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On the continuity of Lyapunov exponents of random walk in random potential

LE THI THU HIEN

Laboratoire de Mathématiques de Bretagne Atlantique (LMBA/UMR 6205), 6 avenue Le Gorgeu, 29200, Brest, France. E-mail: Thi-Thu-Hien.Le@univ-brest.fr

We consider a simple random walk in an i.i.d. nonnegative potential on the d-dimensional cubic lattice $\mathbb{Z}^d$, $d \geq 3$. We prove that the Lyapunov exponents are continuous with respect to the law of the potential. In the quenched case, we assume that the potentials are integrable whilst there are no additional conditions in the annealed case.

Keywords: continuity; Lyapunov exponents; random potential; random walk

1. Introduction

Let $S_n, n \in \mathbb{N}$ be the simple random walk on $\mathbb{Z}^d$. We denote by P_x and E_x the probability measure and the expectation, respectively, of the simple random walk starting from position x. Independently of the random walk, we give ourselves a family of independent and identically distributed random variables $V(x, \omega), x \in \mathbb{Z}^d$ taking values in $[0, \infty]$ that we call the potentials. They are defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with associated expectation $\mathbb{E}$.

For $y \in \mathbb{Z}^d$, let us write $H(y)$ for the hitting time of the walk at site y,

$$H(y) := \inf\{n \geq 0 : S_n = y\}, \tag{1.1}$$

with the convention that $\inf \varnothing = +\infty$. For any $x, y \in \mathbb{Z}^d$, $\omega \in \Omega$ we define:

$$\begin{aligned} &e(x, y, F, \omega) \\ &\quad := E_x\left(\exp\left(-\sum_{m=0}^{H(y)-1} V(S_m, \omega)\right), H(y) < \infty\right) \qquad \big(e(x, y, F, \omega) = 1 \text{ if } x = y\big), \end{aligned} \tag{1.2}$$

where F is the distribution function of $V(0)$. Let us define:

$$a(x, y, F, \omega) := -\ln e(x, y, F, \omega) \in [0, \infty[\,, \tag{1.3}$$

and

$$b(x, y, F) := -\ln \mathbb{E}\big(e(x, y, F, \omega)\big). \tag{1.4}$$

The quantity $a(x, y, F, \omega)$ can be interpreted as measuring the cost of traveling from x to y of the random walk in the potential V. For $x = (x_1, x_2, \dots, x_d) \in \mathbb{Z}^d$, $|x|$ denotes the ℓ_1-norm of x, $|x| = |x_1| + |x_2| + \cdots + |x_d|$. And $|A|$ is the cardinal of the set A.

Theorem A (Zerner [19]). *Let V be a potential such that $\mathbb{E}V(0) < \infty$. Let F be its distribution function. Then there is a nonrandom norm $\alpha_F(\cdot)$ on $\mathbb{R}^d$, such that $\mathbb{P}$-a.s. and in $L^1(\mathbb{P})$, for all $x \in \mathbb{Z}^d$:*

$$\lim_{n\to\infty} \frac{1}{n} a(0, nx, F, \omega) = \lim_{n\to\infty} \frac{1}{n} \mathbb{E}\big(a(0, nx, F, \omega)\big) = \inf_{n\in\mathbb{N}} \frac{1}{n} \mathbb{E}\big(a(0, nx, F, \omega)\big) = \alpha_F(x). \quad (1.5)$$

The norm α_F is called the quenched Lyapunov exponent. Moreover, α_F is monotone with respect to the law of the potential: that is, if F_1, F_2 are distribution functions with finite means such that $F_1 \geq F_2$ (that is $F_1(t) \geq F_2(t)$ for all $t \in \mathbb{R}$), then $\alpha_{F_1}(x) \leq \alpha_{F_2}(x)$ for all $x \in \mathbb{Z}^d$. Moreover, the norm $\alpha_F(\cdot)$ satisfies:

$$-\ln \int e^{-t}\, dF(t) \leq \frac{\alpha_F(x)}{|x|} \leq \ln(2d) + \int t\, dF(t). \quad (1.6)$$

Theorem B (Flury [5]). *Let V be a potential such that $\mathbb{P}(V = \infty) < 1$. Let F be its distribution function. Then there is a nonrandom norm $\beta_F(\cdot)$ on $\mathbb{R}^d$, such that for all $x \in \mathbb{Z}^d$:*

$$\lim_{n\to\infty} \frac{1}{n} b(0, nx, F) = \inf_{n\in\mathbb{N}} \frac{1}{n} b(0, nx, F) = \beta_F(x). \quad (1.7)$$

The norm β_F is called the annealed Lyapunov exponent. β_F is monotone with respect to the law of the potential: that is, if F_1, F_2 are distribution functions such that $F_1 \geq F_2$, then $\beta_{F_1}(x) \leq \beta_{F_2}(x)$ for all $x \in \mathbb{Z}^d$. The norm β_F inherits from $b(0, x)$ the following upper and lower bounds:

$$-\ln \int e^{-t}\, dF(t) \leq \frac{b(0, x, F)}{|x|} \leq \left(\ln 2d - \ln \int e^{-t}\, dF(t)\right). \quad (1.8)$$

By Jensen's equality, $\beta_F \leq \alpha_F$. Moreover, it was shown by Zygouras [20] that for $d \geq 3$, for every $\lambda > 0$ there is $\gamma^*(\lambda) > 0$ such that for all $\gamma \in]0, \gamma^*(\lambda)[: \alpha_F(\cdot) \equiv \beta_F(\cdot)$ (where F is the distribution function of the potential $\lambda + \gamma V$).

Theorems A and B are analogous to the existence of the time constant in first passage percolation. The analogy between first passage percolation and random walk in random potential was first described by Zerner [19]. Moreover, he also proved an analogue of the shape theorem of Cox and Durrett [3]. Recently, Sodin [15] proved two theorems on concentration inequalities for random walk in random potential which are counterparts of Talagrand [18] and Benjamini–Kalai–Schramm [1]. Although first passage percolation and random walk in random potential have many similarities, the techniques used in proofs are often very different. In particular, the use of geodesics is specific to first passage percolation. In this respect, conditions (ii) and (iii) in the abstract theorem below can be simplified when dealing only with first passage percolation.

Many aspects of the Lyapunov exponents of random walk and of Brownian motion in random potential have already been considered. The groundbreaking work appeared in Sznitman's book [17]. Kosygina–Mountford–Zerner [9] considered the behavior of the quenched and the annealed Lyapunov exponents as the potential tends to zero. They showed that both exponents asymptotically behave in the same way. Mourrat [10] gave optimal conditions for the existence of Lyapunov exponents, and for appropriate versions of the shape theorem.

In this paper, we study the continuity of Lyapunov exponents for random walk with respect to the law of the potential assuming independence. The continuity property of the Lyapunov exponents for Brownian motion in a stationary potential was investigated by Rueß [12]. He also provided some interesting counter examples. For first passage percolation, Cox and Kesten [2, 4,7] proved the continuity of the time constant with respect to the law of the passage time. Scholler [13] also studied this question for a random coloring model which is a dependent first passage percolation model. Random walk in a random potential also shares many similarities with random polymer although in this model the average is taken over paths of same length. For random walk in random polymer, when the potential is a function of an ergodic environment and steps of the walk, Lemma 3.1 of [11] showed the L^p continuity ($p > d$) of the quenched point-to-point free energy with respect to the law of the potential.

We now state our main results. We denote by $\mathcal{D}$ the set of distribution functions F which assign probability 1 to $[0, +\infty[$ and such that $F(0) < 1$. And $\mathcal{D}_1$ denotes the subset of $\mathcal{D}$ which contains all distribution functions of finite mean.

Theorem 1.1. *Let $d \geq 3$. Assume that (F_n) is a sequence of distribution functions in $\mathcal{D}_1$ such that there is a distribution function $G \in \mathcal{D}_1$, $G \leq F_n$ for all n. If $F_n \stackrel{w}{\to} F$, then $\lim_{n\to\infty} \alpha_{F_n}(x) = \alpha_F(x)$ for all $x \in \mathbb{R}^d$ and the convergence is uniform on any compact set of $\mathbb{R}^d$.*

Theorem 1.2. *Let $d \geq 3$. Assume that (F_n) is a sequence of distribution functions in $\mathcal{D}$ such that $F_n \stackrel{w}{\to} F$, $F \in \mathcal{D}$. Then $\lim_{n\to\infty} \beta_{F_n}(x) = \beta_F(x)$ for all $x \in \mathbb{R}^d$ and the convergence is uniform on any compact set of $\mathbb{R}^d$.*

Remark 1.1. The simple random walk is transient on $\mathbb{Z}^d$ when $d \geq 3$. This is an important property in our work (see (2.5)). In Propositions 2.6 and 3.4, we consider the cases $d = 1, 2$.

Now, we extract from the work of Cox [2] a set of sufficient conditions in order to formulate his theorem in a very general context. Its proof still holds with some minor modifications. This abstract theorem is given in Theorem 1.4 below. It combines the results of Proposition 4.4, Lemma 4.7 and the proof of Theorem 1.14 in [2]. In our applications of the abstract theorem to Lyapunov exponents, we always take $\mathcal{D}_2 = \mathcal{D}_1$ and the condition (iii) holds for all $0 < c < 1$. Condition (iv) will be trivially satisfied since $\mathcal{D}_2 = \mathcal{D}_1$. But this general formulation allows its use in first passage percolation. $F * G$ denotes the convolution of F and G. The following three truncated distribution functions will appear in condition (iv) of Theorem 1.4.

Definition 1.3. *For a distribution function F, for $\xi > 0$, $t_0 > 0$,*

$$F_\xi(t) := \begin{cases} 0, & \text{if } t < 0, \\ F(\xi), & \text{if } 0 \leq t < \xi, \\ F(t), & \text{if } \xi \leq t, \end{cases} \qquad F^{t_0}(t) := \begin{cases} 0, & \text{if } t < t_0, \\ F(t), & \text{if } t \geq t_0 \end{cases}$$

and for $0 < \xi < t_0$,

$$\hat{F}_{t_0}^{\xi}(t) := \begin{cases} 0, & \text{if } t < t_0 - \xi, \\ F(t_0 + \xi), & \text{if } t_0 - \xi \le t < t_0 + \xi, \\ F(t), & \text{if } t_0 + \xi \le t. \end{cases}$$

Theorem 1.4 (An abstract theorem). *Let* $\mu: \mathcal{D}_1 \longrightarrow \mathbb{R}^+$, $F \mapsto \mu(F)$ *be a map that satisfies*:

(i) $\mu(F) \le \mu(G)$ *for all* $F, G \in \mathcal{D}_1$ *such that* $F \ge G$, *and the following three conditions for some subsets* $\mathcal{D}_2$ *of* $\mathcal{D}_1$.

(ii) *For all* $F \in \mathcal{D}_2$, *there exist* $c_1(F) > 0$ *and* $f_1(F) > 0$ *such that*:
  (1) $\mu(F * G) \le \mu(F) + c_1(F) f_1(F) \int t\, dG(t)$ *for all* $G \in \mathcal{D}_1$,
  (2) $c_1(F) \le c_1(G)$ *for all* $F, G \in \mathcal{D}_2$ *such that* $F \ge G$,
  (3) $\lim_{n\to\infty} f_1(F_n) = f_1(F)$ *for all* $F_n, F \in \mathcal{D}_2$ *such that* $F_n \xrightarrow{w} F$.

(iii) *There exists a positive constant* c *such that for all* $F \in \mathcal{D}_2$ *and for all* $t_0 > 0$ *satisfying* $F(t_0) < c$, *there exist* $c_2(F) > 0$ *and* $f_2(t_0, F)$ *such that* :
  (1) $|\mu(F^{t_0}) - \mu(F)| \le c_2(F) f_2(t_0, F)$,
  (2) $c_2(F) \le c_2(G)$ *for all* $F, G \in \mathcal{D}_2$ *such that* $F \ge G$,
  (3) $\lim_{n\to\infty} f_2(t_0, F_n) = 0$ *for all* $F_n \in \mathcal{D}_2$ *such that* $\lim_{n\to\infty} F_n(t_0-) = 0$.

(iv) *If* $F \in \mathcal{D}_2$, $\hat{F}_{t_0}^{\xi}$, F^{t_0}, F_ξ *are in* $\mathcal{D}_2$ *for all* $t_0 > 0$ *and for* $\xi > 0$ *small enough.*

Then $\liminf_{n\to\infty} \mu(F_n) \ge \mu(F)$ *if* $F_n \xrightarrow{w} F$, $F_n, F \in \mathcal{D}_2$.

We now explain briefly the application of this theorem in first passage percolation. The result of [2] is the continuity of the time constant μ in F with respect to weak convergence in $\mathbb{Z}^2$. Suppose that $F_n \xrightarrow{w} F$, $F_n, F \in \mathcal{D}_1$. If $F(0) \ge p_c$ where p_c is the *critical probability* of percolation, obviously, $\liminf_{n\to\infty} \mu_{F_n} \ge 0 = \mu_F$. Let $\mathcal{D}_2 = \{F \in \mathcal{D}_1, F(0) < p_c\}$. The time constant μ verifies all the conditions of Theorem 1.4. Indeed, by Proposition 4.4 in [2], $\mu_{F*G} \le \mu_F + \mu_F \frac{1}{a(U)} \int t\, dG(t)$ for all $F, U \in \mathcal{D}_2$, $F \le U$, $G \in \mathcal{D}_1$. By Lemma 4.7 in [2], for all $F \in \mathcal{D}_2$ and $t_0 > 0$ such that $p := F(t_0) < \frac{1}{r}$, we have $|\mu_{F^{t_0}} - \mu_F| \le \mu_F \frac{\gamma_p^{-1}(1/r)}{1-\gamma_p^{-1}(1/r)}$ where $r > 2$ is the connectivity constant of $\mathbb{Z}^2$ and $\gamma_p(x) = (\frac{p}{x})^x (\frac{1-p}{1-x})^{1-x}$. Note that $\lim_{p\to 0} \gamma_p^{-1}(x) = 0$ for $0 < x < 1$. Thus, (ii) and (iii) of Theorem 1.4 hold with $c_1(F) = c_2(F) = \mu(F)$, $f_1(F) = \frac{1}{a(U)}$, $f_2(F) = \frac{\gamma_p^{-1}(1/r)}{1-\gamma_p^{-1}(1/r)}$ where $U \in \mathcal{D}_2$ such that $F_n, F \le U$ for all n large enough. By the right-continuity of F, (iv) is verified.

For our model, Mourrat [10] showed that $n^{-1} a(0, nx, \omega)$ converges in probability for all $x \in \mathbb{Z}^d$ if and only if $V(0) < \infty$ a.s. This defines Lyapunov exponents. However, our arguments are not suitable to prove that they are continuous with respect to the law of $V(0)$. Cox and Kesten [4,7] also considered first passage percolation on $\mathbb{Z}^d$, $d \ge 1$ with nonintegrable passage times. But to prove the continuity of the time constant, they used some techniques specific to this model and in particular the existence and properties of geodesics (e.g., Lemma 2 of [4], (5.9) of [7]...).

Consider now a Markov chain on the extended state space $\mathbb{Z}^d \cup \{\triangle\}$ where $\triangle$ is an absorbing state. At each step, the walk jumps to $\triangle$ from x with probability $1 - e^{-V(x)}$. Otherwise, it behaves as a simple symmetric random walk on $\mathbb{Z}^d$. The path measure of this random walk starting at x in a fixed potential $V(x,\omega)$ will be denoted by $\check{P}^F_{x,\omega}$. One can think of $e(x,y,F,\omega)$ as the probability that the random walk reaches y before being killed: $e(x,y,F,\omega) = \check{P}^F_{x,\omega}(H(y) < \infty)$. Let us now introduce as in [8], for $\omega \in \Omega$ and $x, y \in \mathbb{Z}^d$, the quenched path measure: $\hat{P}^{y,F}_{x,\omega}(\cdot) := \check{P}^F_{x,\omega}(\cdot \mid H(y) < \infty)$ and the annealed path measure $\hat{\mathbb{P}}^{y,F}_x(\cdot) := \check{P}^F_x(\cdot \mid H(y) < \infty)$ where $\check{P}^F_x(\cdot) = \mathbb{E}\check{P}_{x,\omega}(\cdot)$. The expectations with respect to $\hat{P}^{y,F}_{x,\omega}$ and $\hat{\mathbb{P}}^{y,F}_x$ are denoted by $\hat{E}^{y,F}_{x,\omega}$ and $\hat{\mathbb{E}}^{y,F}_x$, respectively. This means that, for all $x, y \in \mathbb{Z}^d$, $\omega \in \Omega$, and random variable X,

$$\hat{E}^{y,F}_{x,\omega}(X) = \frac{E_x[X \exp(-\sum_{m=0}^{H(y)-1} V(S_m,\omega)), H(y) < \infty]}{e(x,y,\omega)}, \tag{1.9}$$

$$\hat{\mathbb{E}}^{y,F}_x(X) = \frac{\mathbb{E}E_x[X \exp(-\sum_{m=0}^{H(y)-1} V(S_m,\omega)), H(y) < \infty]}{\mathbb{E}e(x,y,\omega)}. \tag{1.10}$$

We prove Theorem 1.1 in Section 2. The proof is divided in two parts. The first step is to prove that $\limsup \alpha_{F_n}(x) \le \alpha_F(x)$ and the second step is the proof of $\liminf \alpha_{F_n}(x) \ge \alpha_F(x)$. The latter is the most involved. To do so, we will show that the conditions of Theorem 1.4 are actually verified for our model. This is done successively in Proposition 2.1 and Corollary 2.3 by using some properties of the quenched path measure.

Theorem 1.2 which is the continuity of the annealed exponents is shown in Section 3. No condition of finite mean is required here. As in [4], the proof of Theorem 1.2 is done in two steps. First, in Proposition 3.2, we show the continuity of β_F under the hypothesis of Theorem 1.1 and its proof is similar to the quenched case. Next, to eliminate the condition of finite mean, with $t_0 > 0$ arbitrary, in Theorem 3.3 of this paper, we prove that $\beta_{{}^{t_0}F} \to \beta_F$ when $t_0 \to \infty$, where ${}^{t_0}F$ is the distribution function obtained by truncating below at t_0.

2. Quenched exponents: Proof of Theorem 1.1

The following proposition is the main ingredient in the proof of the continuity of Lyapunov exponents. It verifies condition (ii) of Theorem 1.4. Properties of the quenched path measure are important tools here (see (2.2) for example). In particular, if we attach to each trajectory $(S_m)_{m\ge 0}$ which starts at 0, the lattice animal:

$$\mathcal{A}(0,y) = \mathcal{A}\big(0,y,(S_m)_{m\ge 0}\big) := \big\{z \in \mathbb{Z}^d : H(z) < H(y)\big\}, \tag{2.1}$$

by using the strong Markov property and the transience of the simple random walk on $\mathbb{Z}^d$, $d \ge 3$, we can bound $\hat{E}^{nx,F}_{0,\omega}(H(nx))$ by $D\hat{E}^{nx,F}_{0,\omega}|\mathcal{A}(0,nx)|$ where D is a constant that depends only on d (see (2.6)). In the continuous setting of a Brownian motion in a Poissonian potential, the similar question is also considered by [16]. Lemma 3 of [19] will be used to estimate the factor $\mathbb{E}\hat{E}^{nx,F}_{0,\omega}(H(nx))$.

Proposition 2.1. *Let $d \geq 3$. For any distribution functions $F \in \mathcal{D}_1$, there exist $c_1(F) > 0$ and $f_1(F) > 0$ such that*:

(1) $\alpha_{F*G}(x) \leq \alpha_F(x) + c_1(F) f_1(F) \int t\, dG(t)|x|$ *for all* $F, G \in \mathcal{D}_1$ *and* $x \in \mathbb{Z}^d$,
(2) $c_1(F) \leq c_1(G)$ *for all* $F, G \in \mathcal{D}_1$ *such that* $F \geq G$,
(3) $\lim_{n\to\infty} f_1(F_n) = f_1(F)$ *for* $F_n \in \mathcal{D}_1$, $F_n \xrightarrow{w} F$.

Proof. Let $V(x)$, $x \in \mathbb{Z}^d$ be i.i.d. random potentials with distribution F; $W(x)$, $x \in \mathbb{Z}^d$ be i.i.d. random potentials with distribution G such that the two sequences are independent of each other. Then, $(V + W)(x)$, $x \in \mathbb{Z}^d$ are i.i.d. random potentials with distribution $F * G$. By Jensen's inequality, for $x \in \mathbb{Z}^d \setminus \{0\}$, $n \geq 1$, $\omega \in \Omega$:

$$\begin{aligned} a(0, nx, F*G, \omega) &= -\ln E_0\left(\exp\left(-\sum_{m=0}^{H(nx)-1} V(S_m) - \sum_{m=0}^{H(nx)-1} W(S_m)\right), H(nx) < \infty\right) \\ &= -\ln \hat{E}_{0,\omega}^{nx,F}\left(\exp\left(-\sum_{m=0}^{H(nx)-1} W(S_m)\right)\right) + a(0, nx, F, \omega) \\ &\leq \hat{E}_{0,\omega}^{nx,F}\left(\sum_{m=0}^{H(nx)-1} W(S_m)\right) + a(0, nx, F, \omega). \end{aligned} \tag{2.2}$$

We now use Fubini's theorem and the independence of $(W(x))$ and $(V(x))$:

$$\begin{aligned} &\mathbb{E}\hat{E}_{0,\omega}^{nx,F}\left(\sum_{m=0}^{H(nx)-1} W(S_m)\right) \\ &\quad= \mathbb{E}\left(E_0\left(\frac{\sum_{m=0}^{H(nx)-1} W(S_m)\exp(-\sum_{m=0}^{H(nx)-1} V(S_m)), H(nx) < \infty}{e(0, nx, F, \omega)}\right)\right) \\ &\quad= E_0\left(\mathbb{E}\left(\sum_{m=0}^{H(nx)-1} W(S_m)\right)\mathbb{E}\left(\frac{\exp(-\sum_{m=0}^{H(nx)-1} V(S_m)), H(nx) < \infty}{e(0, nx, F, \omega)}\right)\right) \\ &\quad= E_0\left(H(nx)\mathbb{E}\big(W(0)\big)\mathbb{E}\left(\frac{\exp(-\sum_{m=0}^{H(nx)-1} V(S_m)), H(nx) < \infty}{e(0, nx, F, \omega)}\right)\right) \\ &\quad= \int t\, dG(t) \cdot \mathbb{E}\hat{E}_{0,\omega}^{nx,F}\big(H(nx)\big). \end{aligned} \tag{2.3}$$

By using the strong Markov property:

$$\begin{aligned} &\hat{E}_{0,\omega}^{nx,F}\big(H(nx)\big) \\ &\quad= \sum_{z' \in \mathbb{Z}^d} \hat{E}_{0,\omega}^{nx,F}\left(\sum_{m=0}^{H(nx)-1} \mathbf{1}_{\{S_m = z'\}}\right) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{z'\in\mathbb{Z}^d} \frac{1}{e(0,nx,F,\omega)} E_0\left(\sum_{m=0}^{H(nx)-1} \mathbf{1}_{\{S_m=z'\}} \exp\left(-\sum_{m=0}^{H(nx)-1} V(S_m)\right), H(nx)<\infty\right)\\
&= \sum_{z'\in\mathbb{Z}^d} \left[\frac{1}{e(0,nx,F,\omega)} E_0\left(H(z')<H(nx), \exp\left(-\sum_{m=0}^{H(z')-1} V(S_m)\right)\right)\right.\\
&\quad \left.\times E_{z'}\left(\sum_{m=0}^{H(nx)-1} \mathbf{1}_{\{S_m=z'\}} \exp\left(-\sum_{m=0}^{H(nx)-1} V(S_m)\right), H(nx)<\infty\right)\right]\\
&= \sum_{z'\in\mathbb{Z}^d} \left[\frac{1}{e(0,nx,F,\omega)} E_0\left(H(z')<H(nx), \exp\left(-\sum_{m=0}^{H(z')-1} V(S_m)\right)\right) e(z',nx,F,\omega)\right.\\
&\quad \left.\times \frac{1}{e(z',nx,F,\omega)} E_{z'}\left(\sum_{m=0}^{H(nx)-1} \mathbf{1}_{\{S_m=z'\}} \exp\left(-\sum_{m=0}^{H(nx)-1} V(S_m)\right), H(nx)<\infty\right)\right]\\
&= \sum_{z'\in\mathbb{Z}^d} \left[\frac{1}{e(0,nx,F,\omega)} E_0\left(\exp\left(-\sum_{m=0}^{H(nx)-1} V(S_m)\right), H(z')<H(nx)<\infty\right)\right.\\
&\quad \left.\times \hat{E}^{nx,F}_{z',\omega}\left(\sum_{m=0}^{H(nx)-1} \mathbf{1}_{\{S_m=z'\}}\right)\right]\\
&= \sum_{z'\in\mathbb{Z}^d} \hat{P}^{nx,F}_{0,\omega}\big(H(z')<H(nx)\big) \hat{E}^{nx,F}_{z',\omega}\left(\sum_{m=0}^{H(nx)-1} \mathbf{1}_{\{S_m=z'\}}\right).
\end{aligned} \tag{2.4}
$$

By applying the Markov property:

$$
\begin{aligned}
&\hat{E}^{nx,F}_{z',\omega}\left(\sum_{m=0}^{H(nx)-1} \mathbf{1}_{\{S_m=z'\}}\right)\\
&\quad = \frac{1}{e(z',nx,F,\omega)} \sum_{k=0}^{+\infty} E_{z'}\left(\mathbf{1}_{\{S_k=z'\}} \exp\left(-\sum_{m=0}^{H(nx)-1} V(S_m)\right), k<H(nx)<\infty\right)\\
&\quad = \frac{1}{e(z',nx,F,\omega)} \sum_{k=0}^{+\infty} E_{z'}\left(\mathbf{1}_{\{S_k=z'\}} \exp\left(-\sum_{m=0}^{k-1} V(S_m)\right), k<H(nx)\right) \check{P}^{F}_{z',\omega}\big(H(nx)<\infty\big)\\
&\quad \le \sum_{k=0}^{+\infty} E_{z'}(\mathbf{1}_{\{S_k=z'\}}) := D(d) < \infty,
\end{aligned} \tag{2.5}
$$

since the simple random walk is transient on $\mathbb{Z}^d$, $d \geq 3$. From (2.1), (2.4) and (2.5):

$$\hat{E}_{0,\omega}^{nx,F}\big(H(nx)\big) \leq D\hat{E}_{0,\omega}^{nx,F}\big(\big|\mathcal{A}(0,nx)\big|\big). \tag{2.6}$$

Thanks to Lemma 3 in Zerner [19], we have:

$$\mathbb{E}\hat{E}_{0,\omega}^{nx,F}\big(\big|\mathcal{A}(0,nx)\big|\big) \leq \frac{(\ln 2d + \int t\,dF(t))}{-\ln(\int e^{-t}\,dF(t))}|nx|. \tag{2.7}$$

Substitute (2.3) in (2.2) and take the expectation:

$$\begin{aligned}\frac{\mathbb{E}(a(0,nx,F*G,\omega))}{n} &\leq \frac{\mathbb{E}(a(0,nx,F,\omega))}{n} + \int t\,dG(t)\cdot\frac{\mathbb{E}\hat{E}_{0,\omega}^{nx,F}(H(nx))}{n}\\ &\leq \frac{\mathbb{E}(a(0,nx,F,\omega))}{n} + \int t\,dG(t)\cdot D\frac{(\ln 2d + \int t\,dF(t))}{-\ln(\int e^{-t}\,dF(t))}|x|.\end{aligned} \tag{2.8}$$

Remark that the last inequality is from (2.6) and (2.7). Therefore, Proposition 2.1 holds with $c_1(F) = D(\ln 2d + \int t\,dF(t))$ and $f_1(F) = \frac{1}{-\ln(\int e^{-t}\,dF(t))}$. □

We will use the proposition below whose proof is analogous to that of Proposition 2.1 to check condition (iii) of Theorem 1.4.

Proposition 2.2. *Let $d \geq 3$. Let $V(x)$, $x \in \mathbb{Z}^d$ be i.i.d. random variables which have distribution function $F \in \mathcal{D}_1$. For all $t_0 > 0$ such that $p := \mathbb{P}(V(0) < t_0) < 1$, for all $y \in \mathbb{Z}^d$, $|y| > 1$, we have*:

$$\begin{aligned}&\mathbb{E}\hat{E}_{0,\omega}^{y,F}\left(\sum_{m=0}^{H(y)-1}\mathbf{1}_{\{V(S_m)<t_0\}}\right)\\ &\quad\leq D(d)\left(\ln 2d + \int t\,dF(t)\right)\cdot\frac{1}{\ln(1-(1-p)e^{-t_0})/p}\cdot|y|,\end{aligned} \tag{2.9}$$

where $D(d)$ is given in (2.5).

Proof. As in (2.4) and (2.5), we have:

$$\begin{aligned}\hat{E}_{0,\omega}^{y,F}\left(\sum_{m=0}^{H(y)-1}\mathbf{1}_{\{V(S_m)<t_0\}}\right) &= \sum_{z':V(z',\omega)<t_0}\hat{P}_{0,\omega}^{y,F}\big(H(z')<H(y)\big)\hat{E}_{z',\omega}^{y,F}\left(\sum_{m=0}^{H(y)-1}\mathbf{1}_{\{S_m=z'\}}\right)\\ &\leq D\hat{E}_{0,\omega}^{y,F}\left(\sum_{z\in\mathcal{A}(0,y)}\mathbf{1}_{\{V(z)<t_0\}}\right).\end{aligned} \tag{2.10}$$

Take $c = c(t_0, F) := \ln \frac{1-(1-p)e^{-t_0}}{p}$. Note that $c > 0$. We use Jensen's inequality and independence as follows:

$$\begin{aligned}
&c\mathbb{E}\hat{E}_{0,\omega}^{y,F}\Bigg(\sum_{z\in\mathcal{A}(0,y)} \mathbf{1}_{\{V(z)<t_0\}}\Bigg)\\
&\quad\le \mathbb{E}\Bigg(\ln \hat{E}_{0,\omega}^{y,F}\Bigg(\exp\Bigg(c\sum_{z\in\mathcal{A}(0,y)} \mathbf{1}_{\{V(z)<t_0\}}\Bigg)\Bigg)\Bigg)\\
&\quad\le \mathbb{E}\Bigg(a(0,y,F,\omega) + \ln E_0\Bigg(\exp\Bigg(c\sum_{z\in\mathcal{A}(0,y)} \mathbf{1}_{\{V(z)<t_0\}} - \sum_{z\in\mathcal{A}(0,y)} V(z)\Bigg), H(y)<\infty\Bigg)\Bigg)\\
&\quad\le \mathbb{E}\big(a(0,y,F,\omega)\big) + \ln E_0\Bigg(\prod_{z\in\mathcal{A}(0,y)} \mathbb{E}\big(\exp\big(c\mathbf{1}_{\{V(z)<t_0\}} - V(z)\big)\big)\Bigg).
\end{aligned} \tag{2.11}$$

We remark that, for all $z \in \mathbb{Z}^d$:

$$\begin{aligned}
\mathbb{E}\big(\exp\big(c\mathbf{1}_{\{V(z)<t_0\}} - V(z)\big)\big) &\le \mathbb{E}\big(\exp(c\mathbf{1}_{\{V(z)<t_0\}} - t_0\mathbf{1}_{\{V(z)\ge t_0\}})\big)\\
&= \exp(-t_0)\mathbb{E}\big(\exp\big((c+t_0)\mathbf{1}_{\{V(z)<t_0\}}\big)\big) = 1.
\end{aligned} \tag{2.12}$$

From (2.11) and (2.12), $c\mathbb{E}\hat{E}_{0,\omega}^{y,F}(\sum_{z\in\mathcal{A}(0,y)} \mathbf{1}_{\{V(z)<t_0\}}) \le \mathbb{E}(a(0,y,F,\omega))$. Combining this with (2.10) and the fact that $\mathbb{E}(a(0,y,F,\omega)) \le (\ln 2d + \int_0^{+\infty} t\,dF(t))|y|$ (see Lemma 3 of [19]), we obtain:

$$\begin{aligned}
\mathbb{E}\hat{E}_{0,\omega}^{y,F}\Bigg(\sum_{m=0}^{H(y)-1} \mathbf{1}_{\{V(S_m)<t_0\}}\Bigg) &\le D\mathbb{E}\hat{E}_{0,\omega}^{y,F}\Bigg(\sum_{z\in\mathcal{A}(0,y)} \mathbf{1}_{\{V(z)<t_0\}}\Bigg)\\
&\le \frac{D}{c}\mathbb{E}\big(a(0,y,F,\omega)\big)\\
&\le \frac{D}{c}\Bigg(\ln 2d + \int_0^{+\infty} t\,dF(t)\Bigg)|y|. \qquad \square
\end{aligned}$$

We are now ready to verify the condition (iii) of Theorem 1.4. This is done in the corollary below.

Corollary 2.3. *Let $d \ge 3$. For all $F \in \mathcal{D}_1$ and $t_0 > 0$ such that $F(t_0-) < 1$, there exist $c_2(F) > 0$ and $f_2(t_0, F) > 0$ such that, for $x \in \mathbb{Z}^d$:*

1. *$|\alpha_{F^{t_0}}(x) - \alpha_F(x)| \le c_2(F) f_2(t_0, F)|x|$, where F^{t_0} is given in the Definition 1.3,*
2. *$c_2(F) \le c_2(G)$ for all distributions $F, G \in \mathcal{D}_1$ such that $F \ge G$,*
3. *$\lim_{n\to\infty} f_2(t_0, F_n) = 0$ for $F_n \in \mathcal{D}_1$ such that $\lim_{n\to\infty} F_n(t_0-) = 0$.*

Proof. Let $\{V(x)\}_{x\in\mathbb{Z}^d}$ be i.i.d. random variables with distribution function F. Define:

$$W(x) = V(x)\mathbf{1}_{\{V(x)\geq t_0\}} + t_0\mathbf{1}_{\{V(x)<t_0\}}.$$

Then $\{W(x)\}_{x\in\mathbb{Z}^d}$ are i.i.d. random variables with distribution function F^{t_0}.

For $x \in \mathbb{Z}^d$ and $\omega \in \Omega$, we have:

$$\begin{aligned}
&a\big(0, nx, F^{t_0}, \omega\big)\\
&\quad = -\ln\left[\frac{E_0(\exp(-\sum_{m=0}^{H(nx)-1}[V(S_m)\mathbf{1}_{\{V(S_m)\geq t_0\}} + t_0\mathbf{1}_{\{V(S_m)<t_0\}}]), H(nx)<\infty)}{e(0, nx, F, \omega)}\right]\\
&\qquad + a(0, nx, F, \omega)\\
&\quad \leq -\ln \hat{E}^{nx,F}_{0,\omega}\left(\exp\left(-\sum_{m=0}^{H(nx)-1} t_0\mathbf{1}_{\{V(S_m)<t_0\}}\right)\right) + a(0, nx, F, \omega)\\
&\quad \leq t_0\hat{E}^{nx,F}_{0,\omega}\left(\sum_{m=0}^{H(nx)-1}\mathbf{1}_{\{V(S_m)<t_0\}}\right) + a(0, nx, F, \omega).
\end{aligned}$$

Taking expectations, we obtain:

$$\begin{aligned}
\frac{\mathbb{E}a(0, nx, F^{t_0}, \omega)}{n} &\leq t_0\frac{\mathbb{E}\hat{E}^{nx,F}_{0,\omega}(\sum_{m=0}^{H(nx)-1}\mathbf{1}_{\{V(S_m)<t_0\}})}{n} + \frac{\mathbb{E}a(0, nx, F, \omega)}{n},\\
\alpha_{F^{t_0}}(x) &\leq t_0|x|D\left(\ln 2d + \int t\,dF(t)\right)\frac{1}{\ln(1-(1-p)e^{-t_0})/p} + \alpha_F(x).
\end{aligned} \tag{2.13}$$

The last inequality above follows from Proposition 2.2. Recall here $p = F(t_0-)$. Since $F^{t_0} \leq F$, by the monotonicity of Lyapunov exponents, we have: $\alpha_{F^{t_0}} \geq \alpha_F$. Combining this with (2.13), we obtain that Corollary 2.3 holds with $c_2(F) = D(\ln 2d + \int t\,dF(t))$ and $f_2(t_0, F) = \frac{t_0}{\ln(1-(1-p)e^{-t_0})/p}$. □

The inverse of a distribution function G is defined in the usual way, $G^{-1}(t) = \inf\{u \in \mathbb{R} : G(u) > t\}$, $t \in \mathbb{R}$. We will need the following Lemma 2.1 from [2].

Lemma 2.4. *If $(F_n) \in \mathcal{D}$ such that $F_n \leq F$ and $F_n \xrightarrow{w} F$, then $F_n^{-1} \to F^{-1}$ point-wise on $[0, 1)$.*

The proof of the following proposition is analogous to the proof of Theorem 1.13 in [2].

Proposition 2.5. *Let $d \geq 1$. Let $(F_n) \in \mathcal{D}$ such that $F_n \geq G$ for all n and for some $G \in \mathcal{D}_1$. If $F_n \xrightarrow{w} F$, then $\limsup_{n\to\infty} \alpha_{F_n}(x) \leq \alpha_F(x)$ for all $x \in \mathbb{Z}^d$.*

Proof. Because of the monotonicity property of Lyapunov exponents, when dealing with $F_n \xrightarrow{w} F$, it suffices to consider only two cases: $F_n \leq F$ for all n and $F_n \geq F$ for all n. To see this, define $\underline{F}_n(t) = \min\{F_n(t), F(t)\}$ and $\overline{F}_n(t) = \max\{F_n(t), F(t)\}$, so that $\underline{F}_n(t) \le F_n(t) \le \overline{F}_n(t)$. Then $\alpha_{\overline{F}_n} \le \alpha_{F_n} \le \alpha_{\underline{F}_n}$, $\alpha_{\overline{F}_n} \le \alpha_F \le \alpha_{\underline{F}_n}$ and both $\underline{F}_n, \overline{F}_n \xrightarrow{w} F$ whenever $F_n \xrightarrow{w} F$. If $F_n \ge F$, $\alpha_{F_n}(x) \le \alpha_F(x)$ for all n, hence $\limsup \alpha_{F_n}(x) \le \alpha_F(x)$. For the rest of the proof, we shall assume that $F_n \le F$. Let $\xi(x)$, $x \in \mathbb{Z}^d$ be an i.i.d. family of uniform random variables on $(0,1)$. Let $V(x) := F^{-1}(\xi(x))$, $V_n(x) := F_n^{-1}(\xi(x))$ and $W(x) = G^{-1}(\xi(x))$. Then $V(x)$, $V_n(x)$ and $W(x)$ are independent families of i.i.d. random variables with distribution function F, F_n and G, respectively. Furthermore, for each $x \in \mathbb{Z}^d$:

$$V(x) \le V_n(x) \le W(x) \qquad \text{a.s.}$$

and by Lemma 2.4:

$$\lim_{n\to\infty} V_n(x) = V(x) \qquad \text{a.s.}$$

By definition (1.3), we can see easily that $a(0, kx, F_n, \omega) \to a(0, kx, F, \omega)$ when $n \to \infty$ and $a(0, kx, F_n, \omega) \le a(0, kx, G, \omega)$ for all n. Moreover,

$$\mathbb{E}\big(a(0, kx, G, \omega)\big) < k|x|\left(\ln 2d + \int t\, dG(t)\right) < \infty.$$

Apply the dominated convergence theorem:

$$\mathbb{E}a(0, kx, F_n, \omega) \overset{n\to\infty}{\longrightarrow} \mathbb{E}a(0, kx, F, \omega). \tag{2.14}$$

Now fix $\varepsilon > 0$. From (1.5), we can choose K_ε large enough such that:

$$0 \le \frac{\mathbb{E}a(0, K_\varepsilon x, F, \omega)}{K_\varepsilon} - \alpha_F(x) < \varepsilon.$$

From (2.14), choose N_ε such that for all $n \ge N_\varepsilon$:

$$0 \le \frac{\mathbb{E}a(0, K_\varepsilon x, F_n, \omega)}{K_\varepsilon} - \frac{\mathbb{E}a(0, K_\varepsilon x, F, \omega)}{K_\varepsilon} < \varepsilon.$$

We have hence for all $n \ge N_\varepsilon$:

$$\begin{aligned}
0 &\le \alpha_{F_n}(x) - \alpha_F(x) \qquad (\text{since } F_n \le F)\\
&\le \frac{\mathbb{E}a(0, K_\varepsilon x, F_n, \omega)}{K_\varepsilon} - \alpha_F(x) \qquad \left(\text{since } \alpha_{F_n}(x) = \inf_{k\ge 1} \frac{\mathbb{E}a(0, kx, F_n, \omega)}{k}\right)\\
&\le \frac{\mathbb{E}a(0, K_\varepsilon x, F_n, \omega)}{K_\varepsilon} - \frac{\mathbb{E}a(0, K_\varepsilon x, F, \omega)}{K_\varepsilon} + \frac{\mathbb{E}a(0, K_\varepsilon x, F, \omega)}{K_\varepsilon} - \alpha_F(x)\\
&\le \varepsilon + \varepsilon.
\end{aligned}$$

Now let $\varepsilon \to 0$, we have the result. □

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. In our setting, we take $\mathcal{D}_2 = \mathcal{D}_1$. Then, by the monotonicity property of Lyapunov exponents, Proposition 2.1 and Corollary 2.3, we see that all conditions of Theorem 1.4 are satisfied, we have then $\liminf_{n\to\infty} \alpha_{F_n}(x) \geq \alpha_F(x)$ for all $x \in \mathbb{Z}^d$. Combine this with $\limsup_{n\to\infty} \alpha_{F_n}(x) \leq \alpha_F(x)$, $x \in \mathbb{Z}^d$ given by Proposition 2.5 to obtain for all $x \in \mathbb{Z}^d$,

$$\lim_{n\to\infty} \alpha_{F_n}(x) = \alpha_F(x) \qquad \text{if } F_n \xrightarrow{w} F. \tag{2.15}$$

Moreover, it is easy to see that (2.15) holds for all $x \in \mathbb{Q}^d$. Using (1.6) of Theorem A and the fact that α is a norm on $\mathbb{R}^d$, we can extend (2.15) to $x \in \mathbb{R}^d$. Now, we will show that $\lim_{n\to\infty} \alpha_{F_n}(x) = \alpha_F(x)$ uniformly on every compact set of $\mathbb{R}^d$ if $F_n \xrightarrow{w} F$. By contradiction, assume there exists some $\varepsilon > 0$, $R > 0$, $y \in B(0, R)$ and two sequences $n_k \to \infty$ and $x_{n_k} \to y$ such that $|\alpha_{F_{n_k}}(x_k) - \alpha_F(x_k)| > \varepsilon$. Using (1.6) of Theorem A, for k large enough,

$$\begin{aligned}
\varepsilon &< \left|\alpha_{F_{n_k}}(x_k) - \alpha_F(x_k)\right| < \left|\alpha_{F_{n_k}}(x_k) - \alpha_{F_{n_k}}(y)\right| + \left|\alpha_{F_{n_k}}(y) - \alpha_F(y)\right| + \left|\alpha_F(y) - \alpha_F(x_k)\right| \\
&\leq |x_k - y|\left(\ln 2d + \int_0^\infty t\,dG(t)\right) + \left|\alpha_{F_{n_k}}(y) - \alpha_F(y)\right| + |x_k - y|\left(\ln 2d + \int_0^\infty t\,dF(t)\right) \\
&\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3},
\end{aligned}$$

where we recall that G is the distribution function that appears in the hypothesis of Theorem 1.1, $G \in \mathcal{D}_1$ and $F_n \geq G$ for all n. This is a contradiction. $\square$

Proposition 2.6. (i) *Theorem* 1.1 *also holds when* $d = 1$.

(ii) *Let* $d = 2$. *Let* $\lambda > 0$, $\mathcal{D}_\lambda = \{F \in \mathcal{D}_1, F(\lambda) = 0\}$. *Then, Theorem* 1.1 *also holds if we replace* $\mathcal{D}_1$ *by* $\mathcal{D}_\lambda$.

Proof. (i) If $d = 1$, the continuity of the Lyapunov exponents of random walk on the line $\mathbb{Z}$ in a random potential readily follows from Zerner's work. Indeed, thanks to Proposition 10 in [19] and the proof of Proposition 2.5 (see (2.14)), we have for all $m \in \mathbb{N}$:

$$\alpha_F(m) = \mathbb{E}\big(a(0, m, F, \omega)\big) = \lim_{n\to\infty} \mathbb{E}\big(a(0, m, F_n, \omega)\big) = \lim_{n\to\infty} \alpha_{F_n}(m).$$

(ii) In this case, we have $V_n(x), V(x) \geq \lambda$ for all $n > 0$, $x \in \mathbb{Z}^d$ where V_n, V are the potentials with distribution functions F_n and F, respectively. As in (2.5), we also have,

$$\begin{aligned}
\hat{E}^{y,F}_{z,\omega}\left(\sum_{m=0}^{H(y)-1} \mathbf{1}_{\{S_m = z\}}\right) &= \sum_{k=0}^{+\infty} E_z\left(\mathbf{1}_{\{S_k = z\}} \exp\left(-\sum_{m=0}^{k-1} V(S_m)\right), k < H(nx)\right) \\
&\leq \sum_{k=0}^{+\infty} \exp(-k\lambda) = \frac{1}{1 - e^{-\lambda}} := D(\lambda) < \infty.
\end{aligned} \tag{2.16}$$

Then, we can follow the same argument as in the case of $d \geq 3$ to obtain the continuity of the quenched Lyapunov exponents. $\square$

3. Annealed Lyapunov exponents: Proof of Theorem 1.2

We start this section by the following proposition about the ballisticity of the random walk under the conditional annealed path measure. Two different proofs that $\limsup_{|y|\to\infty} \frac{\hat{\mathbb{E}}_0^{y,F}(H(y))}{|y|}$ is finite can be found in [8] (see Theorem 1) and [6] (see Theorem C). Here when $d \geq 3$, we give a simple argument which provides an explicit expression for the upper bound that will be needed in the proof of Theorem 1.2.

Proposition 3.1. *Let $d \geq 3$. Let F be a distribution function whose potential satisfies $\mathbb{P}(V(0) = \infty) < 1$. Then there exists a constant $D(d)$ such that for all $y \in \mathbb{Z}^d$, $y \neq 0$:*

$$\hat{\mathbb{E}}_0^{y,F} H(y) \leq D(d) \frac{1}{-\ln \int \exp(-t)\, dF(t)} \left(\ln 2d - \ln \int \exp(-t)\, dF(t) \right) |y|. \tag{3.1}$$

Proof. As in (2.4), with D given in (2.5), we have:

$$\begin{aligned} E_0&\left(H(y) \exp\left(- \sum_{m=0}^{H(y)-1} V(S_m) \right), H(y) < \infty \right) \\ &= \sum_{z' \in \mathbb{Z}^d} E_0\left(H(z') < H(y), \exp\left(- \sum_{m=0}^{H(y)-1} V(S_m) \right), H(y) < \infty \right) \\ &\quad \times \hat{E}_{z',\omega}^{y,F} \left(\sum_{m=0}^{H(y)-1} \mathbf{1}_{\{S_m = z'\}} \right) \\ &\leq D \cdot E_0\left(\left|\mathcal{A}(0,y)\right| \exp\left(- \sum_{m=0}^{H(y)-1} V(S_m) \right), H(y) < \infty \right). \end{aligned} \tag{3.2}$$

Then,

$$\hat{\mathbb{E}}_0^{y,F}\big(H(y)\big) = \frac{\mathbb{E} E_0(H(y) \exp(-\sum_{m=0}^{H(y)-1} V(S_m)), H(y) < \infty)}{\mathbb{E} e(0, y, F, \omega)} \leq D \hat{\mathbb{E}}_0^{y,F}\big(\left|\mathcal{A}(0,y)\right|\big). \tag{3.3}$$

Take $d_1 := -\ln \int e^{-t}\, dF(t) = -\ln \mathbb{E}(e^{-V(0)})$. By Jensen's equality and independence of $V(x)$, $x \in \mathbb{Z}^d$:

$$\begin{aligned} d_1 &\hat{\mathbb{E}}_0^{y,F}\big(\left|\mathcal{A}(0,y)\right|\big) \\ &\leq \ln \hat{\mathbb{E}}_0^{y,F}\big(\exp\big(d_1 \left|\mathcal{A}(0,y)\right|\big)\big) \\ &\leq b(0,y,F) + \ln \mathbb{E} E_0\left(\exp\left(d_1 \left|\mathcal{A}(0,y)\right| - \sum_{s \in \mathcal{A}(0,y)} V(s) \right), H(y) < \infty \right) \\ &\leq b(0,y,F) + \ln E_0\left(\prod_{s \in \mathcal{A}(0,y)} \mathbb{E}\big(\exp\big(d_1 - V(s)\big)\big) \right) = b(0,y,F). \end{aligned} \tag{3.4}$$

From (1.8), (3.3) and (3.4), for all $y \in \mathbb{R}^d$, $y \neq 0$:

$$\hat{\mathbb{E}}_0^{y,F}\big(H(y)\big) \le \frac{D}{d_1} b(0,y,F) \le \frac{D}{-\ln \int \exp(-t)\,dF(t)} \left(\ln 2d - \ln \int \exp(-t)\,dF(t)\right)|y|. \quad (3.5)$$

□

Proposition 3.2. *Let $d \ge 3$. Assume that (F_n) is a sequence of distribution functions such that $F_n \in \mathcal{D}_1$, $F_n \xrightarrow{w} F$ and $F \in \mathcal{D}_1$. Then $\lim_{n\to\infty} \beta_{F_n}(x) = \beta_F(x)$ for all $x \in \mathbb{Z}^d$.*

Proof. With the same arguments as in Proposition 2.5, we can obtain that for all $x \in \mathbb{Z}^d$:

$$\text{If } (F_n) \in \mathcal{D} \text{ such that } F_n \xrightarrow{w} F, \qquad \text{then } \limsup_{n\to\infty} \beta_{F_n}(x) \le \beta_F(x). \quad (3.6)$$

Note that in this case, the condition of finite mean is not required because the sequence of real numbers $b(0,kx,F_n)$ always converges to $b(0,kx,F)$ for any k and x fixed when $n \to \infty$.

To show that $\liminf_{n\to\infty} \beta_{F_n}(x) \ge \beta_F(x)$, we verify the conditions of Theorem 1.4 with $\mathcal{D}_2 := \mathcal{D}_1$. Condition (i) is the monotonicity property of the annealed Lyapunov exponents referred to in Theorem B. Condition (ii) is verified as in Proposition 2.1 by using the inequality (3.1) of Proposition 3.1. Then we can choose here $c_1(F) = D(\ln 2d - \ln \int e^{-t}\,dF(t))$ and $f_1(F) = \frac{1}{-\ln(\int e^{-t}\,dF(t))}$. And the verification of condition (iii) is analogous to the proofs of Proposition 2.2 and Corollary 2.3. But here we relay on the fact that

$$\hat{\mathbb{E}}_0^{y,F}\left(\sum_{z\in\mathcal{A}(0,y)} \mathbf{1}_{\{V(z)<t_0\}}\right) \le t_0 \frac{b(0,y,F)}{\ln(1-(1-p)e^{-t_0})/p} \le t_0 \frac{(\ln 2d - \ln \int e^{-t}\,dF(t))}{\ln(1-(1-p)e^{-t_0})/p}|y|,$$

whose demonstration combines the idea of (2.11) and (3.4). Then, the constants are chosen as $c_2(F) = D(\ln 2d - \ln \int e^{-t}\,dF(t))$ and $f_2(t_0,F) = \frac{t_0}{\ln(1-(1-p)e^{-t_0})/p}$. □

We now eliminate the condition of finite mean in the annealed case. To do so, we first prove convergence of truncated potentials in Theorem 3.3. The proof is inspired from Theorem 7.12 of [14] and requires only $F \in \mathcal{D}$. Proposition 3.1 is useful for our argument.

Theorem 3.3. *Let $d \ge 3$, $F \in \mathcal{D}$. Then for all $x \in \mathbb{Z}^d$: $\lim_{t_0\to\infty} \beta_{{}^{t_0}F}(x) = \beta_F(x)$, where ${}^{t_0}F$ is defined by*:

$${}^{t_0}F(t) := \begin{cases} F(t), & \text{if } t < t_0, \\ 1, & \text{if } t \ge t_0. \end{cases} \quad (3.7)$$

Proof. Let $V_1(x)$ and $V_2(x)$, $x \in \mathbb{Z}^d$ be two families of i.i.d. random potentials with distribution F, independent of one another. Then, $W_{t_0}(x) := \min\{V_1(x); t_0\}$, $x \in \mathbb{Z}^d$ are i.i.d. random potentials with distribution function ${}^{t_0}F$. Define a distribution function ${}^{t_0}\hat{F}$ by:

$${}^{t_0}\hat{F} := \begin{cases} 0, & \text{if } t < 0, \\ F(t_0), & \text{if } 0 \le t \le t_0, \\ F(t), & \text{if } t > t_0. \end{cases} \quad (3.8)$$

First ${}^{t_0}F \geq F \geq {}^{t_0}F * {}^{t_0}\hat{F}$. Indeed, ${}^{t_0}F * {}^{t_0}\hat{F}(t) = \int_0^t {}^{t_0}F(t-y)\,d^{t_0}\hat{F}(y)$. If $t \leq t_0$, ${}^{t_0}F * {}^{t_0}\hat{F}(t) \leq \int_0^t F(t-y)\,d^{t_0}\hat{F}(y) \leq F(t)$. If $t \geq t_0$, ${}^{t_0}F * {}^{t_0}\hat{F}(t) \leq \int_0^t d^{t_0}\hat{F}(y) = F(t)$. Then for all $x \in \mathbb{Z}^d$:

$$\beta_{{}^{t_0}F}(x) \leq \beta_F(x) \leq \beta_{{}^{t_0}F * {}^{t_0}\hat{F}}(x). \tag{3.9}$$

Take $U_{t_0}(x) = V_2(x)\mathbf{1}_{\{V_2(x) > t_0\}}$. Hence, $U_{t_0}(x), x \in \mathbb{Z}^d$ is an i.i.d. family of random potentials with distribution function ${}^{t_0}\hat{F}$. $(W_{t_0} + U_{t_0})(x), x \in \mathbb{Z}^d$ are i.i.d. random potentials with distribution function ${}^{t_0}F * {}^{t_0}\hat{F}$. Moreover,

$$\begin{aligned} &b\big(0, nx, {}^{t_0}F * {}^{t_0}\hat{F}\big) \\ &\quad = -\ln \frac{\mathbb{E}E_0(\exp(-\sum_{m=0}^{H(nx)-1} W_{t_0}(S_m) - \sum_{m=0}^{H(nx)-1} U_{t_0}(S_m)), H(nx) < \infty)}{\mathbb{E}e(0, nx, \omega, {}^{t_0}F)} \\ &\qquad + b\big(0, nx, {}^{t_0}F\big). \end{aligned} \tag{3.10}$$

As in [5], we now define for $z \in \mathbb{Z}^d$, $n \in \mathbb{N}$ the number of visits to the site z by the random walk up to time n:

$$\ell_z(n) := \big|\{m \in \mathbb{N}_0 : m < n, S_m = z\}\big|.$$

Since the two sequences $(W_{t_0}(x))_{x \in \mathbb{Z}^d}$ and $(U_{t_0}(x))_{x \in \mathbb{Z}^d}$ are independent of each other, the first term on the right-hand side of (3.10) is equal to:

$$\begin{aligned} &-\ln \frac{E_0(\mathbb{E}(\exp(-\sum_{m=0}^{H(nx)-1} U_{t_0}(S_m)))\mathbb{E}(\exp(-\sum_{m=0}^{H(nx)-1} W_{t_0}(S_m))), H(nx) < \infty)}{\mathbb{E}e(0, nx, \omega, {}^{t_0}F)} \\ &\quad = -\ln \frac{E_0(\prod_{z \in \mathbb{Z}^d} \mathbb{E}(\exp(-\ell_z(H(nx))U_{t_0}(z)))\mathbb{E}(\exp(-\sum_{m=0}^{H(nx)-1} W_{t_0}(S_m))), H(nx) < \infty)}{\mathbb{E}e(0, nx, \omega, {}^{t_0}F)} \\ &\quad \leq -\ln \frac{E_0(\prod_{z \in \mathbb{Z}^d} (\mathbb{E}\exp(-U_{t_0}(z)))^{\ell_z(H(nx))}\mathbb{E}(\exp(-\sum_{m=0}^{H(nx)-1} W_{t_0}(S_m))), H(nx) < \infty)}{\mathbb{E}e(0, nx, \omega, {}^{t_0}F)} \\ &\quad = -\ln \hat{\mathbb{E}}_0^{nx, {}^{t_0}F}\big(\mathbb{E}\big(\exp\big(-U_{t_0}(0)\big)\big)^{H(nx)}\big) \leq -\ln \mathbb{E}\exp\big(-U_{t_0}(0)\big)\hat{\mathbb{E}}_0^{nx, {}^{t_0}F}\big(H(nx)\big). \end{aligned} \tag{3.11}$$

We remark that Jensen's inequality has been used for the first and last inequalities of (3.11). From (3.5) of Proposition 3.1:

$$\hat{\mathbb{E}}_0^{nx, {}^{t_0}F}\big(H(nx)\big) \leq \frac{D}{-\ln \mathbb{E}\exp(-W_{t_0}(0))} b\big(0, nx, {}^{t_0}F\big). \tag{3.12}$$

From (3.10), (3.11) and (3.12), for all $x \in \mathbb{Z}^d$:

$$\begin{aligned} \frac{b(0, nx, {}^{t_0}F * {}^{t_0}\hat{F})}{n} &\leq -\ln \mathbb{E}\exp\big(-U_{t_0}(0)\big) \cdot \frac{D}{-\ln \mathbb{E}\exp(-W_{t_0}(0))} \frac{b(0, nx, {}^{t_0}F)}{n} \\ &\quad + \frac{b(0, nx, {}^{t_0}F)}{n}, \end{aligned} \tag{3.13}$$

$$\beta_{t_0 F *^{t_0} \hat{F}}(x) \leq -\ln \mathbb{E} \exp\bigl(-U_{t_0}(0)\bigr) \cdot \frac{D}{-\ln \mathbb{E} \exp(-W_{t_0}(0))} \beta_{t_0 F}(x) + \beta_{t_0 F}(x).$$

Note that $\lim_{t_0 \to \infty} -\ln \mathbb{E} \exp(-U_{t_0}(0)) = 0$ and $\lim_{t_0 \to \infty} -\ln \mathbb{E} \exp(-W_{t_0}(0)) = -\ln \mathbb{E} \times \exp(-V_1(0)) = \text{const}$. From (3.9) and (3.13):

$$\begin{aligned}
&\limsup_{t_0 \to \infty} \beta_{t_0 F}(x) \leq \beta_F(x) \\
&\quad \leq \lim_{t_0 \to \infty} \biggl(-\ln \mathbb{E} \exp\bigl(-U_{t_0}(0)\bigr) \cdot \frac{D}{-\ln \mathbb{E} \exp(-W_{t_0}(0))}\biggr) \beta_F(x) + \liminf_{t_0 \to \infty} \beta_{t_0 F}(x) \qquad (3.14) \\
&\quad \leq \liminf_{t_0 \to \infty} \beta_{t_0 F}(x).
\end{aligned}$$

□

Proof of Theorem 1.2. Fix $t_0 > 0$. By Proposition 3.2, $\lim_{n\to\infty} \beta_{t_0 F_n} = \beta_{t_0 F}$. Furthermore, $\liminf_{n\to\infty} \beta_{F_n} \geq \liminf_{n\to\infty} \beta_{t_0 F_n} = \beta_{t_0 F}$ since ${}^{t_0}F_n \geq F_n$ for all n. Now let $t_0 \to \infty$ and apply Theorem 3.3, $\liminf_{n\to\infty} \beta_{F_n} \geq \lim_{t_0\to\infty} \beta_{t_0 F} = \beta_F$. Combine this with $\limsup_{n\to\infty} \beta_{F_n} \leq \beta_F$ given by (3.6) in Proposition 3.2 to obtain that $\lim_{n\to\infty} \beta_{F_n}(x) = \beta_F(x)$ for all $x \in \mathbb{Z}^d$ when $F_n \xrightarrow{w} F$, $F_n, F \in \mathcal{D}$. The convergence in $\mathbb{R}^d$ and the uniform convergence on any compact set of $\mathbb{R}^d$ follow from an argument as in Theorem 1.1 combined with (1.8). □

The following proposition and its proof are analogous to the quenched case of Proposition 2.6.

Proposition 3.4. (i) *Theorem 1.2 also holds when $d = 1$.*

(ii) *Let $d = 2$. Let $\lambda > 0$, $\mathcal{D}_\lambda = \{F \in \mathcal{D}_1, F(\lambda) = 0\}$. Then, Theorem 1.2 also holds if we replace $\mathcal{D}$ by $\mathcal{D}_\lambda$.*

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References

[1] Benjamini, I., Kalai, G. and Schramm, O. (2003). First passage percolation has sublinear distance variance. *Ann. Probab.* **31** 1970–1978. MR2016607

[2] Cox, J.T. (1980). The time constant of first-passage percolation on the square lattice. *Adv. in Appl. Probab.* **12** 864–879. MR0588407

[3] Cox, J.T. and Durrett, R. (1981). Some limit theorems for percolation processes with necessary and sufficient conditions. *Ann. Probab.* **9** 583–603. MR0624685

[4] Cox, J.T. and Kesten, H. (1981). On the continuity of the time constant of first-passage percolation. *J. Appl. Probab.* **18** 809–819. MR0633228

[5] Flury, M. (2007). Large deviations and phase transition for random walks in random nonnegative potentials. *Stochastic Process. Appl.* **117** 596–612. MR2320951
[6] Ioffe, D. and Velenik, Y. (2012). Self-attractive random walks: The case of critical drifts. *Comm. Math. Phys.* **313** 209–235. MR2928223
[7] Kesten, H. (1986). Aspects of first passage percolation. In *École D'été de Probabilités de Saint-Flour, XIV—1984. Lecture Notes in Math.* **1180** 125–264. Berlin: Springer. MR0876084
[8] Kosygina, E. and Mountford, T. (2012). Crossing velocities for an annealed random walk in a random potential. *Stochastic Process. Appl.* **122** 277–304. MR2860450
[9] Kosygina, E., Mountford, T.S. and Zerner, M.P.W. (2011). Lyapunov exponents of Green's functions for random potentials tending to zero. *Probab. Theory Related Fields* **150** 43–59. MR2800903
[10] Mourrat, J.-C. (2012). Lyapunov exponents, shape theorems and large deviations for the random walk in random potential. *ALEA Lat. Am. J. Probab. Math. Stat.* **9** 165–211. MR2923190
[11] Rassoul-Agha, F. and Seppäläinen, T. (2014). Quenched point-to-point free energy for random walks in random potentials. *Probab. Theory Related Fields* **158** 711–750. MR3176363
[12] Rueß, J. (2014). Continuity results and estimates for the lyapunov exponent of brownian motion in random potential. Preprint. Available at arXiv:1404.1273.
[13] Scholler, J. (2014). On the time constant in a dependent first passage percolation model. *ESAIM Probab. Stat.* **18** 171–184. MR3230873
[14] Smythe, R.T. and Wierman, J.C. (1978). *First-Passage Percolation on the Square Lattice. Lecture Notes in Math.* **671**. Berlin: Springer. MR0513421
[15] Sodin, S. (2014). Positive temperature versions of two theorems on first-passage percolation. In *Geometric Aspects of Functional Analysis* 441–453. Berlin: Springer.
[16] Sznitman, A.-S. (1995). Crossing velocities and random lattice animals. *Ann. Probab.* **23** 1006–1023. MR1349160
[17] Sznitman, A.-S. (1998). *Brownian Motion, Obstacles and Random Media. Springer Monographs in Mathematics*. Berlin: Springer. MR1717054
[18] Talagrand, M. (1995). Concentration of measure and isoperimetric inequalities in product spaces. *Inst. Hautes Études Sci. Publ. Math.* **81** 73–205. MR1361756
[19] Zerner, M.P.W. (1998). Directional decay of the Green's function for a random nonnegative potential on $\mathbf{Z}^d$. *Ann. Appl. Probab.* **8** 246–280. MR1620370
[20] Zygouras, N. (2009). Lyapounov norms for random walks in low disorder and dimension greater than three. *Probab. Theory Related Fields* **143** 615–642. MR2475675

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