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Multilevel path simulation for weak approximation schemes with application to Lévy-driven SDEs

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In this paper, we discuss the possibility of using multilevel Monte Carlo (MLMC) approach for weak approximation schemes. It turns out that by means of a simple coupling between consecutive time discretisation levels, one can achieve the same complexity gain as under the presence of a strong convergence. We exemplify this general idea in the case of weak Euler schemes for Lévy-driven stochastic differential equations. The numerical performance of the new "weak" MLMC method is illustrated by several numerical examples.

Keywords: Lévy-driven stochastic differential equations; multilevel Monte Carlo; weak approximation schemes

1. Introduction

The multilevel path simulation method introduced in Giles [8] has gained huge popularity as a complexity reduction tool in recent times. The main advantage of the MLMC methodology is that it can be simply applied to various situations and requires almost no prior knowledge on the path generating process. Any multilevel Monte Carlo (MLMC) algorithm uses a number of levels of resolution, $l = 0, 1, \ldots, L$, with $l = 0$ being the coarsest, and $l = L$ being the finest. In the context of a SDE simulation on the interval $[0, T]$, the level 0 corresponds to one timestep $\Delta_0 = T$, whereas the level L has 2^L uniform timesteps $\Delta_L = 2^{-L}T$.

Assume that a filtered probability space $(\Omega, \mathcal{F}, \mathrm{P}, (\mathcal{F}_t))$ is given. Consider now a d-dimensional process (X_t) solving the following Lévy-driven SDE

$$X_t = X_0 + \int_0^t a(X_{s-})\, dZ_s, \tag{1.1}$$

where X_0 is a $\mathbb{R}^d$-valued random variable, $Z_t = (Z_t^1, \ldots, Z_t^m)$, $t \geq 0$ is a m-dimensional Lévy process and the mapping $a : \mathbb{R}^d \times \mathbb{R}^m \mapsto \mathbb{R}^d$ is Lipschitz continuous so that the solution of (1.1) is well defined. Our aim is to estimate the expectation $\mathrm{E}[f(X_T)]$, where f is a Lipschitz continuous function from $\mathbb{R}^d$ to $\mathbb{R}$. Let X_T^l be an approximation for X_T by means of a numerical discretisation with time step Δ_l (for various discretisation methods for (1.1) see, for example, Platen and Bruti-Liberati [19] or the recent review of Jourdain and Kohatsu-Higa [13]). The

main idea of the multilevel approach pioneered in Heinrich [10] and Giles [8] consists in writing the expectation of the finest approximation $\mathrm{E}[f(X_T^L)]$ as a telescopic sum

$$\mathrm{E}\big[f\big(X_T^L\big)\big] = \mathrm{E}\big[f\big(X_T^0\big)\big] + \sum_{l=1}^{L} \mathrm{E}\big[f\big(X_T^l\big) - f\big(X_T^{l-1}\big)\big]$$

and then applying Monte Carlo to estimate each expectation in the above telescopic sum. More formally, for any $l = 1, \dots, L$, we independently generate N_l independent copies of the random vector $(X_T^l, X_T^{l-1}) \in \mathbb{R}^d \times \mathbb{R}^d$ and define the multilevel Monte Carlo estimator $\mathcal{M}$ of $\mathrm{E}[f(X_T)]$ as

$$\mathcal{M} \doteq \underbrace{\frac{1}{N_0} \cdot \sum_{i=1}^{N_0} f\big(X_T^{0,(i)}\big)}_{\approx \mathrm{E}(f(X_T^0))} + \sum_{l=0}^{L} \underbrace{\frac{1}{N_l} \cdot \sum_{i=1}^{N_l} \big(f\big(X_T^{l,(i)}\big) - f\big(X_T^{l-1,(i)}\big)\big)}_{\approx \mathrm{E}[f(X_T^l) - f(X_T^{l-1})]}.$$

One important prerequisite for MLMC to outperform the standard MC method is that X_T^l and X_T^{l-1} are coupled in some way and this can be achieved by using the same discretised trajectories of the underlying Lévy processes to construct (X_T^l, X_T^{l-1}). The degree of coupling is usually measured in terms of the variance $\mathrm{Var}[f(X_T^l) - f(X_T^{l-1})]$. It is shown in Giles [8] (see also Giles and Xia [7]), that under the assumptions

$$\big|\mathrm{E}\big[f\big(X_T^L\big)\big] - \mathrm{E}\big[f(X_T)\big]\big| \le c_1 \Delta_L^{\alpha}, \quad \mathrm{Var}\big[f\big(X_T^l\big) - f\big(X_T^{l-1}\big)\big] \le c_2 \Delta_l^{\beta}, \tag{1.2}$$

with some $\alpha \ge 1/2$, $\beta > 0$, $c_1 > 0$ and $c_2 > 0$, the computational complexity of the resulting multilevel estimate needed to achieve the accuracy ε (in terms of RMSE) is proportional to

$$\mathcal{C} = \begin{cases} \varepsilon^{-2}, & \beta > 1, \\ \varepsilon^{-2} \log^2(\varepsilon), & \beta = 1, \\ \varepsilon^{-2-(1-\beta)/\alpha}, & 0 < \beta < 1. \end{cases}$$

The standard way of checking the assumptions (1.2) is to prove that the underlying approximation scheme has weak convergence of order α and strong convergence of order $\beta/2$. Indeed, in the latter case we have for any Lipschitz continuous function f,

$$\begin{aligned} \mathrm{Var}\big[f\big(X_T^l\big) - f\big(X_T^{l-1}\big)\big] &\le c_f \mathrm{E}\big[\big|X_T^l - X_T\big|^2\big] + c_f \mathrm{E}\big[\big|X_T^{l-1} - X_T\big|^2\big] \\ &\le 2 c_f \Delta_l^{\beta} \end{aligned}$$

with some constant $c_f > 0$ depending on f. However, in recent years the so-called weak approximation schemes, that is, schemes that, in general, fulfil only the first assumption in (1.2) became quite popular. The weak Euler scheme is a first-order scheme with $\alpha = 1$, and has been studied by many researchers. Talay and Tubaro [22] show the first-order convergence of the weak Euler scheme. The fact that the convergence rate of the Euler scheme also holds for certain irregular functions under a Hörmander type condition has been proved by Bally and Talay [2] using

Malliavin calculus. The Itô-Taylor (weak-Taylor) high-order scheme is a natural extension of the weak Euler scheme. In the continuous diffusion case, some new discretization schemes (also called Kusuoka type schemes) which are of order $\alpha \geq 2$ without the Romberg extrapolation have been introduced by Kusuoka [14], Lyons and Victoir [15], Ninomiya and Victoir [17], and Ninomiya and Ninomiya [16]. A general class of weak approximation methods, comprising many well known discretisation schemes, was constructed in Kohatsu-Higa and Tanaka [23]. The main advantage of the weak approximation schemes is that simple discrete random variables can be used instead of the Lévy increments. Unfortunately, due to the absence of the strong convergence, the MLMC methodology can not be directly used with the weak approximation schemes. In this paper we make an attempt to overcome this difficulty and develop a kind of "weak" MLMC approach which can be applied to various weak approximation schemes.

The plan of the paper is as follows. First, we recall the Euler scheme for (1.1) and discuss its convergence properties. Next, we show how to construct the corresponding MLMC algorithm, which is able to reduce the complexity of the standard MC to order $\varepsilon^{-2}\log^2(\varepsilon)$ under only requirement that the Euler scheme converges weakly. Finally, we analyse the numerical performance of the presented weak MLMC algorithms.

2. Euler scheme for Lévy-driven SDEs

Fix some $n \in \mathbb{N}$ and set $\Delta = T/n$. Denote $\Delta Z_j = Z_{j\Delta} - Z_{(j-1)\Delta}$, $j = 1, \ldots, n$. For a fixed random vector X_0, the Euler scheme for (1.1) reads as follows

$$\begin{aligned} X_0^\Delta &= X_0, \\ X_{j\Delta}^\Delta &= X_{(j-1)\Delta}^\Delta + a\big(X_{(j-1)\Delta}^\Delta\big)\Delta Z_j, \qquad j = 1, \ldots, n. \end{aligned} \tag{2.1}$$

The convergence of the scheme (2.1) was extensively studied in the literature. The first convergence result is due to Talay and Tubaro [22], who proved that in the case of a diffusion processes with Z being a Brownian motion plus drift, the scheme weakly converges with order 1. In the case of the general Lévy processes, the convergence of (2.1) was studied in Protter and Talay [20], where it is shown that, under some assumption on the function a and the driving Lévy process Z, the weak convergence rate $1/n$ can be recovered. In fact, the main drawback of the scheme (2.1) is the necessity to sample from the distribution of ΔZ_j exactly. Although such exact sampling can be possible for particular Lévy processes (see [20] for some examples), in general this turns out to be a hard numerical problem. This is why Jacod *et al.* [12] proposed to replace the increments ΔZ_j of the original Lévy process by simple random vectors ζ_j which are easy to simulate. It is shown in [12] that if the distributions of ΔZ_j and ζ_j are sufficiently close, then the weak convergence rate $1/n$ continues to hold. These results on weak convergence should be compared with ones on pathwise or strong convergence. In fact, the strong convergence rates usually depend on the characteristics of the Lévy process Z. For example, Rubenthaler [21] studied the strong error when neglecting jumps smaller than δ. In particular, Rubenthaler [21] obtained the estimate of the form

$$\mathrm{E}\Big[\max_{j=0,\ldots,n} \big|X_{j\Delta}^\Delta - X_{j\Delta}\big|^2\Big] \lesssim \left(n^{-1} + \int_{|z|\leq\delta} z^2 \nu(dz)\right)$$

with ν being the Lévy measure of Z. So the rates become quite poor if ν diverges at zero like $z^{-\alpha}$ with α close to 2. Recently, Fournier [6] has proposed a coupling method which allows to get better rates of pathwise convergence in a one-dimensional case. He constructed an approximation $X^{n,\delta}$, satisfying

$$\mathrm{E}\Big[\max_{j=0,\dots,n}\big|X^{n,\delta}_{j\Delta}-X_{j\Delta}\big|^2\Big]\lesssim\left(n^{-1}+n\frac{m_{4,\delta}(\nu)}{m_{2,\delta}(\nu)}\right) \tag{2.2}$$

with $m_{k,\delta}(\nu):=\int_{|z|\le\delta}|z|^k\nu(dz)$. The approximation $X^{n,\delta}$ is constructed by replacing the jumps of Z smaller than δ by an independent Brownian motion. Note that the second summand in the right-hand side of (2.2) represents the approximation error for one increment of the driving Lévy process times the number of such increments. In order to prove a bound for the Wasserstein distance between X and $X^{n,\delta}$, a suitable coupling was used. Note that since X is unknown, such coupling is not implementable. A similar coupling idea in a multidimensional setting was used in Dereich [3] to design a multilevel path simulation approach for (1.1). In particular, Dereich [3] established a new estimate for the Wasserstein metric between an approximative solution $X^{n,\delta}$ with Gaussian approximation and the genuine solution X, based on a Zaitsev's generalization of the well-known Komlós–Major–Tusnády coupling. This new estimate allowed the author in [3] to extend the results of [6] to a multidimensional case.

3. Multilevel path simulation for weak Euler schemes

In order to successfully apply the multilevel approach, one needs to ensure that (1.2) hold. If the scheme (2.1) has strong convergence of order $\beta/2$, that is,

$$\mathrm{E}\Big[\max_{j=0,\dots,n}\big|X^{\Delta}_{j\Delta}-X_{j\Delta}\big|^2\Big]\lesssim\Delta^{\beta},$$

then the conditions (1.2) hold with $\alpha=\beta/2$. However, if some approximations ζ_j, $j=1,\dots,n$, are used instead of the genuine increments ΔZ_j, strong convergence is not any longer guaranteed. Here we propose a general approach how to couple two approximations of X on different time scales in order to guarantee that the second condition in (1.2) still holds with $\beta=1$. In fact, this would lead to a complexity estimate $\varepsilon^{-2}\log^2(\varepsilon)$, does not matter how small is $\alpha\ge 1/2$.

3.1. Coupling idea

Let us fix two natural numbers n_c ("coarse" discretisation level) and n_f ("fine" discretisation level) with $n_f=2\cdot n_c$ and set $\Delta_c=T/n_c$, $\Delta_f=T/n_f$. In order to couple the Euler approximations X^{Δ_c} and X^{Δ_f}, we are going to couple the random matrices $\boldsymbol{\zeta}^c\doteq(\zeta^c_1,\dots,\zeta^c_{n_c})\in\mathbb{R}^{n_c}\otimes\mathbb{R}^m$ and $\boldsymbol{\zeta}^f\doteq(\zeta^f_1,\dots,\zeta^f_{n_f})\in\mathbb{R}^{n_f}\otimes\mathbb{R}^m$. We define the approximation $\boldsymbol{\zeta}^c$ for the increments on the coarse level in such a way that the differences

$$\zeta^c_j-\zeta^f_{2j-1}-\zeta^f_{2j},\qquad j=1,\dots,n_c \tag{3.1}$$

are small. In particular, we can take $\zeta_j^c = \zeta_{2j-1}^f + \zeta_{2j}^f$. The idea behind this coupling is very simple: in the case of the genuine Lévy increments we would get

$$\Delta_f Z_{2j-1} + \Delta_f Z_{2j} = Z_{2j\Delta_f} - Z_{2(j-1)\Delta_f} = \Delta_c Z_j$$

with $\Delta_f Z_j := Z_{(j+1)\Delta_f} - Z_{j\Delta_f}$ and $\Delta_c Z_j := Z_{(j+1)\Delta_c} - Z_{j\Delta_c}$. Suppose that $\zeta_1^f, \dots, \zeta_{n_f}^f$ are i.i.d. random vectors with moments $m_{f,1} \doteq |\mathrm{E}[\zeta_j^f]|$ and $m_{f,2} \doteq \mathrm{E}[|\zeta_j^f|^2]$, where $|\cdot|$ stands for the Euclidian norm. The following proposition holds.

Proposition 1. *Suppose that the coefficient function a in (1.1) is uniformly Lipschitz and has at most linear growth, that is,*

$$\big\|a(x) - a\big(x'\big)\big\| \le L_a\big|x - x'\big|, \qquad \big\|a(x)\big\|^2 \le B_a^2\big(1 + |x|^2\big) \tag{3.2}$$

for any $x, x' \in \mathbb{R}^d$ and some positive constants L_a and B_a, where for any matrix A, $\|A\|$ stands for the operator norm of the matrix A. Denote $\mathcal{R}_j \doteq \zeta_j^c - \zeta_{2j-1}^f - \zeta_{2j}^f$ and suppose that $\mathcal{R}_j$, $j = 1, \dots, n_c$, are zero mean i.i.d. random vectors. Moreover, assume that $\mathrm{E}[|X_0|^2] < \infty$, then the following estimate holds

$$\begin{aligned}\mathrm{E}\Big[\max_{j=0,\dots,n_c}\big|X_{j\Delta}^{\Delta_f} - X_{j\Delta}^{\Delta_c}\big|^2\Big] &\le c_1\big(n_f m_{f,2}^2 + n_f^2 m_{f,1}^2 m_{f,2} + n_f \mathrm{E}\big[|\mathcal{R}_1|^2\big]\big) \\ &\quad \times \exp\big[c_2\big(n_f m_{f,2} + n_f^2 m_{f,1}^2\big)\big]\end{aligned} \tag{3.3}$$

for some constants $c_1 > 0$, $c_2 > 0$ depending on L_a and B_a.

Corollary 2. *If $m_{f,2} = O(\Delta_f)$, $m_{f,1} = O(\Delta_f)$ and $\mathrm{E}[|\mathcal{R}_1|^2] = O(\Delta_f^2)$ for $\Delta_f \to 0$, then*

$$\mathrm{E}\Big[\max_{j=0,\dots,n_c}\big|X_{j\Delta}^{\Delta_f} - X_{j\Delta}^{\Delta_c}\big|^2\Big] = O(\Delta_f), \qquad \Delta_f \to 0.$$

Discussion

First, note that the conditions for (3.3) to hold are formulated not in terms of the original increments (ΔZ_j), but rather in terms of their approximations (ζ_j). For the case of the exact increments, we obviously have $m_{f,2} = O(\Delta_f)$ and $m_{f,1} = O(\Delta_f)$, provided

$$\int_{\mathbb{R}^d} |z|^2 \nu(dz) < \infty,$$

where ν is a Lévy measure of Z. Furthermore, observe that under the assumptions of Corollary 2, the second condition in (1.2) holds with $\beta = 1$ independently of the strong convergence order for the corresponding Euler scheme. Finally, let us stress that the assumptions on the coefficient function a are quite weak and standard in the framework of Lévy-driven SDEs. In fact, they are needed to guarantee existence and uniqueness of the solution of (1.1) (see, e.g., Ikeda and Watanabe [11]).

3.2. MLMC algorithm

Fix some $L > 0$ and set $\Delta_l = 2^{-l}T$, $l = 0, \dots, L$. Denote

$$\boldsymbol{\zeta}_L \doteq (\zeta_{L,1}, \dots, \zeta_{L,2^L}) \in \mathbb{R}^{2^L} \otimes \mathbb{R}^m,$$

where the columns of the matrix $\boldsymbol{\zeta}_L$ are i.i.d. random vectors in $\mathbb{R}^m$. Now we define recursively the independent random matrices $\boldsymbol{\zeta}_{L-1}, \dots, \boldsymbol{\zeta}_0$ with $\boldsymbol{\zeta}_l \in \mathbb{R}^{2^l} \otimes \mathbb{R}^m$ via $\boldsymbol{\zeta}_{l-1} \sim \boldsymbol{\varsigma}(\boldsymbol{\zeta}_l) = (\varsigma_1(\boldsymbol{\zeta}_l), \dots, \varsigma_{l-1}(\boldsymbol{\zeta}_l))$, where each vector $\varsigma_j(\boldsymbol{\zeta}_l)$ is coupled with $\zeta_{l,2j-1}$ and $\zeta_{l,2j}$ in such a way that all differences

$$\varsigma_j(\boldsymbol{\zeta}_l) - \zeta_{l,2j-1} - \zeta_{l,2j}, \qquad j = 1, \dots, 2^{l-1},$$

are small. For example, one can simply put

$$\varsigma_j(\boldsymbol{\zeta}_l) = \zeta_{l,2j-1} + \zeta_{l,2j}, \qquad j = 1, \dots, 2^{l-1}. \tag{3.4}$$

Next, for any $l = 1, \dots, L$, and any random matrix $\boldsymbol{\zeta} \in \mathbb{R}^{2^l} \otimes \mathbb{R}^m$, consider the approximations

$$\begin{aligned} X_0^l(\boldsymbol{\zeta}) &= X_0, \\ X_{j\Delta_l}^l(\boldsymbol{\zeta}) &= X_{(j-1)\Delta_l}^l(\boldsymbol{\zeta}) + a\big(X_{(j-1)\Delta_l}^l(\boldsymbol{\zeta})\big)\zeta_j \end{aligned}$$

with $j = 1, \dots, 2^l$, and some r.v. $X_0 \in \mathbb{R}^d$. Finally, fix a vector of natural numbers $\mathbf{N} = (N_0, \dots, N_L)$ and define a weak MLMC estimate for $\mathrm{E}[f(X_T)]$ as follows

$$Y_{L,\mathbf{N}} \doteq \frac{1}{N_0}\sum_{n=1}^{N_0}\big[f(X_T^0(\boldsymbol{\zeta}^{(n)})\big] + \sum_{l=1}^{L}\frac{1}{N_l}\sum_{n=1}^{N_l}\big[f\big(X_T^l(\boldsymbol{\zeta}_l^{(n)})\big) - f\big(X_T^{l-1}(\overline{\boldsymbol{\zeta}}_l^{(n)})\big)\big],$$

where $\overline{\boldsymbol{\zeta}}_l^{(n)} = \boldsymbol{\varsigma}(\boldsymbol{\zeta}_l^{(n)})$ and $\boldsymbol{\zeta}_l^{(n)}$, $n = 1, \dots, N_l$, are i.i.d. copies of $\boldsymbol{\zeta}_l$.

Proposition 3. *Suppose that the function f is Lipschitz continuous and that the distribution of $\boldsymbol{\zeta}_L$ is chosen in such a way that*

$$\big|\mathrm{E}\big[f\big(X_T^L(\boldsymbol{\zeta}_L)\big)\big] - \mathrm{E}\big[f(X_T)\big]\big| \le c\Delta_L^\alpha \tag{3.5}$$

for some $c > 0$ and $\alpha \ge 1/2$. Then under the assumptions of Proposition 1 and Corollary 2, and under a proper choice of $\mathbf{N}$ and L, the complexity of the estimate $Y_{L,\mathbf{N}}$ needed to achieve the accuracy ε (as measured by RMSE) is of order $\varepsilon^{-2}\log^2(\varepsilon)$.

Remark 4. We start with the finest level L and go backwardly by coupling at each level l the approximations on time scales Δ_l and Δ_{l-1}. The main difficulty lies in the fact that such a coupling changes the distribution of the approximative increments and we need to keep track of these changes, since the MLMC approach essentially relies on fact that the distribution does not

change between two consecutive levels (telescopic sum property). This gives rise to a sequence of distributions for approximative increments.

The distribution of the matrix $\boldsymbol{\zeta}_l$ under coupling (3.4) can be found explicitly in many interesting cases (see examples below). In general, one can compute the characteristic function of each vector $\zeta_{l,j}$ in a closed form, provided the characteristic function of $\zeta_{L,j}$ is known explicitly. Using the Fourier inversion formula, one can then compute the density of each $\zeta_{l,j}$. Let us also note that there is a lot of freedom in the choice of the finest approximation $\boldsymbol{\zeta}_L$ satisfying (3.5).

The case where (3.4) holds only approximately will play an important role for the general Lévy processes (see Section 4.3).

4. Examples

4.1. Diffusion processes

Consider now a d-dimensional diffusion process (X_t) solving the SDE

$$X_t = b(X_t)\,dt + \sigma(X_t)\,dW_t, \qquad t \in [0,T], \tag{4.1}$$

where $W_t = (W^1, \dots, W^m)$ is a m-dimensional Brownian motion, $b : \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^m$ are Lipschitz continuous functions. Although the increments of Wiener process can be simulated exactly, we can consider the following (essentially) weak Euler scheme

$$\begin{aligned} X_0^\Delta &= X_0, \\ X_{j\Delta}^\Delta &= X_{(j-1)\Delta}^\Delta + b\bigl(X_{(j-1)\Delta}^\Delta\bigr)\Delta + \sum_{k=1}^m \sigma_k\bigl(X_{(j-1)\Delta}^\Delta\bigr)\xi_j^k, \end{aligned}$$

where $j = 1, \dots, n$, and i.i.d. random variables (ξ_j^k) satisfy

$$\bigl|\mathrm{E}\bigl[\xi_j^1\bigr]\bigr| + \bigl|\mathrm{E}\bigl[\bigl(\xi_j^1\bigr)^3\bigr]\bigr| + \bigl|\mathrm{E}\bigl[\bigl(\xi_j^1\bigr)^2\bigr] - \Delta\bigr| \le c\Delta^2 \tag{4.2}$$

for some $c > 0$. Under some additional assumptions on the coefficient functions b and σ, and the output function f spelled out in Talay and Tubaro [22] and Bally and Talay [2], it holds

$$\bigl|\mathrm{E}\bigl[f\bigl(X_T^\Delta\bigr)\bigr] - \mathrm{E}\bigl[f(X_T)\bigr]\bigr| \le c\Delta$$

for some $c > 0$. The simplest way of constructing a r.v. ξ with the property (4.2) is to take

$$\mathrm{P}\bigl(\xi_j^i = \pm\sqrt{\Delta}\bigr) = \tfrac{1}{2}, \qquad i = 1, \dots, m. \tag{4.3}$$

Observe that distribution of the components of the vector $\boldsymbol{\xi}_l$ under coupling (3.4) in the ML algorithm, is closely related to the Binomial distribution, namely

$$\frac{\xi_{l,j}^i}{2\sqrt{\Delta_L}} + 2^{L-l-1} \sim \mathrm{Bi}\biggl(2^{L-l}, \frac{1}{2}\biggr). \tag{4.4}$$

Hence the generation of variates $\xi^i_{l,j}$ is straightforward when a generator of binomially distributed random variates is available. For a fixed L, the weak MLMC algorithm implies generation of $\boldsymbol{\xi}_l$ for l starting from 0 up to L. Since all probabilities of distributions $\mathrm{Bi}(2^{L-l}, \frac{1}{2})$ for $l \in \{0, 1, \dots, L\}$ are rational numbers, table look-up or alias methods (see [4]) can be used to achieve fast single random number generation. Since the distributions of $\boldsymbol{\xi}_l$ do not change between different runs of the MLMC method, all the preprocessing required can be done only once and the resulting tables can be stored. In problem-specific hardware (FPGA or ASIC) these tables can be kept in permanent shared constant storage, which is often cheap, fast and abundant. With table lookup methods this would provide $O(L)$ worst-case single random variate generation time at the price of storing $O(2^{L+1})$ items of preprocessing data, and with alias methods it is possible to attain $O(1)$ worst-case single random variate generation time at the price of storing $O(2^{L+1})$ items of preprocessing data. Note that the whole procedure of binomial increments generation can be implemented with the use of integer numbers only. These algorithms can be used for MLMC implementation in problem-specific hardware (see [18]) to increase the performance of the MLMC method due to the faster random increments generation.

4.2. Jump diffusion processes

Consider now a d-dimensional jump diffusion process (X_t) solving the SDE

$$X_t = X_0 + \int_0^t b(X_s)\,ds + \int_0^t \sigma(X_s)\,dW_s + \int_0^t \int_{\mathcal{Z}} \rho(X_{s-}, z) N(ds, dz), \tag{4.5}$$

where $W_t = (W_t^1, \dots, W_t^m)$ is a standard m-dimensional $\mathcal{F}_t$-adapted Brownian motion and $N(ds, dz)$ is a Poisson counting measure on $\mathbb{R}^+ \times \mathcal{Z}$ with a finite intensity measure $ds\,\nu(dz)$. We assume W and N are independent, and that the mappings $b\colon \mathbb{R}^d \mapsto \mathbb{R}^d$, $\sigma : \mathbb{R}^d \mapsto \mathbb{R}^d \otimes \mathbb{R}^m$ and $\rho : \mathbb{R}^d \times \mathcal{Z} \mapsto \mathbb{R}^d$ are Lipschitz continuous and have at most linear growth on $[0, T]$ so that the solution of (4.5) is well defined.

Let $\Gamma = \{-1, 0, \dots, m\}$, $M = \{\gamma = (\gamma_1, \dots, \gamma_l) : \gamma_i \in \Gamma, l \geq 0\}$, and $\varnothing$ stands for the empty set. For any nonempty $\gamma = (\gamma_1, \dots, \gamma_l) \in M$, denote $\_\gamma = (\gamma_2, \dots, \gamma_l)$, $\gamma\_ = (\gamma_1, \dots, \gamma_{l-1})$, $|\gamma| = l$, $\|\gamma\| = |\gamma|+$ number of zero components of γ and $\langle\gamma\rangle$ is the number of negative components of γ. For any mapping $\phi \in C^2(\mathbb{R}^d)$, define the operators associated with (4.5)

$$\begin{aligned}
\mathcal{L}_{-1}[\phi](x; z) &\doteq \phi\big(x + \rho(x, z)\big) - \phi(x), \\
\mathcal{L}_0[\phi](x) &\doteq \sum_{i=1}^d b_i(x)\partial_i\phi(x) + \frac{1}{2}\sum_{i,j=1}^d \varsigma_{ij}(x)\partial_{ij}\phi(x), \\
\mathcal{L}_j[\phi](x) &\doteq \sum_{i=1}^d \sigma_{ij}(x)\partial_i\phi(x), \qquad j = 1, \dots, m,
\end{aligned}$$

where

$$\varsigma_{ij}(x) \doteq \sum_{l=1}^{m} \sigma_{il}(x)\sigma_{jl}(x) \quad \text{and} \quad \partial_j \doteq \frac{\partial}{\partial x^j}.$$

The composite operator is defined recursively as

$$\mathcal{L}_\gamma[\phi](x; z_1, \dots, z_{\langle\gamma\rangle}) = \mathcal{L}_{\gamma_1}\big(\mathcal{L}_{-\gamma}[\phi](x; z_1, \dots, z_{\langle\gamma\rangle})\big)$$

if $\gamma_1 \geq 0$ and via

$$\begin{aligned}\mathcal{L}_\gamma[\phi](x; z_1, \dots, z_{\langle\gamma\rangle}) = \mathcal{L}_{\_\gamma}[\phi]\big(x + \rho(x, z_1); z_2, \dots, z_{\langle\gamma\rangle}\big)\\ - \mathcal{L}_{\_\gamma}[\phi](x; z_2, \dots, z_{\langle\gamma\rangle})\end{aligned}$$

otherwise. Denote $\Delta N_j := N([(j-1)\Delta, j\Delta], \mathcal{Z})$, then the Euler scheme for (4.5) reads as follows

$$\begin{aligned}X_0^\Delta &= X_0,\\ X_{j\Delta}^\Delta &= X_{(j-1)\Delta}^\Delta + a\big(X_{(j-1)\Delta}^\Delta\big)\Delta + \sum_{k=1}^{m} b_k\big(X_{(j-1)\Delta}^\Delta\big)\Delta W_j^k\\ &\quad + \sum_{k=1}^{\Delta N_j} \rho\big(X_{(j-1)\Delta}^\Delta, Z_{jk}\big), \qquad j = 1, \dots, n,\end{aligned}$$

where $Z_{jk}, k = 1, \dots, N_j$, are independent random variables with the law $\frac{\nu(dz)}{\nu(\mathcal{Z})}$. The (essentially) weak Euler scheme can be constructed by replacing the random variables ΔW_j^k and ΔN_j by simple approximations ξ_j^k and η_j, respectively which satisfy

$$\begin{aligned}\big|\mathrm{E}\big[\xi_j^k\big]\big| + \big|\mathrm{E}\big[\big(\xi_j^k\big)^3\big]\big| + \big|\mathrm{E}\big[\big(\xi_j^k\big)^2\big] - \Delta\big| &= O\big(\Delta^2\big),\\ \big|\mathrm{E}\big[(\eta_j)^l\big] - \nu(\mathcal{Z})\Delta\big| &= O\big(\Delta^2\big), \qquad l = 1, 2, 3,\end{aligned}$$

for some $c > 0$. In particular, one can take

$$\begin{aligned}\mathrm{P}\big(\xi_j^i = \sqrt{\Delta}\big) &= \tfrac{1}{2}, \qquad \mathrm{P}\big(\xi_j^i = -\sqrt{\Delta}\big) = \tfrac{1}{2},\\ \mathrm{P}(\eta_j = 1) &= p, \qquad \mathrm{P}(\eta_j = 0) = 1 - p,\end{aligned} \tag{4.6}$$

where $|p - \Delta\nu(\mathcal{Z})| = O(\Delta^2)$. Moreover, the random variables $(Z_{j,k})$ can be replaced by i.i.d. random variables $(\zeta_{j,k})$ satisfying

$$\mathrm{E}\big[\mathcal{L}_\gamma[\Phi](x, \zeta_{1,l_1} \dots, \zeta_{\langle\gamma\rangle,l_{\langle\gamma\rangle}})\big] = \mathrm{E}\big[\mathcal{L}_\gamma[\Phi](x, Z_{1,l_1} \dots, Z_{\langle\gamma\rangle,l_{\langle\gamma\rangle}})\big] \tag{4.7}$$

and

$$\begin{aligned}&\mathrm{E}\big[\mathcal{L}_\gamma[\Phi](x,\zeta_{1,l_1},\dots,\zeta_{\langle\gamma\rangle,l_{\langle\gamma\rangle}})\mathcal{L}_\gamma^\top[\Phi](x,\zeta_{1,l_1},\dots,\zeta_{\langle\gamma\rangle,l_{\langle\gamma\rangle}})\big]\\&\quad=\mathrm{E}\big[\mathcal{L}_\gamma[\Phi](x,Z_{1,l_1},\dots,Z_{\langle\gamma\rangle,l_{\langle\gamma\rangle}})\mathcal{L}_\gamma^\top[\Phi](x,Z_{1,l_1},\dots,Z_{\langle\gamma\rangle,l_{\langle\gamma\rangle}})\big]\end{aligned}\tag{4.8}$$

for $\Phi(x)\equiv x$, all $x\in\mathbb{R}^d$, $|\gamma|\le 2$ with $\langle\gamma\rangle>0$ and $k=1,2$, where Z is distributed according to $\nu(dz)/\nu(\mathcal{Z})$.

Remark 5. If $d=m=1$, then coefficients functions of order $|\gamma|=2$ with $\langle\gamma\rangle>0$ take the form

$$\begin{aligned}\mathcal{L}_{(0,-1)}[\Phi](x,\zeta_{1,l_1})&=b(x)\partial_x\rho(x,\zeta_{1,l_1}),\\\mathcal{L}_{(1,-1)}[\Phi](x,\zeta_{1,l_1})&=\sigma(x)\partial_x\rho(x,\zeta_{1,l_1}),\\\mathcal{L}_{(-1,0)}[\Phi](x,\zeta_{1,l_1})&=b\big(x+\rho(x,\zeta_{1,l_1})\big)-a(x),\\\mathcal{L}_{(-1,1)}[\Phi](x,\zeta_{1,l_1})&=\sigma\big(x+\rho(x,\zeta_{1,l_1})\big)-b(x),\\\mathcal{L}_{(-1,-1)}[\Phi](x,\zeta_{1,l_1},\zeta_{1,l_2})&=\rho\big(x+\rho(x,\zeta_{1,l_1}),\zeta_{1,l_2}\big)-\rho(x,\zeta_{1,l_2}).\end{aligned}$$

If moreover $\rho(x,u)=xu$, $a(x)=a_0+a_1x^\alpha$, $b(x)=b_0+b_1x^\beta$, we get

$$\begin{aligned}\mathcal{L}_{(0,-1)}[\Phi](x,\zeta_{1,l_1})&=\big(a_0+a_1x^\alpha\big)\zeta_{1,l_1},\\\mathcal{L}_{(1,-1)}[\Phi](x,\zeta_{1,l_1})&=\big(b_0+b_1x^\beta\big)\zeta_{1,l_1},\\\mathcal{L}_{(-1,0)}[\Phi](x,\zeta_{1,l_1})&=a_1x^\alpha\big[(1+\zeta_{1,l_1})^\alpha-1\big],\\\mathcal{L}_{(-1,1)}[\Phi](x,\zeta_{1,l_1})&=b_1x^\beta\big[(1+\zeta_{1,l_1})^\beta-1\big],\\\mathcal{L}_{(-1,-1)}[\Phi](x,\zeta_{1,l_1},\zeta_{1,l_2})&=x\zeta_{1,l_1}\zeta_{1,l_2}.\end{aligned}$$

Similar results hold for multidimensional case as well. Hence for a large class of stochastic processes, including affine and polynomial processes, the conditions (4.7) and (4.8) can be viewed as generalised moment conditions.

In the corresponding ML algorithm, we can use the approximation $\eta_{L,j}$ for $\Delta N_{L,j}$ of the form:

$$\mathrm{P}(\eta_{L,j}=1)=\Delta_L\nu(\mathcal{Z}),\qquad\mathrm{P}(\eta_{L,j}=0)=1-\Delta_L\nu(\mathcal{Z}).$$

Then the random variables $\eta_{l,j}$ for $l<L$ have a binomial distribution which can be easily simulated as described in Section 4.1.

4.3. General Lévy processes

Consider a one-dimensional square integrable Lévy process $(Z_t)_{t\ge0}$ of the form

$$Z_t=bt+\sigma B_t+\int_0^t\int_{\mathbb{R}}z\tilde{N}(ds,dz),$$

for some $\sigma \geq 0$, where $\tilde{N}(ds, dz)$ is a compensated Poisson random measure on $\mathbb{R}_+ \otimes \mathbb{R}$ with intensity measure $ds\, \nu(dz)$, where $\int |z|^2 \nu(dz) < \infty$. In order to apply the Euler approximation scheme to (1.1), we need to approximate the increments ΔZ_j. Asmussen and Rosinski [1] (see also [12]) suggested to replace the small jumps in L by an appropriate Gaussian random variable. So we define

$$\zeta^{\delta}_{\Delta,j} \doteq \Delta Z^{\delta}_j + U^{\delta}_{\Delta,j},$$

where Z^{δ} is the same Lévy process as Z without its (compensated) jumps smaller than δ and $U^{\delta}_{\Delta,j}$ is Gaussian random variable with the same mean and variance as the neglected jumps. The resulting Euler scheme takes the form

$$\begin{aligned} X^{\Delta,\delta}_0 &= X_0, \\ X^{\Delta,\delta}_{j\Delta} &= X^{\Delta,\delta}_{(j-1)\Delta} + a\big(X^{\Delta,\delta}_{(j-1)\Delta}\big)\zeta^{\delta}_{\Delta,j}, \qquad j = 1, \ldots, n. \end{aligned} \tag{4.9}$$

Let us discuss the first condition in (1.2) (weak convergence). As was shown in [5] (see also [12]),

$$\big|\mathrm{E}\big[f\big(X^{\Delta,\delta}_T\big)\big] - \mathrm{E}\big[f(X_T)\big]\big| \lesssim \Delta \vee \delta^{3-\alpha}, \tag{4.10}$$

provided $f \in C^2(\mathbb{R})$ and

$$\nu\big(\big\{|z| > t\big\}\big) \lesssim t^{-\alpha}, \qquad t \to +0. \tag{4.11}$$

Note that each r.v. $\zeta^{\delta}_{\Delta,j}$ can be represented as

$$\zeta^{\delta}_{\Delta,j} = \Delta b + \sigma_{\Delta,\delta} \cdot \xi_j + \sum_{i=1}^{N^{\delta}_{\Delta,j}} \big(Z^{\delta}_{i,j} - \mathrm{E}\big[Z^{\delta}_{i,j}\big]\big), \qquad j = 1, \ldots, n,$$

where $\sigma^2_{\Delta,\delta} \doteq \Delta(\sigma^2 + \int_{|z|\leq\delta} z^2 \nu(dz))$, $\xi_j \sim \mathcal{N}(0,1)$, $N^{\delta}_{\Delta,j} \sim \mathrm{Poiss}(\Delta\nu(\{|z| > \delta\}))$ and $Z^{\delta}_{1,j}$, $Z^{\delta}_{2,j}, \ldots$ are i.i.d. random variables with the distribution

$$1_{|z|>\delta}\nu(dz)/\nu\big(\big\{|z| > \delta\big\}\big).$$

Hence the cost of generating one trajectory by means of (4.9) is of order $\Delta^{-1} + \nu(\{|z| > \delta\})$. Let us now fix two natural numbers n_f, $n_c = 2 \cdot n_f$, two positive real numbers δ_f, δ_c and describe the coupling between $\zeta^{\delta_c}_{\Delta_c}$ and $\zeta^{\delta_f}_{\Delta_f}$. Set

$$\begin{aligned} \zeta^{\delta_c}_{\Delta_c,j} &= 2\Delta_f b + \sigma_{\Delta_f,\delta_f} \cdot (\xi_{2j} + \xi_{2j-1}) \\ &\quad + \sum_{i=1}^{N^{\delta_f}_{\Delta_f,2j}} \big(Z^{\delta_f}_{i,2j} 1_{|Z_{i,2j}|>\delta_c} - \mathrm{E}\big[Z^{\delta_f}_{i,2j} 1_{|Z_{i,2j}|>\delta_c}\big]\big) \\ &\quad + \sum_{i=1}^{N^{\delta_f}_{\Delta_f,2j-1}} \big(Z^{\delta_f}_{i,2j-1} 1_{|Z_{i,2j-1}|>\delta_c} - \mathrm{E}\big[Z^{\delta_f}_{i,2j-1} 1_{|Z_{i,2j-1}|>\delta_c}\big]\big), \end{aligned} \tag{4.12}$$

then

$$\begin{aligned}\mathcal{R} &\doteq \zeta^{\delta_c}_{\Delta_c,j} - \zeta^{\delta_f}_{\Delta_f,2j} - \zeta^{\delta_f}_{\Delta_f,2j-1}\\ &= \sum_{i=1}^{N^{\delta_f}_{\Delta_f,2j}} \big(Z^{\delta_f}_{i,2j} 1_{\delta_f<|Z_{i,2j}|\le\delta_c} - \mathrm{E}\big[Z^{\delta_f}_{i,2j} 1_{|Z_{i,2j}|\le\delta_c}\big]\big)\\ &\quad + \sum_{i=1}^{N^{\delta_f}_{\Delta_f,2j-1}} \big(Z^{\delta_f}_{i,2j-1} 1_{|Z_{i,2j-1}|\le\delta_c} - \mathrm{E}\big[Z^{\delta_f}_{i,2j-1} 1_{|Z_{i,2j-1}|\le\delta_c}\big]\big).\end{aligned}$$

As a result, $\mathrm{E}[\mathcal{R}] = 0$ and

$$\mathrm{E}\big[|\mathcal{R}|^2\big] \le 2\Delta_f \int_{|z|\le\delta_c} |z|^2 \nu(dz). \tag{4.13}$$

Hence the assumptions of Corollary 2 are fulfilled, provided

$$\int_{|z|\le\delta_c} |z|^2 \nu(dz) \le c\Delta_f$$

for some $c > 0$. Under (4.11), this is equivalent to the relation $\delta_c \lesssim \Delta_f^{1/(2-\alpha)}$. Using the estimate (4.10), we derive the complexity of the resulting coupled multilevel scheme.

Proposition 6. *If $\alpha \le 3 - \sqrt{3}$ in (4.11), then the complexity of the coupled multilevel algorithm presented in Section 3.2 with the coupling (4.12) is of order*

$$\begin{cases} \varepsilon^{-2} \cdot (\log \varepsilon)^2, & \alpha \le 1,\\ \varepsilon^{-2/(2-\alpha)}, & 1 < \alpha \le 3 - \sqrt{3}, \end{cases}$$

provided $\delta_l = \Delta_l^{1/(2-\alpha)}$. For $\alpha > 3 - \sqrt{3}$ we can use the simplest coupling $\zeta^{\delta_c}_{\Delta_c,j} = \zeta^{\delta_f}_{\Delta_f,2j} + \zeta^{\delta_f}_{\Delta_f,2j-1}$ and constant $\delta_l = \varepsilon^{1/(3-\alpha)}$ to get upper estimate $\varepsilon^{-(6-\alpha)/(3-\alpha)}$ for the complexity of the corresponding coupled multilevel algorithm.

Discussion

Observe that the complexity of the standard MC algorithm for estimating $\mathrm{E}[f(X_T)]$ is bounded above via

$$\begin{cases} \varepsilon^{-3}, & \alpha \le 3/2,\\ \varepsilon^{-(6-\alpha)/(3-\alpha)}, & 3/2 < \alpha \le 2. \end{cases}$$

So the coupled MLMC approach is superior to the standard MC algorithm as long as $\alpha \le 3 - \sqrt{3}$, see Figure 1.

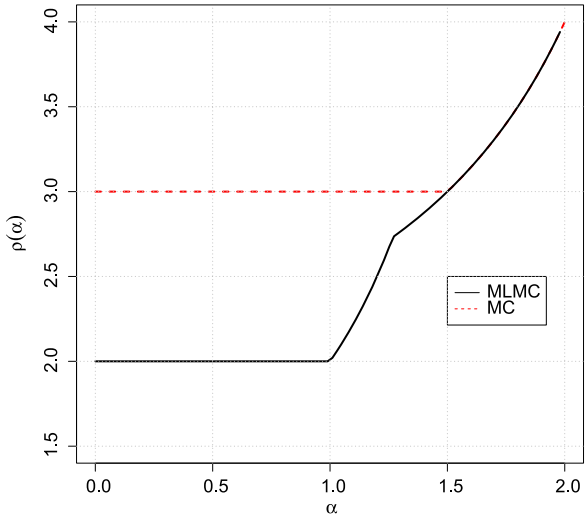

Figure 1. The functions $\rho(\alpha)$, plotted here as functions of α represent the complexity of standard MC and coupled MLMC algorithms in the following sense: for a given α, the MC (MLMC) algorithm has complexity $O(\varepsilon^{-\rho(\alpha)})$ for a prescribed precision ε.

A similar behaviour can be observed in Dereich [3] (at least for $\alpha \leq 1$). We can further replace the restricted Lévy jump sizes $(Z_{i,j}^{\delta_f})$ by some simple random variables using the approach presented in Section 4.2. Note that in the latter case the above complexity bounds continue to hold.

5. Numerical experiments

In this section, we present numerical examples corresponding to process classes discussed in Section 4. The MLMC algorithm is implemented according to the [8], with some changes, due to the specific structure of the simulated process. Recall, that the MLMC estimator has the form:

$$Y_{L,\mathbf{N}} = \frac{1}{N_0}\sum_{n=1}^{N_0}\left[f(X_T^0(\boldsymbol{\zeta}^{(n)})\right] + \sum_{l=1}^{L}\frac{1}{N_l}\sum_{n=1}^{N_l}\left[f\left(X_T^l(\boldsymbol{\zeta}_l^{(n)})\right) - f\left(X_T^{l-1}(\overline{\boldsymbol{\zeta}}_l^{(n)})\right)\right]$$
$$= \hat{Y}_0 + \sum_{l=0}^{L}\hat{Y}_l.$$

But the general scheme is the same for all considered problems and can be summarized in the following algorithm:

Input: Requested accuracy ϵ and set the final level $\hat{L}$.

1. Set $L := 2$.
2. Compute $N_l := 100$ samples on levels $l = 0, 1, 2$.

3. Estimate $\mathrm{Var}(\hat{Y}_l)$ and update N_l for each level $l = 0, \ldots, L$:

$$N_l := \max\left\{N_l, \left\lceil 2 \cdot \epsilon^{-2} \cdot \sqrt{\mathrm{Var}(\hat{Y}_l) 2^{-l}} \cdot \sum_{k=0}^{L} \sqrt{\mathrm{Var}(\hat{Y}_k) \cdot 2^k} \right\rceil\right\}.$$

   If the update N_l is increased less than 1% on the levels, then go to step 5.
4. Compute the additional number of samples and Go to step 3.
5. If $L < 2$ or $\max\{|\hat{Y}_{L-1}|/2, |\hat{Y}_L|\} \geq \epsilon/\sqrt{2}$:

$$L := L + 1, \qquad \mathrm{Var}(\hat{Y}_L) = \mathrm{Var}(\hat{Y}_{L-1})/2.$$

   Else: Return $\sum_{l=0}^{L} \hat{Y}_l$.
6. If $L > \hat{L}$, then
   Display error: The final level $\hat{L}$ is insufficient for the convergence.
   Return $\sum_{l=0}^{L} \hat{Y}_l$.
7. Goto step 3.

In all of our numerical experiments we have chosen $\hat{L}$ to be sufficiently large, so that $\hat{L} \geq L$ was always satisfied.

5.1. Diffusion process

5.1.1. *European max-call option*

Consider a three dimensional process $X_t = (X_t^1, X_t^2, X_t^3)$, $t \in [0, T]$, with independent components where each process X_t^i solves one-dimensional SDE of the form (4.1) with $b(x) = r \cdot x$ and $\sigma(x) = \sigma \cdot x$ for some $r, \sigma \in \mathbb{R}$. We are interested in computing the expectation of

$$f(X_T) = e^{-r \cdot T} \max\bigl(\max\bigl(X_T^1, X_T^2, X_T^3\bigr) - K, 0\bigr).$$

We chose the following parameters:

$$r = 0.05, \qquad \sigma = 0.2, \qquad T = 1, \qquad K = 1, \qquad X_0^i = 1, \qquad i = 1, 2, 3,$$

and $\hat{L} = 9$. In fact in this case the exact solution is available and for above parameter values, we have $\mathrm{E}[f(X_T)] \approx 0.2276799594$. The variance decay is presented on Figure 2. In particular, the line $\alpha_1 - \alpha_2 \cdot l$ with $\alpha_2 = 0.9753$ fits the estimated log-variances best and this is in agreement with Corollary 2. The corresponding RMSE is presented in Figure 3.

5.1.2. *Geometric Asian option*

Consider a one dimensional process X_t, $t \in [0, T]$, where each coordinate process X_t^i solves one-dimensional SDE of the form (4.1) with $b(x) = r \cdot x$ and $\sigma(x) = \sigma \cdot x$ for some $r, \sigma \in \mathbb{R}$. We are interested in computing the expectation of the functional

$$f(X_\cdot) = e^{-r \cdot T} \max\left(\exp\left(\frac{1}{T} \int_0^T \log(X_t)\, dt\right) - K, 0\right).$$

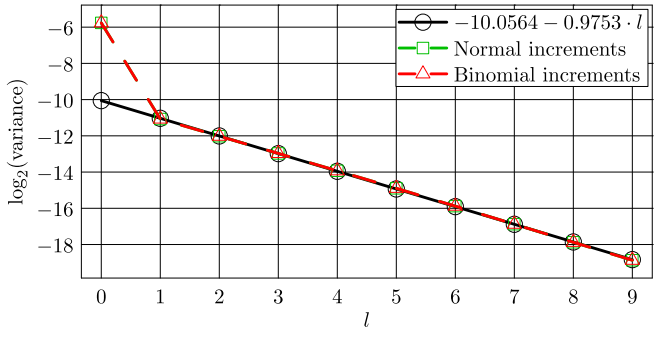

Figure 2. Three dimensional European max-call option: level variances for schemes with binomial and normal increments.

The parameter values are

$$r = 0.05, \qquad \sigma = 0.2, \qquad T = 1, \qquad K = 1.$$

In this case, the exact value of the expectation is given by

$$\mathrm{E}\big[f(X_\cdot)\big] \approx 0.05546818634.$$

The variance decay is presented on Figure 4. Due the fact, that at first levels the variance decays faster than predicted, we have fitted the variance decay only on the last 6 levels with the line $\alpha_1 - \alpha_2 \cdot l$ and got $\alpha_2 = 1.0059$. The corresponding RMSE is presented in Figure 5.

5.2. Jump diffusions

Consider a jump SDE

$$dX_t = \big(r - \lambda \cdot \big(e^{m+0.5\cdot\theta^2} - 1\big)\big) \cdot X_t \cdot dt + \sigma \cdot X_t \cdot W_t + X_t \cdot dJ(t),$$

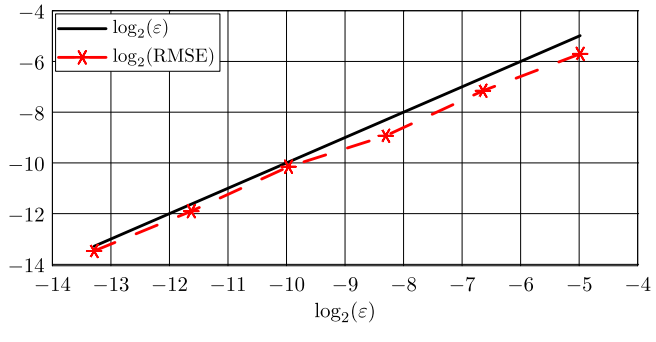

Figure 3. Three dimensional European max-call option: estimated RMSE against the required precision ε for different values of ε.

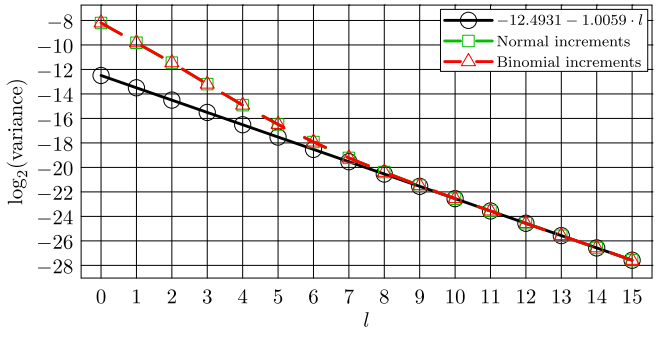

Figure 4. Geometric Asian option: level log-variances for binomial and normal increments.

where

$$J(t) = \sum_{j=1}^{N(t)} (Y_j - 1), \qquad \log(Y_j) \sim \mathcal{N}\big(m, \theta^2\big)$$

and $N(t)$ is a Poisson process with rate λ. We are interested in computing the expectation of

$$f(X_T) = e^{-r\cdot T} \max\big(X(T) - K, 0\big).$$

The parameters' values are

$$r = 0.05, \qquad \sigma = 0.2, \qquad \lambda = 0.5, \qquad m = 0.05, \qquad \theta = 0.25, \qquad T = 1, \qquad K = 1.$$

It follows from [9] (Section 3.5) that, for above parameter values $\mathrm{E}[f(X_T)] \approx 0.153065585$. We have performed two types of simulations with the fixed top level $\hat{L} = 8$:

- Y was sampled from the lognormal distribution, while the increments of the Brownian motion were modelled as normal random variables;

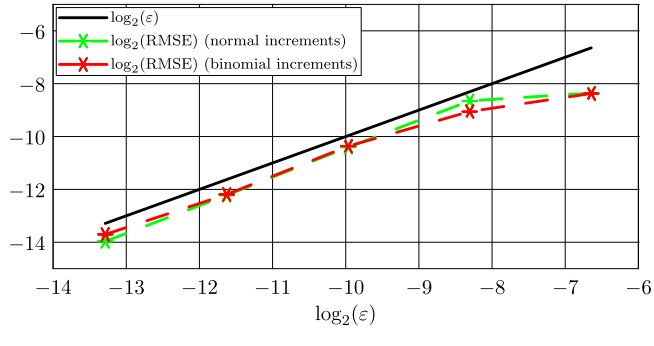

Figure 5. Geometric Asian option: RMSE for binomial and normal increments.

- Y was sampled as a discrete random variable $\hat{Y}$ according to the Remark 4.8, with moments matching first 6 moments of the lognormal distribution, while the increments of the Brownian motion were modelled as discrete random variable defined by (4.4).

In both of those cases, the number of jumps $\eta_{l,j}$ at the level l and step j is generated via

$$\eta_{l,j} \sim \mathrm{Bi}\big(2^{\hat{L}-l}, 2^{-\hat{L}} \cdot \lambda\big). \tag{5.1}$$

One can see, that (5.1) can be implemented in the same spirit as (4.4). On the finest level L, we allow only for two jumps 0 or 1. Let us denote by μ_i the ith moment of the lognormal distribution with parameters m and θ. The random variable $\hat{Y}$ takes 4 values with probabilities $p_1, \ldots, p_4$. The values and probabilities are obtained by solving the optimization problem:

$$\text{Minimize} \left(\sum_{k=1}^{4} p_k \cdot x_k^7 - \mu_7 \right)^2$$

$$\text{subject to} \quad \sum_{k=1}^{4} p_k \cdot x_k^i = \mu_i, \qquad i = 1, \ldots, 6.$$

The solution is

$$\begin{aligned} p_1 &= 0.608176614910593, & x_1 &= 1.081500568717563, \\ p_2 &= 0.003503326771883, & x_2 &= 2.376117006693613, \\ p_3 &= 0.226782660300013, & x_3 &= 0.719559222085786, \\ p_4 &= 0.161537398017512, & x_4 &= 1.581001071314797. \end{aligned}$$

The variance decay is shown in Figure 6 for both types of simulations. We estimated RMSE of the ML estimate based on the weak Euler scheme based on 50 independent runs, see Figure 7.

6. Proofs

Lemma 7. *Suppose that the coefficient function a in (1.1) is uniformly Lipschitz and has at most linear growth, that is,*

$$\big\|a(x) - a\big(x'\big)\big\| \le L_a \big|x - x'\big|, \qquad \big\|a(x)\big\|^2 \le B_a^2\big(1 + |x|^2\big)$$

for any $x, x' \in \mathbb{R}^d$ and some positive constants L_a and B_a. Moreover, assume that $\mathrm{E}[|X_0|^2] < \infty$, then the following estimates hold

$$\mathrm{E}\big[\big|X^f_{n\Delta_f}\big|^2\big] \le 3B_a^2 \cdot \big(n \cdot m_{f,2} + n^2 \cdot m_{f,1}^2\big) \cdot \exp\big(3B_a^2 \cdot \big(n \cdot m_{f,2} + n^2 \cdot m_{f,1}^2\big)\big),$$

$$\mathrm{E}\big[\big|X^c_{n\Delta_c}\big|^2\big] \le 3B_a^2 \cdot \big(n \cdot m_{f,2} + n^2 \cdot m_{f,1}^2\big) \cdot \exp\big(3B_a^2 \cdot \big(n \cdot m_{f,2} + n^2 \cdot m_{f,1}^2\big)\big),$$

for $n = 1, \ldots, n_c$.

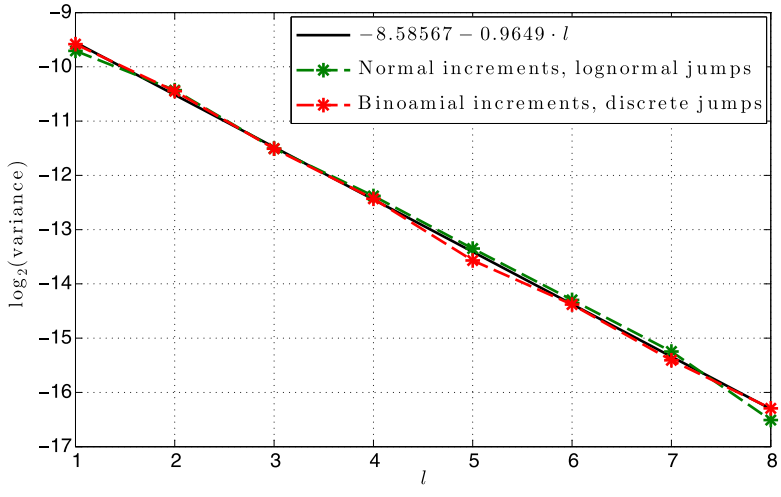

Figure 6. European option: level log-variances for binomial increments and discrete jumps, normal increments and lognormal jumps.

Proof. Since

$$X^f_{n\Delta_f} = X_0 + \sum_{i=1}^{n} \big(X^f_{i\Delta_f} - X^f_{(i-1)\Delta_f}\big),$$

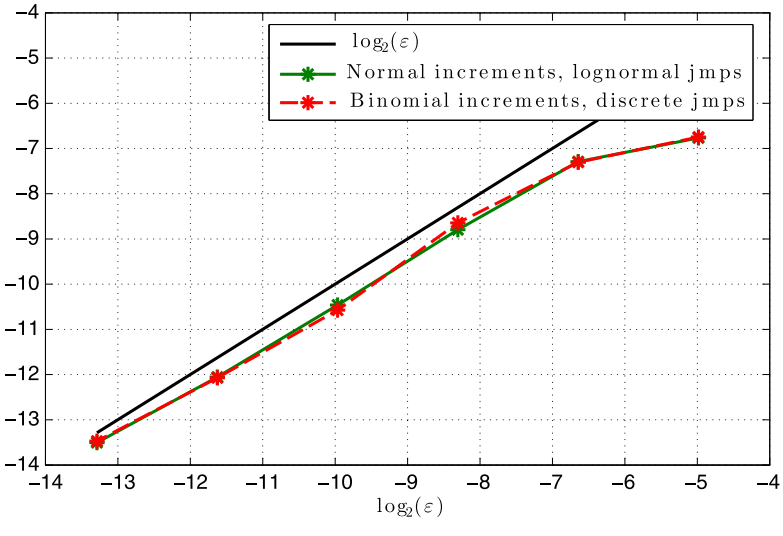

Figure 7. European option: RMSE for binomial increments and discrete jumps, normal increments and lognormal jumps.

we have, due to independence of the increments

$$\begin{aligned}
\mathrm{E}\big[|X^f_{n\Delta_f}|^2\big] &= \mathrm{E}\left[\left|X_0 + \sum_{i=1}^{n} a\big(X^f_{(i-1)\Delta_f}\big)\cdot\big(\zeta_i^f - \mathrm{E}[\zeta_i^f]\big) + \sum_{i=1}^{n} a\big(X^f_{(i-1)\Delta_f}\big)\cdot\mathrm{E}[\zeta_i^f]\right|^2\right] \\
&\le 3\mathrm{E}\big[|X_0|^2\big] + 3\sum_{i=1}^{n}\mathrm{E}\big[\big\|a\big(X^f_{(i-1)\Delta_f}\big)\big\|^2\big]m_{f,2} \\
&\quad + 3n\sum_{i=1}^{n}\mathrm{E}\big[\big\|a\big(X^f_{(i-1)\Delta_f}\big)\big\|^2\big]m_{f,1}^2 \\
&\le 3\mathrm{E}\big[|X_0|^2\big] + 3B_a^2\cdot m_{f,2}\cdot\left(n + \sum_{i=1}^{n}\mathrm{E}\big[|X^f_{i-1}|^2\big]\right) \\
&\quad + 3B_a^2\cdot n\cdot m_{f,1}^2\cdot\left(n + \sum_{i=1}^{n}\mathrm{E}\big[|X^f_{i-1}|^2\big]\right).
\end{aligned}$$

Using the discrete version of the Gronwall inequality (see Appendix), we get

$$\mathrm{E}\big[|X^f_{n\Delta_f}|^2\big] \le 3B_a^2\cdot\big(n\cdot m_{f,2} + n^2\cdot m_{f,1}^2\big)\cdot\exp\big(3B_a^2\cdot\big(n\cdot m_{f,2} + n^2\cdot m_{f,1}^2\big)\big).$$

The second inequality of the lemma can be proved in the same way. □

6.1. Proof of Proposition 1

Due to the Lemma 7, we have

$$\mathrm{E}\big[|X^f_{n\Delta_f}|^2\big] < A_1, \qquad \mathrm{E}\big[|X^c_{n\Delta_c}|^2\big] < A_2 \tag{6.1}$$

for $n = 1, \dots, n_c$, and constants A_1, A_2 not depending on n. We have

$$\begin{aligned}
X^f_{r\Delta_c} - X^c_{r\Delta_c} &= X^f_{(r-1)\Delta_c} - X^c_{(r-1)\Delta_c} + \big[a\big(X^f_{(2r-1)\Delta_f}\big) - a\big(X^c_{(r-1)\Delta_c}\big)\big]\zeta^f_{2r} \\
&\quad + \big[a\big(X^f_{(r-1)\Delta_c}\big) - a\big(X^c_{(r-1)\Delta_c}\big)\big]\zeta^f_{2r-1} - a\big(X^c_{(r-1)\Delta_c}\big)\big[\zeta^c_r - \zeta^f_{2r} - \zeta^f_{2r-1}\big].
\end{aligned}$$

Denote $D_r \doteq X^f_{r\Delta_c} - X^c_{r\Delta_c}$, then we have the representation

$$D_r = D_{r-1} + \delta_r + \varepsilon_r$$

with

$$\begin{aligned}
\varepsilon_r &= \big[a\big(X^f_{(2r-1)\Delta_f}\big) - a\big(X^c_{(r-1)\Delta_c}\big)\big]\big(\zeta^f_{2r} - \mathrm{E}[\zeta^f_{2r}]\big) \\
&\quad + \big[a\big(X^f_{(r-1)\Delta_c}\big) - a\big(X^c_{(r-1)\Delta_c}\big)\big]\big(\zeta^f_{2r-1} - \mathrm{E}[\zeta^f_{2r-1}]\big) \\
&\quad - a\big(X^c_{(r-1)\Delta_c}\big)\big[\zeta^c_r - \zeta^f_{2r} - \zeta^f_{2r-1}\big]
\end{aligned}$$

and

$$\delta_r = \big[a\big(X^f_{(2r-1)\Delta_f}\big) - a\big(X^c_{(r-1)\Delta_c}\big)\big]\mathrm{E}\big[\zeta^f_{2r}\big] + \big[a\big(X^f_{(r-1)\Delta_c}\big) - a\big(X^c_{(r-1)\Delta_c}\big)\big]\mathrm{E}\big[\zeta^f_{2r-1}\big].$$

The Lipschitz continuity of the function a implies

$$\begin{aligned}\mathrm{E}\big[\big\|a\big(X^f_{(2r-1)\Delta_f}\big) - a\big(X^c_{(r-1)\Delta_c}\big)\big\|^2\big] &\le L_a^2\mathrm{E}\big[\big|X^f_{(2r-1)\Delta_f} - X^c_{(r-1)\Delta_c}\big|^2\big]\\ &\le 2L_a^2\mathrm{E}\big[\big|D_{r-1}\big|^2\big]\\ &\quad + 2L_a^2B_a^2\big(1+\mathrm{E}\big|X^f_{(r-1)\Delta_c}\big|^2\big)\mathrm{E}\big[\big|\zeta^f_{2r-1}\big|^2\big]\end{aligned}$$

and

$$\mathrm{E}\big[\big\|a\big(X^f_{(r-1)\Delta_c}\big) - a\big(X^c_{(r-1)\Delta_c}\big)\big\|^2\big] \le L_a^2\mathrm{E}\big[\big|D_{r-1}\big|^2\big].$$

As a result

$$\begin{aligned}\mathrm{E}\big[|\varepsilon_r|^2\big] &\le 3\mathrm{E}\big[\big\|a\big(X^f_{(2r-1)\Delta_f}\big) - a\big(X^c_{(r-1)\Delta_c}\big)\big\|^2\big]\mathrm{E}\big[\big|\xi^f_{2r}\big|^2\big]\\ &\quad + 3\mathrm{E}\big[\big\|a\big(X^f_{(r-1)\Delta_c}\big) - a\big(X^c_{(r-1)\Delta_c}\big)\big\|^2\big]\mathrm{E}\big[\big|\xi^f_{2r-1}\big|^2\big]\\ &\quad + 3\mathrm{E}\big[\big\|a\big(X^c_{(r-1)\Delta_c}\big)\big\|^2\big]\mathrm{E}\big[\big|\zeta^c_r - \zeta^f_{2r} - \zeta^f_{2r-1}\big|^2\big]\\ &\le c_1\big(1+\mathrm{E}\big[\big|X^f_{(r-1)\Delta_c}\big|^2\big]\big)\mathrm{E}\big[\big|\xi^f_{2r-1}\big|^2\big]\mathrm{E}\big[\big|\xi^f_{2r}\big|^2\big]\\ &\quad + c_2\mathrm{E}\big[|D_{r-1}|^2\big]\big(\mathrm{E}\big[\big|\xi^f_{2r}\big|^2\big] + \mathrm{E}\big[\big|\xi^f_{2r-1}\big|^2\big]\big)\\ &\quad + c_3\big(1+\mathrm{E}\big[\big|X^c_{(r-1)\Delta_c}\big|^2\big]\big)\mathcal{R}\\ &\le c_4\big[m_{f,2}\mathrm{E}\big[|D_{r-1}|^2\big] + m^2_{f,2} + \mathcal{R}\big]\end{aligned}$$

for some constants c_1, c_2, c_3, c_4 and $\mathcal{R} \doteq \mathrm{E}[|\zeta^c_r - \zeta^f_{2r} + \zeta^f_{2r-1}|^2]$. Analogously

$$\mathrm{E}\big[|\delta_r|^2\big] \le c_5\big[m^2_{f,1}\mathrm{E}\big[|D_{r-1}|^2\big] + m^2_{f,1}m_{f,2}\big]$$

for some $c_5 > 0$. Define

$$M_r = \sum_{j=1}^{r}\varepsilon_j$$

and note that M_r is martingale with respect to the filtration

$$\mathcal{F}_r \doteq \sigma\big(X^f_{(2j-1)\Delta_f}, X^c_{(j-1)\Delta_c}, X^f_{(j-1)\Delta_c}, j \le r\big), \qquad r = 1, \dots, n_c + 1.$$

Hence, the Doob inequality implies for any $n \le n_c$:

$$\begin{aligned}\mathrm{E}\Big[\Big|\sup_{r=1,\dots,n} M_r\Big|^2\Big] &\le \mathrm{E}\big[|M_n|^2\big]\\ &\le c_6 m_{f,2}\sum_{j=1}^{n}\mathrm{E}\big[|D_{j-1}|^2\big] + c_6 n m^2_{f,2} + c_6 n\mathcal{R}.\end{aligned}$$

So we have for $\overline{D}_n \doteq \max_{j=1,\dots,n} D_j$

$$\begin{aligned}\mathrm{E}\big[|\overline{D}_n|^2\big] &\le 2\cdot n\sum_{j=1}^{n}\mathrm{E}\big[|\delta_j|^2\big] + 2\cdot\mathrm{E}\big[|M_n|^2\big]\\ &\le c_7\big(m_{f,2}+nm_{f,1}^2\big)\sum_{j=1}^{n}\mathrm{E}\big[|\overline{D}_{j-1}|^2\big] + c_7 n\big(m_{f,2}^2+nm_{f,1}^2m_{f,2}+\mathcal{R}\big).\end{aligned}$$

Finally, a discrete version of Gronwall lemma (see Appendix) implies

$$\mathrm{E}\big[|\overline{D}_n|^2\big] \le c_8 n\big(m_{f,2}^2+m_{f,1}^2m_{f,2}+\mathcal{R}\big)\exp\big(c_7\big(nm_{f,2}+n^2m_{f,1}^2\big)\big).$$

6.2. Proof of Proposition 6

We aim to minimize

$$\sum_{l=0}^{L} N_l\cdot\big(\delta_l^{-\alpha}+\Delta_l^{-1}\big)$$

subject to

$$\min\big(\Delta_L,\delta_L^{3-\alpha}\big)\le\varepsilon,\qquad \sum_{l=0}^{L}\frac{\Delta_l}{N_l}\le\varepsilon^2.$$

We denote

$$a_l=\big(\delta_l^{-\alpha}+\Delta_l^{-1}\big),\qquad l=0,\dots,L.$$

From Lagrange principle, we get

$$\begin{aligned}&a_l=-\lambda\cdot N_l^{-2}\cdot\Delta_l \quad\Rightarrow\quad N_l=\sqrt{(-\lambda)\cdot\Delta_l\cdot a_l^{-1}} \quad\Rightarrow\\ &\sum_{l=0}^{L}\frac{\Delta_l}{N_l}=\frac{1}{\sqrt{-\lambda}}\cdot\sum_{l=0}^{L}\frac{\Delta_l}{\sqrt{\Delta_l\cdot a_l^{-1}}}=\varepsilon^2 \quad\Rightarrow\quad \sqrt{-\lambda}=\varepsilon^{-2}\cdot\sum_{l=0}^{L}\sqrt{\Delta_l\cdot a_l} \quad\Rightarrow\\ &N_l=\sqrt{\Delta_l\cdot a_l^{-1}}\cdot\varepsilon^{-2}\cdot\sum_{l=0}^{L}\sqrt{\Delta_l\cdot a_l}.\end{aligned}$$

So the cost has the representation

$$\begin{aligned}\sum_{l=0}^{L}N_l\cdot a_l &= \sum_{l=0}^{L}a_l\cdot\sqrt{\Delta_l\cdot a_l^{-1}}\cdot\varepsilon^{-2}\cdot\sum_{k=0}^{L}\sqrt{\Delta_k\cdot a_k}\\ &= \varepsilon^{-2}\cdot\left(\sum_{l=0}^{L}\sqrt{\Delta_l\cdot a_l}\right)^2=\varepsilon^{-2}\cdot\left(\sum_{l=0}^{L}\sqrt{1+\frac{\Delta_l}{\delta_l^{\alpha}}}\right)^2.\end{aligned}$$

According to the restrictions on the bias, we have

$$\Delta_L = M^{-L} = \varepsilon, \qquad \delta_L = \varepsilon^{1/(3-\alpha)} = \Delta_L^{1/(3-\alpha)}.$$

We now consider two cases.

1. We set $\delta_l = \varepsilon^{1/(3-\alpha)}$ constant on all the levels. Then the cost is bounded from above by

$$\sum_{l=0}^{L} N_l \cdot \left(\delta_l^{-\alpha} + \Delta_l^{-1}\right) \preceq \varepsilon^{-2} \cdot \varepsilon^{-\alpha/(3-\alpha)} = \varepsilon^{-6-\alpha/(3-\alpha)}.$$

2. In the second case, we will set

$$\delta_l = \Delta_l^{1/(2-\alpha)}.$$

Note, that $\delta_L = \Delta_L^{1/(2-\alpha)} < \Delta_L^{1/(3-\alpha)}$, so the bias condition is fulfilled. Then the overall cost

$$\begin{aligned}\varepsilon^{-2} \cdot \left(\sum_{l=0}^{L} \sqrt{1 + \frac{\Delta_l}{\delta_l^{\alpha}}}\right)^2 &\asymp \varepsilon^{-2} \cdot \left(\sum_{l=0}^{L} \sqrt{1 + \Delta_l^{1-\alpha/(2-\alpha)}}\right)^2 \\ &\asymp \varepsilon^{-2} \cdot \left(\sum_{l=0}^{L} \sqrt{1 + \Delta_l^{(2-2\cdot\alpha)/(2-\alpha)}}\right)^2.\end{aligned}$$

Combining all the cases together we get the statement.

Appendix

Lemma A.8. *Let (y_n) and (g_n) be two nonnegative sequences and let c be a nonnegative constant. If*

$$y_n \le c + \sum_{k=1}^{n} g_k y_k, \qquad n \ge 0,$$

then

$$y_n \le c \exp\left(\sum_{k=1}^{n} g_k\right).$$

Proof. We have

$$\begin{aligned} y_n &\le c + \sum_{0\le k<n} c g_k \prod_{k<j<n} (1+g_j) \\ &= c + c \sum_{0\le k<n} \left[\prod_{k\le j<n} (1+g_j) - \prod_{k+1\le j<n} (1+g_j)\right] \end{aligned}$$

$$= c + c\left[\prod_{0\le j<n}(1+g_j) - \prod_{n+1\le j<n}(1+g_j)\right]$$

$$= c\prod_{0\le j<n}(1+g_j)$$

$$\le c\exp\left(\sum_{0\le j<n} g_j\right).$$

□

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