{\mathcal{G}}(A)$.

Keller was able to prove an analogue of the KKL theorem for h-influences provided h is larger than the entropy function Ent defined by $\mathrm{Ent}(x) = x \log \frac{1}{x} + (1 - x) \log \frac{1}{1-x}$ for $x \in (0, 1)$ and $\mathrm{Ent}(0) = \mathrm{Ent}(1) = 0$. His result is stated for functions on the unit cube, equipped with Lebesgue's measure. Using a standard transportation argument yields the following formulation:

Theorem 6 (Keller [11]). *Let μ be a probability measure on $\mathbb{R}$. Then, for every Borel set $A \subseteq \mathbb{R}^n$*

$$\max_{1 \le i \le n} \mathcal{I}_i^{\mathrm{Ent}}(A) \ge \gamma \mu^n(A) \mu^n\left(A^c\right) \frac{\log n}{n},$$

where $\gamma > 0$ is a universal constant.

Keller, Mossel and Sen [12] establish an analogue of the KKL theorem for geometric influences for Boltzmann measures $d\mu_\rho(t) = \exp(-|t|^\rho)\, dt/Z_\rho\, dt$ with $\rho \ge 1$ (and under mild assumptions for log-concave measures enjoying the same isoperimetric inequality as μ_ρ). Thanks to Theorem 3 we can propose a more general result.

Theorem 7. *Let μ be an even log-concave probability measure on $\mathbb{R}$, with positive and C^1-bounded density φ_μ. Assume that $I_\mu \geq J$ where J is a non-negative function on $[0,1]$, which is symmetric with respect to $1/2$, verifies $J(0)=0$ and is such that $t \mapsto J(t)/(t\log(1/t))$ is essentially non-decreasing on $(0,1)$ with constant D. Then for every Borel set $A \subset \mathbb{R}^n$,*

$$\max_{1\leq i\leq n} \mathcal{I}_i^{\mathcal{G}}(A) \geq \alpha_D \mu^n(A)\mu^n\left(A^c\right) J\left(\frac{1}{n}\right),$$

where $\alpha_D \geq \frac{\kappa}{D^3}$ and $\kappa > 0$ is a universal constant.

Remark 4. Actually the conclusion of Theorem 7 holds under less restrictive conditions on the measure. In particular the log-concavity assumption can be removed, either by using a different symmetrization argument than in Keller, Mossel and Sen [12] or by introducing another definition of the geometric influence based on notions of geometric measure theory. These modifications require a substantial and technical work that will appear in the PhD dissertation of the second-named author. In the present paper, we simply explain how the argument of Keller, Mossel and Sen can be adapted, putting forward the parts of the reasoning where the conditions on J are used.

The next lemma follows from Proposition 1.3 in Keller, Mossel and Sen [12]. It explains the connection between geometric influences and boundary measure for the uniform enlargement.

Lemma 11 (Keller, Mossel and Sen [12]). *Let μ be as in Theorem 7. Let $A \subseteq \mathbb{R}^n$ be a monotone increasing set (in the following sense: if $x \in A$ and for all i, $x_i \leq y_i$, then $y \in A$). Then*

$$\sum_{i=1}^n \mathcal{I}_i^{\mathcal{G}}(A) = \left(\mu^n\right)^+(A).$$

Proof of Theorem 7. We follow the argument of Keller, Mossel and Sen. Assume $n \geq 2$. Let $A \subseteq \mathbb{R}^n$ be as in the statement of the theorem. Lemma 3.7 of Keller, Mossel and Sen [12] ensures that, without loss of generality, one can assume that A is increasing. Set $t = \mu^n(A)$. Since A and A^c have the same influences, we may assume that $t \leq 1/2$ (A^c is monotone decreasing, but passing to its image by the symmetry with respect to the origin we may ensure that we work with an increasing set of measure at most $1/2$). We distinguish two cases:

First case: $t \leq 1/n$. Thanks to Lemma 11 and to the isoperimetric inequality of Theorem 3

$$\sum_{i=1}^n \mathcal{I}_i^{\mathcal{G}}(A) = \left(\mu^n\right)^+(A) \geq \frac{1}{c_D} J\left(\mu^n(A)\right) = \frac{1}{c_D} J(t),$$

with $c_D = \frac{2D^2}{(\log 2)^2}$. Lemma 5 asserts that $s \mapsto J(s)/s$ is essentially non-increasing on $(0,1/2]$ with constant $D/\log 2$, therefore $J(t)/t \geq nJ(1/n)\log 2/D$. Consequently

$$\max_i \mathcal{I}_i^{\mathcal{G}}(A) \geq \frac{1}{n}\sum_{i=1}^n \mathcal{I}_i^{\mathcal{G}}(A) \geq \frac{\log 2}{Dc_D} tJ\left(\frac{1}{n}\right) \geq \frac{(\log 2)^3}{2D^3} t(1-t)J\left(\frac{1}{n}\right).$$

Second case: $t \in (1/n, 1/2]$. The argument uses three main ingredients. The first one is the isoperimetric inequality $I_\mu \geq J$, which implies that for all $i \leq n$,

$$\mathcal{I}_i^{\mathcal{G}}(A) = \int_{\mathbb{R}^{n-1}} \mu^+\big(A_i^z\big)\, d\mu^{n-1}(z) \geq \int_{\mathbb{R}^{n-1}} J\big(\mu\big(A_i^z\big)\big)\, d\mu^{n-1}(z) = \mathcal{I}_i^J(A). \tag{12}$$

The second ingredient is a comparison between J-influences and Ent-influences: observe that Ent is symmetric with respect to $1/2$ and is increasing and one-to-one on $[0, 1/2]$. Let Ent^{-1} : $[0, \log 2] \to [0, 1/2]$ be its reciprocal function. Then for all $i \leq n$,

$$\mathcal{I}_i^J(A) \geq \frac{J(s)}{2D} \qquad \text{with } s := \mathrm{Ent}^{-1}\left(\frac{\mathcal{I}_i^{\mathrm{Ent}}(A)}{2}\right) \in \left[0, \frac{1}{2}\right]. \tag{13}$$

We postpone the proof of this inequality, and explain how to conclude. The last ingredient is Keller's version of the KKL inequality (Theorem 6). It provides an index i such that $\mathcal{I}_i^{\mathrm{Ent}}(A) \geq \gamma t(1-t)\log(n)/n$, where $\gamma > 0$ is a universal constant. Observe for further use that necessarily $\gamma < 8$ (indeed, $\mathrm{E}nt \leq \log 2$ and one can choose $n = 2$ and $t = 1/2$ in Keller's theorem). Let us show that the ith coordinate has a large geometric influence.

Observe that for every $y \in [0, \frac{1}{2}] \subset [0, \log 2]$, $\theta(y) := \frac{y}{2\log 1/y} \leq \mathrm{Ent}^{-1}(y)$. Since $\mathcal{I}_i^{\mathrm{Ent}}(A)/2 \leq \log(2)/2 \leq 1/2$, and θ is increasing on $(0, 1)$,

$$\begin{aligned} s = \mathrm{Ent}^{-1}\left(\frac{\mathcal{I}_i^{\mathrm{Ent}}(A)}{2}\right) &\geq \theta\left(\frac{\mathcal{I}_i^{\mathrm{Ent}}(A)}{2}\right) \geq \theta\left(\frac{\gamma}{2} t(1-t)\frac{\log n}{n}\right) \\ &= \frac{\gamma t(1-t)}{4n}\, \frac{\log n}{\log(2n/(\gamma t(1-t)\log n))} \geq \frac{\gamma t(1-t)}{4n} \times \frac{\log n}{\log(4n^2/(\gamma \log n))}, \end{aligned}$$

where the latter inequality relies on $t(1-t) \geq (1-t)/n \geq 1/(2n)$. Since $\gamma \leq 8$, the last fraction in the lower bound of s is a positive function of $n \geq 2$ with a positive limit when n tends to infinity. Hence there exists $c = c(\gamma) > 0$ such that $s \geq ct(1-t)/n$. It remains to combine this estimate with $\mathcal{I}_i^{\mathcal{G}}(A) \geq J(s)/(2D)$, a consequence of (12) and (13):

If $s \leq \frac{1}{n}$, then we also use Lemma 5, which asserts that $u \mapsto J(u)/u$ is essentially non-increasing on $(0, \frac{1}{2}]$ with constant $\frac{D}{\log 2}$:

$$\mathcal{I}_i^{\mathcal{G}}(A) \geq \frac{J(s)}{2D} \geq \frac{\log 2}{2D^2} s \frac{J(1/n)}{1/n} \geq \frac{\log 2}{2D^2} ct(1-t) J\left(\frac{1}{n}\right).$$

If $s > \frac{1}{n}$ then, using the fact that J is essentially non-decreasing on $(0, \frac{1}{2})$ with constant $\frac{2D}{e \log 2}$ (see Lemma 6), we get

$$\mathcal{I}_i^{\mathcal{G}}(A) \geq \frac{J(s)}{2D} \geq \frac{e \log 2}{4D^2} J\left(\frac{1}{n}\right) \geq \frac{e \log 2}{D^2} t(1-t) J\left(\frac{1}{n}\right).$$

Eventually, we give a proof for (13). By hypothesis, J/J_1 is essentially non-decreasing with constant D, where $J_1(x) = x\log(1/x)$. Observe that for $x \in [0, \frac{1}{2}]$,

$$J_1(x) \leq \mathrm{Ent}(x) = J_1(x) + J_1(1-x) \leq 2J_1(x).$$

It follows that $\frac{J}{\mathrm{Ent}}$ is essentially non-decreasing on $(0, \frac{1}{2}]$ with constant $2D$. Recall that $s \in (0, \frac{1}{2})$ verifies $\mathrm{Ent}(s) = \mathcal{I}_i^{\mathrm{Ent}}(A)/2$. Note that, if $x \notin [s, 1-s]$, then $\mathrm{Ent}(x) < \mathcal{I}_i^{\mathrm{Ent}}(A)/2$. This yields

$$\begin{aligned}\int_{\mu(A_i^z)\in[s,1-s]} \mathrm{Ent}\big(\mu\big(A_i^z\big)\big)\, d\mu^{n-1}(z) &= \mathcal{I}_i^{\mathrm{Ent}}(A) - \int_{\mu(A_i^z)\notin[s,1-s]} \mathrm{Ent}\big(\mu\big(A_i^z\big)\big)\, d\mu^{n-1}(z) \\ &\geq \frac{\mathcal{I}_i^{\mathrm{Ent}}(A)}{2}.\end{aligned}$$

Therefore, using in addition the symmetry with respect to $1/2$ of J and Ent and the fact that $\frac{J}{\mathrm{Ent}}$ is essentially non-decreasing on $(0, 1/2]$ with constant $2D$, we get

$$\begin{aligned}\mathcal{I}_i^J(A) &\geq \int_{\mu(A_i^z)\in[s,1-s]} J\big(\mu\big(A_i^z\big)\big)\, d\mu^{n-1}(z) \\ &= \int_{\mu(A_i^z)\in[s,1-s]} J\big(\min\big(\mu\big(A_i^z\big), 1-\mu\big(A_i^z\big)\big)\big)\, d\mu^{n-1}(z) \\ &\geq \frac{1}{2D}\frac{J(s)}{\mathrm{Ent}(s)} \int_{\mu(A_i^z)\in[s,1-s]} \mathrm{Ent}\big(\min\big(\mu\big(A_i^z\big), 1-\mu\big(A_i^z\big)\big)\big)\, d\mu^{n-1}(z) \\ &= \frac{1}{2D}\frac{J(s)}{\mathrm{Ent}(s)} \int_{\mu(A_i^z)\in[s,1-s]} \mathrm{Ent}\big(\mu\big(A_i^z\big)\big)\, d\mu^{n-1}(z) \\ &\geq \frac{1}{2D}\frac{J(s)}{\mathrm{Ent}(s)}\frac{\mathcal{I}_i^{\mathrm{Ent}}(A)}{2} = \frac{J(s)}{2D}.\end{aligned}$$

The proof is complete. □

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References

[1] Ambrosio, L., Fusco, N. and Pallara, D. (2000). *Functions of Bounded Variation and Free Discontinuity Problems*. *Oxford Mathematical Monographs*. New York: The Clarendon Press. MR1857292
[2] Barthe, F. (2002). Log-concave and spherical models in isoperimetry. *Geom. Funct. Anal.* **12** 32–55. MR1904555
[3] Barthe, F. (2004). Infinite dimensional isoperimetric inequalities in product spaces with the supremum distance. *J. Theoret. Probab.* **17** 293–308. MR2053705
[4] Barthe, F., Cattiaux, P. and Roberto, C. (2006). Interpolated inequalities between exponential and Gaussian, Orlicz hypercontractivity and isoperimetry. *Rev. Mat. Iberoam.* **22** 993–1067. MR2320410
[5] Bobkov, S.G. (1997). Isoperimetric problem for uniform enlargement. *Studia Math.* **123** 81–95. MR1438305

[6] Bobkov, S.G. and Houdré, C. (1997). Some connections between isoperimetric and Sobolev-type inequalities. *Mem. Amer. Math. Soc.* **129** viii+111. MR1396954

[7] Bobkov, S.G. and Houdré, C. (2000). Weak dimension-free concentration of measure. *Bernoulli* **6** 621–632. MR1777687

[8] Bollobás, B. and Leader, I. (1991). Edge-isoperimetric inequalities in the grid. *Combinatorica* **11** 299–314. MR1137765

[9] Kahn, J., Kalai, G. and Linial, N. (1988). The influence of variables on Boolean functions. In *Proceedings of* 29*th IEEE Symposium on Foundations of Computer Sciences* 68–80.

[10] Kalai, G. and Safra, S. (2006). Threshold phenomena and influence: Perspectives from mathematics, computer science, and economics. In *Computational Complexity and Statistical Physics*. *St. Fe Inst. Stud. Sci. Complex*. 25–60. New York: Oxford Univ. Press. MR2208732

[11] Keller, N. (2011). On the influences of variables on Boolean functions in product spaces. *Combin. Probab. Comput.* **20** 83–102. MR2745679

[12] Keller, N., Mossel, E. and Sen, A. (2012). Geometric influences. *Ann. Probab.* **40** 1135–1166. MR2962089

[13] Kuczma, M. (2009). *An Introduction to the Theory of Functional Equations and Inequalities*, 2nd ed. Basel: Birkhäuser. Cauchy's equation and Jensen's inequality, Edited and with a preface by Attila Gilányi. MR2467621

[14] Latała, R. and Oleszkiewicz, K. (2000). Between Sobolev and Poincaré. In *Geometric Aspects of Functional Analysis*. *Lecture Notes in Math.* **1745** 147–168. Berlin: Springer. MR1796718

[15] Milman, E. (2009). On the role of convexity in functional and isoperimetric inequalities. *Proc. Lond. Math. Soc.* (3) **99** 32–66. MR2520350

[16] Morgan, F. (2006). Isoperimetric estimates in products. *Ann. Global Anal. Geom.* **30** 73–79. MR2249614

[17] Peetre, J. (1968). On interpolation functions. II. *Acta Sci. Math.* (*Szeged*) **29** 91–92. MR0240614

[18] Ros, A. (2005). The isoperimetric problem. In *Global Theory of Minimal Surfaces*. *Clay Math. Proc.* **2** 175–209. Providence, RI: Amer. Math. Soc. MR2167260

[19] Talagrand, M. (1991). A new isoperimetric inequality and the concentration of measure phenomenon. In *Geometric Aspects of Functional Analysis* (1989–90). *Lecture Notes in Math.* **1469** 94–124. Berlin: Springer. MR1122615

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