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Aggregation of autoregressive random fields and anisotropic long-range dependence

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We introduce the notions of scaling transition and distributional long-range dependence for stationary random fields Y on $\mathbb{Z}^2$ whose normalized partial sums on rectangles with sides growing at rates $O(n)$ and $O(n^\gamma)$ tend to an operator scaling random field V_γ on $\mathbb{R}^2$, for any $\gamma > 0$. The scaling transition is characterized by the fact that there exists a unique $\gamma_0 > 0$ such that the scaling limits V_γ are different and do not depend on γ for $\gamma > \gamma_0$ and $\gamma < \gamma_0$. The existence of scaling transition together with anisotropic and isotropic distributional long-range dependence properties is demonstrated for a class of α-stable $(1 < \alpha \le 2)$ aggregated nearest-neighbor autoregressive random fields on $\mathbb{Z}^2$ with a scalar random coefficient A having a regularly varying probability density near the "unit root" $A = 1$.

Keywords: α-stable mixed moving average; autoregressive random field; contemporaneous aggregation; isotropic/anisotropic long-range dependence; lattice Green function; operator scaling random field; scaling transition

1. Introduction

Following Biermé *et al.* [7], a scalar-valued random field (RF) $V = \{V(x); x \in \mathbb{R}^\nu\}$ is called *operator scaling random field* (OSRF) if there exist a $H > 0$ and a $\nu \times \nu$ real matrix E whose all eigenvalues have positive real parts, such that for any $\lambda > 0$

$$\{V(\lambda^E x); x \in \mathbb{R}^\nu\} \stackrel{\text{f.d.d.}}{=} \{\lambda^H V(x); x \in \mathbb{R}^\nu\}. \tag{1.1}$$

(See the end of this section for all unexplained notation.) In the case when $E = I$ is the unit matrix, (1.1) agrees with the definition of H-self-similar random field (SSRF), the latter referred to as self-similar process when $\nu = 1$. OSRFs may exhibit strong anisotropy and play an important role in various physical theories; see [7] and the references therein. Several classes of OSRFs were constructed and discussed in [7,9].

It is well known that the class of self-similar processes is very large, SSRFs and OSFRs being even more numerous. According to a popular view, the "value" of a concrete self-similar process depends on its "domain of attraction". In the case $\nu = 1$, the domain of attraction of a self-similar stationary increment process $V = \{V(\tau); \tau \ge 0\}$ is defined in [31] as the class of all stationary processes $Y = \{Y(t); t \in \mathbb{Z}_+\}$ whose normalized partial sums tend to V in the distributional

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sense, namely,

$$B_n^{-1} \sum_{t=1}^{[n\tau]} Y(t) \xrightarrow{\text{f.d.d.}} V(\tau), \qquad \tau \in \mathbb{R}_+, n \to \infty. \tag{1.2}$$

The classical Lamperti's theorem [31] says that in the case of (1.2), the normalizing constants B_n necessarily grow as n^H (modulus a slowly varying factor) and the limit random process in (1.2) is H-self-similar. The limit process V in (1.2) characterizes large-scale and dependence properties of Y, leading to the important concept of *distributional short/long memory* originating in Cox [10]; see also ([11], pages 76–77), [22,41–44]. There exists a large probabilistic literature devoted to studying the partial sums limits of various classes of strongly and weakly dependent processes and RFs. In particular, several works [12,13,16,32,36,47] discussed the partial sums limits of (stationary) RFs indexed by $t \in \mathbb{Z}^\nu$:

$$B_n^{-1} \sum_{t \in K_{[nx]}} Y(t) \xrightarrow{\text{f.d.d.}} V(x), \qquad x = (x_1, \dots, x_\nu) \in \mathbb{R}_+^\nu, n \to \infty, \tag{1.3}$$

where $K_{[nx]} := \{t = (t_1, \dots, t_\nu) \in \mathbb{Z}^\nu : 1 \le t_i \le nx_i\}$ is a sequence of rectangles whose all sides increase as $O(n)$. Related results for Gaussian or linear (shot-noise) and their subordinated RFs, with a particular focus on large-time behavior of statistical solutions of partial differential equations, were obtained in [1,2,35–37]. See also the recent paper Anh *et al.* [3] and the numerous references therein. Most of the above mentioned studies deal with "nearly isotropic" models of RFs characterized by a single memory parameter H and a limiting SSRF $\{V(x)\}$ in (1.3).

Similarly as in the case of random processes indexed by $\mathbb{Z}$, stationary RFs usually exhibit two types of dependence: weak dependence and strong dependence. The second type of dependence is often called long memory or *long-range dependence* (LRD). Although there is no single satisfactory definition of LRD, usually it refers to a stationary RF Y having an unbounded spectral density f: $\sup_{x \in [-\pi,\pi]^\nu} f(x) = \infty$ or a non-summable auto-covariance function: $\sum_{t \in \mathbb{Z}^\nu} |\mathrm{cov}(Y(0), Y(t))| = \infty$; see [5,13,15–17,21,32]. The above definitions of LRD do not apply to RFs with infinite variance and are of limited use since these properties are very hard to test in practice. On the other hand, the characterization of LRD based on partial sums as in the case of distributional long memory is directly related to the asymptotic distribution of the sample mean. As noted in [27], in many applications the auto-covariance of RF decays with different exponents (Hurst indices) in different directions. In the latter case, the partial sums of such RF on rectangles $\prod_{i=1}^\nu [1, n_i]$ may grow at different rate with $n_i \to \infty$, leading to a limiting anisotropic OSRF.

The present paper attempts a systematic study of *anisotropic distributional long-range dependence*, by exhibiting some natural classes of RFs whose partial sums tend to OSRFs. Our study is limited to the case $\nu = 2$ and RFs with anisotropy along the coordinate axes and a diagonal matrix E. Note that for $\nu = 2$ and $E = \mathrm{diag}(1, \gamma), 0 < \gamma \ne 1$, relation (1.1) writes as $\{V(\lambda x, \lambda^\gamma y)\} \overset{\text{f.d.d.}}{=} \{\lambda^H V(x, y)\}$, or

$$\{\lambda V(x, y); (x, y) \in \mathbb{R}^2\} \overset{\text{f.d.d.}}{=} \{V(\lambda^{1/H} x, \lambda^{\gamma/H} y); (x, y) \in \mathbb{R}^2\} \qquad \forall \lambda > 0. \tag{1.4}$$

The OSRFs $V = V_\gamma$ depending on $\gamma > 0$ are obtained by taking the partial sums limits

$$n^{-H(\gamma)} \sum_{(t,s)\in K_{[nx,n^\gamma y]}} Y(t,s) \overset{\text{f.d.d.}}{\longrightarrow} V_\gamma(x,y), \qquad (x,y) \in \mathbb{R}_+^2, n \to \infty \tag{1.5}$$

on rectangles $K_{[nx,n^\gamma y]} := \{(t,s) \in \mathbb{Z}^2 : 1 \le t \le nx, 1 \le s \le n^\gamma y\}$ whose sides grow at possibly different rate $O(n)$ and $O(n^\gamma)$. Somewhat unexpectedly, it turned out that for a large class of RFs $Y = \{Y(t,s); (t,s) \in \mathbb{Z}^2\}$, the limit in (1.5) exists for any $\gamma > 0$. What is more surprising, many LRD RFs Y exhibit a dramatic change of their scaling behavior at some point $\gamma_0 > 0$, in the sense that $V_\gamma \overset{\text{f.d.d.}}{=} V_\pm$ do not depend on γ for $\gamma > \gamma_0$ or $\gamma < \gamma_0$ and $V_+ \overset{\text{f.d.d.}}{\neq} V_-$. This phenomenon which we call *scaling transition* seems to be of general nature, suggesting an exciting new area in spatial research [45]. It occurs for α-stable ($1 < \alpha \le 2$) aggregated autoregressive RFs studied in this paper, for a natural class of LRD Gaussian RFs discussed in [45] and Remark 2.2 below, but also in a very different context of network traffic models; see Remark 2.3. In most of the above mentioned works, the limit V_{γ_0} is different from V_+ and V_-, and the differences between V_{γ_0}, V_+, V_- can be characterized by dependence properties of increments $V_\gamma(K) := V_\gamma(x,y) - V_\gamma(u,y) - V_\gamma(x,v) + V_\gamma(u,v)$ on rectangles $K = (u,x] \times (v,y] \subset \mathbb{R}_+^2$, which may change from independent increments in the vertical direction for $\gamma > \gamma_0$ to independent increments in the horizontal direction (or completely dependent increments in the vertical direction) for $\gamma < \gamma_0$, or vice versa. Further on, depending on whether $\gamma_0 = 1$ or $\gamma_0 \neq 1$, the corresponding RF Y is said to have *isotropic distributional LRD* or *anisotropic distributional LRD* properties.

The main purpose of this work is establishing scaling transition and Type I isotropic and anisotropic distributional LRD properties for a natural class of aggregated nearest-neighbor random-coefficient autoregressive RFs with finite and infinite variance. We recall that the idea of contemporaneous aggregation originates to Granger [26], who observed that aggregation of random-coefficient AR(1) equations with random beta-distributed coefficient can lead to a Gaussian process with long memory and slowly decaying covariance function. Since then, aggregation became one of the most important methods for modeling and studying long memory processes; see Beran [5]. For linear and heteroscedastic autoregressive time series models with one-dimensional time, it was developed in [8,23,24,30,40–44,51,52] and for some RF models in [32–34,38]. Aggregation is also important for understanding and modeling of spatial LRD processes by relating them to short-range dependent random-coefficient autoregressive models in a natural way. The two models of interest are given by equations:

$$X_3(t,s) = \frac{A}{3}\big(X_3(t-1,s) + X_3(t,s+1) + X_3(t,s-1)\big) + \varepsilon(t,s), \tag{1.6}$$

$$X_4(t,s) = \frac{A}{4}\big(X_4(t-1,s) + X_4(t+1,s) + X_4(t,s+1) + X_4(t,s-1)\big) + \varepsilon(t,s), \tag{1.7}$$

where $\{\varepsilon(t,s), (t,s) \in \mathbb{Z}^2\}$ are i.i.d. r.v.'s whose generic distribution ε belongs to the domain of (normal) attraction of α-stable law, $1 < \alpha \le 2$, and $A \in [0,1)$ is a r.v. independent of $\{\varepsilon(t,s)\}$ and having a regularly varying probability density ϕ at $a = 1$: there exist $\phi_1 > 0$ and $\beta > -1$ such that

$$\phi(a) \sim \phi_1(1-a)^\beta, \qquad a \nearrow 1. \tag{1.8}$$

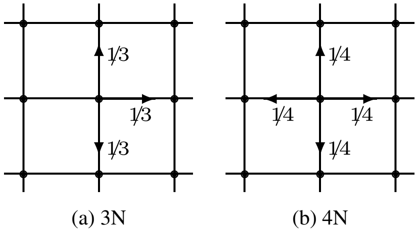

Figure 1. One-step transition probabilities of the random walk underlying models (1.6) and (1.7).

In the sequel, we refer to (1.6) and (1.7) as the 3N and 4N models, N standing for "Neighbor". Let $X_{3j}, X_{4j}, j = 1, \dots, m$ denote m independent copies of X_3, X_4 in (1.6), (1.7), respectively. As shown in Section 5, the aggregated 3N and 4N models defined as $m^{-1}\sum_{j=1}^m X_{ij}(t,s) \xrightarrow{\text{f.d.d.}} \mathfrak{X}_i(t,s), m \to \infty, i = 3, 4$ are written as respective mixed α-stable moving-averages:

$$\mathfrak{X}_i(t,s) = \sum_{(u,v)\in\mathbb{Z}^2} \int_{[0,1)} g_i(t-u, s-v, a) M_{u,v}(\mathrm{d}a), \qquad (t,s) \in \mathbb{Z}^2, i = 3, 4, \tag{1.9}$$

where $\{M_{u,v}(\mathrm{d}a), (u,v) \in \mathbb{Z}^2\}$ are i.i.d. copies of an α-stable random measure M on $[0,1)$ with control measure $\phi(a)\mathrm{d}a$ and g_i is the corresponding (lattice) Green function:

$$g_i(t,s,a) = \sum_{k=0}^{\infty} a^k p_k(t,s), \qquad (t,s) \in \mathbb{Z}^2, a \in [0,1), i = 3, 4, \tag{1.10}$$

where $p_k(t,s) = \mathrm{P}(W_k = (t,s) | W_0 = (0,0))$ is the k-step probability of the nearest-neighbor random walk $\{W_k, k = 0, 1, \dots\}$ on the lattice $\mathbb{Z}^2$ with one-step transition probabilities $p(t,s)$ shown in Figure 1(a)–(b).

The main results of Sections 3 and 4 are Theorems 3.1 and 4.1. The first theorem identifies the scaling limits $V_\gamma, \gamma > 0$ in (1.5) and proves Type I anisotropic LRD property in the sense of Definition 2.4 with $\gamma_0 = 1/2$ for the aggregated 3N model $\mathfrak{X}_3$ in (1.9). Similarly, the second theorem obtains Type I isotropic LRD property ($\gamma_0 = 1$) for the aggregated 4N model $\mathfrak{X}_4$ in (1.9).

The proofs of Theorems 3.1 and 4.1 rely on the asymptotics of the lattice Green function in (1.10) for models 3N and 4N. Particularly, Lemmas 3.1 and 4.1 obtain the following point-wise convergences: as $\lambda \to \infty$,

$$\sqrt{\lambda} g_3\left([\lambda t], [\sqrt{\lambda} s], 1 - \frac{z}{\lambda}\right) \to h_3(t,s,z), \qquad t > 0, s \in \mathbb{R}, z > 0, \tag{1.11}$$

$$g_4\left([\lambda t], [\lambda s], 1 - \frac{z}{\lambda^2}\right) \to h_4(t,s,z), \qquad (t,s) \in \mathbb{R}_0^2, z > 0, \tag{1.12}$$

respectively, together with dominating bounds of the left-hand sides of (1.11)–(1.12). The limit functions h_3 and h_4 in (1.11)–(1.12) (entering stochastic integral representations of the scaling limits V_γ in Theorems 3.1 and 4.1) are given by

$$h_3(t,s,z) := \frac{3}{2\sqrt{\pi t}} \mathrm{e}^{-3zt - s^2/(4t)} \mathbf{1}(t, z > 0),$$
$$h_4(t,s,z) := \frac{2}{\pi} K_0\big(2\sqrt{z(t^2+s^2)}\big) \mathbf{1}(z > 0), \tag{1.13}$$

where K_0 is the modified Bessel function of second kind. Note that h_3 in (1.13) is the Green function of one-dimensional heat equation (modulus constant coefficients), while h_4 is the Green function of the Helmholtz equation in $\mathbb{R}^2$. The proofs of these technical lemmas can be found in the extended version of this paper available at http://arxiv.org/abs/1303.2209v3 and will be published elsewhere. Lemmas 3.1 and 4.1 may also have independent interest for studying the behavior of the autoregressive fields (1.6) and (1.7) with deterministic coefficient A in the vicinity of $A = 1$, particularly, for testing stationarity near the unit root in spatial autoregressive models, cf. [6].

Notation. In what follows, $C, C(K), \ldots$ denote generic constants, possibly depending on the variables in brackets, which may be different at different locations. We write $\xrightarrow{\mathrm{d}}$, $\overset{\mathrm{d}}{=}$, $\xrightarrow{\text{f.d.d.}}$, $\overset{\text{f.d.d.}}{=}$, $\overset{\text{f.d.d.}}{\neq}$ for the weak convergence and equality and inequality of distributions and finite-dimensional distributions, respectively. f.d.d.-lim stands for the limit in the sense of weak convergence of finite-dimensional distributions. For $\lambda > 0$ and a $\nu \times \nu$ matrix $E, \lambda^E := \mathrm{e}^{E \log \lambda}$, where $\mathrm{e}^A = \sum_{k=0}^\infty A^k/k!$ is the matrix exponential. $\mathbb{Z}^\nu_+ := \{(t_1,\ldots,t_\nu) \in \mathbb{Z}^\nu : t_i > 0, i = 1,\ldots,\nu\}, \mathbb{R}^\nu_+ := \{(x_1,\ldots,x_d) \in \mathbb{R}^\nu : x_i > 0, i = 1,\ldots,\nu\}, \bar{\mathbb{R}}^\nu_+ := \{(x_1,\ldots,x_d) \in \mathbb{R}^\nu : x_i \geq 0, i = 1,\ldots,\nu\}, \mathbb{Z}_+ := \mathbb{Z}^1_+, \mathbb{R}_+ := \mathbb{R}^1_+, \bar{\mathbb{R}}_+ := \bar{\mathbb{R}}^1_+, \mathbb{R}^2_0 := \mathbb{R}^2 \setminus \{(0,0)\}$. $E = \operatorname{diag}(\gamma_1,\ldots,\gamma_\nu)$ denotes the diagonal $\nu \times \nu$ matrix with entries $\gamma_1,\ldots,\gamma_\nu$ on the diagonal. $\mathbf{1}_A$ stands for the indicator function of a set A. $\log_+(x) := \log x, x \geq 1, := 0$ otherwise. $[x] = \lfloor x \rfloor := k, x \in [k, k+1), \lceil y \rceil := k+1, y \in (k, k+1], k \in \mathbb{Z}$. $K_{[nx,n^\gamma y]} := \{(t,s) \in \mathbb{Z}^2 : 1 \leq t \leq nx, 1 \leq s \leq n^\gamma y\}$, $K_{(u,v);(x,y)} := \{(t,s) \in \mathbb{R}^2_+ : u < t \leq x, v < s \leq y\}$.

2. Scaling transition and Type I distributional LRD for RFs on $\mathbb{Z}^2$

In this section, by RF on $\bar{\mathbb{R}}^2_+$ we mean a RF $V = \{V(x,y); (x,y) \in \bar{\mathbb{R}}^2_+\}$ such that $V(x,y) = 0$ for any $(x,y) \in \bar{\mathbb{R}}^2_+ \setminus \mathbb{R}^2_+$. A RF V on $\bar{\mathbb{R}}^2_+$ is said *trivial* if $V(x,y) = 0$ for any $(x,y) \in \bar{\mathbb{R}}^2_+$, else V is said *non-trivial*.

Definition 2.1. *Let $Y = \{Y(t,s); (t,s) \in \mathbb{Z}^2\}$ be a stationary RF. Assume that for any $\gamma > 0$ there exist a normalization $A_n(\gamma) \to \infty$ and a non-trivial RF $V_\gamma = \{V_\gamma(x,y); (x,y) \in \bar{\mathbb{R}}^2_+\}$ such that*

$$A_n^{-1}(\gamma) \sum_{(t,s) \in K_{[nx,n^\gamma y]}} Y(t,s) \xrightarrow{\text{f.d.d.}} V_\gamma(x,y), \qquad (x,y) \in \mathbb{R}^2_+, n \to \infty. \tag{2.1}$$

We say that Y exhibits scaling transition *if there exists* $\gamma_0 > 0$ *such that the limits* $V_\gamma \overset{\text{f.d.d.}}{=} V_+, \gamma > \gamma_0$ *and* $V_\gamma \overset{\text{f.d.d.}}{=} V_-, \gamma < \gamma_0$ *do not depend on* γ *for* $\gamma > \gamma_0$ *and* $\gamma < \gamma_0$ *and, moreover,* V_+ *and* V_- *are mutually different RFs, in the sense that for any* $a > 0$

$$V_+ \overset{\text{f.d.d.}}{\neq} aV_-. \tag{2.2}$$

In such case, V_{γ_0} *will be called the* well balanced *and* V_+, V_- *the* unbalanced *scaling limits of* Y*, respectively.*

Note that the fact that (2.2) hold for any $a > 0$ excludes a trivial change of the scaling limit by a linear change of normalization. It follows rather easily that under general set-up scaling limits V_γ satisfy the self-similarity and stationarity of rectangular increments properties stated in Proposition 2.1 below. Let $V = \{V(x, y); (x, y) \in \bar{\mathbb{R}}^2_+\}$ be a RF and $K = K_{(u,v);(x,y)} \subset \mathbb{R}^2_+$ be a rectangle. By *increment of* V *on rectangle* K we mean the difference

$$V(K) := V(x, y) - V(u, y) - V(x, v) + V(u, v).$$

We say that V *has stationary rectangular increments* if for any $(u, v) \in \mathbb{R}^2_+$,

$$\big\{V(K_{(u,v);(x,y)}); x \ge u, y \ge v\big\} \overset{\text{f.d.d.}}{=} \big\{V(K_{(0,0);(x-u,y-v)}); x \ge u, y \ge v\big\}. \tag{2.3}$$

As mentioned in the Introduction, in the case of scaling transition the limits V_{γ_0}, V_+, V_- can be characterized by dependence properties of increments $V(K)$. To define these properties, we introduce some terminology. Let $\ell = \{(x, y) \in \mathbb{R}^2 : ax + by = c\}$ be a line in $\mathbb{R}^2$. A line $\ell' = \{(x, y) \in \mathbb{R}^2 : a'x + b'y = c'\}$ is said *perpendicular to* ℓ (denoted $\ell' \perp \ell$) if $aa' + bb' = 0$. We say that two rectangles $K = K_{(u,v);(x,y)}$ and $K' = K_{(u',v');(x',y')}$ are *separated by line* ℓ' if they lie on different sides of ℓ', in which case K and K' are necessarily disjoint: $K \cap K' = \varnothing$. See Figure 2.

Definition 2.2. *Let* $V = \{V(x, y); (x, y) \in \bar{\mathbb{R}}^2_+\}$ *be a RF with stationary rectangular increments,* $V(x, 0) = V(0, y) \equiv 0, x, y \ge 0,$ *and* $\ell \subset \mathbb{R}^2$ *be a given line* , $(0, 0) \in \ell$. *We say that* V *has*:

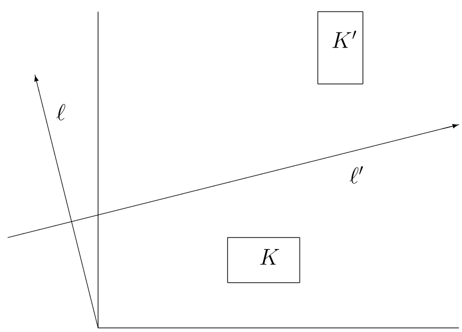

Figure 2. Rectangles K and K' separated by line ℓ'.

(i) independent rectangular increments in direction ℓ *if for any orthogonal line* $\ell' \perp \ell$ *and any two rectangles* $K, K' \subset \mathbb{R}^2_+$ *separated by* ℓ', *increments* $V(K)$ *and* $V(K')$ *are independent*;

(ii) invariant rectangular increments in direction ℓ *if* $V(K) = V(K')$ *for any two rectangles* $K, K' \subset \mathbb{R}^2_+$ *such that* $K' = (x, y) + K$ *for some* $(x, y) \in \ell$;

(iii) properly dependent rectangular increments in direction ℓ *if neither* (i) *nor* (ii) *holds*;

(iv) properly dependent rectangular increments *if* V *has properly dependent rectangular increments in arbitrary direction*;

(v) independent rectangular increments *if* V *has independent rectangular increments in arbitrary direction*.

Example 2.3. *Fractional Brownian sheet* B_{H_1,H_2} with parameters $0 < H_1, H_2 \le 1$ is a Gaussian process on $\bar{\mathbb{R}}^2_+$ with zero mean and covariance

$$
\begin{aligned}
&\mathrm{E}B_{H_1,H_2}(x, y)B_{H_1,H_2}(x', y') \\
&\quad = \frac{1}{4}\left(x^{2H_1} + x'^{2H_1} - |x - x'|^{2H_1}\right)\left(y^{2H_2} + y'^{2H_2} - |y - y'|^{2H_2}\right),
\end{aligned}
\tag{2.4}
$$

where $(x, y), (x', y') \in \bar{\mathbb{R}}^2_+$. It follows (see [4], Corollary 3) that for any rectangles $K = K_{(u,v);(x,y)}$, $K' = K_{(u',v');(x',y')}$

$$
\begin{aligned}
&\mathrm{E}B_{H_1,H_2}(K)B_{H_1,H_2}(K') \\
&\quad = \tfrac{1}{4}\left(|x - x'|^{2H_1} + |u - u'|^{2H_1} - |x - u'|^{2H_1} - |x' - u|^{2H_1}\right) \\
&\qquad \times \left(|y - y'|^{2H_2} + |v - v'|^{2H_2} - |y - v'|^{2H_2} - |y' - v|^{2H_2}\right) \\
&\qquad = \mathrm{E}\left(B_{H_1}(x) - B_{H_1}(u)\right)\left(B_{H_1}(x') - B_{H_1}(u')\right) \\
&\qquad\quad \times \mathrm{E}\left(B_{H_2}(y) - B_{H_2}(v)\right)\left(B_{H_2}(y') - B_{H_2}(v')\right),
\end{aligned}
\tag{2.5}
$$

where $\{B_H(x); x \in \bar{\mathbb{R}}_+\}$ is a fractional Brownian motion on $\bar{\mathbb{R}}_+ = [0, \infty)$ with $\mathrm{E}B_H(x)B_H(x') = (1/2)(x^{2H} + x'^{2H} - |x - x'|^{2H})$, $H \in (0, 1]$. (Recall that $B_{1/2}$ is a standard Brownian motion with variance $\mathrm{E}B^2_{1/2}(x) = x$ and $B_1(x) = xB_1(1)$ is a random line.) In particular, B_{H_1,H_2} has stationary rectangular increments; see [4], Proposition 2. It follows from (2.5) that $B_{1/2,H_2}$ has independent rectangular increments in the horizontal direction since $\mathrm{E}B_{1/2,H_2}(K)B_{1/2,H_2}(K') = 0$ for any K, K' which are separated by a vertical line, or $(u, x] \cap (u', x'] = \varnothing$. Similarly, $B_{H_1,1/2}$ has independent rectangular increments in the vertical direction and $B_{1/2,1/2}$ has independent rectangular increments in arbitrary direction. It is also clear that for $H_1 = 1$ (resp., $H_2 = 1$) B_{H_1,H_2} has invariant rectangular increments in the horizontal (resp., vertical) direction.

Let $H_i \ne 1/2, 1, i = 1, 2$ and ℓ be any line passing through the origin. Let $K = K_{(x-1,y-1);(x,y)}$, $K' = K_{(0,0);(1,1)}$ be two rectangles whose all sides are equal to 1. Clearly, if x and y are large enough, K and K' are separated by an orthogonal line $\ell' \perp \ell$. From (2.5) and Taylor's expansion, it easily follows that

$$
\mathrm{E}B_{H_1,H_2}(K)B_{H_1,H_2}(K') \sim C(H_1, H_2)x^{2H_1-2}y^{2H_2-2} \qquad \text{when } x, y \to \infty,
$$

with

$$C(H_1, H_2) := \prod_{i=1}^{2} (2H_i)(2H_i - 1) \neq 0.$$

This means that for $H_i \notin \{1/2, 1\}, i = 1, 2$, B_{H_1,H_2} has properly dependent rectangular increments in arbitrary direction ℓ.

Using the terminology of Definition 2.2, we conclude that fractional Brownian sheet B_{H_1,H_2} has:

- properly dependent rectangular increments if $H_i \notin \{1/2, 1\}, i = 1, 2$;
- independent rectangular increments in the horizontal (vertical) direction if $H_1 = 1/2$ ($H_2 = 1/2$);
- invariant rectangular increments in the horizontal (vertical) direction if $H_1 = 1$ ($H_2 = 1$);
- independent rectangular increments if $H_1 = H_2 = 1/2$.

Definition 2.4. *Let $Y = \{Y(t, s); (t, s) \in \mathbb{Z}^2\}$ be a stationary RF. Assume that for any $\gamma > 0$ there exist a normalization $A_n(\gamma) \to \infty$ and a non-trivial RF $V_\gamma = \{V_\gamma(x, y); (x, y) \in \bar{\mathbb{R}}_+^2\}$ such that (2.1) holds.*

We say that Y has Type I distributional LRD (*or Y* is a Type I RF) *if there exists $\gamma_0 > 0$ such that*

- *RF V_{γ_0} has properly dependent rectangular increments, and*
- *RFs $V_\gamma, \gamma \neq \gamma_0$ do not have properly dependent rectangular increments; in other words, for any $\gamma \neq \gamma_0, \gamma > 0$ there exists a line $\ell(\gamma) \in \mathbb{R}^2$ such that V_γ has either independent or invariant increments in the direction $\ell(\gamma)$.*

Moreover, a Type I RF Y is said to have isotropic distributional LRD *if $\gamma_0 = 1$ and* anisotropic distributional LRD *if $\gamma_0 \neq 1$.*

Remark 2.1. The above definition does not assume the occurrence of scaling transition at γ_0, although in all cases known to us, Type I distributional LRD property holds simultaneously with scaling transition. On the other hand, Remark 2.3 shows that scaling transition need not lead to Type I distributional LRD. "Type I" indicates that V_γ has properly dependent rectangular increments at a *single* point $\gamma = \gamma_0$. By contrast, "Type II" Gaussian LRD RFs mentioned in Remark 2.2 below have the property that V_γ have properly dependent rectangular increments for *all* $\gamma > 0$.

Remark 2.2. Puplinskaitė and Surgailis [45] established scaling transition and Type I distributional LRD property for stationary Gaussian RFs with spectral density $f(x, y) = g(x, y)(|x|^2 + |y|^{2\mathcal{H}_2/\mathcal{H}_1})^{-\mathcal{H}_1/2}$, $(x, y) \in [-\pi, \pi]^2$, where $\mathcal{H}_i > 0, \mathcal{H}_1\mathcal{H}_2 < \mathcal{H}_1 + \mathcal{H}_2$ are parameters and g is a bounded positive function having nonzero limit at the origin. In this case, $\gamma_0 = \mathcal{H}_1/\mathcal{H}_2$ and the unbalanced scaling limits $V_\pm$ agree with a fractional Brownian sheet B_{H_1,H_2} where at least one of the two parameters H_1, H_2 equals 1/2 or 1. Moreover, $\mathcal{H}_1 = \mathcal{H}_2$ (resp., $\mathcal{H}_1 \neq \mathcal{H}_2$) correspond to Type I isotropic (resp., anisotropic) distributional LRD properties. By contrast, "Type II" Gaussian RFs with spectral density of the form $f(x, y) = g(x, y)|x|^{-2d_1}|y|^{-2d_2}$, $0 < d_1, d_2 < 1/2$ and a similar function g do not exhibit scaling transition since their scaling limits V_γ for any $\gamma > 0$ coincide with a fractional Brownian sheet $B_{d_1+0.5,d_2+0.5}$ up to a multiplicative constant; see [45]. [32,34] discuss scaling limits of Gaussian LRD RFs with general anisotropy axis.

Remark 2.3. Scaling transition different from Type I arises under joint temporal and contemporaneous aggregation of *independent* LRD processes in telecommunication and economics; see [14,20,39,42] and the references therein. In these works, $\{Y(t,s); t \in \mathbb{Z}\}, s \in \mathbb{Z}$ are independent copies of a stationary LRD process $X = \{X(t); t \in \mathbb{Z}\}$ and the scaling limits V_γ of RF $Y = \{Y(t,s); (t,s) \in \mathbb{Z}^2\}$ necessarily have independent increments in the vertical direction for any $\gamma > 0$, meaning that Y cannot have Type I distributional LRD by definition. Nevertheless, for heavy-tailed centered ON/OFF process X and some other duration based models, the results in [39] imply that Y exhibits a scaling transition with some $\gamma_0 \in (0,1)$ and markedly distinct "supercritical" and "subcritical" unbalanced scaling limits $V_\pm$, V_+ being a Gaussian RF with dependent increments in the horizontal direction and V_- having α-stable $(1 < \alpha < 2)$ distributions and independent increments in the horizontal direction. The well-balanced scaling limit V_{γ_0} termed the "intermediate process" is discussed in detail in [19,42].

Proposition 2.1. *Let $Y = \{Y(t,s); (t,s) \in \mathbb{Z}^2\}$ be a stationary RF satisfying (2.1) for some $\gamma > 0$ and $A_n(\gamma) = L(n)n^H$, where $H > 0$ and $L : [1,\infty) \to \mathbb{R}_+$ is a slowly varying function. Then the limit RF V_γ in (1.5) satisfies the self-similarity property (1.4). In particular, V_γ is OSRF corresponding to $E := \operatorname{diag}(1,\gamma)$. Moreover, V_γ has stationary rectangular increments.*

Proof. Fix $\lambda > 0$ and let $m := n\lambda^{1/H}$. Then $L(n)/L(m) \to 1, n \to \infty$ and

$$\begin{aligned} V_\gamma\big(\lambda^{1/H}x, \lambda^{\gamma/H}y\big) &= \text{f.d.d.-}\lim_{n\to\infty} \frac{1}{n^H L(n)} \sum_{(t,s)\in K_{[x\lambda^{1/H}n,\, y\lambda^{\gamma/H}n^\gamma]}} Y(t,s) \\ &= \text{f.d.d.-}\lim_{m\to\infty} \frac{L(m)}{L(n)} \frac{\lambda}{m^H L(m)} \sum_{(t,s)\in K_{[xm,\, ym^\gamma]}} Y(t,s) \stackrel{\text{f.d.d.}}{=} \lambda V_\gamma(x,y). \end{aligned}$$

The fact that V_γ has stationary rectangular increments is an easy consequence of Y being stationary. □

3. Scaling transition in the aggregated 3N model

This section establishes scaling transition and Type I anisotropic distributional LRD property, in the sense of Definitions 2.1 and 2.4 of Section 2, for the aggregated 3N model $\mathfrak{X}_3$ in (1.9). We shall assume that M in (1.9) is symmetric α-stable with characteristic function $\mathrm{E}\mathrm{e}^{\mathrm{i}\theta M(B)} = \mathrm{e}^{-|\theta|^\alpha \Phi(B)}$, $B \subset [0,1)$. The case of general α-stable random measure M (see (5.36)) in (1.9) can be discussed in a similar way. Recall that $g_3(t,s,a)$ in (1.9) is the Green function of the random walk $\{W_k\}$ on $\mathbb{Z}^2$ with one-step transition probabilities shown in Figure 1(a). According to Remark 5.2, RF $\mathfrak{X}_3$ in (1.9) with mixing distribution in (1.8) is well-defined if $1 < \alpha \le 2$, $\beta > -(\alpha-1)/2$.

For given $\gamma > 0$, introduce a RF $V_\gamma = \{V_{3\gamma}(x,y); (x,y) \in \bar{\mathbb{R}}_+^2\}$ written as a stochastic integral

$$V_{3\gamma}(x,y) := \int_{\mathbb{R}^2\times\mathbb{R}_+} F_{3\gamma}(x,y;u,v,z)\mathcal{M}(\mathrm{d}u,\mathrm{d}v,\mathrm{d}z), \tag{3.1}$$

where $F_{3\gamma}(x,y;u,v,z)$ is defined as

$$F_{3\gamma} := \begin{cases} \int_0^x \int_0^y h_3(t-u,s-v,z)\,\mathrm{d}t\,\mathrm{d}s, \\ \qquad \gamma = 1/2, \\ \mathbf{1}(0<v<y)\int_0^x \mathrm{d}t \int_{\mathbb{R}} h_3(t-u,w,z)\,\mathrm{d}w, \\ \qquad \gamma > 1/2, 0<\beta<\alpha-1, \\ x\int_0^y h_3(-u,s-v,z)\,\mathrm{d}s, \\ \qquad \gamma > 1/2, -(\alpha-1)/2<\beta<0, \\ \mathbf{1}(0<u<x)\int_0^y \mathrm{d}s \int_{\mathbb{R}} h_3(w,v-s,z)\,\mathrm{d}w, \\ \qquad \gamma < 1/2, (\alpha-1)/2<\beta<\alpha-1, \\ y\int_0^x h_3(t-u,v,z)\,\mathrm{d}t, \\ \qquad \gamma < 1/2, -(\alpha-1)/2<\beta<(\alpha-1)/2, \end{cases} \tag{3.2}$$

$h_3(t,s,z) = \frac{3}{2\sqrt{\pi t}}\mathrm{e}^{-3zt-s^2/(4t)}\mathbf{1}(t>0,z>0)$ as in (1.13), and $\mathcal{M}$ is an α-stable random measure on $\mathbb{R}^2\times\mathbb{R}_+$ with control measure $\mathrm{d}\mu(u,v,z) := \phi_1 z^\beta\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}z$ and characteristic function $\mathrm{E}\mathrm{e}^{\mathrm{i}\theta\mathcal{M}(B)} = \mathrm{e}^{-|\theta|^\alpha\mu(B)}$, where $\phi_1 > 0, \beta > -1$ are the asymptotic parameters in (1.8) and $B \subset \mathbb{R}^2\times\mathbb{R}_+$ is a measurable set with $\mu(B) < \infty$.

Proposition 3.1. (i) *The RF $V_{3\gamma}$ in (3.1) is well-defined for any $\gamma > 0$, $1 < \alpha \le 2$ and β in (3.2). It has α-stable finite-dimensional distributions and stationary rectangular increments in the sense of (2.3).*

(ii) *$V_{3\gamma}$ is OSRF: for any $\lambda > 0$,*

$$\{V_{3\gamma}(\lambda x,\lambda^\gamma y); (x,y)\in\bar{\mathbb{R}}_+^2\} \overset{\text{f.d.d.}}{=} \{\lambda^{H(\gamma)}V_{3\gamma}(x,y); (x,y)\in\bar{\mathbb{R}}_+^2\},$$

where

$$H(\gamma) := \begin{cases} \dfrac{\gamma+\alpha-\beta}{\alpha}, & \gamma \ge 1/2, \beta > 0, \\ \dfrac{\gamma+\alpha-2\beta\gamma}{\alpha}, & \gamma \ge 1/2, \beta < 0, \\ \dfrac{1-\gamma+2\gamma(\alpha-\beta)}{\alpha}, & \gamma < 1/2, \beta > (\alpha-1)/2, \\ \dfrac{\alpha\gamma+(\alpha+1)/2-\beta}{\alpha}, & \gamma < 1/2, \beta < (\alpha-1)/2. \end{cases} \tag{3.3}$$

(iii) *RF* $V_{3\gamma}$ *has properly dependent rectangular increments for* $\gamma = 1/2$ *and does not have properly dependent rectangular increments for* $\gamma \neq 1/2$.

(iv) *RFs* $V_{3\gamma} = V_{3,+}(\gamma > 1/2)$ *and* $V_{3\gamma} = V_{3,-}(\gamma < 1/2)$ *do not depend on* γ *for* $\gamma > 1/2$ *and* $\gamma < 1/2$.

(v) *For* $\alpha = 2$, *the RFs*

$$\begin{aligned} V_{3,+} &\overset{\text{f.d.d.}}{=} \kappa_{3,+} \begin{cases} B_{1-(\beta/2),1/2}, & 0 < \beta < 1, \\ B_{1,(1/2)-\beta}, & -1/2 < \beta < 0, \end{cases} \\ V_{3,-} &\overset{\text{f.d.d.}}{=} \kappa_{3,-} \begin{cases} B_{1/2,(3/2)-\beta}, & 1/2 < \beta < 1, \\ B_{(3/4)-(\beta/2),1}, & -1/2 < \beta < 1/2, \end{cases} \end{aligned} \tag{3.4}$$

agree, up to some constants $\kappa_{3,\pm} = \kappa_{3,\pm}(\beta) \neq 0$, *with fractional Brownian sheet* B_{H_1,H_2} *where one of the parameters* H_1, H_2 *equals* $1/2$ *or* 1.

Remark 3.1. Similarly, as in the case of fractional Brownian sheet (case $\alpha = 2$), the unbalanced limit RFs $V_{3,\pm}$ have a very special dependence structure, being either "independent" or "deterministic continuations" of random processes with one-dimensional time:

$$\begin{aligned} \mathcal{V}_{11} &:= \{V_{3,+}(t,1); t \geq 0\}, & 0 < \beta < \alpha - 1, \\ \mathcal{V}_{12} &:= \{V_{3,+}(1,t); t \geq 0\}, & -(\alpha-1)/2 < \beta < 0, \\ \mathcal{V}_{21} &:= \{V_{3,-}(1,t); t \geq 0\}, & (\alpha-1)/2 < \beta < \alpha - 1, \\ \mathcal{V}_{22} &:= \{V_{3,-}(t,1); t \geq 0\}, & -(\alpha-1)/2 < \beta < (\alpha-1)/2. \end{aligned} \tag{3.5}$$

The four processes $\mathcal{V}_{ij}, i, j = 1, 2$ in (3.5) are all symmetric α-stable (SαS) and self-similar with stationary increments (SSSI) with corresponding self-similarity parameters:

$$H_{11} := \frac{\alpha - \beta}{\alpha}, \qquad H_{12} := \frac{1 - 2\beta}{\alpha},$$
$$H_{21} := \frac{2(\alpha - \beta) - 1}{\alpha}, \qquad H_{22} := \frac{\alpha + 1 - 2\beta}{2\alpha}.$$

These facts follow from Proposition 3.1, for example, the self-similarity property of $\mathcal{V}_{12}$ follows from the definition of $V_{3,+}$ and Proposition 3.1(ii): $\forall \lambda > 0$,

$$\begin{aligned} \{\mathcal{V}_{12}(\lambda t)\} &= \{V_{3,+}(1, \lambda t)\} = \{V_{3,+}(\lambda^{1/\gamma}\lambda^{-1/\gamma} 1, \lambda t)\} \\ &\overset{\text{f.d.d.}}{=} \lambda^{H(\gamma)/\gamma}\{V_{3,+}(\lambda^{-1/\gamma} 1, t)\} = \lambda^{(H(\gamma)-1)/\gamma}\{V_{3,+}(1,t)\} = \lambda^{H_{12}}\{\mathcal{V}_{12}(t)\}. \end{aligned}$$

For $\alpha = 2$, processes $\mathcal{V}_{ij}, i, j = 1, 2$ are representations of fractional Brownian motion and, for $1 < \alpha < 2$, they belong to the class of SαS SSSI processes discussed in [48]. Note that the self-similarity exponents satisfy $1/\alpha < H_{ij} < 1, i, j = 1, 2$ and fill in all points of the interval $(1/\alpha, 1)$ as β vary in the corresponding intervals in (3.5).

Proof of Proposition 3.1. (i) It suffices to show $J_\gamma(x,y) := \int_{\mathbb{R}^2\times\mathbb{R}_+} |F_{3\gamma}(x,y;u,v,z)|^\alpha \times \mu(\mathrm{d}u,\mathrm{d}v,\mathrm{d}z) < \infty, x,y>0$. For simplicity, we restrict the proof to $x=y=1$, or $J_\gamma < \infty, J_\gamma := J_\gamma(1,1)$.

First, consider the case $\gamma = 1/2$. Write $J_{1/2} = J' + J''$, where $J' := \int_{\mathbb{R}^2\times\mathbb{R}_+} (\int_0^1\int_0^1 h_3(t-u, s-v, z)\,\mathrm{d}t\,\mathrm{d}s)^\alpha \mathbf{1}(|v|\le 2)\,\mathrm{d}\mu$, $J'' := \int_{\mathbb{R}^2\times\mathbb{R}_+} (\int_0^1\int_0^1 h_3(t-u,s-v,z)\,\mathrm{d}t\,\mathrm{d}s)^\alpha \mathbf{1}(|v|>2)\,\mathrm{d}\mu$. Then

$$\begin{aligned} J' &\le C\int_{-\infty}^1 \mathrm{d}u \int_0^\infty z^\beta\,\mathrm{d}z\Big(\int_0^1 \frac{\mathbf{1}(t>u)\,\mathrm{d}t}{\sqrt{t-u}}\mathrm{e}^{-3z(t-u)}\Big)^\alpha = C\Big(\int_{-\infty}^0 \mathrm{d}u + \cdots + \int_0^1 \mathrm{d}u\cdots\Big) \\ &=: C\big(J_1' + J_2'\big). \end{aligned}$$

By Minkowski's inequality,

$$\begin{aligned} J_1' &\le \Big\{\int_0^1 \mathrm{d}t\Big(\int_{-\infty}^0 \frac{\mathrm{d}u}{(t-u)^{\alpha/2}}\int_0^\infty \mathrm{e}^{-(3\alpha/2)z(t-u)}z^\beta\,\mathrm{d}z\Big)^{1/\alpha}\Big\}^\alpha \\ &\le C\Big\{\int_0^1 \mathrm{d}t\Big(\int_0^\infty \frac{\mathrm{d}u}{(t+u)^{1+\beta+(\alpha/2)}}\Big)^{1/\alpha}\Big\}^\alpha = C\Big\{\int_0^1 \frac{\mathrm{d}t}{t^{(1/2)+(\beta/\alpha)}}\Big\}^{1/\alpha} < \infty \end{aligned}$$

since $(1/2)+(\beta/\alpha) < 1$ due to $\beta < \alpha - 1, \alpha \le 2$. We also have

$$J_2' \le C\int_0^\infty z^\beta\,\mathrm{d}z\Big\{\int_0^1 \mathrm{e}^{-(3\alpha/2)zx}\,\mathrm{d}x\Big\}^\alpha = C\int_0^\infty z^{\beta-\alpha}(1-\mathrm{e}^{-z})^\alpha\,\mathrm{d}z < \infty$$

since $\alpha > 1+\beta$. On the other hand, since $(s-v)^2 \ge v^2/4$ for $|s|<1, |v|>2$, so using Minkowski's inequality we obtain

$$\begin{aligned} J'' &\le \Big\{\int_0^1 \mathrm{d}t\Big(\int_{-\infty}^t \frac{\mathrm{d}u}{(t-u)^{\alpha/2}}\int_{|v|>2}\mathrm{e}^{-v^2/4(t-u)}\,\mathrm{d}v\int_0^\infty \mathrm{e}^{-(3\alpha/2)z(t-u)}z^\beta\,\mathrm{d}z\Big)^{1/\alpha}\Big\}^\alpha \\ &\le C\int_0^\infty \frac{\mathrm{d}x}{x^{1+\beta+(\alpha/2)}}\int_{|v|>2}\mathrm{e}^{-v^2/(4x)}\,\mathrm{d}v, \end{aligned}$$

where the last integral is easily seen to be finite. This proves $J_{1/2} < \infty$.

Next, consider J_γ for $\gamma > 1/2, 0<\beta<\alpha-1$. Using $h_\star(u,z) := \int_{\mathbb{R}} h_3(u,v,z)\,\mathrm{d}v = 12\mathrm{e}^{-3uz}\mathbf{1}(u>0)$, similarly as above we obtain

$$\begin{aligned} J_\gamma &\le C\int_{-\infty}^1 \mathrm{d}u\int_0^\infty z^\beta \mathrm{d}z\Big(\int_{u\vee 0}^1 \mathrm{e}^{-3z(t-u)}\,\mathrm{d}t\Big)^\alpha = C\Big\{\int_{-\infty}^0 \mathrm{d}u + \cdots + \int_0^1 \mathrm{d}u\cdots\Big\} \\ &=: C\{J_{\gamma 1} + J_{\gamma 2}\}, \end{aligned}$$

where $J_{\gamma 1} \le C\{\int_0^1 \mathrm{d}t(\int_0^\infty (t+u)^{-1-\beta}\mathrm{d}u)^{1/\alpha}\}^\alpha \le C\{\int_0^1 t^{-\beta/\alpha}\,\mathrm{d}t\}^{1/\alpha} < \infty$ and

$$J_{\gamma 2} \le C\int_0^1 \mathrm{d}u\int_0^\infty z^\beta\,\mathrm{d}z\Big(\int_u^1 \mathrm{e}^{-3z(t-u)}\,\mathrm{d}t\Big)^\alpha \le C\int_0^\infty z^\beta\,\mathrm{d}z\big((1-\mathrm{e}^{-z})/z\big)^\alpha < \infty$$

because of $\beta - \alpha < -1$. This proves $J_\gamma < \infty$ for $\gamma > 1/2, 0<\beta<\alpha-1$.

Next, let $\gamma > 1/2$, $-(\alpha-1)/2 < \beta < 0$. We have

$$\begin{aligned}
J_\gamma &\le C \int_{\mathbb{R}_+\times\mathbb{R}\times\mathbb{R}_+} \mathrm{d}\mu\biggl(\int_0^1 h_3(u, s-v, z)\,\mathrm{d}s\biggr)^\alpha \\
&\le C \int_0^\infty u^{-\alpha/2}\mathrm{d}u \int_{\mathbb{R}} \mathrm{d}v \int_0^\infty \mathrm{e}^{-zu} z^\beta\,\mathrm{d}z\biggl(\int_0^1 \mathrm{e}^{-(s-v)^2/u}\,\mathrm{d}s\biggr)^\alpha \\
&= C \int_0^\infty u^{-(1+\beta+\alpha/2)}\,\mathrm{d}u\biggl\{\int_{|v|\le 2}\mathrm{d}v + \int_{|v|>2}\mathrm{d}v\biggr\}\biggl(\int_0^1 \mathrm{e}^{-(s-v)^2/u}\,\mathrm{d}s\biggr)^\alpha =: C\{J_{\gamma 1} + J_{\gamma 2}\}.
\end{aligned}$$

Here,

$$\begin{aligned}
J_{\gamma 1} &\le C \int_0^\infty u^{-(1+\beta+\alpha/2)}\,\mathrm{d}u\biggl(\int_0^1 \mathrm{e}^{-s^2/u}\,\mathrm{d}s\biggr)^\alpha \\
&\le C\biggl(\int_0^1 u^{-(1+\beta)}\,\mathrm{d}u + \int_1^\infty u^{-(1+\beta+\alpha/2)}\,\mathrm{d}u\biggr) < \infty
\end{aligned}$$

since $\beta < 0$, $\beta > -\alpha/2$, while

$$\begin{aligned}
J_{\gamma 2} &\le C \int_0^\infty u^{-(1+\beta+\alpha/2)}\,\mathrm{d}u \int_1^\infty \mathrm{e}^{-v^2/u}\,\mathrm{d}v \\
&\le C \int_0^\infty u^{-(1/2+\beta+\alpha/2)}\,\mathrm{d}u \int_{u^{1/2}}^\infty \mathrm{e}^{-z^2}\,\mathrm{d}z < \infty
\end{aligned}$$

as $\frac{1}{2} + \beta + \frac{\alpha}{2} > 1$ and $\int_{1/u^{1/2}}^\infty \mathrm{e}^{-z^2}\,\mathrm{d}z$ decays exponentially when $u \to 0$. This proves $J_\gamma < \infty$ for $\gamma > 1/2$, $-(\alpha-1)/2 < \beta < 0$.

Consider the case $0 < \gamma < 1/2$, $(\alpha-1)/2 < \beta < \alpha - 1$. Then using $\int_{\mathbb{R}} h_3(w, v, z)\,\mathrm{d}w = \frac{\sqrt{3}}{2\sqrt{z}}\mathrm{e}^{-\sqrt{3z}|v|}$ we obtain

$$\begin{aligned}
J_\gamma &\le C \int_{\mathbb{R}} \mathrm{d}v \int_0^\infty z^\beta\,\mathrm{d}z\biggl(\int_0^1 z^{-1/2}\mathrm{e}^{-\sqrt{z}|s-v|}\,\mathrm{d}s\biggr)^\alpha = C\biggl\{\int_{|v|\le 2}\mathrm{d}v + \cdots + \int_{|v|>2}\mathrm{d}v\cdots\biggr\} \\
&=: C\{J_{\gamma 1} + J_{\gamma 2}\},
\end{aligned}$$

where $J_{\gamma 1} \le C\int_0^\infty z^{\beta-(\alpha/2)}\,\mathrm{d}z(\int_0^1 \mathrm{e}^{-z|s|}\,\mathrm{d}s)^\alpha \le C\int_0^\infty z^{\beta-\alpha}(1-\mathrm{e}^{-\sqrt{z}})^\alpha\,\mathrm{d}z < \infty$ for $0 < \beta < \alpha - 1$ and

$$J_{\gamma 2} \le C \int_1^\infty \mathrm{d}v \int_0^\infty z^{\beta-\alpha/2}\mathrm{e}^{-\sqrt{z}v}\,\mathrm{d}z = C\int_1^\infty v^{\alpha-2-2\beta}\,\mathrm{d}v < \infty$$

since $2 + 2\beta - \alpha > 1$ for $\beta > (\alpha-1)/2$.

Finally, let $0 < \gamma < 1/2$, $-(\alpha-1)/2 < \beta < (\alpha-1)/2$. Then $J_\gamma = C\int_{-\infty}^1 \mathrm{d}u \int_{\mathbb{R}} \mathrm{d}v \int_0^\infty z^\beta\,\mathrm{d}z \times (\int_0^1 h_3(t-u, v, z)\,\mathrm{d}t)^\alpha = C\{\int_{-\infty}^0 \mathrm{d}u + \cdots + \int_0^1 \mathrm{d}u\cdots\} =: C\{J_{\gamma 1} + J_{\gamma 2}\}$. By Minkowski's inequality,

$$
\begin{aligned}
J_{\gamma 1}^{1/\alpha} &\le C \int_0^1 \mathrm{d}t \left\{ \int_0^\infty \mathrm{d}u \int_0^\infty \mathrm{d}v \int_0^\infty h_3^\alpha(t+u, v, z) z^\beta \,\mathrm{d}z \right\}^{1/\alpha} \\
&= C \int_0^1 \mathrm{d}t \left\{ \int_0^\infty \frac{\mathrm{d}u}{(t+u)^{1+\beta+(\alpha-1)/2}} \right\}^{1/\alpha} = \int_0^1 \mathrm{d}t \left\{ \frac{1}{t^{\beta+(\alpha-1)/2}} \right\}^{1/\alpha} < \infty
\end{aligned}
$$

and, similarly,

$$
J_{\gamma 2}^{1/\alpha} \le C \int_0^1 \mathrm{d}t \left\{ \int_0^\infty \mathrm{d}v \int_0^\infty h_3^\alpha(t, v, z) z^\beta \,\mathrm{d}z \right\}^{1/\alpha} = C \int_0^1 \mathrm{d}t \left\{ \frac{1}{t^{\beta+(\alpha+1)/2}} \right\}^{1/\alpha} < \infty
$$

since $|\beta| < (\alpha-1)/2$. This proves $J_\gamma < \infty$, or the existence of $V_{3\gamma}$, for all choices of α, β, γ in (3.2). The fact that linear combinations of integrals in (3.1) are α-stable is well known ([46]). Stationarity of increments of (3.1) is an easy consequence of the integrand (3.2) and the control measure μ. This proves part (i).

(ii) The OSRF property is immediate from the scaling properties $h_3(\lambda u, \sqrt{\lambda} v, \lambda^{-1} z) = \lambda^{-1/2} h_3(u, v, z)$ of the kernel h_3 in (1.13) and $\{\mathcal{M}(\mathrm{d}\lambda u, \mathrm{d}\lambda^\gamma v, \mathrm{d}\lambda^{-1} z)\} \overset{\text{f.d.d.}}{=} \{\lambda^{(\gamma-\beta)/\alpha} \mathcal{M}(\mathrm{d}u, \mathrm{d}v, \mathrm{d}z)\}$ of the stable random measure $\mathcal{M}$, the last property being a consequence of the scaling property of $\mu(\mathrm{d}\lambda u, \mathrm{d}\lambda^\gamma v, \mathrm{d}\lambda^{-1} z) = \lambda^{\gamma-\beta} \mu(\mathrm{d}u, \mathrm{d}v, \mathrm{d}z)$ of the control measure μ.

(iii) Let $\gamma = \gamma_0 := 1/2$. Consider arbitrary rectangles $K_i = K_{(\xi_i, \eta_i); (x_i, y_i)} \subset \mathbb{R}_+^2, i = 1, 2$, and write $\int = \int_{\mathbb{R}^2 \times \mathbb{R}_+}$. Then $V_{3\gamma_0}(K_i) = \int G_{K_i}(u, v, z) \,\mathrm{d}\mathcal{M}$, where $G_{K_i}(u, v, z) := \int_{K_i} h_3(t-u, s-v, z) \,\mathrm{d}t \,\mathrm{d}s$. Note $G_{K_i} \ge 0$ and $G_{K_i}(u, v, z) > 0$ for any $u < x_i$ implying $\operatorname{supp}(G_{K_1}) \cap \operatorname{supp}(G_{K_2}) \ne \varnothing$. Hence, and from ([46], Theorem 3.5.3, page 128) it follows that the increments $V_{3\gamma_0}(K_i), i = 1, 2$ on arbitrary nonempty rectangles K_1, K_2 are dependent. It is also easy to show that $V_{3\gamma_0}$ does not have invariant rectangular increments in any direction. This proves (iii) for $\gamma = 1/2$.

Next, let $\gamma > 1/2, 0 < \beta < \alpha - 1$. Similarly as above, for any rectangle $K = K_{(\xi, \eta); (x, y)} \subset \mathbb{R}_+^2$, we have $V_{3\gamma}(K) = \int G_{K,\gamma}(u, v, z) \,\mathrm{d}\mathcal{M}$, where $G_{K,\gamma}(u, v, z) := \mathbf{1}(\eta < v \le y) \int_\xi^\eta h_{3\gamma}(t-u, z) \,\mathrm{d}t$. Clearly, if $K_i, i = 1, 2$ are any two rectangles separated by a horizontal line, then $\operatorname{supp}(G_{K_1,\gamma}) \cap \operatorname{supp}(G_{K_2}) = \varnothing$, implying independence of $V_{3\gamma}(K_1)$ and $V_{3\gamma}(K_2)$. Thus, $V_{3\gamma}$ for $0 < \beta < \alpha - 1$ has independent increments in the vertical direction. The fact that $V_{3\gamma}$ for $\gamma > 1/2, -(\alpha-1)/2 < \beta < 0$ has invariant increments in the horizontal direction is obvious from (3.1) and (3.2). The properties of $V_{3\gamma}$ in the case $0 < \gamma < 1/2$ are completely analogous.

(iv) Follows from (3.1) and (3.2).

(v) Since $V_{3,\pm}$ for $\alpha = 2$ are zero mean Gaussian RFs, it suffices to show that their covariances agree with that of fractional Brownian sheet in (2.4). This can be easily verified by using self-similarity and stationarity of increments properties stated in (i) and (ii), as follows.

Let $0 < \beta < 1$ and $\rho_+(x, x') := \mathrm{E} V_{3,+}(x, 1) V_{3,+}(x', 1), x, x' \ge 0$. By (3.1) and (3.2), $\mathrm{E} V_{3,+}(x, y) V_{3,+}(x', y') = (y \wedge y') \rho_+(x, x'), (x, y), (x', y') \in \mathbb{R}_+^2$. According to (ii), for any $\lambda > 0$

$$
\begin{aligned}
\rho_+(\lambda x, \lambda x') &= \mathrm{E} V_{3,+}(\lambda x, 1) V_{3,+}(\lambda x', 1) = \lambda^{2H(\gamma)} \mathrm{E} V_{3,+}(x, \lambda^{-\gamma}) V_{3,+}(x', \lambda^{-\gamma}) \\
&= \lambda^{2H(\gamma)-\gamma} \mathrm{E} V_{3,+}(x, 1) V_{3,+}(x', 1) = \lambda^{2H_+} \rho(x, x'),
\end{aligned} \tag{3.6}
$$

where $H_+ := H(\gamma) - (\gamma/2) = 1 - (\beta/2)$; see (3.3). The stationarity of rectangular increments property of RF $V_{3,+}$ implies that the process $\{V_{3,+}(x,1), x \ge 0\}$ has stationary increments. Together with the scaling property in (3.6), this implies that $\rho_+(x,x') = (\kappa_+^2/2)(x^{2H_+} + x'^{2H_+} - |x - x'|^{2H_+})$, $x, x' \ge 0$, or $\mathrm{E}V_{3,+}(x,y)V_{3,+}(x',y') = \kappa_+^2 \mathrm{E}B_{1-(\beta/2),1/2}(x,y)B_{1-(\beta/2),1/2}(x',y')$, see (2.4). The remaining relations in (v) are analogous. Proposition 3.1 is proved. □

The main result of this section is Theorem 3.1. Its proof is based on the asymptotics of the Green function g_3 in Lemma 3.1, below. The proof of Lemma 3.1 can be found at http://arxiv.org/abs/1303.2209v3.

Lemma 3.1. *For any* $(t,s,z) \in (0,\infty) \times \mathbb{R} \times (0,\infty)$ *the point-wise convergence in* (1.11) *holds. This convergence is uniform on any relatively compact set* $\{\epsilon < t < 1/\epsilon, \epsilon < |s| < 1/\epsilon, \epsilon < z < 1/\epsilon\} \subset (0,\infty) \times \mathbb{R} \times (0,\infty)$, $\epsilon > 0$.

Moreover, there exist constants $C, c > 0$ *such that for all sufficiently large* λ *and any* (t,s,z), $t > 0$, $s \in \mathbb{R}$, $0 < z < \lambda$ *the following inequality holds*:

$$\sqrt{\lambda} g_3\left([\lambda t], [\sqrt{\lambda} s], 1 - \frac{z}{\lambda}\right) < C\left(\bar{h}_3(t,s,z) + \sqrt{\lambda} \mathrm{e}^{-zt - c(\lambda t)^{1/3} - c(\sqrt{\lambda}|s|)^{1/2}}\right), \tag{3.7}$$

where $\bar{h}_3(t,s,z) := \frac{1}{\sqrt{t}} \mathrm{e}^{-zt - s^2/(16t)}$, $(t,s,z) \in (0,\infty) \times \mathbb{R} \times (0,\infty)$.

Theorem 3.1. *Assume that the mixing density* ϕ *is bounded on any interval* $[0, 1-\epsilon)$, $\epsilon > 0$ *and satisfies* (1.8), *where*

$$-(\alpha - 1)/2 < \beta < \alpha - 1, \qquad 1 < \alpha \le 2, \beta \ne 0, \beta \ne (\alpha - 1)/2. \tag{3.8}$$

Let $\mathfrak{X}_3$ *be the aggregated RF in* (1.9). *Then for any* $\gamma > 0$

$$n^{-H(\gamma)} \sum_{t=1}^{[nx]} \sum_{s=1}^{[n^\gamma y]} \mathfrak{X}_3(t,s) \xrightarrow{\text{f.d.d.}} V_{3\gamma}(x,y), \qquad x, y > 0, n \to \infty, \tag{3.9}$$

where $H(\gamma)$ *and* $V_{3\gamma}$ *are given in* (3.3) *and* (3.1), *respectively. As a consequence, the RF* $\mathfrak{X}_3$ *exhibits scaling transition at* $\gamma_0 = 1/2$ *and enjoys Type I anisotropic distributional LRD with* $\gamma_0 = 1/2$ *in the sense of Definition* 2.4.

Remark 3.2. As it follows from the proof of Theorem 3.1, for $\gamma = 1/2$ the limit in (3.9) exists also when $\beta = 0$ or $\beta = (\alpha - 1)/2$ and is given in (3.1) as in the remaining cases. On the other hand, the existence of the scaling limit (3.9) in the cases $\gamma > 1/2, \beta = 0$ and $0 < \gamma < 1/2$ and $\beta = (\alpha - 1)/2$ is an open and delicate question. Note a sharp transition in the dependence structure of the limit fields $V_{3,+}$ and $V_{3,-}$ in the vicinity of $\beta = 0$ and $\beta = (\alpha - 1)/2$, respectively, changing abruptly from independent rectangular increments in one direction to invariant (completely dependent) rectangular increments in the perpendicular direction. For $\alpha = 2$, the above transition may be related to the fact that the covariance functions of the "vertical" and "horizontal sectional processes" $\{\mathfrak{X}_3(0,s); s \in \mathbb{Z}\}$ and $\{\mathfrak{X}_3(t,0); t \in \mathbb{Z}\}$ change their summability properties at respective points $\beta = 0$ and $\beta = 1/2$; see Proposition 3.2 below.

Let $\alpha = 2$ and $r_3(t,s) = \mathrm{E}\mathfrak{X}_3(t,s)\mathfrak{X}_3(0,0)$ be the covariance function of the aggregated Gaussian RF in (1.9). The proof of Proposition 3.2 using Lemma 3.1 can be found in the arXiv version http://arxiv.org/abs/1303.2209v3.

Proposition 3.2. *Assume $\alpha = 2$ and the conditions of Theorem 3.1. Then for any $(t,s) \in \mathbb{R}_0^2$*

$$\lim_{\lambda\to\infty} \lambda^{\beta+1/2} r_3\big([\lambda t], [\sqrt{\lambda} s]\big) = \begin{cases} C_3|s|^{-2\beta-1}\gamma\big(\beta+1/2, s^2/4|t|\big), & t \neq 0, s \neq 0, \\ C_3|s|^{-2\beta-1}\Gamma(\beta+1/2), & t = 0, \\ C_4|t|^{-\beta-1/2}, & s = 0, \end{cases}$$

where $\gamma(\alpha, x) := \int_0^x y^{\alpha-1}\mathrm{e}^{-y}\,\mathrm{d}y$ is incomplete gamma function and $C_3 := \pi^{-1/2}2^{2\beta-1}3^{1-\beta}\sigma^2 \times \phi_1\Gamma(\beta+1)$, $C_4 := 4^{-1/2-\beta}C_3$.

Proof of Theorem 3.1. Write $S_{n\gamma}(x,y)$ for the left-hand side of (3.9). It suffices to prove the convergence of characteristic functions:

$$\mathrm{E}\mathrm{e}^{\mathrm{i}\sum_{j=1}^p \theta_j S_{n\gamma}(x_j,y_j)} \to \mathrm{E}\mathrm{e}^{\mathrm{i}\sum_{j=1}^p \theta_j V_{3\gamma}(x_j,y_j)}, \qquad n \to \infty, \tag{3.10}$$

for any $p \in \mathbb{N}_+$, $\theta_j \in \mathbb{R}$, $(x_j, y_j) \in \mathbb{R}_+^2$, $j = 1, \ldots, p$. We have

$$\mathrm{E}\mathrm{e}^{\mathrm{i}\sum_{j=1}^p \theta_j S_{n\gamma}(x_j,y_j)} = \mathrm{e}^{-J_{n\gamma}}, \qquad \mathrm{E}\mathrm{e}^{\mathrm{i}\sum_{j=1}^p \theta_j V_{3\gamma}(x_j,y_j)} = \mathrm{e}^{-J_\gamma}, \tag{3.11}$$

where

$$\begin{aligned} J_\gamma &:= \int_{\mathbb{R}^2\times\mathbb{R}_+} \big|G_\gamma(u,v,z)\big|^\alpha \,\mathrm{d}\mu, \qquad G_\gamma(u,v,z) := \sum_{j=1}^p \theta_j F_{3\gamma}(x_j,y_j;u,v,z), \\ J_{n\gamma} &:= n^{-H(\gamma)\alpha} \sum_{(u,v)\in\mathbb{Z}^2} \mathrm{E}\left|\sum_{j=1}^p \theta_j \sum_{1\le t\le[nx_j], 1\le s\le[n^\gamma y_j]} g_3(t-u, s-v, a)\right|^\alpha. \end{aligned} \tag{3.12}$$

Thus, (3.10) follows from

$$\lim_{n\to\infty} J_{n\gamma} = J_\gamma. \tag{3.13}$$

To prove (3.13), we write $J_{n\gamma}$ as an integral

$$J_{n\gamma} = \int_{\mathbb{R}^2\times\mathbb{R}_+} \big|G_{n\gamma}(u,v,z)\big|^\alpha \chi_n(z)\mu(\mathrm{d}u,\mathrm{d}v,\mathrm{d}z), \tag{3.14}$$

where the functions χ_n satisfying $\chi_n(z) \to 1 (n \to \infty)$ uniformly in $z > 0$ will be specified later, and where $G_{n\gamma} : \mathbb{R}^2 \times \mathbb{R}_+ \to \mathbb{R}$ are some functions which approach G_γ in (3.12) in the following sense. Let $W_\epsilon := \{(u,v,z) \in \mathbb{R}^2 \times \mathbb{R}_+ : |u| + |v| < 1/\epsilon, \epsilon < z < 1/\epsilon\}$, $W_\epsilon^c := (\mathbb{R}^2 \times \mathbb{R}_+) \setminus W_\epsilon$, $\epsilon > 0$. We will prove that

$$\lim_{n\to\infty} \int_{W_\epsilon} \big|G_{n\gamma}(u,v,z) - G_\gamma(u,v,z)\big|^\alpha \,\mathrm{d}\mu = 0 \qquad \forall\epsilon > 0, \tag{3.15}$$

and

$$\lim_{\epsilon\to 0}\limsup_{n\to\infty}\int_{W_\epsilon^c}\left|G_{n\gamma}(u,v,z)\right|^\alpha \mathrm{d}\mu = 0. \tag{3.16}$$

Since $\mu(W_\epsilon) < \infty$, (3.15) follows from the uniform convergence

$$\lim_{n\to\infty}\sup_{(u,v,z)\in W_\epsilon}\left|G_{n\gamma}(u,v,z) - G_\gamma(u,v,z)\right| = 0 \qquad \forall\epsilon > 0. \tag{3.17}$$

Clearly, (3.15) and (3.16) together with (3.14) and the above mentioned property of χ_n imply (3.13).

The subsequent proof of (3.15) and (3.16) is split into several cases depending on values γ and β.

Case $\gamma = \gamma_0 = 1/2$. In this case, (3.14) holds with

$$\begin{aligned} &G_{n\gamma_0}(u,v,z) \\ &\quad := \sum_{j=1}^p \theta_j \int_0^{\lfloor nx_j\rfloor/n}\int_0^{\lfloor\sqrt{n}y_j\rfloor/\sqrt{n}} \sqrt{n} g_3\left(\lceil nt\rceil - \lceil nu\rceil, \lceil\sqrt{n}s\rceil - \lceil\sqrt{n}v\rceil, 1 - \frac{z}{n}\right)\mathrm{d}t\,\mathrm{d}s \end{aligned} \tag{3.18}$$

and $\chi_n(z) := (z/n)^{-\beta}(\phi(1-z/n)/\phi_1)\mathbf{1}(0<z<n) \to 1$ boundedly on $\mathbb{R}_+$ as $n\to\infty$ according to condition (1.8). To show (3.17), for given $\epsilon_1 > 0$ split $G_{n\gamma_0}(u,v,z) - G_{\gamma_0}(u,v,z) = \sum_{i=1}^3 \Gamma_{ni}(u,v,z)$, where, for $0 < z < n$,

$$\begin{aligned} \Gamma_{n1}(u,v,z) &:= \sum_{j=1}^p \theta_j \int_0^{\lfloor nx_j\rfloor/n}\int_0^{\lfloor\sqrt{n}y_j\rfloor/\sqrt{n}} \left\{\sqrt{n} g_3\left(\lceil nt\rceil - \lceil nu\rceil, \lceil\sqrt{n}s\rceil - \lceil\sqrt{n}v\rceil, 1 - \frac{z}{n}\right)\right. \\ &\qquad \left. - h_3(t-u, s-v, z)\right\}\mathbf{1}\big((t,s)\in D_j(\epsilon_1)\big)\,\mathrm{d}t\,\mathrm{d}s, \\ \Gamma_{n2}(u,v,z) &:= \sum_{j=1}^p \theta_j \int_0^{\lfloor nx_j\rfloor/n}\int_0^{\lfloor\sqrt{n}y_j\rfloor/\sqrt{n}} \sqrt{n} g_3\left(\lceil nt\rceil - \lceil nu\rceil, \lceil\sqrt{n}s\rceil - \lceil\sqrt{n}v\rceil, 1 - \frac{z}{n}\right) \\ &\qquad \times \mathbf{1}\big((t,s)\notin D_j(\epsilon_1)\big)\,\mathrm{d}t\,\mathrm{d}s, \\ \Gamma_{n3}(u,v,z) &:= -\sum_{j=1}^p \theta_j \int_0^{\lfloor nx_j\rfloor/n}\int_0^{\lfloor\sqrt{n}y_j\rfloor/\sqrt{n}} h_3(t-u,s-v,z)\mathbf{1}\big((t,s)\notin D_j(\epsilon_1)\big)\,\mathrm{d}t\,\mathrm{d}s, \end{aligned}$$

and where the sets $D_j(\epsilon_1)$, $j = 1,\dots,p$ (depending on u, v) are defined by

$$D_j(\epsilon_1) := \big\{(t,s)\in(0,x_j]\times(0,y_j] : t - u > \epsilon_1, |s-v| > \epsilon_1\big\}.$$

Relation (3.17) follows from

$$\lim_{n\to\infty}\sup_{(u,v,z)\in W_\epsilon}\left|\Gamma_{n1}(u,v,z)\right| = 0, \tag{3.19}$$

$$\lim_{\epsilon_1\to 0}\limsup_{n\to\infty}\sup_{(u,v,z)\in W_\epsilon}\left|\Gamma_{ni}(u,v,z)\right| = 0, \qquad i = 2,3. \tag{3.20}$$

Here, (3.19) follows from Lemma 3.1. Next, $|\Gamma_{n3}(u,v,z)| \le C\int_0^{\epsilon_1} t^{-1/2}\,\mathrm{d}t + C\int_{\epsilon_1}^1 t^{-1/2}\,\mathrm{d}t \times \int_{|s|<\epsilon_1}\mathrm{d}s = O(\sqrt{\epsilon_1})$, implying (3.20) for $i=3$. Similarly, using (3.7) we obtain $|\Gamma_{n2}(u,v,z)| \le C\sqrt{\epsilon_1} + C\sqrt{n}\int_0^1 \mathrm{e}^{-c(nt)^{1/3}}\,\mathrm{d}t \le C\sqrt{\epsilon_1} + C/\sqrt{n}$. This proves (3.20) for $i=2$, and hence (3.17), too.

Consider (3.16). W.l.g., we can assume $p=1, \theta_1 = x_1 = y_1 = 1$. With (3.18) and (3.7) in mind, we have $0 \le G_{n\gamma_0}(u,v,z) \le C(\bar{G}(u,v,z) + \widetilde{G}_n(u,v,z))$, where

$$
\begin{aligned}
\bar{G}(u,v,z) &:= \int_0^1\int_0^1 \bar{h}_3(t-u, s-v, z)\,\mathrm{d}t\,\mathrm{d}s,\\
\widetilde{G}_n(u,v,z) &:= \sqrt{n}\mathbf{1}(0<z<n)\int_0^1\int_0^1 \mathrm{e}^{-z(t-u)-c(n(t-u))^{1/3}-c(\sqrt{n}|s-v|)^{1/2}}\mathbf{1}(t>u)\,\mathrm{d}t\,\mathrm{d}s,
\end{aligned}
$$

where $c>0$ is the same as in (3.7). Relation (3.16) with $G_{n\gamma}$ replaced by $\bar{G}$ follows from $\bar{G} \in L^\alpha(\mu)$ (see Proposition 5.1, proof of (i)), since $\bar{h}_3(t,s,z)$ and $h_3(t,s,z)$ differ only in constants. Thus, (3.16) follows from

$$
\widetilde{J}_n := \int_{\mathbb{R}^2\times\mathbb{R}_+} \bigl(\widetilde{G}_n(u,v,z)\bigr)^\alpha\,\mathrm{d}\mu = o(1), \qquad n\to\infty. \tag{3.21}
$$

Split $\widetilde{J}_n = \sum_{i=1}^3 I_{ni}$, where

$$
\begin{aligned}
I_{n1} &:= \int_{(-\infty,0]\times\mathbb{R}\times\mathbb{R}_+} (\widetilde{G}_n)^\alpha\,\mathrm{d}\mu,\\
I_{n2} &:= \int_{(0,1]\times[-2,2]\times\mathbb{R}_+} (\widetilde{G}_n)^\alpha\,\mathrm{d}\mu,\\
I_{n3} &:= \int_{(0,1]\times[-2,2]^c\times\mathbb{R}_+} (\widetilde{G}_n)^\alpha\,\mathrm{d}\mu,
\end{aligned}
$$

$[-2,2]^c := \mathbb{R}\setminus[-2,2]$. Using the fact that $\int_{\mathbb{R}} \mathrm{e}^{-cn^{1/4}|s-v|^{1/2}}\,\mathrm{d}v = C/\sqrt{n}$ and Minkowski's inequality,

$$
\begin{aligned}
I_{n1} &\le Cn^{\alpha/2}\biggl\{\int_{(0,1]^2}\mathrm{d}t\,\mathrm{d}s\biggl(\int_{\mathbb{R}_+\times\mathbb{R}\times\mathbb{R}_+} \mathrm{e}^{-\alpha z(t+u)-c\alpha(n(t+u))^{1/3}-c\alpha(\sqrt{n}|s-v|)^{1/2}}\\
&\qquad \times z^\beta\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}z\biggr)^{1/\alpha}\biggr\}^\alpha\\
&\le Cn^{(\alpha-1)/2}\biggl\{\int_0^1\mathrm{d}t\biggl(\int_0^\infty \mathrm{e}^{-c\alpha(n(t+u))^{1/3}}\frac{\mathrm{d}u}{(t+u)^{1+\beta}}\biggr)^{1/\alpha}\biggr\}^\alpha\\
&\le Cn^{-((\alpha+1)/2-\beta)}I,
\end{aligned}
$$

where $\frac{\alpha+1}{2}-\beta>0$ and $I:=\{\int_0^\infty \mathrm{d}t(\int_0^\infty \mathrm{e}^{-c\alpha(t+u)^{1/3}}(t+u)^{-1-\beta}\,\mathrm{d}u)^{1/\alpha}\}^\alpha<\infty$. Next,

$$\begin{aligned}
I_{n2} &\le Cn^{\alpha/2}\int_0^\infty z^\beta\,\mathrm{d}z\left\{\int_{(0,4]^2}\mathrm{e}^{-zt-c(nt)^{1/3}-c(\sqrt{n}|s|)^{1/2}}\,\mathrm{d}t\,\mathrm{d}s\right\}^\alpha \\
&\le C\left\{\int_0^4 \mathrm{e}^{-c(nt)^{1/3}}\,\mathrm{d}t\left(\int_0^\infty \mathrm{e}^{-\alpha zt}z^\beta\,\mathrm{d}z\right)^{1/\alpha}\right\}^\alpha \\
&\le C\left\{\int_0^\infty \mathrm{e}^{-c(nt)^{1/3}}t^{-(1+\beta)/\alpha}\,\mathrm{d}t\right\}^\alpha \le Cn^{-(\alpha-1-\beta)}=o(1).
\end{aligned}$$

Finally, using $\mathrm{e}^{-c(\sqrt{n}|s-v|)^{1/2}}\le \mathrm{e}^{-(c/2)(\sqrt{n}|v|)^{1/2}}$ for $|v|\ge 2, |s|\le 1$ it easily follows $I_{n3}=O(\mathrm{e}^{-c'n^{1/4}})=o(1)(\exists c'>0)$, thus completing the proof of (3.21) and (3.13) for $\gamma=\gamma_0=1/2$.

Case $\gamma>1/2, 0<\beta<\alpha-1$. In this case, (3.14) holds with

$$\begin{aligned}
&G_{n\gamma}(u,v,z) \\
&:=\sum_{j=1}^p\theta_j\int_0^{\lfloor nx_j\rfloor/n}\mathrm{d}t n^{-1/2}\sum_{s=1}^{\lfloor n^\gamma y_j\rfloor} g_3\left(\lceil nt\rceil-\lceil nu\rceil, s-\lceil n^\gamma v\rceil, 1-\frac{z}{n}\right)\mathbf{1}(0<z<n) \\
&=\sum_{j=1}^p\theta_j\int_0^{\lfloor nx_j\rfloor/n}\mathrm{d}t\int_{\mathbb{R}}\mathrm{d}s\sqrt{n}g_3\left(\lceil nt\rceil-\lceil nu\rceil, \lceil\sqrt{n}s\rceil, 1-\frac{z}{n}\right) \qquad (3.22)\\
&\quad\times\mathbf{1}\big(0<z<n, 1-\lceil n^\gamma v\rceil\le\lceil\sqrt{n}s\rceil\le\lfloor n^\gamma y_j\rfloor-\lceil n^\gamma v\rceil\big) \\
&=:\sum_{j=1}^p\theta_j\int_0^{x_j}\mathrm{d}t\int_{\mathbb{R}}\mathrm{d}s f_{nj}(t,s,u,v,z).
\end{aligned}$$

We first check the point-wise convergence: for any $(u,z)\in\mathbb{R}\times\mathbb{R}_+$, $v\in\mathbb{R}\setminus\{0,y_j\}$, $j=1,\dots,p$

$$\begin{aligned}
&G_{n\gamma}(u,v,z)\to G_\gamma(u,v,z) \\
&\qquad:=\sum_{j=1}^p\theta_j\int_0^{x_j}\mathrm{d}t\int_{\mathbb{R}}\mathrm{d}s h_3(t-u,s,z)\mathbf{1}(0<v<y_j), \qquad n\to\infty.
\end{aligned} \qquad (3.23)$$

To prove (3.23), note that from (1.11), (3.7) and $\gamma>1/2$, for any $u<t\in\mathbb{R}$, $v\in\mathbb{R}\setminus\{0,y_j\}$, $j=1,\dots,p$, $s\in\mathbb{R}$, and $z>0$, we have the point-wise convergences (as $n\to\infty$)

$$\begin{aligned}
\sqrt{n}g_3\left(\lceil nt\rceil-\lceil nu\rceil, \lceil\sqrt{n}s\rceil, 1-\frac{z}{n}\right)\mathbf{1}(0<z<n)&\to h_3(t-u,s,z), \\
\mathbf{1}\big(1-\lceil n^\gamma v\rceil\le\lceil\sqrt{n}s\rceil\le\lfloor n^\gamma y_j\rfloor-\lceil n^\gamma v\rceil\big)&\to\mathbf{1}(0<v<y_j)
\end{aligned}$$

and hence

$$f_{nj}(t,s;u,v,z)\to f_j(t,s;u,v,z):=h_3(t-u,s,z)\mathbf{1}(0<v<y_j). \qquad (3.24)$$

Using (3.24), relation (3.23) can be shown similarly as in the case $\gamma = \gamma_0$ above. Namely, write $G_{n\gamma}(u,v,z) - G_\gamma(u,v,z) = \sum_{i=1}^3 \Gamma_{ni}(u,v,z)$, where, for $0 < z < n$,

$$\Gamma_{n1}(u,v,z) := \sum_{j=1}^p \theta_j \int_0^{\lfloor nx_j\rfloor/n} \mathrm{d}t \int_{\mathbb{R}} \bigl\{f_{nj}(t,s;u,v,z) - f_j(t,s;u,v,z)\bigr\}\mathbf{1}\bigl((t,s)\in D_j(\epsilon_1)\bigr)\,\mathrm{d}s,$$

$$\Gamma_{n2}(u,v,z) := \sum_{j=1}^p \theta_j \int_0^{\lfloor nx_j\rfloor/n} \mathrm{d}t \int_{\mathbb{R}} f_j(t,s;u,v,z)\mathbf{1}\bigl((t,s)\notin D_j(\epsilon_1)\bigr)\,\mathrm{d}s,$$

$$\Gamma_{n3}(u,v,z) := \sum_{j=1}^p \theta_j \int_0^{\lfloor nx_j\rfloor/n} \mathrm{d}t \int_{\mathbb{R}} f_{nj}(t,s;u,v,z)\mathbf{1}\bigl((t,s)\notin D_j(\epsilon_1)\bigr)\,\mathrm{d}s,$$

and where $D_j(\epsilon_1) := \{(t,s)\in(0,x_j]\times\mathbb{R} : t-u > \epsilon_1, |s-v| > \epsilon_1, |s| < 1/\epsilon_1\}$. Then (3.23) follows if we show that, for any $(u,z)\in\mathbb{R}\times\mathbb{R}_+$, $v\in\mathbb{R}\setminus\{0,y\}$,

$$\begin{aligned} \lim_{n\to\infty}\bigl|\Gamma_{n1}(u,v,z)\bigr| = 0 \qquad \forall\epsilon_1 > 0 \quad \text{and} \\ \lim_{\epsilon_1\to 0}\limsup_{n\to\infty}\bigl|\Gamma_{ni}(u,v,z)\bigr| = 0, \qquad i = 2,3. \end{aligned} \tag{3.25}$$

Here, the first relation in (3.25) follows from the uniform convergence statement of Lemma 3.1, and the second one from the dominating bound in (3.7); in particular,

$$\begin{aligned} &\int_0^{x_j} \mathrm{d}t \int_{\mathbb{R}} f_{nj}(t,s;u,v,z)\mathbf{1}(t-u\le\epsilon_1)\,\mathrm{d}s \\ &\quad\le \int_0^{\epsilon_1+n^{-1}} \mathrm{d}t \int_{\mathbb{R}} \biggl(\frac{1}{\sqrt{t}}\mathrm{e}^{-cs^2/t} + \sqrt{n}\mathrm{e}^{-c(nt)^{1/3}-c(\sqrt{n}|s|)^{1/2}}\biggr)\,\mathrm{d}s \le C\bigl(\epsilon_1+n^{-1}\bigr) \end{aligned}$$

vanishes as $n\to\infty$ and $\epsilon_1\to 0$.

With (3.23) in mind, the convergence of integrals in (3.15) and (3.16) can be established using the dominated convergence theorem and the bound (3.7) of Lemma 3.1, similarly as in the case $\gamma = 1/2$ above.

Case $\gamma > 1/2$, $-(\alpha-1)/2 < \beta < 0$. In this case, (3.14) holds with $\chi_n(z) := (z/n^{2\gamma})^{-\beta}(\phi(1-(z/n^{2\gamma}))/\phi_1)\mathbf{1}(0<z<n^{2\gamma}) \to 1$ and

$$\begin{aligned} G_{n\gamma}(u,v,z) &:= \sum_{j=1}^p \theta_j \int_0^{\lfloor nx_j\rfloor/n} \mathrm{d}t \int_0^{\lfloor n^\gamma y_j\rfloor/n^\gamma} \mathrm{d}s\, n^\gamma \\ &\qquad\times g_3\biggl(\lceil nt\rceil - \lceil n^{2\gamma}u\rceil, \lceil n^\gamma s\rceil - \lceil n^\gamma v\rceil, 1-\frac{z}{n^{2\gamma}}\biggr)\mathbf{1}\bigl(0<z<n^{2\gamma}\bigr) \\ &=: \sum_{j=1}^p \theta_j \int_0^{\lfloor nx_j\rfloor/n} \mathrm{d}t \int_0^{\lfloor n^\gamma y_j\rfloor/n^\gamma} \mathrm{d}s\, f_n(t,s;u,v,z). \end{aligned}$$

Note that in the above integral, variables t and u are rescaled by n and $n^{2\gamma} \gg n$, respectively. Therefore, by (1.11) the integrand

$$f_n(t, s; u, v, z) \to f(s; u, v, z) := h_3(-u, s - v, z) \qquad \text{as } n \to \infty \tag{3.26}$$

converges point-wise to $f(s; u, v, z)$ independent of t, for any $u < 0$, $s, v \in \mathbb{R}$, $s \in \mathbb{R}$, and $z > 0$ fixed. By using (3.26) and splitting $G_{n\gamma}(u, v, z)$ similarly as in the case $\gamma = \gamma_0$ above, we can show the uniform convergence in (3.17) with

$$G_\gamma(u, v, z) := \sum_{j=1}^p \theta_j \int_0^{x_j} \mathrm{d}t \int_0^{y_j} f(s; u, v, z)\,\mathrm{d}s = \sum_{j=1}^p \theta_j x_j \int_0^{y_j} h_3(-u, s - v, z)\,\mathrm{d}s$$

satisfying (3.12); see the definition of $F_{3\gamma}(x, y; u, v, z) = G_\gamma(u, v, z)$ in (3.2). The proof of (3.16) uses the dominating bound (3.7) of Lemma 3.1 similarly as in the previous cases.

Case $0 < \gamma < 1/2$, $(\alpha - 1)/2 < \beta < \alpha - 1$. We have (3.14) with

$$\begin{aligned} &G_{n\gamma}(u, v, z) \\ &\quad := \sum_{j=1}^p \theta_j \int_0^{\lfloor nx_j \rfloor / n^{2\gamma}} \mathrm{d}t \int_0^{\lfloor n^\gamma y_j \rfloor / n^\gamma} \mathrm{d}s\, n^\gamma g_3\biggl(\lceil n^{2\gamma} t \rceil - \lceil nu \rceil, \lceil n^\gamma s \rceil - \lceil n^\gamma v \rceil, 1 - \frac{z}{n^{2\gamma}}\biggr) \\ &\quad = \sum_{j=1}^p \theta_j \int_0^\infty \mathrm{d}w \int_0^{\lfloor n^\gamma y_j \rfloor / n^\gamma} f_{nj}(w, s; u, v, z)\,\mathrm{d}s, \end{aligned}$$

where

$$\begin{aligned} f_{nj}(w, s; u, v, z) &:= n^\gamma g_3\biggl(\lceil n^{2\gamma} w \rceil, \lceil n^\gamma s \rceil - \lceil n^\gamma v \rceil, 1 - \frac{z}{n^{2\gamma}}\biggr) \\ &\qquad \times \mathbf{1}\biggl(\frac{1 - \lceil nu \rceil}{n^{2\gamma}} < w < \frac{\lfloor nx_j \rfloor - \lceil nu \rceil}{n^{2\gamma}}\biggr) \\ &\quad \to \mathbf{1}(0 < u < x_j) h_3(w, s - v, z), \qquad n \to \infty \end{aligned}$$

point-wise for each $u \in \mathbb{R} \setminus \{0, x_j\}$, $w > 0$, $s \in (0, y_j)$, $v \in \mathbb{R}$, $s \neq v$, $z > 0$ fixed, according to Lemma 3.1. This leads to the point-wise convergence of integrals, namely,

$$\begin{aligned} &G_{n\gamma}(u, v, z) \to G_\gamma(u, v, z) \\ &\qquad = \sum_{j=1}^p \theta_j \mathbf{1}(0 < u < x_j) \int_0^\infty \mathrm{d}w \int_0^{y_j} \mathrm{d}s\, h_3(w, s - v, z), \qquad n \to \infty \end{aligned} \tag{3.27}$$

similarly as in (3.23) above. We omit the rest of the proof of (3.15) and (3.16) which uses (3.27), Lemma 3.1 and the dominated convergence theorem.

Case $0 < \gamma < 1/2$, $-(\alpha-1)/2 < \beta < (\alpha-1)/2$. We have (3.14) with

$$G_{n\gamma}(u,v,z) := \sum_{j=1}^{p} \theta_j \int_0^{\lfloor nx_j \rfloor/n} \mathrm{d}t \int_0^{\lfloor n^\gamma y_j \rfloor/n^\gamma} f_n(t,s;u,v,z)\,\mathrm{d}s$$

and

$$\begin{aligned} f_n(t,s;u,v,z) &:= n^{1/2} g_3\left(\lceil nt \rceil - \lceil nu \rceil, \lceil n^\gamma s \rceil - \lceil n^{1/2} v \rceil, 1 - \frac{z}{n}\right) \mathbf{1}(0 < z \le n) \\ &\to h_3(t-u, -v, z) \end{aligned}$$

tending to a limit independent of s for each $t < u, s \in \mathbb{R}, v \in \mathbb{R}, z > 0$ fixed, according to Lemma 3.1 and using the fact that $\sup_{s\in[0,y]} |\frac{\lceil n^\gamma s\rceil - \lceil n^{1/2} v\rceil}{n^{1/2}} - v| \to 0$ for any $y > 0$ as $\gamma < 1/2$. Whence, the point-wise convergence, as $n \to \infty$,

$$\begin{aligned} G_{n\gamma}(u,v,z) \to G_\gamma(u,v,z) &= \sum_{j=1}^{p} \theta_j y_j \int_0^{x_j} h_3(t-u,-v,z)\,\mathrm{d}t \\ &= \sum_{j=1}^{p} \theta_j F_{3\gamma}(x_j, y_j; u, v, z), \end{aligned} \tag{3.28}$$

can be obtained. The details of the proof of (3.28) and subsequently (3.17) and (3.16) are similar as in other cases above.

This proves (3.13), and hence the limit in (3.9) in all cases of γ and β under consideration. The second statement of the theorem follows from (3.9) and Proposition 3.1. Theorem 3.1 is proved. □

4. Scaling transition in the aggregated 4N model

In this section, we discuss scaling transition and Type I isotropic distributional LRD property for the aggregated 4N model $\mathfrak{X}_4$ in (1.9). Recall that $g_4(t,s,a)$ in (1.9) is the Green function of the random walk $\{W_k\}$ on $\mathbb{Z}^2$ with one-step transition probabilities shown in Figure 1(b). Recall that

$$h_4(t,s,z) = \frac{2}{\pi} K_0\big(2\sqrt{z(t^2+s^2)}\big) = \frac{2}{\pi} \int_0^\infty w^{-1} \mathrm{e}^{-zw-(t^2+s^2)/w}\,\mathrm{d}w, \qquad (t,s) \in \mathbb{R}_0^2, z > 0$$

is the potential of the Brownian motion in $\mathbb{R}^2$ with covariance matrix diag$(1/2, 1/2)$, written via K_0, the modified Bessel function of second kind. See [29], Chapter 7.2.

For any $\gamma > 0$, introduce a RF $V_{4\gamma} = \{V_{4\gamma}(x,y); (x,y) \in \bar{\mathbb{R}}_+^2\}$ as a stochastic integral

$$V_{4\gamma}(x,y) := \int_{\mathbb{R}^2 \times \mathbb{R}_+} F_{4\gamma}(x,y;u,v,z) \mathcal{M}(\mathrm{d}u, \mathrm{d}v, \mathrm{d}z), \tag{4.1}$$

where $F_{4\gamma}(x, y; u, v, z)$ is defined as

$$F_{4\gamma} := \begin{cases} \int_0^x \int_0^y h_4(t-u, s-v, z)\,\mathrm{d}t\,\mathrm{d}s, & \gamma = 1, \\ \mathbf{1}(0 < v < y) \int_0^x \mathrm{d}t \int_{\mathbb{R}} h_4(t-u, w, z)\,\mathrm{d}w, & \gamma > 1, \beta > (\alpha-1)/2, \\ \mathbf{1}(0 < u < x) \int_{\mathbb{R}} \mathrm{d}w \int_0^y h_4(w, s-v, z)\,\mathrm{d}s, & \gamma < 1, \beta > (\alpha-1)/2, \\ x \int_0^y h_4(u, s-v, z)\,\mathrm{d}s, & \gamma > 1, 0 < \beta < (\alpha-1)/2, \\ y \int_0^x h_4(t-u, v, z)\,\mathrm{d}t, & \gamma < 1, 0 < \beta < (\alpha-1)/2, \end{cases} \tag{4.2}$$

and where $\mathcal{M}$ is the same α-stable random measure on $\mathbb{R}^2 \times \mathbb{R}_+$ as in (3.1).

Proposition 4.1. (i) $V_{4\gamma}$ *in* (4.1) *is well-defined for any* $\gamma > 0, 1 < \alpha \le 2, 0 < \beta < \alpha - 1$ *with exception of* $\gamma \neq 1, \beta = (\alpha-1)/2$. *It has* α*-stable finite-dimensional distributions and stationary rectangular increments in the sense of* (2.3).

(ii) $V_{4\gamma}$ *is OSRF*: *for any* $\lambda > 0$, $\{V_{4\gamma}(\lambda x, \lambda^\gamma y); (x, y) \in \mathbb{R}_+^2\} \overset{\text{f.d.d.}}{=} \{\lambda^{H(\gamma)} V_{4\gamma}(x, y); (x, y) \in \mathbb{R}_+^2\}$, *with*

$$H(\gamma) := \begin{cases} \dfrac{2(\alpha-\beta)}{\alpha}, & \gamma = 1, \\ \dfrac{\gamma - 1 + 2(\alpha-\beta)}{\alpha}, & \gamma > 1, \beta > (\alpha-1)/2, \\ \dfrac{\alpha + \alpha\gamma - 2\beta\gamma}{\alpha}, & \gamma > 1, \beta < (\alpha-1)/2, \\ \dfrac{1 - \gamma + 2\gamma(\alpha-\beta)}{\alpha}, & \gamma < 1, \beta > (\alpha-1)/2, \\ \dfrac{\alpha + \alpha\gamma - 2\beta}{\alpha}, & \gamma < 1, \beta < (\alpha-1)/2. \end{cases} \tag{4.3}$$

(iii) *RFs* $V_{4\gamma} = V_{4,+}(\gamma > 1)$ *and* $V_{4\gamma} = V_{4,-}(\gamma < 1)$ *do not depend on* γ *for* $\gamma > 1$ *and* $\gamma < 1$.

(iv) *RF* $V_{4\gamma}$ *has properly dependent rectangular increments for* $\gamma = 1$ *and does not have properly dependent rectangular increments for* $\gamma \neq 1$.

(v) *For* $\alpha = 2$, *the RFs*

$$\begin{aligned} V_{4,+} &\overset{\text{f.d.d.}}{=} \kappa_{4,+} \begin{cases} B_{(3/2)-\beta, 1/2}, & 1/2 < \beta < 1, \\ B_{1, 1-\beta}, & 0 < \beta < 1/2, \end{cases} \\ V_{4,-} &\overset{\text{f.d.d.}}{=} \kappa_{4,-} \begin{cases} B_{1/2, (3/2)-\beta}, & 1/2 < \beta < 1, \\ B_{1-\beta, 1}, & 0 < \beta < 1/2, \end{cases} \end{aligned} \tag{4.4}$$

agree, up to some constants $\kappa_{4,\pm} = \kappa_{4,\pm}(\beta) \neq 0$, *with fractional Brownian sheet* B_{H_1, H_2} *where one of the parameters* H_1, H_2 *equals* $1/2$ *or* 1.

Proof. (i) As in the proof of Proposition 3.1(i), we show $J_\gamma := \int_{\mathbb{R}^2\times\mathbb{R}_+}(F_{4\gamma}(1,1;u,v,z))^\alpha\,\mathrm{d}\mu < \infty$ only. First, consider the case $\gamma = 1$. We have $J_1 = C\int_{\mathbb{R}^2\times\mathbb{R}_+}(\int_{(0,1]^2} K_0(2\sqrt{z}\|v-w\|)\,\mathrm{d}v)^\alpha z^\beta\,\mathrm{d}w\,\mathrm{d}z < \infty$. Here, $\|x\|^2 := x_1^2 + x_2^2$, for $x = (x_1,x_2)\in\mathbb{R}^2$. Split $J_1 = J' + J''$, where $J' := \int_{\{\|w\|\le\sqrt{2}\}\times\mathbb{R}_+}\cdots$, $J'' := \int_{\{\|w\|>\sqrt{2}\}\times\mathbb{R}_+}\cdots$. By Minkowski's inequality,

$$
\begin{aligned}
J'' &\le C\biggl\{\int_{\{\|v\|\le\sqrt{2}\}}\mathrm{d}v\biggl[\int_{\{\|w\|>\sqrt{2}\}\times\mathbb{R}_+} K_0^\alpha\bigl(2\sqrt{z}\|v-w\|\bigr)z^\beta\,\mathrm{d}z\,\mathrm{d}w\biggr]^{1/\alpha}\biggr\}^\alpha \\
&\le C\biggl\{\int_{\{\|v\|\le\sqrt{2}\}}\mathrm{d}v\biggl[\int_{\{\|w\|>\sqrt{2}\}}\|v-w\|^{-2-2\beta}\,\mathrm{d}w\biggr]^{1/\alpha}\biggr\}^\alpha \\
&\le C\biggl\{\int_{\{\|v\|\le\sqrt{2}\}}\bigl(\sqrt{2}-\|v\|\bigr)^{-2\beta/\alpha}\,\mathrm{d}v\biggr\}^\alpha < \infty,
\end{aligned}
$$

where we used the facts that $\int_0^\infty K_0^\alpha(2\sqrt{z})z^\beta\,\mathrm{d}z < \infty$ and $0 < \beta < \alpha - 1 \le 2$. Next,

$$
\begin{aligned}
J' &\le C\int_{\{\|w\|\le\sqrt{2}\}}\mathrm{d}w\int_0^\infty z^\beta\,\mathrm{d}z\biggl(\int_{\{\|v\|\le\sqrt{2}\}} K_0\bigl(2\sqrt{z}\|v\|\bigr)\,\mathrm{d}v\biggr)^\alpha \\
&\le C\int_0^\infty z^\beta\,\mathrm{d}z\biggl(\int_0^{\sqrt{2}} K_0(2\sqrt{z}r)r\,\mathrm{d}r\biggr)^\alpha \\
&\le C\int_0^\infty z^\beta\bigl(z^{-\alpha/2}\mathbf{1}(0<z<1) + z^{-\alpha}\mathbf{1}(z\ge 1)\bigr)\,\mathrm{d}z < \infty,
\end{aligned}
$$

where we used $0 < \beta < \alpha - 1$ and the inequality

$$
\int_0^{\sqrt{2}} K_0(2\sqrt{z}r)r\,\mathrm{d}r \le C\begin{cases} z^{-1/2}, & 0 < z \le 1,\\ z^{-1}, & z > 1,\end{cases}
$$

which is a consequence of the fact that the function $r\mapsto rK_0(r)$ is bounded and integrable on $(0,\infty)$. This proves $J_1 < \infty$.

Next, let $\gamma > 1$, $(\alpha-1)/2 < \beta < \alpha - 1$. Using $h_{4\star}(u,z) := \int_{\mathbb{R}} h_4(u,w,z)\,\mathrm{d}w = \frac{2}{\pi}\int_{\mathbb{R}} K_0(2\times\sqrt{z(u^2+w^2)})\,\mathrm{d}w = \frac{2}{\pi}\sqrt{\frac{u}{4z^{1/2}}}K_{-1/2}(2\sqrt{z}|u|) = \sqrt{\frac{1}{4\pi z}}\mathrm{e}^{-2\sqrt{z}|u|}$ ([25], 6.596, 8.469), we obtain $J_\gamma \le C\int_{\mathbb{R}}\mathrm{d}u\int_{\mathbb{R}_+} z^\beta\,\mathrm{d}z(\int_0^1 h_{4\star}(t-u,z)\,\mathrm{d}t)^\alpha \le C\{\int_{|u|\le 2}\cdots + \int_{|u|>2}\cdots\} =: C\{J'_\gamma + J''_\gamma\}$, where

$$
\begin{aligned}
J'_\gamma &\le C\int_0^\infty z^\beta\,\mathrm{d}z\biggl(\int_0^1 h_{4\star}(t,z)\,\mathrm{d}t\biggr)^\alpha \le C\int_0^\infty z^{\beta-(\alpha/2)}\,\mathrm{d}z\biggl(\int_0^1 \mathrm{e}^{-2\sqrt{z}t}\,\mathrm{d}t\biggr)^\alpha \\
&\le C\int_0^\infty z^{\beta-\alpha}\,\mathrm{d}z\bigl(1-\mathrm{e}^{-2\sqrt{z}}\bigr)^\alpha,
\end{aligned}
$$

where the last integral converges for any $0 < \beta < \alpha - 1$, $1 < \alpha \le 2$. Next,

$$
J''_\gamma \le C\int_1^\infty \mathrm{d}u\int_0^\infty z^{\beta-(\alpha/2)}\mathrm{e}^{-2\sqrt{z}u}\,\mathrm{d}z \le C\int_0^\infty z^{\beta-(1+\alpha)/2}\mathrm{e}^{-2z}\,\mathrm{d}z < \infty
$$

provided $\beta > (\alpha-1)/2$ holds. Hence, $J_\gamma < \infty$.

Consider J_γ for $\gamma > 1, 0 < \beta < (\alpha - 1)/2$. We have $J_\gamma \le C \int_{\mathbb{R}} \mathrm{d}u \int_{\mathbb{R}} \mathrm{d}v \int_{\mathbb{R}_+} z^\beta \,\mathrm{d}z (\int_0^1 h_4(u, s - v, z)\,\mathrm{d}s)^\alpha \le C\{\int_{|v|\le 2} \cdots + \int_{|v|>2} \cdots\} =: C\{J'_\gamma + J''_\gamma\}$. By Minkowski's inequality,

$$
\begin{aligned}
J'_\gamma &\le C \int_0^\infty \mathrm{d}u \int_0^\infty z^\beta \mathrm{d}z \biggl(\int_0^1 h_4(u, s, z)\,\mathrm{d}s \biggr)^\alpha \\
&\le C \biggl\{ \int_0^1 \mathrm{d}s \biggl[\int_0^\infty \mathrm{d}u \int_0^\infty z^\beta K_0^\alpha \bigl(2\sqrt{z\bigl(t^2 + u^2\bigr)}\bigr)\,\mathrm{d}z \biggr]^{1/\alpha} \biggr\}^\alpha \\
&\le C \biggl\{ \int_0^1 \mathrm{d}s \biggl[\int_0^\infty \frac{\mathrm{d}u}{(t^2 + u^2)^{\beta+1}} \biggr]^{1/\alpha} \biggr\}^\alpha \le C \biggl\{ \int_0^1 \mathrm{d}s \biggl[\frac{1}{s^{2\beta+1}} \biggr]^{1/\alpha} \biggr\}^\alpha < \infty
\end{aligned}
$$

since $\beta < (\alpha - 1)/2$. Next,

$$
\begin{aligned}
J''_\gamma &\le C \int_1^\infty \mathrm{d}v \int_0^\infty \mathrm{d}u \int_0^\infty z^\beta \,\mathrm{d}z h_4^\alpha(u, v, z) \\
&\le C \int_1^\infty \mathrm{d}v \int_0^\infty \frac{\mathrm{d}u}{(u^2 + v^2)^{\beta+1}} \le C \int_1^\infty \frac{\mathrm{d}v}{v^{1+2\beta}} < \infty.
\end{aligned}
$$

Hence, $J_\gamma < \infty$ for $\gamma > 1$. The case $0 < \gamma < 1$ follows by symmetry. This proves the existence of $V_{4\gamma}$ for all choices of α, β, γ in (4.2). The remaining facts in (i) are similar as in Proposition 3.1.

(ii) Follows analogously as in Proposition 3.1(ii).

(iii) Follows from the definition of the integrand $F_{4\gamma}$ in (4.2).

(iv) The proof is completely similar to that of Proposition 3.1(iii), taking into account the form of $V_{4\gamma}$ in (4.1) and the fact that $h_4(u, v, z)$ is everywhere positive on $\mathbb{R}^2 \times \mathbb{R}_+$.

(v) Follows from the OSRF property in (ii) analogously as in Proposition 3.1(v). Proposition 4.1 is proved. □

The main result of this section is Theorem 4.1. Its proof is based on the asymptotics of the Green function g_4 in Lemma 4.1, below. The proof of Lemma 4.1 can be found at http://arxiv.org/abs/1303.2209v3.

Lemma 4.1. *For any* $(t, s, z) \in \mathbb{R}_0^2 \times (0, \infty)$

$$
\lim_{\lambda\to\infty} g_4\biggl([\lambda t], [\lambda s], 1 - \frac{z}{\lambda^2}\biggr) = h_4(t, s, z) = \frac{2}{\pi} K_0\bigl(2\sqrt{z\bigl(t^2 + s^2\bigr)}\bigr). \tag{4.5}
$$

The convergence in (4.5) *is uniform on any relatively compact set* $\{\epsilon < |t| + |s| < 1/\epsilon\} \times \{\epsilon < z < 1/\epsilon\} \subset \mathbb{R}_0^2 \times \mathbb{R}_+, \epsilon > 0$.

Moreover, there exists constants $C, c > 0$ *such that for all sufficiently large* λ *and any* $(t, s, z) \in \mathbb{R}_0^2 \times (0, \lambda^2)$ *the following inequality holds*:

$$
g_4\biggl([\lambda t], [\lambda s], 1 - \frac{z}{\lambda^2}\biggr) < C\bigl\{h_4(t, s, z) + \mathrm{e}^{-c\sqrt{\lambda}(|t|^{1/2}+|s|^{1/2})}\bigr\}. \tag{4.6}
$$

Theorem 4.1. *Assume that the mixing density ϕ is bounded on $[0, 1)$ and satisfies (1.8), where*

$$0 < \beta < \alpha - 1, \qquad 1 < \alpha \le 2, \qquad \beta \ne (\alpha - 1)/2. \tag{4.7}$$

Let $\mathfrak{X}_4$ be the aggregated 4N model in (1.9). Then for any $\gamma > 0$

$$n^{-H(\gamma)} \sum_{t=1}^{[nx]} \sum_{s=1}^{[n^\gamma y]} \mathfrak{X}_4(t, s) \xrightarrow{\text{f.d.d.}} V_{4\gamma}(x, y), \qquad x, y > 0, n \to \infty, \tag{4.8}$$

where $H(\gamma)$ and $V_{4\gamma}$ are given in (4.3) and (4.1), respectively. As a consequence, the RF $\mathfrak{X}_4$ exhibits scaling transition at $\gamma_0 = 1$ and enjoys Type I isotropic distributional LRD property in the sense of Definition 2.4.

Proof. Similarly, as in the proof of Theorem 3.1, it suffices to prove the limit

$$\lim_{n\to\infty} J_{n\gamma} = J_\gamma, \tag{4.9}$$

where

$$\begin{aligned} J_{n\gamma} &:= n^{-\alpha H(\gamma)} \sum_{(u,v)\in\mathbb{Z}^2} \mathrm{E}\left| \sum_{j=1}^p \theta_j \sum_{1\le t\le [nx_j], 1\le s\le [n^\gamma y_j]} g_4(t-u, s-v, A)\right|^\alpha, \\ J_\gamma &:= \int_{\mathbb{R}^2\times\mathbb{R}_+} \left|G_\gamma(u, v, z)\right|^\alpha \mathrm{d}\mu, \qquad G_\gamma(u, v, z) := \sum_{j=1}^p \theta_j F_{4\gamma}(x_j, y_j; u, v, z), \end{aligned} \tag{4.10}$$

for any $p \in \mathbb{N}_+, \theta_j \in \mathbb{R}, (x_j, y_j) \in \mathbb{R}_+^2, j = 1, \dots, p$. The proof of (4.9) follows the same strategy as in the case of Theorem 3.1, that is, we write $J_{n\gamma}$ as a Riemann sum approximation

$$J_{n\gamma} = \int_{\mathbb{R}^2\times\mathbb{R}_+} \left|G_{n\gamma}(u, v, z)\right|^\alpha \chi_n(z) \mu(\mathrm{d}u, \mathrm{d}v, \mathrm{d}z), \tag{4.11}$$

to the integral J_γ, where $\chi_n(z) \to 1 (n \to \infty)$ boundedly in $z > 0$, and $G_{n\gamma} : \mathbb{R}^2 \times \mathbb{R}_+ \to \mathbb{R}$ are some functions tending to G_γ in (4.10). We use Lemma 4.1 and the dominated convergence theorem to deduce the convergence in (4.9). Because of the differences in the form of the integrand in (4.2), several cases of γ and β need to be discussed separately. The approximation is similar as in the proof of Theorem 3.1 and is discussed briefly below.

For $\epsilon > 0$, denote $W_\epsilon := \{(u, v, z) \in \mathbb{R}^2 \times \mathbb{R}_+ : |u| + |v| < 1/\epsilon, \epsilon < z < 1/\epsilon\}$, $W_\epsilon^c := (\mathbb{R}^2 \times \mathbb{R}_+) \setminus W_\epsilon$. Similarly, as in Theorem 3.1, (4.9) follows from

$$\lim_{n\to\infty} \int_{W_\epsilon} \left|G_{n\gamma}(u, v, z) - G_\gamma(u, v, z)\right|^\alpha \mathrm{d}\mu = 0 \qquad \forall\epsilon > 0, \tag{4.12}$$

and

$$\lim_{\epsilon\to 0} \limsup_{n\to\infty} \int_{W_\epsilon^c} \left|G_{n\gamma}(u, v, z)\right|^\alpha \mathrm{d}\mu = 0. \tag{4.13}$$

Case $\gamma = \gamma_0 = 1$. In this case, (4.10) and (4.11) hold with $G_{\gamma_0}(u,v,z) := \sum_{j=1}^p \theta_j \times \int_0^{x_j}\int_0^{y_j} h_4(t-u, s-v, z)\,\mathrm{d}t\,\mathrm{d}s$ and

$$G_{n\gamma_0}(u,v,z) := \sum_{j=1}^p \theta_j \int_0^{\lfloor nx_j\rfloor/n}\int_0^{\lfloor ny_j\rfloor/n} g_4\biggl(\lceil nt\rceil - \lceil nu\rceil, \lceil ns\rceil - \lceil nv\rceil, 1 - \frac{z}{n^2}\biggr) \times \mathbf{1}\bigl(0 < z < n^2\bigr)\,\mathrm{d}t\,\mathrm{d}s.$$

Then, by splitting $G_{n\gamma_0}(u,v,z) - G_{\gamma_0}(u,v,z) = \sum_{i=1}^3 \Gamma_{ni}(u,v,z)$ and using Lemma 4.1 similarly as in the proof of Theorem 3.1, Case $\gamma = 1/2$, relation (4.12) can be obtained.

Consider (4.13). Since $G_{\gamma_0} \in L^\alpha(\mu)$, see the proof of Proposition 4.1(i), relation (4.13) holds with $G_{n\gamma_0}$ replaced by G_{γ_0}. Hence and with (4.6) in mind, it suffices to check (4.13) with $G_{n\gamma_0}$ replaced by $\widetilde{G}_n(u,v,z) := \mathbf{1}(0 < z < n^2)\int_0^1\int_0^1 \mathrm{e}^{-c(\sqrt{n|t-u|}+\sqrt{n|s-v|})}\,\mathrm{d}t\,\mathrm{d}s$, which follows from

$$\widetilde{J}_n := \int_{\mathbb{R}^2\times\mathbb{R}_+} \bigl(\widetilde{G}_n(u,v,z)\bigr)^\alpha\,\mathrm{d}\mu = O\bigl(n^{2(\beta-\alpha+1)}\bigr) = o(1). \tag{4.14}$$

We have $\widetilde{J}_n \le Cn^{2\beta+2}\{\int_{\mathbb{R}}(\int_0^1 \mathrm{e}^{-c\sqrt{n|t-u|}}\,\mathrm{d}t)^\alpha\,\mathrm{d}u\}^2$, where $\int_{\mathbb{R}}(\int_0^1 \mathrm{e}^{-c\sqrt{n|t-u|}}\,\mathrm{d}t)^\alpha\,\mathrm{d}u \le \int_{\{|u|<2\}}(\cdots)^\alpha\,\mathrm{d}u + \int_{\{|u|\ge 2\}}(\cdots)^\alpha\,\mathrm{d}u =: i_n' + i_n''$. Here, $i_n' \le C(\int_0^3 \mathrm{e}^{-c\sqrt{nv}}\,\mathrm{d}v)^\alpha \le C/n^\alpha$ and $i_n'' \le C\int_2^\infty \mathrm{e}^{-c\alpha\sqrt{n(u-1)}}\,\mathrm{d}u = O(\mathrm{e}^{-c'\sqrt{n}})$, $c' > 0$. This proves (4.14) and (4.13).

Case $\gamma > 1$, $(\alpha-1)/2 < \beta < \alpha - 1$. We have (4.11) with $G_\gamma(u,v,z) = \sum_{j=1}^p \theta_j \mathbf{1}(0 < v < y_j)\int_0^{x_j}\mathrm{d}t\int_{\mathbb{R}} h_4(t-u, s, z)\,\mathrm{d}s$ and

$$\begin{aligned}
G_{n\gamma}(u,v,z) &:= \sum_{j=1}^p \theta_j \int_0^{\lfloor nx_j\rfloor/n} \mathrm{d}t n^{-1} \sum_{s=1}^{\lfloor n^\gamma y_j\rfloor} g_4\biggl(\lceil nt\rceil - \lceil nu\rceil, s - \bigl\lceil n^\gamma v\bigr\rceil, 1 - \frac{z}{n^2}\biggr) \\
&\quad \times \mathbf{1}\bigl(0 < z < n^2\bigr) \\
&= \sum_{j=1}^p \theta_j \int_0^{\lfloor nx_j\rfloor/n} \mathrm{d}t \int_{\mathbb{R}} g_4\biggl(\lceil nt\rceil - \lceil nu\rceil, \lceil ns\rceil, 1 - \frac{z}{n^2}\biggr) \\
&\quad \times \mathbf{1}\bigl(0 < z < n^2, 1 - \bigl\lceil n^\gamma v\bigr\rceil \le \lceil ns\rceil \le \bigl\lfloor n^\gamma y_j\bigr\rfloor - \bigl\lceil n^\gamma v\bigr\rceil\bigr)\,\mathrm{d}s \\
&=: \sum_{j=1}^p \theta_j \int_0^{x_j} \mathrm{d}t \int_{\mathbb{R}} f_{nj}(t,s;u,v,z)\,\mathrm{d}s,
\end{aligned}$$

cf. (3.22). From (1.12), (4.6) and $\gamma > 1$, for any $u, t \in \mathbb{R}$, $u \ne t$, $v \in \mathbb{R}\setminus\{0, y_j\}$, $j = 1,\dots,p$, s, and $z > 0$, we have point-wise convergences

$$\begin{aligned}
&g_4\biggl(\lceil nt\rceil - \lceil nu\rceil, \lceil n\rceil, 1 - \frac{z}{n^2}\biggr)\mathbf{1}\bigl(0 < z < n^2\bigr) \to h_4(t-u, s, z), \\
&\mathbf{1}\bigl(1 - \bigl\lceil n^\gamma v\bigr\rceil \le \lceil ns\rceil \le \bigl\lfloor n^\gamma y_j\bigr\rfloor - \bigl\lceil n^\gamma v\bigr\rceil\bigr) \to \mathbf{1}(0 < v < y_j)
\end{aligned}$$

implying $f_{nj}(t, s, u, v, z) \to h_4(t - u, s, z)\mathbf{1}(0 < v < y_j)$ similarly as in (3.24) in the proof of Theorem 3.1. The remaining details of the proof of (4.12) and (4.13) are similar as in Theorem 3.1, Case $\gamma > 1/2, 0 < \beta < \alpha - 1$.

Case $\gamma > 1, 0 < \beta < (\alpha-1)/2$. We have (4.11) with $G_\gamma(u, v, z) = \sum_{j=1}^{p} \theta_j x_j \int_0^{y_j} h_4(-u, s - v, z)\,\mathrm{d}s$ and

$$
\begin{aligned}
G_{n\gamma}(u, v, z) &:= \sum_{j=1}^{p} \theta_j \int_0^{\lfloor nx_j \rfloor/n} \mathrm{d}t \int_0^{\lfloor n^\gamma y_j \rfloor/n^\gamma} g_4\biggl(\lceil nt \rceil - \lceil n^\gamma u \rceil, \lceil n^\gamma s \rceil - \lceil n^\gamma v \rceil, 1 - \frac{z}{n^{2\gamma}}\biggr) \\
&\quad \times \mathbf{1}\bigl(0 < z < n^{2\gamma}\bigr) \\
&=: \sum_{j=1}^{p} \theta_j \int_0^{\lfloor nx_j \rfloor/n} \mathrm{d}t \int_0^{\lfloor n^\gamma y_j \rfloor/n^\gamma} f_n(t, s; u, v, z)\,\mathrm{d}s,
\end{aligned}
$$

where $f_n(t, s; u, v, z) \to f(s; u, v, z) := h_4(-u, s - v, z)$ tends to a limit independent of t, as $n \to \infty$. Again, we omit the details of the proof of (4.12) and (4.13) which are similar as in Theorem 3.1, Case $\gamma > 1/2, -(\alpha - 1)/2 < \beta < (\alpha - 1)/2$.

Case $0 < \gamma < 1$ in (4.8) follows from case $\gamma > 1$ by lattice isotropy of the 4N model. This ends the proof of (4.8). The second statement of the theorem follows from Proposition 4.1. Theorem 4.1 is proved. □

The following proposition obtains the asymptotic behavior of the covariance function $r_4(t, s) = \mathrm{E}\mathfrak{X}_4(t, s)\mathfrak{X}_4(0, 0)$ of the aggregated Gaussian RF $\mathfrak{X}_4$ in (1.9) ($\alpha = 2$). The proof of Proposition 4.2 uses Lemma 4.1 and is omitted. □

Proposition 4.2. *Assume $\alpha = 2$ and the conditions of Theorem 4.1. Then for any $(t, s) \in \mathbb{R}_0^2$*

$$
\lim_{\lambda\to\infty} \lambda^{2\beta} r_4\bigl([\lambda t], [\lambda s]\bigr) = \frac{\sigma^2 \phi_1 \Gamma(\beta + 1)\Gamma(\beta)}{\pi} \bigl(t^2 + s^2\bigr)^{-\beta}. \tag{4.15}
$$

5. Auxiliary results

This section obtains conditions for the existence of a stationary solution of a general random-coefficient nearest-neighbor autoregressive RF in (5.1). We also discuss contemporaneous aggregation of (5.1) under the assumption that the innovations belong the domain of attraction of α-stable law, $0 < \alpha \le 2$.

5.1. Existence of random-coefficient autoregressive RF

Consider a general random-coefficient nearest-neighbor autoregressive RF on $\mathbb{Z}^2$:

$$
X(t, s) = \sum_{|u|+|v|=1} a(u, v)X(t + u, s + v) + \varepsilon(t, s), \qquad (t, s) \in \mathbb{Z}^2, \tag{5.1}
$$

where $\{\varepsilon(t,s); (t,s) \in \mathbb{Z}^2\}$ are i.i.d. r.v.'s with finite pth moment, $p \in (0,2]$, and $a(t,s) \ge 0$, $|t| + |s| = 1$ are *random* coefficients independent of $\{\varepsilon(t,s)\}$ and satisfying

$$A := \sum_{|t|+|s|=1} a(t,s) \in (0,1) \qquad \text{a.s.} \tag{5.2}$$

Set also $a(t,s) := 0$, $(t,s) \in \mathbb{Z}^2$, $|t| + |s| \ne 1$. Clearly, the 3N and 4N models in (1.6) and (1.7) are particular cases of (5.1).

Let us discuss solvability of (5.1). We will show that under certain conditions this equation admits a stationary solution given by the convergent series

$$X(t,s) = \sum_{(u,v)\in\mathbb{Z}^2} g(t-u, s-v, \mathbf{a})\varepsilon(u,v), \qquad (t,s) \in \mathbb{Z}^2, \tag{5.3}$$

where $g(t,s,\mathbf{a})$, $(t,s) \in \mathbb{Z}^2$, $\mathbf{a} = (a(t,s); |t| + |s| = 1) \in [0,1)^4$ is the (random) Green function defined as

$$g(t,s,\mathbf{a}) := \sum_{k=0}^{\infty} a^{\star k}(t,s), \tag{5.4}$$

where $a^{\star k}(t,s)$ is the k-fold convolution of $a(t,s)$, $(t,s) \in \mathbb{Z}^2$ defined recursively by

$$a^{\star 0}(t,s) = \delta(t,s) := \begin{cases} 1, & (t,s) = (0,0), \\ 0, & (t,s) \ne (0,0), \end{cases}$$

$$a^{\star k}(t,s) = \sum_{(u,v)\in\mathbb{Z}^2} a^{\star(k-1)}(u,v)a(t-u, s-v), \qquad k \ge 1.$$

Note that (5.4) can be rewritten as

$$g(t,s,\mathbf{a}) = \sum_{k=0}^{\infty} A^k p_k(t,s), \tag{5.5}$$

cf. (1.10), where A is defined in (5.2) and $p_k(t,s) = \mathrm{P}(W_k = (t,s)|W_0 = (0,0))$ is the k-step probability of nearest-neighbor random walk $\{W_k, k = 0, 1, \ldots\}$ on $\mathbb{Z}^2$ with one-step transition probabilities

$$p(t,s) := \frac{a(t,s)}{A} \ge 0, \qquad (t,s) \in \mathbb{Z}^2. \tag{5.6}$$

Generally, the $p_k(t,s)$'s depend also on $\mathbf{a} = (a(t,s); |t| + |s| = 1) \in [0,1)^4$ but this dependence is suppressed below for brevity. Note that the series in (5.5) absolutely converges a.s., moreover,

$$\sum_{(t,s)\in\mathbb{Z}^2} g(t,s,\mathbf{a}) = \sum_{k=0}^{\infty} A^k \sum_{(t,s)\in\mathbb{Z}^2} p_k(t,s) = \sum_{k=0}^{\infty} A^k = \frac{1}{1-A} < \infty \qquad \text{a.s.} \tag{5.7}$$

according to (5.2). From (5.7), it follows that the Fourier transforms $\hat{p}(x,y) := \sum_{|t|+|s|=1} \mathrm{e}^{-\mathrm{i}(tx+sy)} p(t,s)$ and

$$\begin{aligned}\hat{g}(x,y,\mathbf{a}) &:= \sum_{(t,s)\in\mathbb{Z}^2} \mathrm{e}^{-\mathrm{i}(tx+sy)} g(t,s,\mathbf{a}) = \sum_{k=0}^{\infty} A^k \sum_{(t,s)\in\mathbb{Z}^2} \mathrm{e}^{-\mathrm{i}(tx+sy)} p_k(t,s) \\ &= \sum_{k=0}^{\infty} A^k \bigl(\hat{p}(x,y)\bigr)^k = \frac{1}{1-A\hat{p}(x,y)}\end{aligned}$$

are well-defined and continuous on $\Pi^2 := [-\pi,\pi]^2$, a.s. From Parseval's identity,

$$\sum_{(t,s)\in\mathbb{Z}^2} \bigl|g(t,s,\mathbf{a})\bigr|^2 = (2\pi)^{-2} \int_{\Pi^2} \frac{\mathrm{d}x\,\mathrm{d}y}{|1-A\hat{p}(x,y)|^2}. \tag{5.8}$$

Let

$$\begin{aligned} q_1 &:= p(0,1)+p(0,-1) = 1-p(1,0)-p(-1,0) =: 1-q_2, \qquad q := \min(q_1,q_2), \\ \mu_1 &:= p(1,0)-p(-1,0), \qquad \mu_2 := p(0,1)-p(0,-1), \qquad \mu := \sqrt{\mu_1^2+\mu_2^2}. \end{aligned} \tag{5.9}$$

Note $q_i \in [0,1]$ and $q_1 = 0$ (resp., $q_2 = 0$) means that random walk $\{W_k\}$ is concentrated on the horizontal (resp., vertical) axis of the lattice $\mathbb{Z}^2$. Condition $\mu = 0$ means that $\{W_k\}$ has zero mean. Denote

$$\Psi(A,q,\mu) := \min\biggl(\frac{1}{q(1-A)}, \frac{1}{\mu\sqrt{q(1-A)}}\biggr)\biggl(1+\log_+\biggl(\frac{\mu^2}{q(1-A)}\biggr)\biggr). \tag{5.10}$$

The main result of this section is Theorem 5.1, below, which provides sharp sufficient conditions for the convergence of the series in (5.3) involving the quantity $\Psi(A,q,\mu)$ in (5.10). The proof of Theorem 5.1 uses the following Lemma 5.1. The proof of this lemma is given at the end of this subsection.

Lemma 5.1. *There exists a (non-random) constant $C < \infty$ such that*

$$\int_{\Pi^2} \frac{\mathrm{d}x\,\mathrm{d}y}{|1-A\hat{p}(x,y)|^2} \le C\Psi(A,q,\mu). \tag{5.11}$$

Theorem 5.1. (i) *Assume there exists $0 < p \le 2$ such that*

$$\mathrm{E}\bigl|\varepsilon(0,0)\bigr|^p < \infty \quad \textit{and} \quad \mathrm{E}\varepsilon(0,0) = 0 \qquad \textit{for } 1 \le p \le 2. \tag{5.12}$$

Then there exists a stationary solution of random-coefficient equation (5.1) given by (5.3), where the series converges conditionally a.s. and in L^p for any $\mathbf{a} = (a(t,s) \ge 0, |t|+|s| = 1) \in [0,1)^4$ satisfying (5.2).

(ii) *In addition to* (5.12), *assume that* $q > 0$ *a.s. and*

$$\begin{cases} \mathrm{E}\big[\Psi(A,q,\mu)^{p-1}(1-A)^{p-2}\big] < \infty, & \text{if } 1 < p \le 2, \\ \mathrm{E}\big[(1-A)^{2p-3}\big] < \infty, & \text{if } 0 < p \le 1. \end{cases} \tag{5.13}$$

Then the series in (5.3) *converges unconditionally in* L^p, *moreover,*

$$\mathrm{E}\big[|X(t,s)|^p\big] \le C \begin{cases} \mathrm{E}\big[\Psi(A,q,\mu)^{p-1}(1-A)^{p-2}\big] < \infty, & 1 < p \le 2, \\ \mathrm{E}\big[(1-A)^{2p-3}\big] < \infty, & 0 < p \le 1. \end{cases} \tag{5.14}$$

Proof. Part (i) follows similarly as in [43], proof of Proposition 1. Let us prove part (ii). We shall use the following inequality; see [50], also [43], (2.7). Let $0 < p \le 2$, and let $\xi_1, \xi_2, \dots$ be random variables with $\mathrm{E}|\xi_i|^p < \infty$. For $1 \le p \le 2$, assume in addition that the ξ_i's are independent and have zero mean $\mathrm{E}\xi_i = 0$. Then $\mathrm{E}|\sum_i \xi_i|^p \le 2\sum_i \mathrm{E}|\xi_i|^p$. The last inequality and the fact that (5.3) converges conditionally in L^p (see part (i)) imply that

$$\mathrm{E}\big[|X(t,s)|^p|\mathbf{a}\big] \le 2\mathrm{E}|\varepsilon(0,0)|^p \sum_{(u,v)\in\mathbb{Z}^2} |g(u,v,\mathbf{a})|^p. \tag{5.15}$$

Accordingly, it suffices to prove that

$$\mathrm{E} \sum_{(t,s)\in\mathbb{Z}^2} |g(t,s,\mathbf{a})|^p < \infty. \tag{5.16}$$

For $p = 2$, (5.16) is immediate from (5.8) and (5.11). Next, using (5.8), (5.11) and Hölder's inequality, for any $1 < p < 2$ we obtain

$$\begin{aligned} \sum_{(t,s)\in\mathbb{Z}^2} |g(t,s,\mathbf{a})|^p &= \sum_{(t,s)\in\mathbb{Z}^2} |g(t,s,\mathbf{a})|^{2(p-1)} |g(t,s,\mathbf{a})|^{2-p} \\ &\le \Big(\sum_{(t,s)\in\mathbb{Z}^2} |g(t,s,\mathbf{a})|^2\Big)^{p-1} \Big(\sum_{(t,s)\in\mathbb{Z}^2} |g(t,s,\mathbf{a})|\Big)^{2-p} \\ &\le C\Psi(A,q,\mu)^{p-1}(1-A)^{p-2}. \end{aligned} \tag{5.17}$$

Next, consider the case $0 < p \le 1$. Using (5.5), the inequality $|\sum_i x_i|^p \le \sum_i |x_i|^p$ and Hölder's inequality, we obtain

$$\begin{aligned} \sum_{(t,s)\in\mathbb{Z}^2} |g(t,s,\mathbf{a})|^p &\le \sum_{k=0}^{\infty} A^{kp} \sum_{|t|+|s|\le k} p_k^p(t,s) \\ &\le \sum_{k=0}^{\infty} A^{kp} \Big\{ \sum_{|t|+|s|\le k} p_k(t,s) \Big\}^p \Big\{ \sum_{|t|+|s|\le k} 1 \Big\}^{1-p} \\ &\le C \sum_{k=0}^{\infty} A^{kp} k^{2(1-p)} \le \frac{C}{(1-A^p)^{3-2p}} \le \frac{C}{(1-A)^{3-2p}}, \end{aligned} \tag{5.18}$$

where the last inequality follows from $1 - x^p \ge p(1-x), x \in [0,1]$. Note that C in (5.17)–(5.18) are non-random. Hence, (5.16) follows from (5.13) and the bounds in (5.17)–(5.18), proving the unconditional convergence of (5.3). Inequality (5.14) is a consequence of (5.17)–(5.18) and (5.15). □

Proof of Lemma 5.1. Write I for the left-hand side of (5.11). Since (5.11) holds trivially for $0 \le A \le 1/2$, we assume $1/2 < A < 1$ in the sequel. We have

$$\begin{aligned} 1 - A\hat{p}(x,y) &= (1-A) + A \sum_{|t|+|s|=1} p(t,s)\big(1 - \mathrm{e}^{\mathrm{i}(tx+sy)}\big) \\ &= (1-A) + A\big[q_2\big(1-\cos(x)\big) + q_1\big(1-\cos(y)\big)\big] - \mathrm{i}A\big(\mu_1 \sin(x) + \mu_2 \sin(y)\big) \end{aligned}$$

and

$$\begin{aligned} \big|1 - A\hat{p}(x,y)\big|^2 &= \big((1-A) + A\big[q_2\big(1-\cos(x)\big) + q_1\big(1-\cos(y)\big)\big]\big)^2 \\ &\quad + A^2\big(\mu_1\sin(x) + \mu_2\sin(y)\big)^2 \\ &\ge (1/4)\big\{\big((1-A) + q\big[\big(1-\cos(x)\big) + \big(1-\cos(y)\big)\big]\big)^2 \\ &\quad + \mu^2\big(\nu_1\sin(x) + \nu_2\sin(y)\big)^2\big\}, \end{aligned} \tag{5.19}$$

where $\nu_i := \mu_i/\mu, i = 1,2, \nu_1^2 + \nu_2^2 = 1$. Split $I = I_1 + I_2$, where $I_1 := \int_{[-\pi/4,\pi/4]^2}$, $I_2 := \int_{\Pi^2 \setminus [-\pi/4,\pi/4]^2}$. Changing the coordinates $\sin(x) = u$, $\sin(y) = v$, $\nu_1 u + \nu_2 v = s$, $-\nu_2 u + \nu_1 v = t$, $r^2 = t^2 + s^2$, $s = r\sin(\phi)$ we get

$$\begin{aligned} I_1 &= C \int_{[-1/\sqrt{2},1/\sqrt{2}]^2} \frac{1}{\sqrt{(1-u^2)(1-v^2)}} \\ &\quad \times \frac{\mathrm{d}u\,\mathrm{d}v}{\{((1-A) + q[(1-\sqrt{1-u^2}) + (1-\sqrt{1-v^2})])^2 + \mu^2(\nu_1 u + \nu_2 v)^2\}} \\ &\le C \int_{u^2+v^2 \le 1} \frac{\mathrm{d}u\,\mathrm{d}v}{((1-A) + q[u^2+v^2])^2 + \mu^2(\nu_1 u + \nu_2 v)^2} \\ &= C \int_{t^2+s^2\le 1} \frac{\mathrm{d}s\,\mathrm{d}t}{((1-A) + q[s^2+t^2])^2 + \mu^2 s^2} \\ &= C \int_0^1 \int_0^{\pi/2} \frac{r\mathrm{d}r\,\mathrm{d}\phi}{((1-A) + qr^2)^2 + \mu^2 r^2 \sin^2(\phi)}. \end{aligned}$$

Using $\sin(\phi) \ge (1/2)\phi, \phi \in [0,\pi/2)$ with $W := \frac{((1-A)+qx)^2}{x}$, we obtain

$$\begin{aligned} I_1 &\le C \int_0^1 \int_0^1 \frac{\mathrm{d}x\,\mathrm{d}y}{((1-A)+qx)^2 + \mu^2 x y^2} \\ &\le C \int_0^1 \frac{\mathrm{d}x}{x} \int_0^1 \frac{\mathrm{d}y}{W + \mu^2 y^2} \end{aligned}$$

$$
\begin{aligned}
&\le C\int_0^1 \frac{\mathrm{d}x}{x\sqrt{W}}\int_0^{1/\sqrt{W}}\frac{\mathrm{d}u}{1+\mu^2u^2}\\
&\le \frac{C}{\mu}\int_0^1 \frac{\mathrm{d}x}{x\sqrt{W}}\min\biggl(1,\frac{\mu}{\sqrt{W}}\biggr)=C\bigl(I_1'+I_1''\bigr),
\end{aligned}
$$

where

$$
I_1' := \frac{1}{\mu}\int_0^1 \frac{\mathrm{d}x}{x\sqrt{W}}\mathbf{1}(\mu>\sqrt{W}),\qquad I_1'' := \int_0^1 \frac{\mathrm{d}x}{xW}\mathbf{1}(\mu<\sqrt{W}).
$$

Here,

$$
\begin{aligned}
I_1'' &\le \min\biggl(\int_0^\infty \frac{\mathrm{d}x}{((1-A)+qx)^2},\frac{1}{\mu}\int_0^\infty \frac{\mathrm{d}x}{x^{1/2}((1-A)+qx)}\biggr)\\
&\le C\min\biggl(\frac{1}{q(1-A)},\frac{1}{\mu\sqrt{q(1-A)}}\biggr).
\end{aligned}
\tag{5.20}
$$

Since $I_1'=0$ for $\mu^2\le q(1-A)$ we obtain

$$
\begin{aligned}
I_1' &\le \frac{1}{\mu}\int_0^1 \frac{\mathrm{d}x}{x^{1/2}((1-A)+qx)}\mathbf{1}\bigl(\mu^2>q(1-A)\bigr)\\
&\le \frac{C}{\mu\sqrt{q(1-A)}}\mathbf{1}\bigl(\mu^2>q(1-A)\bigr).
\end{aligned}
\tag{5.21}
$$

Relations (5.20) and (5.21) yield

$$
I_1\le C\min\biggl(\frac{1}{q(1-A)},\frac{1}{\mu\sqrt{q(1-A)}}\biggr).
\tag{5.22}
$$

Below we prove the bound

$$
I_2\le C\begin{cases}(1-A+q)^{-2}, & \mu\le 1-A+q,\\ \mu^{-1}(1-A+q)^{-1}\bigl(1+\log\bigl(\mu/(1-A+q)\bigr)\bigr), & \mu>1-A+q,\end{cases}
\tag{5.23}
$$

with C independent of A,q,μ, as elsewhere in this proof. Since $1-A+q\ge\sqrt{q(1-A)}$, the desired inequality (5.11), viz., $I\le C\Psi(A,q,\mu)$, follows from (5.22) and (5.23).

Let us prove (5.23). For $\mu\le 1-A+q$ it follows trivially from (5.19). Let $\mu>1-A+q$ in the rest of the proof. From (5.19), we obtain that

$$
\begin{aligned}
I_2 &\le C\int_{\Pi^2\setminus[-\pi/4,\pi/4]^2}\frac{\mathrm{d}x\,\mathrm{d}y}{(1-A+q)^2+\mu^2(\nu_1\sin(x)+\nu_2\sin(y))^2}\\
&\le C\int_{[0,\pi/2]^2}\frac{\mathrm{d}x\,\mathrm{d}y}{(1-A+q)^2+\mu^2(\tilde\nu_1\sin(x)+\tilde\nu_2\sin(y))^2},
\end{aligned}
\tag{5.24}
$$

where $|\tilde{\nu}_i| = |\nu_i|, i = 1, 2$ satisfy $\tilde{\nu}_1^2 + \tilde{\nu}_2^2 = 1$. Then

$$I_2 \le C \int_{[0,1]^2} \frac{(1-u^2)^{-1/2}(1-v^2)^{-1/2}\,\mathrm{d}u\,\mathrm{d}v}{(1-A+q)^2 + \mu^2(\tilde{\nu}_1 u + \tilde{\nu}_2 v)^2}$$
$$\le \frac{C}{\mu^2} \int_{[0,1]^2} \frac{\mathrm{d}u\,\mathrm{d}v}{(\epsilon^2 + (\tilde{\nu}_1 u + \tilde{\nu}_2 v)^2)\sqrt{(1-u)(1-v)}} \qquad \text{with } \epsilon := \frac{1-A+q}{\mu} \ge 0.$$

We claim that

$$\int_{[0,1]^2} \frac{\mathrm{d}u\,\mathrm{d}v}{(\epsilon^2 + (\tilde{\nu}_1 u + \tilde{\nu}_2 v)^2)\sqrt{(1-u)(1-v)}} \le \frac{C}{\epsilon}\bigl(1 + \log_+(1/\epsilon)\bigr) \tag{5.25}$$

with $C < \infty$ independent of $\epsilon > 0$ and $\tilde{\nu}_i, i = 1, 2, \tilde{\nu}_1^2 + \tilde{\nu}_2^2 = 1$. Bound (5.25) proves (5.23) and hence (5.11) and the lemma, too. Therefore, it remains to prove (5.25).

By symmetry, it suffices to prove (5.25) for $\tilde{\nu}_1 \ge 1/\sqrt{2}, 0 \ge \tilde{\nu}_2 \ge -1/\sqrt{2}$, or

$$J := \int_{[0,1]^2} \frac{\mathrm{d}u\,\mathrm{d}v}{(\epsilon^2 + (u - rv)^2)\sqrt{(1-u)(1-v)}} \le \frac{C}{\epsilon}\bigl(1 + \log_+(1/\epsilon)\bigr) \tag{5.26}$$

uniformly in $r \in [0, 1]$.

We have $J = \frac{1}{\sqrt{r}} \int_0^r \frac{\mathrm{d}v}{\sqrt{r-v}} \int_0^1 \frac{\mathrm{d}u}{(\epsilon^2 + (u-v)^2)\sqrt{1-u}} = \frac{1}{\sqrt{r}} \int_0^r \frac{\mathrm{d}v}{\sqrt{r-v}} \int_0^1 \frac{\mathrm{d}z}{(\epsilon^2 + (1-z-v)^2)\sqrt{z}} = \sum_{i,j=1}^2 J_{ij}$, where

$$J_{11} := \frac{1}{\sqrt{r}} \int_0^r \frac{\mathbf{1}(1-v > 2\epsilon)\,\mathrm{d}v}{\sqrt{r-v}} \int_0^1 \frac{\mathbf{1}(|1-z-v| > \epsilon)\,\mathrm{d}z}{(\epsilon^2 + (1-z-v)^2)\sqrt{z}},$$
$$J_{12} := \frac{1}{\sqrt{r}} \int_0^r \frac{\mathbf{1}(1-v > 2\epsilon)\,\mathrm{d}v}{\sqrt{r-v}} \int_0^1 \frac{\mathbf{1}(|1-z-v| < \epsilon)\,\mathrm{d}z}{(\epsilon^2 + (1-z-v)^2)\sqrt{z}},$$
$$J_{21} := \frac{1}{\sqrt{r}} \int_0^r \frac{\mathbf{1}(1-v < 2\epsilon)\,\mathrm{d}v}{\sqrt{r-v}} \int_0^1 \frac{\mathbf{1}(|1-z-v| > \epsilon)\,\mathrm{d}z}{(\epsilon^2 + (1-z-v)^2)\sqrt{z}},$$
$$J_{22} := \frac{1}{\sqrt{r}} \int_0^r \frac{\mathbf{1}(1-v < 2\epsilon)\,\mathrm{d}v}{\sqrt{r-v}} \int_0^1 \frac{\mathbf{1}(|1-z-v| < \epsilon)\,\mathrm{d}z}{(\epsilon^2 + (1-z-v)^2)\sqrt{z}}.$$

Bound (5.26) will be proved for each $J_{ij}, i, j = 1, 2$.

Estimation of J_{22}. We have

$$\begin{aligned}
J_{22} &\le \frac{1}{\sqrt{r}} \int_0^r \frac{\mathbf{1}(1-v < 2\epsilon)\,\mathrm{d}v}{\sqrt{r-v}} \frac{1}{\epsilon^2} \int_0^1 \frac{\mathbf{1}(|1-z-v| < \epsilon)\,\mathrm{d}z}{\sqrt{z}} \\
&\le \frac{1}{\sqrt{r}} \int_0^r \frac{\mathbf{1}(1-v < 2\epsilon)\,\mathrm{d}v}{\sqrt{r-v}} \frac{1}{\epsilon^2} \int_0^{3\epsilon} \frac{\mathrm{d}z}{\sqrt{z}} \le \frac{C}{\sqrt{r}\epsilon^{3/2}} \int_0^r \frac{\mathbf{1}(1-v < 2\epsilon)\,\mathrm{d}v}{\sqrt{r-v}} \\
&\le \frac{C}{\sqrt{r}\epsilon^{3/2}} \int_{r-2\epsilon}^r \frac{\mathrm{d}v}{\sqrt{r-v}} \mathbf{1}(r > 1-2\epsilon) \le \frac{C}{\epsilon}\mathbf{1}(r > 1-2\epsilon).
\end{aligned} \tag{5.27}$$

Estimation of J_{21}. We have

$$
\begin{aligned}
J_{21} &\le \int_0^1 \frac{\mathbf{1}(1-rv<2\epsilon)\,\mathrm{d}v}{\sqrt{1-v}} \int_0^1 \frac{\mathbf{1}(|1-z-rv|>\epsilon)\,\mathrm{d}z}{(1-z-rv)^2\sqrt{z}} \\
&= \int_0^{2\epsilon} \frac{\mathrm{d}x}{\sqrt{x}} \int_0^1 \frac{\mathbf{1}(|x-z|>\epsilon)\,\mathrm{d}z}{(x-z)^2\sqrt{z}} \le \frac{1}{\epsilon}\int_0^2 \frac{\mathrm{d}x}{\sqrt{x}} \int_0^\infty \frac{\mathbf{1}(|x-z|>1)\,\mathrm{d}z}{(x-z)^2\sqrt{z}} \le \frac{C}{\epsilon}
\end{aligned}
\tag{5.28}
$$

since the last double integral converges.

Estimation of J_{12}. We have

$$
\begin{aligned}
J_{12} &\le \frac{1}{\sqrt{r}}\int_0^r \frac{\mathbf{1}(1-v>2\epsilon)\,\mathrm{d}v}{\sqrt{r-v}} \frac{1}{\epsilon^2}\int_0^1 \frac{\mathbf{1}(|(1-v)-z|<\epsilon)\,\mathrm{d}z}{\sqrt{z}} \\
&\le \frac{C}{\sqrt{r}}\int_0^r \frac{\mathbf{1}(1-v>2\epsilon)\,\mathrm{d}v}{\sqrt{r-v}} \frac{1}{\epsilon^2}\int_{1-v-\epsilon}^{1-v+\epsilon} \frac{\mathrm{d}z}{\sqrt{z}} \\
&\le \frac{C}{\sqrt{r}\epsilon^2}\int_0^r \frac{\mathbf{1}(1-v>2\epsilon)\,\mathrm{d}v}{\sqrt{r-v}} (\sqrt{1-v+\epsilon}-\sqrt{1-v-\epsilon}) \\
&\le \frac{C}{\sqrt{r}\epsilon^2}\int_0^r \frac{\mathbf{1}(1-v>2\epsilon)\,\mathrm{d}v}{\sqrt{r-v}} \frac{\epsilon}{\sqrt{1-v}} \le \frac{C\log(1/\epsilon)}{\epsilon}.
\end{aligned}
\tag{5.29}
$$

Indeed, if $r\in[0,1/2]$ then $\frac{1}{\sqrt{r}}\int_0^r \frac{\mathbf{1}(1-v>2\epsilon)\,\mathrm{d}v}{\sqrt{r-v}\sqrt{1-v}} \le \frac{C}{\sqrt{r}}\int_0^r \frac{\mathrm{d}w}{\sqrt{w}} \le C$, and if $r\in[1/2,1]$, $\epsilon\le 1/2$ then with $z=w-(2\epsilon+r-1)$

$$
\begin{aligned}
\frac{1}{\sqrt{r}}\int_0^r \frac{\mathbf{1}(1-v>2\epsilon)\,\mathrm{d}v}{\sqrt{r-v}\sqrt{1-v}} &\le C\int_0^r \frac{\mathbf{1}(w>2\epsilon+r-1)\,\mathrm{d}w}{\sqrt{w}\sqrt{1-r+w}} \\
&\le C\begin{cases} \displaystyle\int_0^1 \frac{\mathrm{d}z}{\sqrt{z}\sqrt{z+2\epsilon}}, & 2\epsilon>1-r, \\ \displaystyle\int_0^r \frac{\mathrm{d}w}{\sqrt{w}\sqrt{1-r+w}}, & 1-r\ge 2\epsilon, \end{cases} \le C\log(1/\epsilon).
\end{aligned}
$$

Estimation of J_{11}. We have

$$
\begin{aligned}
J_{11} &\le \frac{1}{\sqrt{r}}\int_{1-r}^1 \frac{\mathbf{1}(w>2\epsilon)\,\mathrm{d}w}{\sqrt{w-(1-r)}}\int_0^1 \frac{\mathbf{1}(|z-w|>\epsilon)\,\mathrm{d}z}{(z-w)^2\sqrt{z}} \\
&\le \frac{C}{\epsilon\sqrt{r}}\int_{(1-r)/\epsilon}^{1/\epsilon} \frac{L(w)\mathbf{1}(w>2)\,\mathrm{d}w}{\sqrt{w-(1-r)/\epsilon}},
\end{aligned}
\tag{5.30}
$$

where $L(w):=\int_0^\infty \frac{\mathbf{1}(|z-w|>1)\,\mathrm{d}z}{(z-w)^2\sqrt{z}} \le Cw^{-1/2}$ for $w\ge 1$. W.l.g., let $\epsilon\in(0,1/2]$. First, let $(1-r)/\epsilon<1$, then $r\in(1/2,1]$ and $w-\frac{1-r}{\epsilon}>w/2$ for $w>2$. The above facts imply that

$$
J_{11} \le \frac{C}{\epsilon}\int_1^{1/\epsilon}\frac{\mathrm{d}w}{w} \le \frac{C\log(1/\epsilon)}{\epsilon} \qquad \text{when } (1-r)/\epsilon<1, \tag{5.31}
$$

with C independent of r, ϵ. Next, let $(1-r)/\epsilon \ge 1$ then from (5.30), $L(w) = O(w^{-1/2})$ and the change of variables $w - \frac{1-r}{\epsilon} = \frac{1-r}{\epsilon} x$ we obtain

$$\begin{aligned} J_{11} &\le \frac{C}{\epsilon\sqrt{r}} \int_0^{r/(1-r)} \frac{\mathrm{d}x}{\sqrt{x}\sqrt{1+x}} \le \frac{C}{\epsilon\sqrt{r}} \begin{cases} \bigl(r/(1-r)\bigr)^{1/2}, & r \in [0, 3/4], \\ \log\bigl(r/(1-r)\bigr), & r \in [3/4, 1], \end{cases} \\ &\le \frac{C \log(1/\epsilon)}{\epsilon} \qquad \text{when } (1-r)/\epsilon \ge 1. \end{aligned} \tag{5.32}$$

Bounds (5.31), (5.32) prove (5.26) for J_{11}, thereby completing the proof of (5.26). Lemma 5.1 is proved. □

5.2. Aggregation of autoregressive RF

Definition 5.2. *Write $\varepsilon \in D(\alpha)$, $0 < \alpha \le 2$ if*

(i) $\alpha = 2$ *and* $\mathrm{E}\varepsilon = 0, \sigma^2 := \mathrm{E}\varepsilon^2 < \infty$, *or*
(ii) $0 < \alpha < 2$ *and there exist some constants* $c_1, c_2 \ge 0, c_1 + c_2 > 0$ *such that*

$$\lim_{x\to\infty} x^\alpha \mathrm{P}(\varepsilon > x) = c_1 \quad \textit{and} \quad \lim_{x\to-\infty} |x|^\alpha \mathrm{P}(\varepsilon \le x) = c_2;$$

moreover, $\mathrm{E}\varepsilon = 0$ *whenever* $1 < \alpha < 2$, *while for* $\alpha = 1$ *we assume that the distribution of* ε *is symmetric.*

Remark 5.1. Condition $\varepsilon \in D(\alpha)$ implies that the r.v. ε belongs to the domain of normal attraction of an α-stable law; in other words,

$$N^{-1/\alpha} \sum_{i=1}^N \varepsilon_i \xrightarrow{\mathrm{d}} Z, \qquad N \to \infty, \tag{5.33}$$

where Z is an α-stable r.v.; see [18], pages 574–581. The characteristic function of r.v. Z in (5.33) is given by

$$\mathrm{E}\mathrm{e}^{\mathrm{i}\theta Z} = \mathrm{e}^{-|\theta|^\alpha \omega(\theta)}, \qquad \theta \in \mathbb{R}, \tag{5.34}$$

where $\omega(\theta)$ depends only on $\mathrm{sign}(\theta)$ and α, c_1, c_2, σ in Definition 5.2. See, for example, [18], pages 574–581.

Let $\{X_i(t,s)\}, i = 1, 2, \ldots$ be independent copies of (5.3) with i.i.d. innovations $\varepsilon(t,s) \in D(\alpha), 0 < \alpha \le 2$. The aggregated field $\{\mathfrak{X}(t,s); (t,s) \in \mathbb{Z}^2\}$ is defined as the limit in distribution:

$$N^{-1/\alpha} \sum_{i=1}^N X_i(t,s) \xrightarrow{\text{f.d.d.}} \mathfrak{X}(t,s), \qquad (t,s) \in \mathbb{Z}^2, N \to \infty. \tag{5.35}$$

Introduce an independently scattered α-stable random measure M on $\mathbb{Z}^2 \times [0,1)^4$ with characteristic functional

$$\mathrm{E}\exp\biggl\{\mathrm{i}\sum_{(t,s)\in\mathbb{Z}^2}\theta_{t,s}M_{t,s}(B_{t,s})\biggr\} = \exp\biggl\{-\sum_{(t,s)\in\mathbb{Z}^2}|\theta_{t,s}|^\alpha\omega(\theta_{t,s})\Phi(B_{t,s})\biggr\}, \tag{5.36}$$

where $\Phi(B) := \mathrm{P}(\mathbf{a} = (a(t,s), |t|+|s| = 1) \in B)$ is the mixing distribution, $\theta_{t,s} \in \mathbb{R}$, B, $B_{t,s} \subset [0,1)^4$ are arbitrary Borel sets, and ω is the same as in (5.34). According to the terminology in [46], Definition 3.3.1, M is called an α-stable measure with control measure $\mathrm{Re}(\omega(1))\Phi(\mathrm{d}\mathbf{a})$ proportional to the mixing distribution Φ, and a constant skewness intensity $\mathrm{Im}(\omega(1))/\mathrm{Re}(\omega(1))\tan(\pi\alpha/2)$.

Proposition 5.1. *Let $\varepsilon(0,0) \in D(\alpha)$, $0 < \alpha \le 2$. Assume that the mixing distribution satisfies the following condition: there exists $\epsilon > 0$ such that*

$$\begin{cases} \mathrm{E}\bigl[\Psi(A,q,\mu)\bigr] < \infty, & \text{if } \alpha = 2, \\ \mathrm{E}\bigl[\Psi^{p-1}(A,q,\mu)(1-A)^{p-2}\bigr] < \infty, & \text{if } 1 < \alpha < 2,\ p = \alpha \pm \epsilon, \\ \mathrm{E}\bigl[(1-A)^{2\alpha-3-\epsilon}\bigr] < \infty, & \text{if } 0 < \alpha \le 1, \end{cases} \tag{5.37}$$

where $\Psi(A,q,\mu)$ is defined in (5.10). Then the limit aggregated RF in (5.35) exists and has the stochastic integral representation

$$\mathfrak{X}(t,s) = \sum_{(u,v)\in\mathbb{Z}^2}\int_{[0,1)^4} g(t-u, s-v, \mathbf{a})M_{u,v}(\mathrm{d}\mathbf{a}), \qquad (t,s) \in \mathbb{Z}^2. \tag{5.38}$$

Remark 5.2. Note for the 3N and 4N models, we have $\mu = 1, q = 1/3, \Psi(A,1/3,1) \le \frac{C}{\sqrt{1-A}}(1 + |\log(1-A)|)$ and $\mu = 0, q = 1/4, \Psi(A,1/4,0) \le C/(1-A)$, respectively. As a consequence, for the aggregated 3N and 4N models and a regularly varying (mixing) density of A in (1.8), condition (5.37) for $1 < \alpha \le 2$ reduces to $\beta > -(\alpha-1)/2$ and $\beta > 0$, respectively.

Proof of Proposition 5.1. Let $T \subset \mathbb{Z}^2$ be a finite set, $\theta_{t,s} \in \mathbb{R}, (t,s) \in T$, and $S_N = N^{-1/\alpha}\sum_{i=1}^N U_i$ be a sum of i.i.d. r.v.'s with common distribution

$$U := \sum_{(t,s)\in T}\theta_{t,s}X(t,s) = \sum_{(u,v)\in\mathbb{Z}^2} G(u,v,\mathbf{a})\varepsilon(u,v),$$

$$G(u,v,\mathbf{a}) := \sum_{(t,s)\in T}\theta_{t,s}g(t-u,s-v,\mathbf{a}).$$

It suffices to prove that $S_N \xrightarrow{\mathrm{d}} S (N \to \infty)$, where $S := \sum_{(t,s)\in T}\theta_{t,s}\mathfrak{X}(t,s)$ is a α-stable r.v. with characteristic function

$$\mathrm{E}e^{\mathrm{i}wS} = \exp\biggl\{-|w|^\alpha\sum_{(u,v)\in\mathbb{Z}^2}\mathrm{E}\bigl[\bigl|G(u,v,\mathbf{a})\bigr|^\alpha\omega\bigl(wG(u,v,\mathbf{a})\bigr)\bigr]\biggr\}.$$

For this, it suffices to prove that r.v. U belongs to the domain of attraction of r.v. S (in the sense of (5.33)) or $U \in D(\alpha)$, see Remark 5.1; in other words, that

$$\mathrm{E}U^2 = \mathrm{E}S^2 < \infty \qquad \text{for } \alpha = 2, \tag{5.39}$$

and, for $0 < \alpha < 2$,

$$\begin{aligned} \lim_{x\to\infty} x^\alpha \mathrm{P}(U > x) &= \sum_{(u,v)\in\mathbb{Z}^2} \mathrm{E}\big[\big|G(u,v,\mathbf{a})\big|^\alpha \big\{c_1 \mathbf{1}\big(G(u,v,\mathbf{a}) > 0\big) + c_2 \mathbf{1}\big(G(u,v,\mathbf{a}) < 0\big)\big\}\big], \\ \lim_{x\to-\infty} |x|^\alpha \mathrm{P}(U \le x) &= \sum_{(u,v)\in\mathbb{Z}^2} \mathrm{E}\big[\big|G(u,v,\mathbf{a})\big|^\alpha \big\{c_1 \mathbf{1}\big(G(u,v,\mathbf{a}) < 0\big) + c_2 \mathbf{1}\big(G(u,v,\mathbf{a}) > 0\big)\big\}\big], \end{aligned} \tag{5.40}$$

where $c_i, i = 1, 2$ are the asymptotic constants in Definition 5.2 satisfied by $\varepsilon(0,0) \sim D(\alpha)$. Here, (5.39) follows from definitions of U and S and Theorem 5.1 with $p = 2$. To prove (5.40), we use [28], Theorem 3.1. Accordingly, it suffices to check that there exists $\epsilon > 0$ such that for $0 < \alpha < 2, \alpha \neq 1$,

$$\sum_{(u,v)\in\mathbb{Z}^2} \mathrm{E}\big|G(u,v,\mathbf{a})\big|^{\alpha+\epsilon} < \infty \quad \text{and} \quad \sum_{(u,v)\in\mathbb{Z}^2} \mathrm{E}\big|G(u,v,\mathbf{a})\big|^{\alpha-\epsilon} < \infty, \tag{5.41}$$

and

$$\mathrm{E}\Big(\sum_{(u,v)\in\mathbb{Z}^2} \big|G(u,v,\mathbf{a})\big|^{\alpha-\epsilon}\Big)^{(\alpha+\epsilon)/(\alpha-\epsilon)} < \infty \qquad \text{for } \alpha = 1.$$

Since $T \subset \mathbb{Z}^2$ is a finite set, it suffices to show (5.41) with $G(u,v,\mathbf{a})$ replaced by $g(u,v,\mathbf{a})$. Let $1 < \alpha < 2$ and $p = \alpha \pm \epsilon \in (1,2)$ in (5.37). Then $\sum_{(u,v)\in\mathbb{Z}^2} \mathrm{E}|g(u,v,\mathbf{a})|^p \le C\mathrm{E}[\Psi(A,q,\mu)^{p-1}(1-A)^{p-2}] < \infty$ follows from (5.17) and (5.37). In the case $0 < \alpha < 1$, relations (5.41) immediately follow from (5.18) and (5.37) with $p = \alpha \pm \epsilon \in (0,1)$. For $\alpha = 1$, (5.41) follows from (5.18) in a similar way. □

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