

Integration theory for infinite dimensional volatility modulated Volterra processes

FRED ESPEN BENTH¹ and ANDRÉ SÜSS²

¹Department of Mathematics, University of Oslo, PO Box 1053 Blindern, N-0316 Oslo, Norway.

E-mail: fredb@math.uio.no

²Facultat de Matemàtiques, Universitat de Barcelona, Gran Via de les Corts Catalanes, 585, E-08007 Barcelona, Spain. E-mail: suess.andre@freenet.de

We treat a stochastic integration theory for a class of Hilbert-valued, volatility-modulated, conditionally Gaussian Volterra processes. We apply techniques from Malliavin calculus to define this stochastic integration as a sum of a Skorohod integral, where the integrand is obtained by applying an operator to the original integrand, and a correction term involving the Malliavin derivative of the same altered integrand, integrated against the Lebesgue measure. The resulting integral satisfies many of the expected properties of a stochastic integral, including an Itô formula. Moreover, we derive an alternative definition using a random-field approach and relate both concepts. We present examples related to fundamental solutions to partial differential equations.

Keywords: Gaussian random fields; Malliavin calculus; stochastic integration; Volterra processes

1. Introduction

Let throughout this article $0 < T < \infty$ be a finite time horizon, fix $t \in [0, T]$ and let \mathcal{H}_1 , \mathcal{H}_2 and \mathcal{H}_3 be three separable Hilbert spaces. As the main object of investigation in this article, we introduce the following process:

$$X(t) = \int_0^t g(t, s)\sigma(s)\delta B(s), \tag{1.1}$$

where B is a cylindrical Wiener process on \mathcal{H}_1 , σ is stochastic process on a time interval $[0, T]$ with values in $L(\mathcal{H}_1, \mathcal{H}_2)$, not necessarily adapted to the Wiener process B and g is a deterministic function depending on two time parameters such that $g(t, s) \in L(\mathcal{H}_2, \mathcal{H}_2)$ for all $0 \leq s < t \leq T$. In order for the stochastic integral to be well-defined, one has to assume that $g(t, \cdot)\sigma(\cdot)$ is Skorohod integrable on $[0, t]$ so that $X(t)$ exists as a random element in \mathcal{H}_2 . The aim of this article is to define a stochastic integral with respect to the stochastic process $X = (X(t))_{t \in [0, T]}$, that is we want to derive an integration theory for the integral

$$Z(t) = \int_0^t Y(s) dX(s), \tag{1.2}$$

where we assume that $Y(t) \in L(\mathcal{H}_2, \mathcal{H}_3)$ for all $t \in [0, T]$. With the integration concept we develop, we will see that naturally $Z(t) \in \mathcal{H}_3$ for all $t \in (0, T]$.

We want to point out some remarkable facts about the objects we have just introduced. First, $g(s, s)$ does not have to be defined for any $s \in [0, T]$, it can be singular on the diagonal. Note, moreover, that one could put suitable measurability conditions on σ , such as predictability, but in general this is not necessary. In such a case, the integral in (1.1) would turn out to be an Itô integral in a Hilbert space and the condition for the existence of the integral would be

$$\mathbb{E} \left[\int_0^t \|g(t, s)\sigma(s)\|_{L_2(\mathcal{H}_1, \mathcal{H}_2)}^2 ds \right] < \infty.$$

At this point, we note that our considerations go beyond the classical semimartingale case as treated in [17] for real-valued and [7] for Hilbert-valued (semi)martingales. We will however see in Section 4.2 that under some conditions on g (in particular that $g(s, s)$ exists for all $s \in [0, T]$), X will turn out to be a semimartingale and that the integral in (1.2) and the classical integral with respect to a semimartingale, denoted by $Y \cdot X$ coincide in some cases. In fact, if X is a semimartingale, then the difference between these two integrals can be compared to the situation of the Itô and Skorohod integral with respect to the one-dimensional Brownian motion. Next, we list some examples in order to show the wide range of processes X that can be used as integrators in (1.2).

Example 1.1 (Ambit fields). The situation that motivates our problem of defining (1.2) comes from ambit processes; see [3]. There we deal with random fields consisting of a stochastic integral over a random field defined as follows:

$$X(t, x) = \int_0^t \int_{D(t,x)} g(t, s, x, y)\sigma(s, y)W(ds, dy),$$

where $D(t, x) \subseteq \mathbb{R}^d$, W is a Gaussian noise white in time and (possibly) correlated in space, g is a deterministic function and σ is a random field. These processes are included in our setting if interpret these equations in the Hilbert space sense of [7] where \mathcal{H}_2 is interpreted as $L^2(\mathbb{R}^d)$ and $X(t, \cdot)$ is assumed to be in some L^2 with respect to the spatial parameter. In Section 6, we come back to this example and also derive an integral with respect to a random field, similar to [19].

Example 1.2 (Gaussian processes and VMBV). A one-dimensional subclass of X has already been treated in [1], and more generally in [2]. In the former paper, the authors study integration with respect to \mathbb{R} -valued Gaussian processes where they assumed $\sigma(s) = 1$ for all $s \in [0, T]$. In the latter paper, the authors considered the possibility of a nontrivial σ and they referred to those processes as in (1.1) in one dimension as volatility modulated Volterra processes driven by Gaussian noise (VMBV). In this article, we generalize both Gaussian and VMBV processes to infinite dimensions. A particular example is fractional Brownian motion in infinite dimensions. Choose for all $s \in [0, T]$, $\sigma(s) = Q^{1/2}$ where Q is a nonnegative, self-adjoint, trace-class operator, let $H \in (0, 1)$ and set

$$g(t, s) = c_H(t - s)^{H-1/2} + c_H\left(\frac{1}{2} - H\right) \int_s^t (u - s)^{H-3/2}(1 - (s/u)^{1/2-H}) du.$$

Then X is a Hilbert-valued fractional Brownian motion with Hurst parameter H and covariance operator Q .

Example 1.3 (Solutions to $S(P)DE$). Another application is stochastic integration with respect to the solution to a stochastic differential equation in a Hilbert space. This includes solutions to SPDEs interpreted in the sense of [7]. Let, for instance, X be the mild solution to

$$dX(t) = -AX(t) + \sigma(X(t)) dB(t),$$

with $X_0 = 0$, where A is an unbounded linear operator, σ is a deterministic function subject to some regularity conditions and B is again a cylindrical Wiener process on some Hilbert space; see [7], Chapter 6, for a detailed treatment of these equations. Then if $-A$ generates a strongly continuous semigroup of linear operators $(g(t))_{t \in [0, T]}$, the mild solution to this equation is given by the following integral equation:

$$X(t) = \int_0^t g(t-s)\sigma(X(s)) dB(s),$$

which has the form of (1.1). With the help of the theory we develop in this article, we are then able to define a stochastic integral with respect to this solution X . A particular example here are Ornstein–Uhlenbeck processes in infinite dimensions given by the SDE $dX(t) = -AX(t) + F dB(t)$, where F is a bounded linear operator. Let the C_0 -semigroup generated by $-A$ be denoted by $g(t, s) = \exp(-(t-s)A)$ so that in this case

$$X(t) = \int_0^t \exp(-(t-s)A)F dB(s).$$

The paper is structured in the following way. In Section 2, we list some fundamental results and in the subsequent Section 3 we give the motivation for the definition of our integral. Sections 4.1 and 5 are dedicated to showing properties of the integral and an Itô formula. In the final Sections 6 and 7, we provide a random-field integration approach to the integral and show their equivalence.

Throughout this article, C denotes a positive generic constant, which may change from line to line without further notice.

2. Preliminaries

2.1. Vector measures

This subsection deals with the generalization of measures to set functions taking values in a Banach space, so called *vector measures*. The case which is most important for us is when the vector measure is defined on subsets of \mathbb{R}_+ taking values in the space of the bounded linear operators $L(\mathcal{H}_2, \mathcal{H}_2)$, where \mathcal{H}_2 is a separable Hilbert space. After providing the definition and the important concept of total variation of a vector measure, we list the most relevant properties here and refer to [8,9,12] for more details.

Definition 2.1. Let $(\mathcal{F}, \mathcal{F})$ be a measurable space and let \mathcal{B} be a Banach space. A set function $\mu : \mathcal{F} \rightarrow \mathcal{B}$ is called a finitely additive vector measure, or in short vector measure, if $\mu(F_1 \cup F_2) = \mu(F_1) + \mu(F_2)$ for any two disjoint sets $F_1, F_2 \in \mathcal{F}$. Moreover if, for any sequence $(F_n)_{n \in \mathbb{N}} \subseteq \mathcal{F}$ of pairwise disjoint subsets of \mathcal{F} , we have $\mu(\bigcup_{n=1}^{\infty} F_n) = \sum_{n=1}^{\infty} \mu(F_n)$, then μ is called a countably additive vector measure. Note that the convergence of the sum takes place in the norm topology of \mathcal{B} .

The (total) variation of a vector measure $|\mu|$ is the set function on $(\mathcal{F}, \mathcal{F})$ with values in $\mathbb{R}_+ \cup \{\infty\}$ defined for all $F \in \mathcal{F}$ by

$$|\mu|(F) := \sup_{\pi} \sum_{A \in \pi} \|\mu(A)\|_{\mathcal{B}},$$

where the supremum is taken over all partitions π of F into a finite number of pairwise disjoint sets $A \in \mathcal{F}$. If $|\mu|(F) < \infty$, then μ is said to be a vector measure of finite variation.

The total variation of a vector measure is the smallest of all nonnegative, additive set functions λ such that $\|\mu(F)\|_{\mathcal{B}} \leq \lambda(F)$ for all $F \in \mathcal{F}$. If a countably additive vector measure has finite variation, then $|\mu|$ is also countably additive. For vector measures (not necessarily having finite variation) an integration theory similar to that for \mathbb{R}_+ -valued measures can be developed; see [9], Section III.

We briefly list some properties of the Lebesgue–Stieltjes integral with respect to a Banach-valued function which are important in the remaining paper. Let g be a \mathcal{B} -valued function on a finite or infinite interval of \mathbb{R} , which is assumed to have bounded variation. Then we can define a vector measure μ_g on all finite subintervals $[a, b]$ by $\mu_g([a, b]) := g(b) - g(a)$. Then we extend the measure μ_g onto the Borel σ -field of \mathbb{R}_+ as its Lebesgue extension. We denote the extension by μ_g , too. Next, we fix a function $f : \mathbb{R}_+ \rightarrow L(\mathcal{B}, \mathcal{B}_1)$, where \mathcal{B}_1 is another Banach space. Then, if f is a μ_g -integrable function over an interval $[a, b]$, we denote the integral with respect to μ_g throughout this article by $\int_a^b f(s)g(ds)$ instead of $\int_a^b f(s)\mu_g(ds)$. A special choice for g is the identity on \mathbb{R} . In this case, the integral is also known as *Bochner integral* or *Pettis integral* depending on measurability properties of f and integrability properties of $\|f\|$; see [8], Chapter II.

Finally, we provide a notion of absolute continuity of the vector measure μ_g with respect to the one-dimensional Lebesgue measure λ . For this, consider the measure space $([0, T], \mathcal{B}([0, T]), \lambda)$. Let $g : [0, T] \rightarrow \mathcal{B}$ be a function of finite variation and assume that there exists some function $\phi : [0, T] \rightarrow \mathcal{B}$ such that $g(t) = \int_0^t \phi(s) ds$. This holds, for instance, when g is Fréchet differentiable with derivative ϕ . Assume moreover that f is a μ_g -integrable function taking values in $L(\mathcal{B}, \mathcal{B}_1)$, where \mathcal{B}_1 is another Banach space. Then we have that for every $A \in \mathcal{B}([0, T])$

$$\int_A f(s)g(ds) = \int_A f(s)\mu_g(ds) = \int_A f(s)\phi(s) ds;$$

see [9], Section III.11, for more details. Throughout the rest of this article, we will apply the facts stated in this subsection to the measure generated by the $L(\mathcal{H}_2, \mathcal{H}_2)$ -valued function g used in (1.1), always integrating with respect to the first time argument while leaving the second one fixed.

2.2. Multidimensional Stieltjes integration

In this subsection, we give a quick reminder about an extension of the integration theory treated in the previous one. We want to define the Lebesgue–Stieltjes integral with respect to a function that has more than one argument; see [9], Chapter VII. For this, we need the concept of (locally) bounded variation in the case of functions with several variables, so-called BV functions. Let for this $U \subseteq \mathbb{R}^d$ be an open subset and let $\mathcal{O}_c(U)$ be the set of all precompact open subsets of U . Then a function $g : \mathbb{R}^d \rightarrow \mathbb{R}$ is said to be of locally bounded variation in U if $g \in L^1_{\text{loc}}(U)$ and

$$V(g, O) := \sup_{\phi} \left\{ \int_U g(x) \operatorname{div} \phi(x) \, dx; \phi \in C^1_c(O; \mathbb{R}^d), \|\phi\|_{L^\infty(U)} \leq 1 \right\}$$

is finite for all $O \in \mathcal{O}_c(U)$. Then the set of all functions with (locally) bounded variation forms a Banach space, which is nonseparable. For our needs, the most important property of BV functions is that they are precisely those integrators with respect to which one can define a Stieltjes integral of all continuous functions. So one defines an integral like that by starting with the simple functions $f(x) = 1_A(x) = \prod_{j=1}^d 1_{[a_j, b_j]}(x_j)$, that is, indicator functions of sets $A = \times_{j=1}^d [a_j, b_j]$, where $a, b \in \mathbb{R}^d$ and $a \leq b$ (coordinatewise). For these functions, we define the Stieltjes integral with respect to g and define the notation $g(A)$ by

$$g(A) := \int_{\mathbb{R}^d} f(x)g(\mathrm{d}x) = \sum_{j=0}^d (-1)^j \sum_{\substack{x \in \mathbb{R}^d \\ x_i \in \{b_i, a_i\}; i=1, \dots, d \\ |\{i: x_i = a_i\}| = j}} g(x). \tag{2.1}$$

Note that the sum inside is a finite sum with at most $\binom{d}{j}$ summands and it means that we sum over all those $g(x)$ where there are exactly j arguments that come from the lower point a and the other ones come from b . As an example, one has for $d = 1$ the usual result $g(A) = g(b) - g(a)$, for $d = 2$ one has

$$g(A) = g(b_1, b_2) - g(b_1, a_2) - g(a_1, b_2) + g(a_1, a_2),$$

and for $d = 3$, (2.1) becomes

$$g(A) = g(b_1, b_2, b_3) - g(b_1, b_2, a_3) - g(b_1, a_2, b_3) - g(a_1, b_2, b_3) + g(b_1, a_2, a_3) + g(a_1, b_2, a_3) + g(a_1, a_2, b_3) - g(a_1, a_2, a_3).$$

These formulas become much simpler if g is the product of d functions with one argument each, that is, $g(x) = \prod_{j=1}^d g_j(x_j)$. Then (2.1) can be easily seen to reduce to

$$g(A) = \prod_{j=1}^d \int_{\mathbb{R}} 1_{[a_j, b_j]}(x_j)g(\mathrm{d}x_j) = \prod_{j=1}^d (g_j(b_j) - g_j(a_j)),$$

which is what one expects. This yields a measure μ_g on the Borel σ -field on \mathbb{R}^d by considering the Lebesgue extension of g in (2.1). With respect to this measure, one can now derive an integration theory for real valued functions in $f \in L^p(\mu_g)$ where $p \in [1, \infty]$, and for such a function we denote the integral by

$$\int_O f(x)g(dx) = \int_O f(x)\mu_g(dx),$$

where $O \in \mathcal{O}_c(U)$. This integration theory will be used in Section 6, where we define a stochastic integral with respect to a random field, which has $d + 1$ variables, d being the spatial dimension and one the temporal dimension.

2.3. Hilbert-valued Malliavin calculus

In this subsection, we provide some ideas and the main results we need related to Malliavin calculus. It will, however, not be sufficient to only look at Malliavin calculus for real-valued random variables or \mathbb{R}^d -valued random vectors as treated extensively in [14]. Instead we have to deal with random elements taking values in some separable Hilbert space. Some sources for this are [4], Section 5, or [11] and references therein. A more general setting is the one treated in [16] where the authors treat Malliavin calculus for random elements taking values in some UMD Banach space. In this subsection, we will without further notice identify the dual of a separable Hilbert space \mathcal{G}^* with \mathcal{G} .

Let \mathcal{G} be a separable Hilbert space and let $(W(h), h \in \mathcal{G})$ be an isonormal Gaussian process; see [14], Section 1.1.1, for some of its properties. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be the probability space induced by the isonormal process. Furthermore, we choose another separable Hilbert space \mathcal{G}_1 , and we consider the class of Hilbert-valued smooth random elements $F \in L^2(\Omega; \mathcal{G}_1)$ given by $F = f(W(h_1), \dots, W(h_n))$ for $h_1, \dots, h_n \in \mathcal{G}$, $n \in \mathbb{N}$ and $f: \mathbb{R}^n \rightarrow \mathcal{G}_1$ which is infinitely Fréchet differentiable with some boundedness condition, polynomially bounded or bounded. These functions f are dense in $L^2(\Omega; \mathcal{G}_1)$. For these random elements F , the Malliavin derivative is given by

$$DF := \sum_{j=1}^n \frac{\partial f}{\partial x_j}(W(h_1), \dots, W(h_n)) \otimes h_j.$$

Consequently, for smooth random elements F we can interpret its Malliavin derivative as another random element with values in $L_2(\mathcal{G}, \mathcal{G}_1)$, the space of Hilbert–Schmidt operators from \mathcal{G} to \mathcal{G}_1 , or equivalently in the tensor product $\mathcal{G}_1 \otimes \mathcal{G}$ since $L_2(\mathcal{G}, \mathcal{G}_1)$ is isomorphic to this tensor product. Throughout this article, we will mainly work with the first approach, but occasionally use the second one when it is more convenient. We can also apply projections onto the coordinates of \mathcal{G}_1 leading to one-dimensional Malliavin calculus. In fact, for some $l \in \mathcal{G}_1$

$$D^l F := \langle DF, l \rangle_{\mathcal{G}_1} = \sum_{j=1}^n \left\langle \frac{\partial f}{\partial x_j}(W(h_1), \dots, W(h_n)), l \right\rangle_{\mathcal{G}_1} h_j.$$

A special choice for l in the previous equality is an element from a CONS of \mathcal{G}_1 , denoted by $(e_k)_{k \in \mathbb{N}}$ yielding $D_k F := D^{e_k} F$. A similar calculation for a CONS in \mathcal{G} leads to directional Malliavin derivatives as in the one-dimensional case.

As in the real-valued Malliavin calculus, the operator D is closable in $L^2(\Omega; \mathcal{G}_1)$ and we define the Malliavin derivative of an element $F \in L^2(\Omega; \mathcal{G}_1)$ which can be represented as a limit of a sequence of smooth \mathcal{G}_1 -valued random elements $(F_n)_{n \in \mathbb{N}}$ to be the limit of the Malliavin derivatives of the elements of the sequence, that is, $DF := \lim_{n \rightarrow \infty} DF_n$. This convergence takes place in $L^2(\Omega; L_2(\mathcal{G}, \mathcal{G}_1))$. The space of all such elements will be denoted by $\mathbb{D}^{1,2}(\mathcal{G}_1)$ and it has the norm

$$\|F\|_{1,2,\mathcal{G}_1}^2 = \mathbb{E}[\|F\|_{\mathcal{G}_1}^2] + \mathbb{E}[\|DF\|_{L_2(\mathcal{G}, \mathcal{G}_1)}^2],$$

where the index \mathcal{G}_1 will be dropped if this does not cause any confusion. One can define, as in the real-valued case, the spaces $\mathbb{D}^{k,p}(\mathcal{G}_1)$ for $k \in \mathbb{N}$ and $p \geq 1$.

We also need to define a Hilbert-valued equivalent to the divergence operator δ . This operator $\delta_{\mathcal{G}_1} : L^2(\Omega; L_2(\mathcal{G}, \mathcal{G}_1)) \rightarrow L^2(\Omega; \mathcal{G}_1)$ is defined to be the adjoint of D , that is,

$$\mathbb{E}[\langle DF, G \rangle_{L_2(\mathcal{G}, \mathcal{G}_1)}] = \mathbb{E}[\langle F, \delta_{\mathcal{G}_1}(G) \rangle_{\mathcal{G}_1}],$$

for $F \in \mathbb{D}^{1,2}(\mathcal{G}_1)$ and all $G \in L^2(\Omega; L_2(\mathcal{G}, \mathcal{G}_1))$ for which

$$|\mathbb{E}[\langle DF, G \rangle_{L_2(\mathcal{G}, \mathcal{G}_1)}]| \leq C(\mathbb{E}[\|F\|_{\mathcal{G}_1}^2])^{1/2}.$$

From now on, we drop the index \mathcal{G}_1 from the divergence operator if this does not cause any confusion.

Having defined these two operators we will now collect some calculus rules which we will rely on in the subsequent sections. First, we see that D and δ are unbounded linear operators. This implies that one can pull bounded linear deterministic operators or functionals in and out of the Malliavin derivative and the divergence operator. As in the real-valued Malliavin calculus, there is a product and chain rule for the Malliavin derivative which in the Hilbert-valued case need some explications. Let $\mathcal{G}_1, \mathcal{G}_2$ and \mathcal{G}_3 be separable Hilbert spaces and let $F \in L_2(\mathcal{G}_1, \mathcal{G}_2)$ and $G \in L_2(\mathcal{G}_2, \mathcal{G}_3)$ be two random linear operators which are Malliavin differentiable. Then $GF \in \mathbb{D}^{1,2}(L_2(\mathcal{G}_1, \mathcal{G}_3))$ and

$$D(GF) = (DG)F + GDF, \tag{2.2}$$

where this equality has to be interpreted as $(D(GF))h = (DG)F(h) + G(DF)(h)$ for all $h \in \mathcal{G}_1$. A similar rule applies for directional Malliavin derivatives. For smooth Hilbert-valued random elements F and G this is shown in [11], Lemma 2.1, and the general case follows by an approximation procedure by Hilbert-valued smooth random elements. The chain rule in Hilbert-valued Malliavin calculus is defined for functions $\phi : \mathcal{G}_1 \rightarrow \mathcal{G}_2$ which are either Fréchet differentiable or Lipschitz continuous. Let $F \in \mathbb{D}^{1,2}(\mathcal{G}_1)$. Then $\phi(F) \in \mathbb{D}^{1,2}(\mathcal{G}_2)$ and

$$D\phi(F) = \phi'(F)DF, \tag{2.3}$$

where ϕ is Fréchet differentiable and ϕ' denotes the Fréchet derivative of ϕ . If ϕ is only Lipschitz continuous, then $D\phi(F) = \bar{\phi}DF$ where $\bar{\phi}$ is a random linear operator from \mathcal{G}_1 to \mathcal{G}_2 whose norm is almost surely bounded by the smallest Lipschitz constant of ϕ .

In our setting with a cylindrical Wiener process B on \mathcal{G}_0 , we can make some simplifications. First, we note that in this setting, the Hilbert space \mathcal{G} on which the isonormal Gaussian process is defined is equal to $L^2([0, T]; \mathcal{G}_0)$ and one can reinterpret the Malliavin derivative DF as some $L_2(\mathcal{G}_0, \mathcal{G}_1)$ -valued stochastic process $(D_t F)_{t \in [0, T]}$ on the time interval $[0, T]$ given by

$$D_t F := \sum_{j=1}^n \frac{\partial f}{\partial x_j} (W(h_1), \dots, W(h_n)) \otimes h_j(t),$$

where $h_j \in L^2([0, T]; \mathcal{G}_0)$ for all $1 \leq j \leq n$. Therefore, DF actually denotes an equivalence class of functions from $\Omega \times [0, T]$ with values in $L_2(\mathcal{G}_0, \mathcal{G}_1)$, but one can find a representative such that $D_t F$ is measurable in ω for all $t \in [0, T]$ and that $(DF)(\omega)$ is measurable in t for all $\omega \in \Omega$, which we denote as *the* Malliavin derivative of F . As in the general case, one can define the spaces $\mathbb{D}^{k,p}(\mathcal{G}_1)$, but moreover one can also define the spaces $\mathbb{L}^{k,p}(\mathcal{G}_1)$ to be $L^p([0, T]; \mathbb{D}^{k,p}(\mathcal{G}_1))$. In the classical real-valued Malliavin calculus $\mathbb{L}^{1,2} := \mathbb{L}^{1,2}(\mathbb{R})$. If $k = 1$, then the norm is given by

$$\|F\|_{\mathbb{L}^{1,p}(\mathcal{G}_1)}^p = \int_0^T \mathbb{E}[\|F_t\|_{\mathcal{G}_1}^p] dt + \int_0^T \int_0^T \mathbb{E}[\|D_s F_t\|_{L_2(\mathcal{G}_0, \mathcal{G}_1)}^p] ds dt. \tag{2.4}$$

If $k \geq 2$, then iterated Malliavin derivatives and further integrals are added to this expression.

For the \mathcal{G}_1 -valued divergence operator $\delta_{\mathcal{G}_1}$ this has the consequence that it reduces to the \mathcal{G}_1 -valued Skorohod integral and for all $G \in L^2([0, T] \times \Omega; L_2(\mathcal{G}_0, \mathcal{G}_1))$ we write $\int_0^T G_s \delta B_s$ instead of $\delta(G)$. If moreover G is predictable, then this integral turns out to be the \mathcal{G}_1 -valued Itô integral.

The last issue we focus on in this subsection is the interplay between the Hilbert-valued Malliavin derivative and Skorohod integral. First, we have the general commutator relation $D\delta(u) = u + \delta(Du)$, similar to [14], equation (1.46). Let now u be a stochastic process in $\mathbb{L}^{1,2}(L_2(\mathcal{G}_0, \mathcal{G}_1))$ and we assume that for all $t \in [0, T]$ the process $(D_t u(s))_{s \in [0, T]}$ is Skorohod integrable and the process $(\int_0^T D_t u(s) \delta B(s))_{t \in [0, T]}$ has a version which is in $L^2(\Omega \times [0, T]; L_2(\mathcal{G}_0, \mathcal{G}_1))$. This condition holds, for instance, if u is twice Malliavin differentiable. Then $\int_0^T u(s) \delta B(s) \in \mathbb{D}^{1,2}(\mathcal{G}_1)$ and for all $t \in [0, T]$

$$D_t \int_0^T u(s) \delta B(s) = u(t) + \int_0^T D_t u(s) \delta B(s).$$

Finally, we provide a Hilbert-valued integration by parts formula which is inspired by [4], Theorem 5.2. However, we need it in a slightly more general setting which is why we include a quick proof here. Before we start, we fix a notation. Let throughout this article $\text{tr}_{\mathcal{G}_0}$ denote the trace of a linear operator $A : \mathcal{G}_0 \rightarrow L_2(\mathcal{G}_0, \mathcal{G}_1)$ taken only over \mathcal{G}_0 , that is,

$$\text{tr}_{\mathcal{G}_0}(A) := \sum_{k \in \mathbb{N}} \langle A e_k, e_k \rangle_{\mathcal{G}_0},$$

where $(e_k)_{k \in \mathbb{N}}$ is a CONS of \mathcal{G}_0 . Consequently, the object $\text{tr}_{\mathcal{G}_0}(A)$ takes values in \mathcal{G}_1 . From the definition of the Skorohod integral as the adjoint of the Malliavin derivative one has that for all

$u \in L^2(\Omega \times [0, T]; L_2(\mathcal{G}_0, \mathcal{G}_1))$ and $A \in \mathbb{D}^{1,2}(L_2(\mathcal{G}_1, \mathcal{G}_2))$

$$\mathbb{E} \left[A \int_0^T u(s) \delta B(s) \right] = \mathbb{E} \left[\int_0^T \text{tr}_{\mathcal{G}_1}((D_s A)u(s)) \, ds \right], \tag{2.5}$$

where the integrand is \mathcal{G}_2 -valued and the integral is understood as a Bochner integral. Similarly, one could write the trace outside the integral, which would yield an $L_2(\mathcal{G}_0, \mathcal{G}_0 \otimes \mathcal{G}_2)$ -valued integrand and integral. Now we are in the position to formulate the integration by parts formula.

Proposition 2.2. *Let $u \in L^2(\Omega \times [0, T]; L_2(\mathcal{G}_0, \mathcal{G}_1))$ be in the domain of the Skorohod integral $\delta_{\mathcal{G}_1}$ and let $A \in \mathbb{D}^{1,2}(L_2(\mathcal{G}_1, \mathcal{G}_2))$. Then $Au \in \text{Dom}(\delta_{\mathcal{G}_2})$ and*

$$\int_0^t Au(s) \delta B(s) = A \int_0^t u(s) \delta B(s) - \text{tr}_{\mathcal{G}_0} \int_0^t D_s(A)u(s) \, ds, \tag{2.6}$$

for all $t \in [0, T]$. Note that under the conditions above the right-hand side of this equality is an element in $L^2(\Omega; \mathcal{G}_2)$.

Proof. Assume that $\Psi \in \mathbb{D}^{1,2}(L_2(\mathcal{G}_2, \mathbb{R}))$. We have to show that

$$\mathbb{E} \left[\Psi \int_0^t Au(s) \delta B(s) \right] = \mathbb{E} \left[\Psi \left(A \int_0^t u(s) \delta B(s) - \text{tr}_{\mathcal{G}_0} \int_0^t D_s(A)u(s) \, ds \right) \right], \tag{2.7}$$

for all such Ψ . Then a calculation similar to [4], Proposition 5.3, yields

$$\begin{aligned} \mathbb{E} \left[\Psi \int_0^t Au(s) \delta B(s) \right] &= \mathbb{E} \left[\text{tr}_{\mathcal{G}_0} \int_0^t D_s(\Psi)Au(s) \, ds \right] \\ &= \mathbb{E} \left[\text{tr}_{\mathcal{G}_0} \int_0^t (D_s(\Psi A) - \Psi D_s A)u(s) \, ds \right] \\ &= \mathbb{E} \left[\Psi \left(A \int_0^t u(s) \delta B(s) - \text{tr}_{\mathcal{G}_0} \int_0^t D_s Au(s) \, ds \right) \right], \end{aligned}$$

where we used (2.5) in the first equality. This implies the assertion. □

3. Stochastic integration

In this section, we provide an exact definition for the stochastic integral in (1.2) with respect to an integrator as in (1.1). We are keen on deriving an integration theory that also covers singular g , that is, where $g(t, t)$ is not well-defined. In order to motivate the definition of the stochastic integral, we provide a *heuristic* calculation that shows how each term comes into play. Throughout this and the following sections, we work under the following assumption which have already been mentioned in Section 1.

Assumption 3.1. Fix $T > 0$ and let $t \in [0, T]$, and let $B = (B_t)_{t \geq 0}$ be a cylindrical Wiener process on \mathcal{H}_1 . Furthermore, let $g(t, s) \in L(\mathcal{H}_2, \mathcal{H}_2)$ (nonrandom) for all $0 \leq s < t \leq T$ and let $(\sigma(t))_{t \geq 0}$ be an $L(\mathcal{H}_1, \mathcal{H}_2)$ -valued stochastic process such that $g(t, s)\sigma(s) \in L_2(\mathcal{H}_1, \mathcal{H}_2)$ for all $s \in [0, t)$, and $1_{[0,t]}(\cdot)g(t, \cdot)\sigma(\cdot) \in \text{Dom}(\delta)$ for all $t \in [0, T]$. Assume that for all $s \in [0, t)$ the $L(\mathcal{H}_2, \mathcal{H}_2)$ -valued vector measure $g(du, s)$ has bounded variation on $[u, v]$ for all $0 \leq s < u < v \leq t$.

We note again that unlike in [2] we have not assumed that σ is predictable with respect to B , so that the integral in (1.1) is a genuine Skorohod integral. In the following derivation, we will first assume that $\mathcal{H}_3 = \mathbb{R}$, the general case will be discussed shortly after. The basic idea for the calculations that follow is to expand all operators which appear in (1.1) and (1.2) into their coordinates, perform similar calculations as in [2] and then reassemble the original operators to get closed-form expressions for Hilbert-valued random elements. To this end, we fix $(e_k)_{k \in \mathbb{N}}$ to be a CONS of \mathcal{H}_2 and $(f_l)_{l \in \mathbb{N}}$ to be a CONS of \mathcal{H}_1 . Then $(B^l)_{l \in \mathbb{N}} := (\langle B, f_l \rangle_{\mathcal{H}_1})_{l \in \mathbb{N}}$ is a sequence of independent, one-dimensional Brownian motions. With the help of these two CONS, we can expand $X^k(t) := \langle X(t), e_k \rangle$ for all $k \in \mathbb{N}$ in the following way:

$$\begin{aligned} X^k(t) &= \left\langle \sum_{l \in \mathbb{N}} \int_0^t g(t, s)\sigma(s)(f_l)\delta B^l(s), e_k \right\rangle_{\mathcal{H}_2} \\ &= \sum_{l \in \mathbb{N}} \int_0^t \langle g(t, s), e_k \rangle_{\mathcal{H}_2} \sigma(s)(f_l)\delta B^l(s), \end{aligned} \tag{3.1}$$

where in the last term $\langle g(t, s), e_k \rangle_{\mathcal{H}_2}$ denotes the linear functional from \mathcal{H}_2 to \mathbb{R} which is defined by $\langle g(t, s), e_k \rangle_{\mathcal{H}_2}(x) := \langle g(t, s)x, e_k \rangle_{\mathcal{H}_2}$ for all $x \in \mathcal{H}_2$. The reason why we introduce this notation is to perform calculations in a more intuitive manner. In fact, this is a deterministic linear functional which commutes with the Skorohod integral and the Malliavin derivative as mentioned in Section 2.3.

Now we are in the position to motivate the definition of the stochastic integral. In what follows, we first assume that the random integrand $Y : [0, T] \rightarrow L(\mathcal{H}_2, \mathcal{H}_3)$ is differentiable, but this assumption will be removed afterward. In the following $Y(t)(e_k) = Y(t)e_k$ is just applying the linear operator $Y(t)$ to $e_k \in \mathcal{H}_2$. In this case, we obtain

$$\begin{aligned} &\int_0^t Y(s) dX(s) \\ &= \sum_{k \in \mathbb{N}} \int_0^t Y(s)(e_k) dX^k(s) \\ &= \sum_{k \in \mathbb{N}} \left(Y(t)(e_k)X^k(t) - \int_0^t \frac{\partial Y}{\partial u}(u)(e_k)X^k(u) du \right) \\ &= \sum_{k \in \mathbb{N}} Y(t)(e_k)X^k(t) - \sum_{k, l \in \mathbb{N}} \int_0^t \frac{\partial Y}{\partial u}(u)(e_k) \int_0^u \langle g(u, s), e_k \rangle_{\mathcal{H}_2} \sigma(s)(f_l) dB^l(s) du \end{aligned} \tag{3.2}$$

$$\begin{aligned}
 &= \sum_{k \in \mathbb{N}} Y(t)(e_k) X^k(t) - \sum_{k, l \in \mathbb{N}} \left(\int_0^t \int_0^u \frac{\partial Y}{\partial u}(u)(e_k) \langle g(u, s), e_k \rangle_{\mathcal{H}_2} \sigma(s)(f_l) \delta B^l(s) du \right. \\
 &\quad \left. + \int_0^t \int_0^u D_{s,l} \left(\frac{\partial Y}{\partial u}(u)(e_k) \right) \langle g(u, s), e_k \rangle_{\mathcal{H}_2} \sigma(s)(f_l) ds du \right) \\
 &= \sum_{k \in \mathbb{N}} Y(t)(e_k) X^k(t) \\
 &\quad - \sum_{k, l \in \mathbb{N}} \left(\int_0^t \left(\int_s^t \frac{\partial Y}{\partial u}(u)(e_k) \langle g(u, s), e_k \rangle_{\mathcal{H}_2} du \right) \sigma(s)(f_l) \delta B^l(s) \right. \\
 &\quad \left. + \int_0^t D_{s,l} \left(\int_s^t \frac{\partial Y}{\partial u}(u)(e_k) \langle g(u, s), e_k \rangle_{\mathcal{H}_2} du \right) \sigma(s)(f_l) ds \right),
 \end{aligned}$$

where we first expanded the stochastic integral in \mathcal{H}_2 along the coordinates of integrand and integrator, substituted (3.1) and did a series expansion in \mathcal{H}_1 . Then we pulled the linear operator $\partial Y(u)/\partial u$ inside the stochastic integral using Proposition 2.2, used the stochastic Fubini's theorem; see [14], Exercise 3.2.7, pulled the deterministic bounded linear operator $\langle g(t, s), e_k \rangle$ inside the Malliavin derivative and commuted the Malliavin derivative and the deterministic integral. Using similar steps and Proposition 2.2 again, we calculate the first term on the right-hand side of the last expression to be equal to

$$\begin{aligned}
 \sum_{k \in \mathbb{N}} Y(t)(e_k) X^k(t) &= \sum_{k, l \in \mathbb{N}} Y(t)(e_k) \int_0^t \langle g(t, s), e_k \rangle_{\mathcal{H}_2} \sigma(s)(f_l) \delta B^l(s) \\
 &= \sum_{k, l \in \mathbb{N}} \left(\int_0^t Y(t)(e_k) \langle g(t, s), e_k \rangle_{\mathcal{H}_2} \sigma(s)(f_l) \delta B^l(s) \right. \\
 &\quad \left. + \int_0^t D_{s,l} (Y(t)(e_k)) \langle g(t, s), e_k \rangle_{\mathcal{H}_2} \sigma(s)(f_l) ds \right).
 \end{aligned}$$

Now we substitute this term into (3.2) to obtain

$$\begin{aligned}
 &\int_0^t Y(s) dX(s) \\
 &= \sum_{k, l \in \mathbb{N}} \left(\int_0^t \left(Y(t)(e_k) \langle g(t, s), e_k \rangle_{\mathcal{H}_2} \right. \right. \\
 &\quad \left. \left. - \int_s^t \frac{\partial Y}{\partial u}(u)(e_k) \langle g(u, s), e_k \rangle_{\mathcal{H}_2} du \right) \sigma(s)(f_l) \delta B^l(s) \right. \\
 &\quad \left. + \int_0^t D_{s,l} \left(Y(t)(e_k) \langle g(t, s), e_k \rangle_{\mathcal{H}_2} \right. \right. \\
 &\quad \left. \left. - \int_s^t \frac{\partial Y}{\partial u}(u)(e_k) \langle g(u, s), e_k \rangle_{\mathcal{H}_2} du \right) \sigma(s)(f_l) ds \right) \quad (3.3)
 \end{aligned}$$

$$\begin{aligned}
 &= \sum_{k,l \in \mathbb{N}} \left(\int_0^t \left(Y(s)(e_k) \langle g(s, s), e_k \rangle_{\mathcal{H}_2} \right. \right. \\
 &\quad \left. \left. + \int_s^t Y(u)(e_k) \left\langle \frac{\partial g}{\partial u}(u, s), e_k \right\rangle_{\mathcal{H}_2} du \right) \sigma(s)(f_l) \delta B^l(s) \right. \\
 &\quad \left. + \int_0^t D_{s,l} \left(Y(s)(e_k) \langle g(s, s), e_k \rangle_{\mathcal{H}_2} \right. \right. \\
 &\quad \left. \left. + \int_s^t Y(u)(e_k) \left\langle \frac{\partial g}{\partial u}(u, s), e_k \right\rangle_{\mathcal{H}_2} du \right) \sigma(s)(f_l) ds \right) \\
 &= \int_0^t \left(Y(s)g(s, s) + \int_s^t Y(u)g(du, s) \right) \sigma(s) \delta B(s) \\
 &\quad + \text{tr}_{\mathcal{H}_1} \int_0^t D_s \left(Y(s)g(s, s) + \int_s^t Y(u)g(du, s) \right) \sigma(s) ds,
 \end{aligned}$$

by performing a deterministic integration by parts procedure using the fact that we can commute Fréchet differentiation and the projection onto the k th coordinate since they are bounded linear operators which commute with the Fréchet derivative. Then we summed up over both CONS $(e_k)_{k \in \mathbb{N}}$ and $(f_l)_{l \in \mathbb{N}}$.

Next, we briefly treat the case of a general separable Hilbert space \mathcal{H}_3 . For this, fix a CONS of \mathcal{H}_3 , denoted by $(d_k)_{k \in \mathbb{N}}$, and use the expansion $Y(t) = \sum_{k \in \mathbb{N}} \langle Y(t), d_k \rangle d_k$. Then we perform all the calculations above on each coordinate of \mathcal{H}_3 separately and at the end sum up again to obtain a closed expression for the integral. Note that for this summing up to be true, certain summability conditions on the elements in each coordinate have to be assumed.

These derivations motivate the definition of the following linear operator for every $h \in L(\mathcal{H}_2, \mathcal{H}_3)$:

$$\mathcal{K}_g(h)(t, s) := h(s)g(t, s) + \int_s^t (h(u) - h(s))g(du, s), \tag{3.4}$$

where the integral is defined as an integral with respect to an $L(\mathcal{H}_2, \mathcal{H}_2)$ -valued vector measure whenever it makes sense; note that $g(du, s)$ has finite variation on all subintervals $[v, t]$ where $v > s$ by definition. For all such h , $\mathcal{K}_g(h)(t, s) \in L(\mathcal{H}_2, \mathcal{H}_3)$. Sometimes we will call the operator in (3.4) the *kernel* associated to X . Under the Assumptions 3.1 and the ones that follow in Definition 3.2, this linear operator is well-defined. We remark that this operator is the infinite-dimensional analogon to the one which appears in [2], which in turn already appeared in [1]. In some special cases, this operator can be written in a different way. In fact, if $g(s, s)$ is a well-defined linear operator from \mathcal{H}_2 to \mathcal{H}_2 , then

$$\mathcal{K}_g(h)(t, s) = h(s)g(s, s) + \int_s^t h(u)g(du, s). \tag{3.5}$$

Note that this is the kernel appearing in (3.3), and now it is obvious that (3.4) is a generalization of it. If $g(\cdot, s)$ is absolutely continuous with respect to the one-dimensional Lebesgue measure

on $[0, t]$ with density $\phi(\cdot, s)$, then obviously

$$\mathcal{K}_g(h)(t, s) = h(s)g(t, s) + \int_s^t (h(u) - h(s))\phi(u, s) du,$$

where the integral is understood as a Bochner integral. This situation applies in particular if $g(\cdot, s)$ is Fréchet differentiable. If g is homogeneous in its arguments, that is, g depends on t and s only through their difference, the kernel (3.4) can be rewritten as

$$\mathcal{K}_g(h)(t, s) = h(s)g(t - s) + \int_0^{t-s} h(u + s)g(du).$$

Going back to (3.3), we see that we can define the stochastic integral in (1.2) as

$$\int_0^t Y(s) dX(s) := \int_0^t \mathcal{K}_g(Y)(t, s)\sigma(s)\delta B(s) + \text{tr}_{\mathcal{H}_1} \int_0^t D_s(\mathcal{K}_g(Y)(t, s))\sigma(s) ds. \tag{3.6}$$

From the definition in the last line, one can see that the integral $\int_0^t Y(s) dX(s)$ does not depend on the particular choices of the bases $(e_k)_{k \in \mathbb{N}}$ and $(f_l)_{l \in \mathbb{N}}$. In fact, this can be shown for the stochastic integral as for the usual Itô integrals in separable Hilbert spaces, and the trace terms is also independent of the choice of basis.

Next, we describe the domain of this integral.

Definition 3.2. Fix $t \geq 0$, let X be defined by (1.1) and assume Assumption 3.1. We say that a stochastic process $(Y(s))_{s \in [0, t]}$ belongs to the domain of the stochastic integral with respect to X , if:

- (i) the process $(Y(u) - Y(s))_{u \in (s, t]}$ is integrable with respect to $g(du, s)$ almost surely,
- (ii) $s \mapsto \mathcal{K}_g(Y)(t, s)\sigma(s)1_{[0, t]}(s)$ is in the domain of the \mathcal{H}_3 -valued divergence operator δB , and
- (iii) $\mathcal{K}_g(Y)(t, s)$ is Malliavin differentiable with respect to D_s for all $s \in [0, t]$ and the \mathcal{H}_3 -valued stochastic process $s \mapsto \text{tr}_{\mathcal{H}_1} D_s(\mathcal{K}_g(Y)(t, s))\sigma(s)$ is Bochner integrable on $[0, t]$ almost surely.

We denote this by $Y \in \mathcal{I}^X(0, t)$ and the integral $\int_0^t Y(s) dX(s)$ is defined by (3.6).

Now one may ask about the concrete form of the domain of this integral. So far, we have been unable to derive a characterization of it which is a similar problem as in anticipating calculus, where the domain of the divergence operator δ is not completely characterized. Instead, one can identify a subset $\mathbb{L}^{1,2}$ in this domain. This is also the case for the X -integral that for some cases, we can identify a subset of its domain. In fact, if σ is assumed to be Malliavin differentiable then we can define the subset $\mathcal{I}_{1,2}^X(0, t)$ which is given by the set of processes for which the seminorm

$$\|Y\|_{\mathcal{I}_{1,2}^X(0, t)} := \|\mathcal{K}_g(Y)(t, \cdot)\sigma(\cdot)\|_{\mathbb{L}^{1,2}(L_2(\mathcal{H}_1, \mathcal{H}_3))}$$

is finite. The equivalence classes of this seminorm depend on the exact shape of g and σ . Similarly, one can define a semi-inner product in an obvious way. The set $\mathcal{I}_{1,2}^X(0, t)$ is then included

in $\mathcal{I}^X(0, t)$, since for any $Y \in \mathcal{I}_{1,2}^X(0, t)$ we have

$$\begin{aligned} \mathbb{E} \left[\left\| \int_0^t Y(s) dX(s) \right\|_{\mathcal{H}_3}^2 \right] &\leq 2\mathbb{E} \left[\left\| \int_0^t \mathcal{K}_g(Y)(t, s)\sigma(s)\delta B(s) \right\|_{\mathcal{H}_3}^2 \right] \\ &\quad + 2T\mathbb{E} \left[\int_0^t \|\text{tr}_{\mathcal{H}_1} D_s \mathcal{K}_g(Y)(t, s)\sigma(s)\|_{\mathcal{H}_3}^2 ds \right] \quad (3.7) \\ &\leq C_T \|\mathcal{K}_g(Y)(t, \cdot)\sigma(\cdot)\|_{\mathbb{L}^{1,2}(L_2(\mathcal{H}_1, \mathcal{H}_3))}^2, \end{aligned}$$

where we have used the continuity of the Skorohod integral on $\mathbb{L}^{1,2}(\mathcal{H}_3)$; see [14], equation (1.47). Note that the second term in the second line is a part of the Malliavin derivative of the integrand of the stochastic integral and, therefore, already included in the norm estimate for the Skorohod integral. This space $\mathcal{I}_{1,2}^X(0, t)$ takes the role of $\mathbb{L}^{1,2}$ from classical Malliavin calculus.

4. Calculus with respect to the integral

In the first two subsections, we present some general properties of the stochastic integral defined in the previous section. Afterward, in Sections 4.3 and 4.4 we investigate some particular cases for the integrand.

4.1. Basic calculus rules

At first, we can conclude that the integral defined in the previous section is linear. This follows immediately from the linearity of the Malliavin derivative, divergence operator and the Lebesgue(–Stieltjes) integral. Formally, we have for $Y, Z \in \mathcal{I}^X(0, t)$ and two constants $a, b \in \mathbb{R}$ that $aY + bZ \in \mathcal{I}^X(0, t)$ and

$$\int_0^t (aY(s) + bZ(s)) dX(s) = a \int_0^t Y(s) dX(s) + b \int_0^t Z(s) dX(s).$$

Another immediate property follows from integrating constants. In fact, if we choose $\mathcal{H}_2 = \mathcal{H}_3$ and $Y \equiv \text{id}_{\mathcal{H}_2}$, which is easily seen to be in $\mathcal{I}^X(0, t)$ for all $t \geq 0$, then $\mathcal{K}_g(Y)(t, s) = g(t, s)$ for all $s < t$ and since g is deterministic, we have $D\mathcal{K}_g(Y)(t, s) \equiv 0$. This implies

$$\int_0^t dX(s) = \int_0^t \text{id}_{\mathcal{H}_2} dX(s) = \int_0^t g(t, s)\sigma(s)\delta B(s) = X(t). \quad (4.1)$$

By combining this with the linearity of the integral for the deterministic integrand $Y(s) = 1_{[u,v]}\text{id}_{\mathcal{H}_2}$ we obtain

$$\int_0^t 1_{[u,v]}(s) dX(s) = \int_0^t 1_{[u,v]}(s)\text{id}_{\mathcal{H}_2} dX(s) = X(v) - X(u), \quad (4.2)$$

which gives us the intuitive property that the integral over an indicator function is the increment of the integrator. We have furthermore that if $0 < t < T$ and $Y \in \mathcal{I}^X(0, t)$ then $Y1_{[0,t]} \in \mathcal{I}^X(0, T)$ and

$$\int_0^T 1_{[0,t]}(s)Y(s) dX(s) = \int_0^t Y(s) dX(s). \tag{4.3}$$

This can be seen by splitting the integral $\int_s^T \dots g(du, s)$ into two integrals $\int_s^t \dots g(du, s)$ and $\int_t^T \dots g(du, s)$. Note that this equality does not hold if we only assume that $Y \in \mathcal{I}^X(0, T)$ because the fact that X is Skorohod integrable over $[0, T]$ does not in general imply that $X1_{[0,t]}$ is Skorohod integrable over $[0, T]$; see [14], Exercise 3.2.1. However, if σ is assumed to be Malliavin differentiable, this does hold. In fact, in this case $\mathcal{K}_g(Y)(T, s)\sigma(s)$ is Malliavin differentiable for all $s \in [0, t]$ which implies that $\mathcal{K}_g(Y)(t, s)\sigma(s)$ is Skorohod integrable over $[0, t]$ for all $t \in [0, T]$. Combining (4.3) with the linearity of the integral, we have immediately that for $0 \leq u < v \leq t$ and $Y \in \mathcal{I}^X(0, u) \cap \mathcal{I}^X(0, v)$, that $Y1_{[u,v]} \in \mathcal{I}^X(0, t)$ and

$$\int_0^t Y(s)1_{[u,v]}(s) dX(s) = \int_0^v Y(s) dX(s) - \int_0^u Y(s) dX(s).$$

Using these basic rules, we can derive more interesting properties for the stochastic integral with respect to X .

Proposition 4.1. *Assume that Assumption 3.1 holds and that X is defined by (1.1). Let $t > 0$ and assume $Y \in \mathcal{I}^X(0, t)$.*

(i) *Let Z be a random linear operator from \mathcal{H}_3 to another separable Hilbert space \mathcal{H}_4 which is almost surely bounded. Then $ZY \in \mathcal{I}^X(0, t)$ and*

$$\int_0^t ZY(s) dX(s) = Z \int_0^t Y(s) dX(s) \quad \text{almost surely.}$$

(ii) *The X -integral is local, that is, if $Y = 0$ on a measurable set $A \subseteq \Omega$, then*

$$\int_0^t Y(s) dX(s) = 0 \quad \text{on } A.$$

(iii) *Let Y be a simple process, that is, $Y = \sum_{j=1}^{n-1} Z_j 1_{(t_j, t_{j+1}]}$ where Z_j is a random linear operator from \mathcal{H}_2 to \mathcal{H}_3 which is almost surely bounded for all $j = 1, \dots, n - 1$ and $0 \leq t_1 < \dots < t_n \leq t$ is a partition of the interval $[0, t]$. Then $Y \in \mathcal{I}^X(0, t)$ and*

$$\int_0^t Y(s) dX(s) = \sum_{j=1}^{n-1} Z_j (X(t_{j+1}) - X(t_j)).$$

(iv) *Let furthermore σ be Malliavin differentiable. Then the X -integral is a continuous linear operator from $\mathcal{I}_{1,2}^X(0, t)$ to $L^2(\Omega; \mathcal{H}_3)$.*

Proof. (i) Note that by definition of \mathcal{K}_g we have $\mathcal{K}_g(ZY)(t, s) = Z\mathcal{K}_g(Y)(t, s)$ and by the product rule of Malliavin calculus (2.2)

$$D_s(Z\mathcal{K}_g(Y)(t, s)) = D_s(Z)\mathcal{K}_g(Y)(t, s) + ZD_s\mathcal{K}_g(Y)(t, s).$$

This and the Hilbert-valued integration by parts formula (2.6) yield the assertion

$$\begin{aligned} & \int_0^t ZY(s) dX(s) \\ &= \int_0^t Z\mathcal{K}_g(Y)(t, s)\sigma(s)\delta B(s) \\ & \quad + \text{tr}_{\mathcal{H}_1} \int_0^t (ZD_s\mathcal{K}_g(Y)(t, s) + D_s(Z)\mathcal{K}_g(Y)(t, s))\sigma(s) ds \\ &= Z\left(\int_0^t \mathcal{K}_g(Y)(t, s)\sigma(s)\delta B(s) + \text{tr}_{\mathcal{H}_1} \int_0^t D_s\mathcal{K}_g(Y)(t, s)\sigma(s) ds\right) \\ &= Z \int_0^t Y(s) dX(s). \end{aligned}$$

(ii) This claim follows immediately from the fact that the Malliavin derivative, the Skorohod integral and the Lebesgue(–Stieltjes) integral are local operators.

(iii) This is a combination of (i) and (4.2).

(iv) This is an obvious conclusion from (i) and the continuity in (3.7). □

Another property we want to investigate in this subsection are two projection equalities with respect to the CONS $(e_k)_{k \in \mathbb{N}}$ of \mathcal{H}_2 . The first one is given by the application $\mathcal{K}_g(Y)(t, s)$ to e_k , which yields

$$\mathcal{K}_g(Y)(t, s)(e_k) = \mathcal{K}_{g(e_k)}(Y)(t, s) \tag{4.4}$$

which is easy to see from (3.4), where $g(e_k) \in \mathcal{H}_2$ is defined by $g(t, s)e_k$ for all $0 \leq s < t \leq T$. The other projection equality is integration with respect to $X^k = \langle X, e_k \rangle_{\mathcal{H}_2}$, which is a real-valued stochastic process. Applying the definition of the stochastic integral (3.6) invoking (3.1) with $\mathcal{H}_2 = \mathbb{R}$, we see

$$\int_0^t Y(s) dX^k(s) = \int_0^t \mathcal{K}_{\langle g, e_k \rangle}(Y)(t, s)\delta B(s) + \text{tr}_{\mathcal{H}_1} \int_0^t D_s\mathcal{K}_{\langle g, e_k \rangle}(Y)(t, s) ds,$$

where Y is a stochastic process taking values in a Hilbert space \mathcal{H}_3 . Note that if $\mathcal{H}_3 = \mathbb{R}$, the function $s \mapsto \mathcal{K}_{\langle g, e_k \rangle}(Y)(t, s)$ is real-valued, so we can relate this to the one-dimensional case treated in [2]. In order to recover exactly that situation, one has to assume $\mathcal{H}_1 = \mathbb{R}$, too.

4.2. Semimartingale condition

In this subsection, we focus on stating a condition under which the stochastic integral process $t \mapsto X(t)$ is a semimartingale. This condition is a regularity and smoothness assumption on g . The following proposition is inspired by [2], Proposition 5.

Proposition 4.2. *Let $t > 0$ and assume that $g(t, s)$ is well-defined for all $0 \leq s \leq t$. Furthermore, assume that there is a bi-measurable function $\phi : [0, T] \rightarrow L(\mathcal{H}_2, \mathcal{H}_2)$ such that*

$$g(t, s) = g(s, s) + \int_s^t \phi(v, s) \, dv, \tag{4.5}$$

for all $0 \leq s \leq t$, where this integral is defined in the sense of Bochner and

$$\int_0^t \|g(s, s)\|_{L(\mathcal{H}_2, \mathcal{H}_2)}^2 \, ds < \infty \quad \text{and} \quad \int_0^t \int_0^u \|\phi(u, s)\|_{L(\mathcal{H}_2, \mathcal{H}_2)}^2 \, ds \, du < \infty.$$

Suppose furthermore that σ is adapted to B and pathwise locally bounded almost surely. Then X defined by (1.1) is a semimartingale with decomposition

$$X(t) = \int_0^t g(s, s)\sigma(s) \, dB(s) + \int_0^t \int_0^s \phi(s, u)\sigma(u) \, dB(u) \, ds. \tag{4.6}$$

Proof. Substituting the definition of g and applying a stochastic Fubini theorem, we obtain

$$\begin{aligned} X(t) &= \int_0^t g(t, s)\sigma(s) \, dB(s) \\ &= \int_0^t g(s, s)\sigma(s) \, dB(s) + \int_0^t \int_0^s \phi(s, u)\sigma(s) \, dB(u) \, ds, \end{aligned}$$

which can be seen to be a semimartingale by [13], Remark 26.4. □

From (4.5), we see that ϕ can be interpreted as some sort of derivative of g . In fact, if $g(\cdot, s)$ is Fréchet differentiable, then $\phi(t, s) = \frac{\partial g}{\partial t}(t, s)$. Note that one can also relax the condition on σ to be adapted, leading to so-called Skorohod semimartingales; see [15].

In the case when X is a semimartingale, the question arises in which relationship our integral and the classical integral with respect to the semimartingale X (see [17]) stand to each other. The following proposition provides a partial answer.

Proposition 4.3. *Assume that Assumption 3.1 holds and that X is defined by (1.1). Assume that the semimartingale in Proposition 4.2 hold. Suppose that $Y = (Y(s))_{s \in [0, T]}$ is a predictable stochastic process, which is integrable with respect to the semimartingale X . Assume moreover that either $D_s Y(s) = 0$ for almost all $s \in [0, T]$ or $g(s, s) = 0$ for almost all $s \in [0, T]$. Then $Y \in \mathcal{I}^X(0, t)$ for all $t \in [0, T]$ and $(Y \cdot X)_t = \int_0^t Y(s) \, dX(s)$.*

Proof. Using the decomposition of X given in (4.6), the stochastic integral of Y with respect to X , denoted by $Y \cdot X$, is given by

$$(Y \cdot X)_t = \int_0^t Y(s)g(s, s)\sigma(s) dB(s) + \int_0^t Y(s) \int_0^s \phi(s, u)\sigma(u) dB(u) ds$$

However, by applying integration by parts and the stochastic Fubini theorem, we get,

$$\begin{aligned} &= \int_0^t Y(s)g(s, s)\sigma(s) dB(s) + \int_0^t \int_s^t Y(u)\phi(u, s) du \sigma(s)\delta B(s) \\ &\quad + \text{tr}_{\mathcal{H}_1} \int_0^t \int_s^t D_s(Y(u))\phi(u, s)\sigma(s) du ds \\ &= \int_0^t Y(s) dX(s) - \text{tr}_{\mathcal{H}_1} \int_0^t D_s(Y(s))g(s, s)\sigma(s) ds. \end{aligned} \tag{4.7}$$

Therefore, these integrals are equal if $D_s Y(s) = 0$ or $g(s, s) = 0$ for almost all $s \in [0, t]$. □

The additional condition that $D_s Y(s) = 0$ for almost all $s \in [0, t]$ holds true, for instance, when Y is a simple process as in Proposition 4.1(iii), where Z_{t_i} is assumed to be measurable with respect to $(B_t)_{t \in [0, t_i]}$. Similarly, this also holds when Y is deterministic, or independent of the driving Wiener process or lags behind the driving Wiener process.

The difference of the integrals comes from the following fact. Starting from the predictable simple processes for which $D_s Y(s) = 0$ for almost all $s \in [0, t]$, we define the semimartingale integral as the closure of the simple processes under the $L^2([X]^c)$ -norm. For the X -integral, this is not what one does. In fact, predictable simple functions do not even approximate all the processes in the domain of the X -integral with respect to the $L^2([X]^c)$ -norm.

Example 4.4. For instance, take $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}_3 = \mathbb{R}$, let Y be a Brownian motion on $[0, T]$ and let $g \equiv 1$ and $\sigma \equiv 1$. In this setup, $\mathcal{K}_g(Y)(t, s) = B(s)$ and $D_s \mathcal{K}_g(Y)(t, s) = 1$. Therefore, $Y \in \mathcal{I}^X(0, t)$ for all $t \in [0, T]$ and

$$\int_0^t Y(s) dX(s) = \int_0^t B(s) dB(s) + \int_0^t ds = \frac{1}{2}(B(t)^2 + t). \tag{4.8}$$

But the Malliavin derivative of each predictable simple process Y_n in Proposition 4.1(iii) approximating the Brownian motion is equal to zero and, therefore, does not converge to 1.

Another set of integrands on which the integrals coincide are semimartingales of the form

$$Y(t) = A_t + \int_0^t u(s) dB(s),$$

where A is a process of bounded variation and we assume that $D_s A(s) = 0$ and $D_s u(s) = 0$ for almost all $s \in [0, T]$.

4.3. Deterministic integrands and OU processes

In this subsection, we investigate the integral $\int_0^t h(t, s) dX(s)$ if h is chosen to be deterministic. To this end, fix $t > 0$ and let $s \mapsto h(t, s)$ be a measurable function from $[0, t]$ into $L(\mathcal{H}_2, \mathcal{H}_3)$ such that $h(t, u) - h(t, s)$ is integrable with respect to $g(du, s)$ over $[s, t]$ and Definition 3.2(ii) holds. In this case, the kernel $\mathcal{K}_g(h)$ is deterministic and has therefore a Malliavin derivative equal to zero, which immediately implies condition (iii) in Definition 3.2. Then, by the definition of the integral we have

$$\begin{aligned} \int_0^t h(t, s) dX(s) &= \int_0^t \mathcal{K}_g(h(t, \cdot))(t, s) \sigma(s) \delta B(s) \\ &= \int_0^t \left(h(t, s) g(t, s) + \int_s^t (h(t, u) - h(t, s)) g(du, s) \right) \sigma(s) \delta B(s). \end{aligned}$$

Note that this is a new process of the same type as X in (1.1) where now $\mathcal{K}_g(h)$ plays the role of g .

Now we apply this to a specific deterministic integrand which is stationary in time in order to show that Ornstein–Uhlenbeck (OU) processes driven by X are representable as a stochastic integral of a deterministic integrand with respect to X . By an OU process, we mean the solution $(Y(t))_{t \geq 0}$ to the infinite-dimensional SDE $dY(t) = -AY(t) dt + F dX(t)$, or in integral terms

$$Y(t) = - \int_0^t AY(s) ds + FX(t), \tag{4.9}$$

where $(X(t))_{t \geq 0}$ is the process defined in (1.1) and $-A$ is an unbounded linear operator from \mathcal{H}_3 to \mathcal{H}_3 whose domain is dense in \mathcal{H}_3 and that generates a C_0 -semigroup on \mathcal{H}_3 denoted by $S(t) = e^{-tA}$ for all $t \geq 0$. Moreover, F is a bounded linear operator from \mathcal{H}_2 to \mathcal{H}_3 . Note that this is a first step into the investigation of SDEs driven by a Volterra-type process. As suggested from the classical situation where the driving noise in (4.9) is a Wiener process, we obtain that the solution to (4.9) is given by a stochastic convolution.

Proposition 4.5. *Suppose that Assumption 3.1 holds. Assume furthermore that $t \mapsto \sigma(t)$ is pathwise locally bounded almost surely. Assume for all $s \in [0, T]$ that the map $(s, T) \ni u \mapsto (e^{-(u-s)A} - e^{-sA})$ is integrable with respect to $g(du, s)$, that*

$$\mathbb{E} \left[\int_0^t \left\| \int_0^s F \mathcal{K}_g(S)(s, u) \sigma(u) \delta B(u) \right\|_{\mathcal{H}_3}^2 ds \right] < \infty, \tag{4.10}$$

and

$$\lim_{t \downarrow s} \left\| (\text{id}_{\mathcal{H}_2} - e^{-(t-s)A}) g(t, s) \right\|_{L_2(\mathcal{H}_2, \mathcal{H}_3)} = 0. \tag{4.11}$$

Then $S \in \mathcal{I}^X(0, t)$ and

$$Y(t) = \int_0^t S(t-s) F dX(s) = \int_0^t e^{-(t-s)A} F dX(s) \tag{4.12}$$

solves the SDE (4.9).

First, let us explain the two conditions (4.10) and (4.11). The former one is the Hilbert valued equivalent of the one in [14], Exercise 3.2.7, which is sufficient for an application of the stochastic Fubini theorem with the divergence operator that we will use in the following proof. If σ is assumed to be predictable, the Skorohod integral becomes an Itô integral since $\mathcal{K}_g(h)$ is deterministic. In this case, one can derive similarly as in [2], Lemma 2, that

$$\int_0^T \int_s^T \|g(u, s)\|_{L(\mathcal{H}_2, \mathcal{H}_2)}^2 du ds < \infty$$

is a sufficient condition for the stochastic Fubini theorem to hold. The latter condition (4.11) can be interpreted as a kind of weighted C_0 -semigroup condition. In fact, we demand that the semigroup $S(t) = e^{-tA}$ goes to the identity as $t \downarrow 0$ in a family of norms with weights given by the functions $g(\cdot, s)$.

Proof of Proposition 4.5. Let throughout the proof $\zeta \in \text{Dom}(A^*)$. By substituting (4.12) and the definition of the stochastic integral into (4.9) and applying the stochastic Fubini theorem, one obtains

$$\begin{aligned} \left\langle \zeta, \int_0^t AY(u) du \right\rangle_{\mathcal{H}_3} &= \int_0^t \left\langle A^* \zeta, \int_0^u \mathcal{K}_g(SF)(u, s) \sigma(s) \delta B(s) \right\rangle_{\mathcal{H}_2} du \\ &= \int_0^t \left\langle \left(\int_s^t A \mathcal{K}_g(SF)(u, s) du \right)^* \zeta, \sigma(s) \delta B(s) \right\rangle_{\mathcal{H}_2}, \end{aligned}$$

and similarly for the first term of (4.9)

$$\langle \zeta, Y(t) \rangle_{\mathcal{H}_3} = \int_0^t \langle \mathcal{K}_g(SF)(t, s)^* \zeta, \sigma(s) \delta B(s) \rangle_{\mathcal{H}_2}.$$

These two equalities yield that

$$\begin{aligned} \left\langle \zeta, Y(t) + \int_0^t AY(u) du \right\rangle_{\mathcal{H}_3} & \\ &= \int_0^t \left\langle \left(\mathcal{K}_g(SF)(t, s) + \int_s^t A \mathcal{K}_g(SF)(u, s) du \right)^* \zeta, \sigma(s) \delta B(s) \right\rangle_{\mathcal{H}_2}. \end{aligned} \tag{4.13}$$

Now we investigate the term in the brackets in the previous equality. Written out, this term is equal to

$$\begin{aligned} e^{-(t-s)A} Fg(t, s) + \int_s^t (e^{-(t-u)A} - e^{-(t-s)A}) Fg(du, s) & \\ + \int_s^t \left(A e^{-(u-s)A} Fg(u, s) + \int_s^u (A e^{-(u-v)A} - A e^{-(u-s)A}) Fg(dv, s) \right) du. & \end{aligned} \tag{4.14}$$

Using Fubini’s theorem and [10], Lemma II.1.3(ii), we see that

$$\begin{aligned} & \int_s^t \int_s^u (Ae^{-(u-v)A} - Ae^{-(u-s)A}) Fg(dv, s) du \\ &= \int_s^u \int_v^t \left(-\frac{\partial}{\partial u} e^{-(u-v)A} + \frac{\partial}{\partial u} e^{-(u-s)A} \right) du Fg(dv, s) \\ &= - \int_s^t (e^{-(t-v)A} - e^{-(t-s)A}) Fg(dv, s) + \int_s^t (1 - e^{-(v-s)A}) Fg(dv, s). \end{aligned} \tag{4.15}$$

Note that by an integration by parts procedure and using [10], Lemma II.1.3(ii), again, the second term on the right-hand side of the last equality may be calculated to be

$$\begin{aligned} & \int_s^t (1 - e^{-(v-s)A}) Fg(dv, s) \\ &= (1 - e^{-(t-s)A}) Fg(t, s) - \int_s^t Ae^{-(v-s)A} Fg(v, s) ds, \end{aligned} \tag{4.16}$$

where we used the condition (4.11) so that the boundary term at s does not appear. Assembling (4.14)–(4.16), we obtain

$$\mathcal{K}_g(SF)(t, s) + \int_u^t A\mathcal{K}_g(SF)(s, u) ds = Fg(t, s),$$

which in turn, by substituting this into (4.13) and using the fact that F is bounded, implies the assertion

$$\left\langle \zeta, Y(t) + \int_0^t AY(u) du \right\rangle_{\mathcal{H}_3} = \left\langle \zeta, \int_0^t Fg(t, s)\sigma(s)\delta B(s) \right\rangle_{\mathcal{H}_3} = \langle \zeta, FX(t) \rangle_{\mathcal{H}_3}.$$

The proof ends by the fact that A and hence A^* are densely defined in \mathcal{H}_3 which implies (4.12) □

Note that (4.12) has the exact same form as in the classical case when the driving process is a Wiener process. We however have to remark that this does not show uniqueness of the solution to the SDE (4.9).

4.4. Volterra processes as integrands

In this subsection, we want to calculate the stochastic integral with respect to X when the integrand is a Volterra process itself. This will also lead us to an integral of type $\int_0^t X(s) dX(s)$. Throughout this subsection, we assume that $\sigma \equiv \text{id}_{\mathcal{H}_2}$ and assume that B is a (cylindrical) Wiener process on \mathcal{H}_2 . However, one can reintroduce σ into the calculations by formally considering σdB as the noise. Moreover, we assume in the following without loss of generality that $\mathcal{H}_3 = \mathbb{R}$. The case of general separable Hilbert spaces then follows by the expansion

$Y(s) = \sum_k \langle Y(s), f_k \rangle f_k$, where $(f_k)_{k \in \mathbb{N}}$ is a CONS of \mathcal{H}_3 , and performing the subsequent calculations coordinatewise.

Let in the following h be an $L(\mathcal{H}_2, L(\mathcal{H}_2, \mathbb{R}))$ -valued deterministic function in $\mathcal{I}^X(0, t)$ and by the results of the previous subsection

$$Y(s) = \int_0^s h(u) dX(u) = \int_0^s \mathcal{K}_g(h)(s, v) dB(v). \tag{4.17}$$

Here, the integral is in fact a classical Itô integral. Furthermore, let $(e_k)_{k \in \mathbb{N}}$ be a CONS of \mathcal{H}_2 .

Proposition 4.6. *Suppose that Assumption 3.1 holds and assume that σ, B, h, Y and \mathcal{H}_3 are as stated above. Assume furthermore that for almost all $s \in [0, T]$, $g(s, s)$ is well-defined as a linear operator on \mathcal{H}_2 . Then $Y \in \mathcal{I}^X(0, t)$ for all $t \in [0, T]$ and $\int_0^t Y(s) dX(s)$ is a sum of an element in the second and zeroth Wiener chaos.*

Proof. Using the definition of the stochastic integral with respect to X and the notion of the projection in (4.4), we have

$$\begin{aligned} & \int_0^t Y(s) dX(s) \\ &= \sum_{m \in \mathbb{N}} \left(\int_0^t \mathcal{K}_g(Y)(t, s)(e_m) \delta B^m(s) + \int_0^t D_{s,m} \mathcal{K}_g(Y)(t, s)(e_m) ds \right). \end{aligned} \tag{4.18}$$

Now, taking advantage of the specific form of Y , we calculate the integrand in the stochastic integral in (4.18) to be

$$\begin{aligned} & \mathcal{K}_{g(e_m)}(Y)(t, s) \\ &= Y(s)g(t, s)(e_m) + \int_s^t (Y(u) - Y(s))g(du, s)(e_m) \\ &= \int_0^s \mathcal{K}_g(h)(t, s) dB(v)g(t, s)(e_m) \\ & \quad + \int_s^t \int_0^s (\mathcal{K}_g(h)(u, v) - \mathcal{K}_g(h)(s, v)) dB(v)g(du, s)(e_m) \\ & \quad + \int_s^t \int_s^u \mathcal{K}_g(h)(u, v) dB(v)g(du, s)(e_m) \\ &= \sum_{l \in \mathbb{N}} \left(\int_0^s \mathcal{K}_{g(e_l)}(h)(t, s)g(t, s)(e_m) dB^l(v) \right. \\ & \quad + \int_s^t \int_v^t \mathcal{K}_{g(e_l)}(h)(u, v)g(du, s)(e_m) dB^l(v) \\ & \quad \left. + \int_0^s \int_s^t (\mathcal{K}_{g(e_l)}(h)(u, v) - \mathcal{K}_{g(e_l)}(h)(s, v))g(du, s)(e_m) dB^l(v) \right) \end{aligned} \tag{4.19}$$

$$\begin{aligned}
 &= \sum_{l \in \mathbb{N}} \left(\int_0^s \mathcal{K}_{g(e_m)}(\mathcal{K}_{g(e_l)}(h)(\cdot, v))(t, s) \, dB^l(v) \right. \\
 &\quad \left. + \int_s^t \int_v^t \mathcal{K}_{g(e_l)}(h)(u, v) g(du, s)(e_m) \, dB^l(v) \right) \\
 &= \sum_{l \in \mathbb{N}} \int_0^t \tilde{\mathcal{K}}_{g,g}^{l,m}(h)(s, v, t) \, dB^l(v),
 \end{aligned}$$

where we have set

$$\begin{aligned}
 \tilde{\mathcal{K}}_{g_1, g_2}^{l,m}(h)(s, v, t) &= 1_{\{v \leq s\}} \mathcal{K}_{g_2(e_m)}(\mathcal{K}_{g_1(e_l)}(h)(\cdot, v))(t, s) \\
 &\quad + 1_{\{v > s\}} \int_v^t \mathcal{K}_{g_1(e_l)}(h)(u, v) g_2(du, s)(e_m).
 \end{aligned} \tag{4.20}$$

Applying the Malliavin derivative operator in (4.19) yields

$$\begin{aligned}
 D_{s,m} \mathcal{K}_{g(e_m)}(Y)(t, s) &= D_{s,m} \left(\sum_{l \in \mathbb{N}} \int_0^t \tilde{\mathcal{K}}_{g,g}^{l,m}(h)(s, v, t) \, dB^l(v) \right) \\
 &= \tilde{\mathcal{K}}_{g,g}^{m,m}(h)(s, s, t).
 \end{aligned} \tag{4.21}$$

Note that on the right-hand side the argument (s, s, t) appears which implies the explicit appearance of $g(s, s)$ in the formulas; see below for a comment. Assembling all the terms (4.18)–(4.21), we obtain

$$\begin{aligned}
 \int_0^t Y(s) \, dX(s) &= \sum_{m \in \mathbb{N}} \left(\int_0^t \sum_{l \in \mathbb{N}} \int_0^t \tilde{\mathcal{K}}_{g,g}^{l,m}(h)(s, v, t) \, dB^l(v) \delta B^m(s) + \int_0^t \tilde{\mathcal{K}}_{g,g}^{l,m}(h)(s, s, t) \, ds \right) \\
 &= \sum_{l,m \in \mathbb{N}} \int_0^t \int_0^t \tilde{\mathcal{K}}_{g,g}^{l,m}(h)(s, v, t) \, dB^l(v) \delta B^m(s) + \sum_{m \in \mathbb{N}} \int_0^t \tilde{\mathcal{K}}_{g,g}^{m,m}(h)(s, s, t) \, ds \\
 &= \sum_{l,m \in \mathbb{N}} \int_0^t \int_0^s (\tilde{\mathcal{K}}_{g,g}^{l,m}(h)(s, v, t) + \tilde{\mathcal{K}}_{g,g}^{m,l}(h)(v, s, t)) \, dB^l(v) \, dB^m(s) \\
 &\quad + \sum_{m \in \mathbb{N}} \int_0^t \tilde{\mathcal{K}}_{g,g}^{m,m}(h)(s, s, t) \, ds.
 \end{aligned}$$

If we now define the linear operator $\tilde{\mathcal{K}}_{g,g}(h)$ by $\tilde{\mathcal{K}}_{g,g}(h)(s, v, t)(e_l \otimes e_m) := \tilde{\mathcal{K}}_{g,g}^{l,m}(h)(s, v, t)$, we see in total that

$$\begin{aligned}
 \int_0^t Y(s) \, dX(s) &= \int_0^t \int_0^s (\tilde{\mathcal{K}}_{g,g}(h)(s, v, t) + \tilde{\mathcal{K}}_{g,g}(h)(v, s, t)) \, dB(v) \otimes dB(s) \\
 &\quad + \text{tr}_{\mathcal{H}_2} \int_0^t \tilde{\mathcal{K}}_{g,g}(h)(s, s, t) \, ds.
 \end{aligned} \tag{4.22}$$

From this identity, we can see that every integrator in this formula is deterministic and, therefore, we can conclude that the integral $\int_0^t Y(s) dX(s)$ belongs to the second Wiener chaos plus a term from the zeroth Wiener chaos. \square

Note that the term $\tilde{\mathcal{K}}_{g,g}(h)(s, s, t)$ appears in the correction term in the formula (4.22). This implies that the integral term in (4.20) does not appear, but in the first summand in (4.20) the terms $g(s, s)$ will appear explicitly. This is why we have to assume sufficient regularity for the kernel g , in particular that $g(t, t)$ exists as a linear operator for all $t \in [0, T]$. The reason for g to be regular is that it appears in the integrand (4.17) as well as in the integrator. Otherwise, both might have a singularity at the same point which would cause problems when integrating.

Remark 4.7. Similarly to these calculations above, one can easily extend the results to iterated X integrals of higher order. As long as σ is deterministic, the X -integral of an integrand Y , which is itself an iterated X -integral, consists of a Skorohod integral over $\mathcal{K}_g(Y)$ with respect to a cylindrical Wiener process, and a pathwise integral of the Malliavin derivative of $\mathcal{K}_g(Y)$. This increases the number of the Wiener chaos of the integrand by one (Wiener integral) and decreases the number of the Wiener chaos of the integrand by one (pathwise integral). Therefore, for k odd (even), the k th iterated X -integral of a deterministic function is an element of all odd (even and zeroth) Wiener chaoses up to order k , respectively.

Now we apply this to the problem of calculating $\int_0^t X(s) dX(s)$. Since both elements are \mathcal{H}_2 -valued, we cannot apply (4.22) verbatim. However, by the Riesz representation theorem, we interpret the integrand as a linear functional on \mathcal{H}_2 and, therefore,

$$\int_0^t X(s) dX(s) := \int_0^t X(s)^* dX(s) = \int_0^t \langle X(s), dX(s) \rangle = \sum_{k \in \mathbb{N}} \int_0^t X^k(s) dX^k(s).$$

Proposition 4.8. *Suppose that Assumption 3.1 holds and assume that σ, B and \mathcal{H}_3 are as stated at the beginning of this section. Assume furthermore that for almost all $s \in [0, T]$, $g(s, s)$ is well-defined as a linear operator on \mathcal{H}_2 . Then $X \in \mathcal{I}^X(0, t)$ for all $t \in [0, T]$. Moreover, this integral can be written as the sum of $\frac{1}{2} \langle X(t), X(t) \rangle_{\mathcal{H}_2}$ and some correction term.*

Proof. In this case, we choose $h(s) = \text{id}_{\mathcal{H}_2}$ which implies $\mathcal{K}_g(h)(t, s) = g(t, s)$. Since $X^k(s) = \int_0^s \langle g(s, u), e_k \rangle_{\mathcal{H}_2} dB(s)$, we obtain that $D_{s,l} X^k(u) = \langle g(u, s) e_l, e_k \rangle_{\mathcal{H}_2}$. Then

$$\begin{aligned} & \sum_{k \in \mathbb{N}} \int_0^t X^k(s) dX^k(s) \\ &= \sum_{k \in \mathbb{N}} \int_0^t \left(X^k(s) \langle g(s, s), e_k \rangle_{\mathcal{H}_2} + \int_s^t X^k(r) \langle g(dr, s), e_k \rangle_{\mathcal{H}_2} \right) dB(s) \\ & \quad + \sum_{k, l \in \mathbb{N}} \int_0^t D_{s,l} \left(X^k(s) \langle g(s, s), e_k \rangle_{\mathcal{H}_2} + \int_s^t X^k(r) \langle g(dr, s), e_k \rangle_{\mathcal{H}_2} \right) (e_l) ds \end{aligned} \tag{4.23}$$

$$\begin{aligned}
 &= \int_0^t \int_0^s \langle g(s, u) dB(u), g(s, s) dB(s) \rangle_{\mathcal{H}_2} \\
 &\quad + \int_0^t \int_s^t \left\langle \int_0^r g(r, u) dB(u), g(dr, s) dB(s) \right\rangle_{\mathcal{H}_2} \\
 &\quad + \text{tr}_{\mathcal{H}_2} \int_0^t \left(\langle g(s, s), g(s, s) \rangle_{\mathcal{H}_2} + \int_s^t \langle g(r, s), g(dr, s) \rangle_{\mathcal{H}_2} \right) ds.
 \end{aligned}$$

We want to find a formula that links this term with $\langle X(t), X(t) \rangle_{\mathcal{H}_2}$, which can be calculated to be

$$\begin{aligned}
 &\frac{1}{2} \langle X(t), X(t) \rangle_{\mathcal{H}_2} \\
 &= \int_0^t \int_0^s \langle g(t, u) dB(u), g(t, s) dB(s) \rangle_{\mathcal{H}_2} \\
 &= \int_0^t \int_0^s \langle g(s, u) dB(u), g(s, s) dB(s) \rangle_{\mathcal{H}_2} \\
 &\quad + \int_0^t \int_s^t \left\langle \int_0^r g(r, u) dB(u), g(dr, s) dB(s) \right\rangle_{\mathcal{H}_2} \tag{4.24} \\
 &\quad + \int_0^t \int_0^s \left\langle \int_s^t g(dv, u) dB(u), g(s, s) dB(s) \right\rangle_{\mathcal{H}_2} \\
 &\quad - \int_0^t \int_s^t \left\langle \int_s^r g(t, u) dB(u), g(dr, s) dB(s) \right\rangle_{\mathcal{H}_2} \\
 &\quad + \int_0^t \int_s^t \left\langle \int_0^r (g(t, u) - g(r, u)) dB(u), g(dr, s) dB(s) \right\rangle_{\mathcal{H}_2}
 \end{aligned}$$

using the order $u \leq s \leq r \leq t$. Note that the first and second term on the right-hand side of this equality are equal to first and second term on the right-hand side of (4.23). The three last terms in (4.24) can be rewritten by grouping the first and third one together as

$$\begin{aligned}
 &\int_0^t \int_0^s \left(\left\langle \int_s^t g(dv, u) dB(u), g(s, s) dB(s) \right\rangle_{\mathcal{H}_2} \right. \\
 &\quad \left. + \int_s^t \left\langle \int_r^t g(dv, u) dB(u), g(dr, s) dB(s) \right\rangle_{\mathcal{H}_2} \right) \\
 &\quad - \int_0^t \int_s^t \left\langle \int_s^r g(r, u) dB(u), g(dr, s) dB(s) \right\rangle_{\mathcal{H}_2},
 \end{aligned}$$

which we can calculate to be, using Fubini's theorem,

$$\begin{aligned}
 & \sum_{k,l,m \in \mathbb{N}} \int_0^t \int_0^s \left(\left\langle \int_s^t g(dv, u) e_l, e_k \right\rangle_{\mathcal{H}_2} \left\langle g(s, s) e_m, e_k \right\rangle_{\mathcal{H}_2} \right. \\
 & \quad \left. + \int_s^t \left\langle \int_r^t g(dv, u) e_l, e_k \right\rangle_{\mathcal{H}_2} \left\langle g(dr, s) e_m, e_k \right\rangle_{\mathcal{H}_2} \right) dB^l(u) dB^m(s) \\
 & - \sum_{k,l,m \in \mathbb{N}} \int_0^t \int_s^t \int_s^r \left\langle g(r, u) e_l, e_k \right\rangle_{\mathcal{H}_2} dB^l(u) \left\langle g(dr, s) e_m, e_k \right\rangle_{\mathcal{H}_2} dB^m(s) \\
 & = \sum_{k,l,m \in \mathbb{N}} \int_0^t \int_0^s \mathcal{K}_{(g(e_l), e_k)} \left(\int_s^t \left\langle g(dv, u) e_m, e_k \right\rangle_{\mathcal{H}_2} \right) (t, s) dB^l(u) dB^m(s) \\
 & - \sum_{k,l,m \in \mathbb{N}} \int_0^t \int_s^t \int_u^t \left\langle g(r, u) e_l, e_k \right\rangle_{\mathcal{H}_2} \left\langle g(dr, s) e_m, e_k \right\rangle_{\mathcal{H}_2} dB^l(u) dB^m(s) \\
 & = \int_0^t \int_0^s \mathcal{K}_g \left(\int_s^t g(dv, u)^* \right) (t, s) dB(u) \otimes dB(s) \\
 & - \int_0^t \int_s^t \int_u^t \left\langle g(r, u), g(dr, s) \right\rangle_{\mathcal{H}_2} dB(u) \otimes dB(s),
 \end{aligned}$$

where $g(\cdot, s)^*$ means the adjoint linear operator of $g(\cdot, s)$. The third term in (4.23) can be seen to be equal to

$$\begin{aligned}
 & \sum_{k,l \in \mathbb{N}} \int_0^t \left(\left\langle g(s, s) e_l, e_k \right\rangle_{\mathcal{H}_2} \left\langle g(s, s) e_l, e_k \right\rangle_{\mathcal{H}_2} + \int_s^t \left\langle g(r, s) e_l, e_k \right\rangle_{\mathcal{H}_2} \left\langle g(dr, s) e_l, e_k \right\rangle_{\mathcal{H}_2} \right) ds \\
 & = \sum_{k,l \in \mathbb{N}} \int_0^t \mathcal{K}_{(g(e_l), e_k)} \left(\left\langle g(\cdot, s) e_l, e_k \right\rangle_{\mathcal{H}_2} \right) (t, s) ds \tag{4.25} \\
 & = \text{tr}_{\mathcal{H}_2} \int_0^t \mathcal{K}_g(g(\cdot, s)^*) (t, s) ds.
 \end{aligned}$$

One could also reformulate the integral term in the first line of (4.25), which has the form $\int_s^t f(u) f(du) = \frac{1}{2}(f^2(t) - f^2(s))$, to be

$$\begin{aligned}
 & \sum_{k \in \mathbb{N}} \int_0^t \left(\left\langle g(s, s) e_l, e_k \right\rangle_{\mathcal{H}_2} \left\langle g(s, s) e_l, e_k \right\rangle_{\mathcal{H}_2} + \int_s^t \left\langle g(r, s) e_l, e_k \right\rangle_{\mathcal{H}_2} \left\langle g(dr, s) e_l, e_k \right\rangle_{\mathcal{H}_2} \right) ds \\
 & = \frac{1}{2} \int_0^t \left(\|g(t, s)\|_{L_2(\mathcal{H}_1)}^2 + \|g(s, s)\|_{L_2(\mathcal{H}_1)}^2 \right) ds.
 \end{aligned}$$

So, collecting all the terms above we finally have an equality

$$\begin{aligned} & \int_0^t X(s) dX(s) \\ &= \frac{1}{2} \langle X(t), X(t) \rangle_{\mathcal{H}_2} - \int_0^t \int_0^s \mathcal{K}_g \left(\int_0^t g(dv, u)^* \right) (t, s) dB(u) \otimes dB(s) \\ & \quad - \int_0^t \int_s^t \int_u^t \langle g(r, u), g(dr, s) \rangle_{\mathcal{H}_2} dB(u) \otimes dB(s) + \text{tr}_{\mathcal{H}_2} \int_0^t \mathcal{K}_g(g(\cdot, s)^*) (t, s) ds, \end{aligned}$$

which gives us the desired link between the X -integral of X and $\|X(t)\|_{\mathcal{H}_2}^2$. □

5. An Itô formula

In this section, we derive Itô formulas for the processes X and Z defined in (1.1) and (1.2). In order to do this, we rely on the Itô formula in Hilbert spaces with anticipating integrands in [11], Proposition 4.10.

Proposition 5.1. *Let \mathcal{H}, \mathcal{K} be separable Hilbert spaces and B be a \mathcal{H} -valued cylindrical Wiener process. Let moreover $F \in \mathcal{C}^2(\mathcal{K}; \mathbb{R})$ (the twice Fréchet differentiable functionals) and let $(V(t))_{t \geq 0}$ be the stochastic process defined by*

$$V(t) = V(0) + \int_0^t A(s) ds + \int_0^t C(s) \delta B(s),$$

where $V(0) \in \mathbb{D}^{1,4}(\mathcal{K})$, $A \in \mathbb{L}^{1,4}(\mathcal{K})$ and $C \in \mathbb{L}^{2,p}(\mathcal{H}, \mathcal{K})$ for some $p > 4$, see (2.4). Then

$$\begin{aligned} F(V(t)) &= F(V(0)) + \int_0^t F'(V(s))A(s) ds + \int_0^t F'(V(s))C(s)\delta B(s) \\ & \quad + \frac{1}{2} \text{tr}_{\mathcal{H}} \int_0^t F''(V(s))(D^-V)(s)(C(s)) ds \\ & \quad + \frac{1}{2} \text{tr}_{\mathcal{H}} \int_0^t F''(V(s))(C(s))(C(s)) ds, \end{aligned}$$

where

$$(D^-V)(s) = 2D_s V(0) + 2 \int_0^s D_s A(r) dr + 2 \int_0^s D_s C(r) \delta B(r).$$

We give some remarks on the definition of the various terms in the above formula. Since $F' \in L(\mathcal{K}, L(\mathcal{K}, \mathbb{R}))$, the first two integral terms take values in \mathbb{R} , as $F(V(0))$ does. For the fourth term on the right-hand side, we use the definition of F'' to be an element of $L(\mathcal{K}, L(\mathcal{K}, L(\mathcal{K}, \mathbb{R})))$, whereas for the fifth term we use the equivalent formulation $F'' \in L(\mathcal{K} \otimes \mathcal{K}, L_2(\mathcal{K}, \mathbb{R}))$. Then,

since $(D^-V)(s) \in L_2(\mathcal{H}, \mathcal{K})$ and $C(s) \in L_2(\mathcal{H}, \mathcal{K})$, we have that $F''(V(s))(D^-V)(s)(C(s)) \in L_2(\mathcal{H}, L_2(\mathcal{H}, \mathbb{R}))$ and $F''(V(s))(D^-V)(s)(C(s)) \in L_2(\mathcal{H}, L_2(\mathcal{H}, \mathbb{R}))$ so that the trace over \mathcal{H} is in both cases well-defined. Note that this trace is our way of writing the inner products in the original formula in [11], Proposition 4.10.

The first issue in this section is to extend the above proposition to functions $F \in \mathcal{C}^2(\mathcal{K}; \mathcal{K}_1)$ where \mathcal{K}_1 is another separable Hilbert space, not necessarily \mathbb{R} . We can reduce this to applying Proposition 5.1 coordinatewise in the following sense: set $F^k := \langle F, e_k \rangle_{\mathcal{K}_1}$ and observe

$$\begin{aligned} &F^k(V(t)) \\ &= F^k(V(0)) + \int_0^t (F^k)'(V(s))A(s) \, ds + \int_0^t (F^k)'(V(s))C(s)\delta B(s) \\ &\quad + \frac{1}{2} \operatorname{tr}_{\mathcal{H}} \int_0^t (F^k)''(V(s))(D^-V)(s)C(s) \, ds + \frac{1}{2} \operatorname{tr}_{\mathcal{H}} \int_0^t (F^k)''(V(s))(C(s))C(s) \, ds \\ &= \left\langle F(V(0)) + \int_0^t F'(V(s))A(s) \, ds + \int_0^t F'(V(s))C(s)\delta B(s) \right. \\ &\quad \left. + \frac{1}{2} \operatorname{tr}_{\mathcal{H}} \int_0^t F''(V(s))(D^-V)(s)C(s) \, ds + \frac{1}{2} \operatorname{tr}_{\mathcal{H}} \int_0^t F''(V(s))(C(s))C(s) \, ds, e_k \right\rangle_{\mathcal{K}_1}. \end{aligned}$$

This holds true since one can commute Fréchet derivatives and projections, as they are bounded linear operators. Hence, we have identified each coordinate and summing over these coordinates yields an Itô formula in the Hilbert-valued case.

Now we derive an Itô formula for processes like Z in (1.2) which may be Hilbert-valued or real-valued. In order to apply Proposition 5.1 to our case, we will have to assume that $g(s, s)$ is a well-defined linear operator for all $s \in [0, T]$ and that there exists a function ϕ as in (4.5). Therefore, the Itô formula in this section will only hold for the case when X is a Skorohod semimartingale, that is, with g satisfying the conditions of Proposition 4.2 but with an anticipating integrand in the stochastic integral. So we will apply the above proposition to the following stochastic process:

$$\begin{aligned} &\int_0^t Y(s) \, dX(s) \\ &= \int_0^t \mathcal{K}_g(Y)(t, s)\sigma(s)\delta B(s) + \int_0^t \operatorname{tr}_{\mathcal{H}_1}(D_s \mathcal{K}_g(Y)(t, s)\sigma(s)) \, ds \\ &= \int_0^t Y(s)g(s, s)\sigma(s)\delta B(s) + \int_0^t \int_s^t Y(u)\phi(u, s) \, du\sigma(s)\delta B(s) \\ &\quad + \int_0^t \operatorname{tr}_{\mathcal{H}_1} D_s(Y(s))g(s, s)\sigma(s) \, ds + \int_0^t \operatorname{tr}_{\mathcal{H}_1} \int_s^t D_u(Y(u))\phi(u, s) \, du\sigma(s) \, ds \\ &= \int_0^t b(s)\delta B(s) + \int_0^t \left(\int_0^s a(s, u)\delta B(u) + \operatorname{tr}_{\mathcal{H}_1}(b_D(s)) + \operatorname{tr}_{\mathcal{H}_1} \int_0^s a_D(s, u) \, du \right) ds, \end{aligned}$$

where we have used the stochastic Fubini theorem and

$$\begin{aligned} b(s) &= Y(s)g(s, s)\sigma(s), & a(s, u) &= Y(s)\phi(s, u)\sigma(u), \\ b_D(s) &= D_s(Y(s))g(s, s)\sigma(s), & a_D(s, u) &= D_s(Y(s))\phi(s, u)\sigma(u). \end{aligned}$$

So we apply Proposition 5.1 to the processes $C(s) = b(s)$ and

$$A(s) = \int_0^s a(s, u)\delta B(u) + \text{tr}_{\mathcal{H}_1} b_D(s) + \text{tr}_{\mathcal{H}_1} \int_0^s a_D(s, u) du. \tag{5.1}$$

After the following theorem which sets up an Itô formula for the integral with respect to X , we will provide some sufficient conditions so that these processes satisfy the conditions in Proposition 5.1.

Theorem 5.2. *Let \mathcal{H}_4 be a separable Hilbert space and let $F : \mathcal{H}_3 \rightarrow \mathcal{H}_4$ be twice Fréchet differentiable. Let X and Z be defined as in (1.1) and (1.2) where we suppose that Assumption 3.1 holds. Furthermore, assume that g satisfies the semimartingale conditions in Proposition 4.2. Assume that Y and σ are twice Malliavin differentiable and*

$$C(s) \in \mathbb{L}^{2,p}(\mathcal{H}_1, \mathcal{H}_3) \quad \text{and} \quad A(s) \in \mathbb{L}^{1,4}(\mathcal{H}_3), \tag{5.2}$$

where $A(s)$ and $C(s)$ are as in (5.1). Then $F'(Z)Y \in \mathcal{I}^X(0, t)$ for all $t \in [0, T]$ and

$$\begin{aligned} F(Z(t)) &= F(0) + \int_0^t \mathcal{K}_g(F'(Z)Y)(t, s)\sigma(s)\delta B(s) \\ &\quad + \text{tr}_{\mathcal{H}_1} \int_0^t D_s(\mathcal{K}_g(F'(Z)Y)(t, s))\sigma(s) ds \\ &\quad - \frac{1}{2} \text{tr}_{\mathcal{H}_1} \int_0^t F''(Z(s))(Y(s)g(s, s)\sigma(s))(Y(s)g(s, s)\sigma(s)) ds \\ &= F(0) + \int_0^t F'(Z(s))Y(s) dX(s) \\ &\quad - \frac{1}{2} \text{tr}_{\mathcal{H}_1} \int_0^t F''(Z(s))(Y(s)g(s, s)\sigma(s))(Y(s)g(s, s)\sigma(s)) ds. \end{aligned} \tag{5.3}$$

Proof. Applying Proposition 5.1 yields

$$\begin{aligned} F(Z(t)) &= F(0) + \int_0^t F'(Z(s))b(s)\delta B(s) + \int_0^t F'(Z(s)) \int_0^s a(s, u)\delta B(u) ds \\ &\quad + \int_0^t F'(Z(s)) \text{tr}_{\mathcal{H}_1} b_D(s) ds + \int_0^t F'(Z(s)) \text{tr}_{\mathcal{H}_1} \int_0^s a_D(s, u) du ds \\ &\quad + \text{tr}_{\mathcal{H}_1} \int_0^t F''(Z(s)) \int_0^s D_s b(u)\delta B(u)b(s) ds \end{aligned} \tag{5.4}$$

$$\begin{aligned}
 & + \operatorname{tr}_{\mathcal{H}_1} \int_0^t F''(Z(s)) \int_0^s D_s \int_0^u a(u, r) \delta B(r) \, du \, b(s) \, ds \\
 & + \operatorname{tr}_{\mathcal{H}_1} \int_0^t F''(Z(s)) \int_0^s D_s \operatorname{tr}_{\mathcal{H}_1} b_D(u) \, du \, b(s) \, ds \\
 & + \operatorname{tr}_{\mathcal{H}_1} \int_0^t F''(Z(s)) \int_0^s D_s \operatorname{tr}_{\mathcal{H}_1} \int_0^u a_D(u, r) \, dr \, du \, b(s) \, ds \\
 & + \frac{1}{2} \operatorname{tr}_{\mathcal{H}_1} \int_0^t F''(Z(s)) b(s) b(s) \, ds.
 \end{aligned}$$

Note that the third term on the right-hand side of (5.4) can be rewritten using (2.6) as

$$\begin{aligned}
 & \int_0^t F'(Z(s)) \int_0^s a(s, u) \delta B(u) \, ds \\
 & = \int_0^t \int_0^s F'(Z(s)) a(s, u) \delta B(u) \, ds + \int_0^t \int_0^s \operatorname{tr}_{\mathcal{H}_1} ((D_u F'(Z(s))) a(s, u)) \, du \, ds.
 \end{aligned} \tag{5.5}$$

By the Itô formula in Proposition 5.1, we know that all the terms on the right-hand side of (5.4) are well-defined. Now we calculate the terms on the right-hand side of (5.3) assuming that they exist. We will see that they are equal to a sum of terms on the right-hand side of (5.4), and we already know that the latter terms are well-defined. Therefore, we can conclude that all the terms in (5.3) are also well-defined and that the equality in (5.3) holds. In fact, we have by definition

$$\begin{aligned}
 & \int_0^t \mathcal{K}_g(F'(Z)Y)(t, s) \sigma(s) \delta B(s) \\
 & = \int_0^t F'(Z(s)) b(s) \delta B(s) + \int_0^t \int_0^s F'(Z(s)) a(s, u) \delta B(u) \, ds
 \end{aligned} \tag{5.6}$$

and

$$\begin{aligned}
 & \operatorname{tr}_{\mathcal{H}_1} \int_0^t (D_s \mathcal{K}_g(F'(Z)Y)(t, s)) \sigma(s) \, ds \\
 & = \operatorname{tr}_{\mathcal{H}_1} \int_0^t \left(D_s F'(Z(s)) b(s) \, ds + F'(Z(s)) b_D(s) \right. \\
 & \quad \left. + \int_s^t ((D_s F'(Z(u))) a(u, s) + F'(Z(u)) a_D(u, s)) \, du \right) ds \\
 & = \operatorname{tr}_{\mathcal{H}_1} \int_0^t (D_s F'(Z(s))) b(s) \, ds + \operatorname{tr}_{\mathcal{H}_1} \int_0^t F'(Z(s)) b_D(s) \, ds \\
 & \quad + \operatorname{tr}_{\mathcal{H}_1} \int_0^t \int_0^s (D_u F'(Z(s))) a(s, u) \, du \, ds + \operatorname{tr}_{\mathcal{H}_1} \int_0^t \int_0^s F'(Z(s)) a_D(s, u) \, du \, ds.
 \end{aligned} \tag{5.7}$$

Note that by the chain rule (2.3)

$$D_s F'(Z(s)) = F''(Z(s)) \left(b(s) + \int_0^s D_s b(u) \delta B(u) + \text{tr}_{\mathcal{H}_1} \int_0^s D_s \int_0^u a(u, r) \delta B(r) du + \int_0^s D_s \text{tr}_{\mathcal{H}_1} b_D(u) du + \int_0^s D_s \text{tr}_{\mathcal{H}_1} \int_0^u a_D(u, r) dr du \right).$$

With this, we can rewrite the first term on the right-hand side of (5.7) to

$$\begin{aligned} & \text{tr}_{\mathcal{H}_1} \int_0^t (D_s F'(Z(s))) b(s) ds \\ &= \text{tr}_{\mathcal{H}_1} \int_0^t F''(Z(s)) b(s) b(s) ds + \text{tr}_{\mathcal{H}_1} \int_0^t F''(Z(s)) \int_0^s D_s b(u) \delta B(u) b(s) ds \\ & \quad + \text{tr}_{\mathcal{H}_1} \int_0^t F''(Z(s)) \int_0^s D_s \int_0^u a(u, r) \delta B(r) du b(s) ds \tag{5.8} \\ & \quad + \text{tr}_{\mathcal{H}_1} \int_0^t F''(Z(s)) \int_0^s D_s \text{tr}_{\mathcal{H}_1} b_D(u) du b(s) ds \\ & \quad + \text{tr}_{\mathcal{H}_1} \int_0^t F''(Z(s)) \int_0^s D_s \text{tr}_{\mathcal{H}_1} \int_0^u a_D(u, r) dr du b(s) ds. \end{aligned}$$

Now we collect terms. The two terms on the right-hand side of (5.6) are the second term in (5.4) and the first term in (5.5). Moreover, the last three terms on the right-hand side of (5.7) are equal to the fourth term in (5.4), the second term in (5.5) and the fifth term in (5.4). Furthermore, the last four terms in (5.8) are the sixth to ninth term in (5.4). Finally, the first term on the right-hand side of (5.8) is twice the last term in (5.4). Hence, by adding and subtracting the last term in (5.4) to this equality, we obtain the assertion. \square

Next, we provide a sufficient condition under which (5.2) holds relying on Hölder’s inequality.

Remark 5.3. For the term $C(s)$ in (5.2), we have to deal with the following three terms which constitute its $\mathbb{L}^{2,p}$ -norm. We will apply Hölder’s Inequality with respect to ω (note that g is assumed to be deterministic) with the two conjugate exponents $q_1, q_2 \in [1, \infty]$ (where if one of them is infinite, the supremum norm has to be used). We will now assume that $Y(s) \in L_2(\mathcal{H}_2, \mathcal{H}_3)$, but a similar calculation would also be possible with $\sigma(s) \in L_2(\mathcal{H}_1, \mathcal{H}_2)$. We obtain

$$\begin{aligned} & \int_0^T \mathbb{E} \left[\| Y(s) g(s, s) \sigma(s) \|_{L_2(\mathcal{H}_1, \mathcal{H}_3)}^p \right] ds \\ & \leq \int_0^T \| g(s, s) \|_{L(\mathcal{H}_2, \mathcal{H}_2)}^p \left(\mathbb{E} \left[\| Y(s) \|_{L_2(\mathcal{H}_2, \mathcal{H}_3)}^{pq_1} \right] \right)^{1/q_1} \left(\mathbb{E} \left[\| \sigma(s) \|_{L(\mathcal{H}_1, \mathcal{H}_2)}^{pq_2} \right] \right)^{1/q_2} ds, \end{aligned}$$

similarly for the Malliavin derivative using (2.2)

$$\begin{aligned} & \int_0^T \int_0^T \mathbb{E} \left[\left\| D_t(Y(s)g(s, s)\sigma(s)) \right\|_{L_2(L_2(\mathcal{H}_0, \mathcal{H}_1), \mathcal{H}_3)}^p \right] ds dt \\ & \leq \int_0^T \|g(s, s)\|_{L(\mathcal{H}_2, \mathcal{H}_2)}^p \\ & \quad \times \left((\mathbb{E}[\|\sigma(s)\|_{L(\mathcal{H}_1, \mathcal{H}_2)}^{pq_2}])^{1/q_2} \int_0^T (\mathbb{E}[\|D_t Y(s)\|_{L_2(\mathcal{H}_1 \otimes \mathcal{H}_2, \mathcal{H}_3)}^{pq_1}])^{1/q_1} dt \right. \\ & \quad \left. + (\mathbb{E}[\|Y(s)\|_{L_2(\mathcal{H}_2, \mathcal{H}_3)}^{pq_1}])^{1/q_2} \int_0^T (\mathbb{E}[\|D_t \sigma(s)\|_{L_2(\mathcal{H}_1 \otimes \mathcal{H}_1, \mathcal{H}_2)}^{pq_2}])^{1/q_1} dt \right) ds, \end{aligned}$$

and finally for the second Malliavin derivative

$$\begin{aligned} & \int_0^T \int_0^T \int_0^T \mathbb{E} \left[\left\| D_r D_t(Y(s)g(s, s)\sigma(s)) \right\|_{L_2(L_2(\mathcal{H}_0, \mathcal{H}_1) \otimes \mathcal{H}_1, \mathcal{H}_3)}^p \right] ds dt dr \\ & \leq \int_0^T \|g(s, s)\|_{L(\mathcal{H}_2, \mathcal{H}_2)}^p \\ & \quad \times \left((\mathbb{E}[\|\sigma(s)\|_{L(\mathcal{H}_1, \mathcal{H}_2)}^{pq_2}])^{1/q_2} \int_0^T \int_0^T (\mathbb{E}[\|D_r D_t Y(s)\|_{L_2(\mathcal{H}_1 \otimes \mathcal{H}_1 \otimes \mathcal{H}_2, \mathcal{H}_3)}^{pq_1}])^{1/q_1} dt dr \right. \\ & \quad + 2 \int_0^T (\mathbb{E}[\|D_r Y(s)\|_{L_2(\mathcal{H}_2, \mathcal{H}_3)}^{pq_1}])^{1/q_1} dr \\ & \quad \times \int_0^T (\mathbb{E}[\|D_t \sigma(s)\|_{L_2(L_2(\mathcal{H}_0, \mathcal{H}_1) \otimes \mathcal{H}_1, \mathcal{H}_2)}^{pq_2}])^{1/q_2} dt \\ & \quad + (\mathbb{E}[\|Y(s)\|_{L_2(\mathcal{H}_2, \mathcal{H}_3)}^{pq_1}])^{1/q_1} \\ & \quad \left. \times \int_0^T \int_0^T (\mathbb{E}[\|D_r D_t \sigma(s)\|_{L_2(L_2(\mathcal{H}_0, \mathcal{H}_1) \otimes \mathcal{H}_1, \mathcal{H}_2)}^{pq_2}])^{1/q_2} dt dr \right) ds. \end{aligned}$$

From the last equality, we see that $Y, DY, DDY, \sigma, D\sigma$ and $DD\sigma$ appear and have to satisfy some integrability conditions with respect to the temporal and spatial argument. Therefore, we conclude that a sufficient condition for the conditions in (5.2) is that Y and σ belong to a Sobolev-type space with respect to all arguments (including the random argument), that is, in some $\mathbb{L}^{2,q}$ -spaces for q sufficiently large, possibly with a different q for each of the two. Moreover, $g(s, s)$ has to be in some L^p -space for $p > 4$ with respect to $\lambda|_{[0, T]}$.

In some special situations, we can further manipulate the terms in the equality above. For instance, one can assume that σ is independent of the noise (what would imply its Malliavin derivatives to be zero), or σ might be independent of Y in which case the expectations involving Y and σ would factorize. An investigation of the term $A(s)$ in (5.2) does not yield to any further conditions, except that also the function $(t, s) \mapsto \|\phi(t, s)\|_{L(\mathcal{H}_2, \mathcal{H}_2)}^4$ has to be integrable with respect to $\lambda^2|_{[0, T] \times [0, T]}$.

A first immediate application of Theorem 5.2 is to derive an Itô formula for X . Note that due to (4.1), $Y(s) \equiv \text{id}_{\mathcal{H}_2}$ and, therefore,

$$X(t) = \int_0^t dX(s) = \int_0^t \mathcal{K}_g(\text{id}_{\mathcal{H}_2})(t, s)\sigma(s)\delta B(s) + \text{tr}_{\mathcal{H}_1} \int_0^t (D_s \mathcal{K}_g(\text{id}_{\mathcal{H}_2})(t, s))\sigma(s) ds.$$

Using Theorem 5.2, we get the following result.

Corollary 5.4. *Suppose that X is defined as in (1.1), where the conditions in Assumption 3.1 hold, and that $F : \mathcal{H}_2 \rightarrow \mathcal{H}_3$ is twice Fréchet differentiable. Assume moreover that σ is twice Malliavin differentiable and that g has the form (4.5). Then $F'(X) \in \mathcal{L}^X(0, t)$ for all $t \in [0, T]$ and*

$$F(X(t)) = F(0) + \int_0^t F'(X(s)) dX(s) - \frac{1}{2} \text{tr}_{\mathcal{H}_1} \int_0^t F''(X(s))(g(s, s)\sigma(s))(g(s, s)\sigma(s)) ds.$$

Note that this is consistent with the Itô formulas in the one-dimensional case treated in [1], Theorems 1, 2. In fact, in that setting we have $\sigma \equiv 1$, $\mathcal{H}_1 = \mathcal{H}_2 = \mathbb{R}$, $X(t) = \int_0^t g(t, s) dW(s)$ and $R(s) := \int_0^s g(s, r)g(s, r) dr$. Therefore, Corollary 5.4 implies

$$\begin{aligned} F(X(t)) &= F(0) + \int_0^t \left(F'(X(s))g(s, s) + \int_s^t F'(X(u))\phi(u, s) du \right) \delta B(s) \\ &\quad + \int_0^t \left(D_s F'(X(s))g(s, s) + \int_s^t D_s F'(X(u))\phi(u, s) du \right) ds \\ &\quad - \frac{1}{2} \int_0^t F''(X(s))g(s, s)g(s, s) ds. \end{aligned} \tag{5.9}$$

Note that $D_s F'(X(u)) = F''(X(s))g(u, s)$ for $u \geq s$. Note furthermore that

$$\frac{dR(s)}{ds} = g(s, s)^2 + 2 \int_0^s g(s, u)\phi(s, u) du.$$

This and Fubini's theorem yield that the last three terms in (5.9) are equal to

$$\frac{1}{2} \int_0^t F''(X(s)) \left(g(s, s)^2 + 2 \int_0^s g(s, u)\phi(s, u) du \right) ds = \frac{1}{2} \int_0^t F''(X(s)) dR(s).$$

This yields

$$F(X(t)) = F(0) + \int_0^t F'(X(s)) dX(s) + \int_0^t F''(X(s)) dR(s),$$

which is the formula in [1], Theorems 1, 2.

A second application of Theorem 5.2 is to calculate Z^2 in the case $\mathcal{H}_3 = \mathbb{R}$. For this, we suppose the same conditions as in Theorem 5.2, assume $\mathcal{H}_3 = \mathbb{R}$ and $F(x) = x^2$. Then applying

Theorem 5.2 yields

$$\begin{aligned}
 \frac{1}{2}(Z(t))^2 &= \int_0^t Z(s)Y(s)g(s, s)\sigma(s)\delta B(s) + \int_0^t \int_0^s Z(s)Y(s)\phi(s, u)\sigma(u)\delta B(u) ds \\
 &\quad + \text{tr}_{\mathcal{H}_1} \int_0^t (D_s Z(s))Y(s)g(s, s)\sigma(s) ds \\
 &\quad + \text{tr}_{\mathcal{H}_1} \int_0^t Z(s)(D_s Y(s))g(s, s)\sigma(s) ds \\
 &\quad + \text{tr}_{\mathcal{H}_1} \int_0^t \int_0^s (D_s Z(s))Y(s)\phi(s, u)\sigma(u) du ds \\
 &\quad + \text{tr}_{\mathcal{H}_1} \int_0^t \int_0^s Z(s)(D_s Y(s))\phi(s, u)\sigma(u) du ds \\
 &\quad - \frac{1}{2} \int_0^t \|Y(s)g(s, s)\sigma(s)\|_{L_2(\mathcal{H}_1, \mathcal{H}_2)}^2 ds.
 \end{aligned}
 \tag{5.10}$$

If we now introduce the symbolic notation

$$\begin{aligned}
 dZ(s) &= \mathcal{K}_g(Y)(\cdot, s)\sigma(s)\delta B(s) + \text{tr}_{\mathcal{H}_1} D_s \mathcal{K}_g(Y)(\cdot, s)\sigma(s) ds \\
 &= Y(s)g(s, s)\sigma(s)\delta B(s) + \int_0^s Y(s)\phi(s, u)\sigma(u)\delta B(u) ds \\
 &\quad + \text{tr}_{\mathcal{H}_1} (D_s Y(s))g(s, s)\sigma(s) ds + \text{tr}_{\mathcal{H}_1} \int_0^s (D_s Y(s))\phi(s, u)\sigma(u) du ds,
 \end{aligned}$$

then we see that the term $\int_0^t Z(s) dZ(s)$ accounts for the first, second, fourth, fifth and sixth term (use the rule in Proposition 2.2) in (5.10). The third and last term remain as correction terms and one has the formal result

$$\begin{aligned}
 \frac{1}{2}(Z(t))^2 &= \int_0^t Z(s) dZ(s) + \text{tr}_{\mathcal{H}_1} \int_0^t (D_s Z(s))Y(s)g(s, s)\sigma(s) ds \\
 &\quad - \frac{1}{2} \int_0^t \|Y(s)g(s, s)\sigma(s)\|_{L_2(\mathcal{H}_1, \mathcal{H}_2)}^2 ds.
 \end{aligned}
 \tag{5.11}$$

One could also alter the correction terms. For instance, one could also expand the second term on the right-hand side of (5.11) by calculating $D_s Z(s)$. This would reverse the sign of the third term on the right-hand side of (5.11), but one would get other correction terms.

6. A random-field approach to the X -integral

In this section, we investigate whether the random-field approach pioneered in [19] gives a reasonable interpretation of the X -integral, that is, whether we can derive an integral which has the

form $\int_0^t \int_{\mathbb{R}^d} Y(s, y)X(ds, dy)$, where

$$X(t, x) = \int_0^t \int_{\mathbb{R}^d} g(t, s; x, y)\sigma(s, y)M(\delta s, dy). \tag{6.1}$$

Here, similar to the terms in (1.1) and (1.2), g is a deterministic function, σ and Y are random fields and M is a martingale measure and the integral in (6.1) is understood in the Walsh sense. In the first part of this section, we quickly review the concept of Walsh integration and summarize the main ideas of a minor generalization for anticipating integrands. In the second part, we present the random-field X -integral in the special case of homogeneous noise and show that this integral and the one derived in Section 3 coincide.

6.1. Walsh integration

Let $(M_t(A); t \in [0, T], A \in \mathcal{B}_b(\mathbb{R}^d))$ be a worthy Gaussian martingale measure, where \mathcal{B}_b are the bounded Borel sets. This means that each $M_t(A)$ has a Gaussian distribution on \mathbb{R} . Fix the filtration $(\mathcal{F}_t)_{t \in [0, T]}$ as the one generated by the martingale measure and augmented by the sets of probability zero. The martingale measure takes values in $L^2(\Omega)$. Since the martingale measure is a martingale in t for all fixed sets $A \in \mathcal{B}_b(\mathbb{R}^d)$, we can associate a quadratic covariance functional to it, denoted by $Q_M(t, A, B) = \langle M(A), M(B) \rangle_t$. We can define a stochastic integral of elementary processes $f(s, x, \omega) = 1_{(a, b)}(s)1_A(x)Z(\omega)$, where $0 \leq a < b \leq T$, $A \in \mathcal{B}_b(\mathbb{R}^d)$ and Z is a bounded \mathcal{F}_a -measurable random variable, with respect to the martingale measure by

$$\int_0^t \int_{\mathbb{R}^d} f(s, y)M(ds, dy) = (M_{t \wedge b}(A) - M_{t \wedge a}(A))Z,$$

and for simple processes (linear combinations of elementary processes) by an obvious linear combination. Then, as in Itô integration, this stochastic integral is extended to a larger class of integrands by using the isometry

$$\begin{aligned} & \mathbb{E} \left[\left(\int_0^t \int_{\mathbb{R}^d} f(s, y)M(ds, dy) \right)^2 \right] \\ &= \mathbb{E} \left[\int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(s, x)f(s, y)Q_M(ds, dx, dy) \right] =: \|f\|_0^2, \end{aligned} \tag{6.2}$$

where we extended the covariance functional to a measure on $[0, T] \times \mathbb{R}^d \times \mathbb{R}^d$. In order to be able to do that, one needs the fact that the martingale measure is assumed to be worthy. This means that there exists a dominating measure K_M , that is a positive definite measure with some regularity conditions for which $|Q_M([0, t], A, B)| \leq K_M([0, t], A, B)$ for all $t \in [0, T]$ and $A, B \in \mathcal{B}_b(\mathbb{R}^d)$. In this case one gets an upper bound for the isometry (6.2)

$$\begin{aligned} & \mathbb{E} \left[\int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(s, x)f(s, y)Q_M(ds, dx, dy) \right] \\ & \leq \mathbb{E} \left[\int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |f(s, x)||f(s, y)|K_M(ds, dx, dy) \right] =: \|f\|_+^2. \end{aligned}$$

The norm $\| \cdot \|_0$ is actually induced by an inner product and one defines two possible domains \mathcal{P}_+ and \mathcal{P}_0 of the stochastic integral. The former one is defined to be the set of all functions f with $\|f\|_+ < \infty$ and the latter one is defined as the completion of the space of simple processes with respect to the inner product $\langle \cdot, \cdot \rangle_0$. Note that by [19], Proposition 2.3, \mathcal{P}_+ is complete and the simple processes form a dense subset in this space. \mathcal{P}_0 on the other hand is a Hilbert space, but does not coincide with the set of all processes such that $\|f\|_0 < \infty$. Note that for particular choices of M , there might be even Schwartz distributions in \mathcal{P}_0 . A particular example for this construction is the case of spatially homogeneous noise, where

$$\begin{aligned} Q_M([0, t], A, B) &= K_M([0, t], A, B) \\ &= \int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} 1_{[0,t]}(s) 1_A(y) 1_B(y-x) \, dy \, \Gamma(dx) \, ds, \end{aligned} \tag{6.3}$$

for a nonnegative, nonnegative definite tempered measure Γ . Then one can find a spectral representation of the $\| \cdot \|_0$ -norm so that in this case \mathcal{P}_0 contains distributions. Note however, that in this article we only use the integration concept by Walsh for functions in \mathcal{P}_+ since we are exclusively interested in the case when g is a function, as this is the case with ambit fields in Example 1.1. However, in Section 7 we will also admit distributions for g which calls for the Dalang integral introduced in [5].

For a correct treatment, we need to provide a certain extension of the Walsh integral in the case of homogeneous noise so that it can handle anticipating integrands. This is, however, not very difficult if one keeps in mind the reformulation of the Walsh integral as a sum of independent Brownian motions on a Hilbert space; see [6] or as the divergence operator of Malliavin calculus, see [18].

We define \mathcal{H} to be the completion of the simple functions on \mathbb{R}^d by the scalar product

$$\langle f, g \rangle_{\mathcal{H}} = \int_{\mathbb{R}^d} (f * g)(z) \Gamma(dz) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(y) g(y-z) \, dy \, \Gamma(dz),$$

with Γ as above and $\mathcal{H}_T := L^2([0, T]; \mathcal{H})$. In contrast to [5] we do not need to introduce the spectral representation of this scalar product since we need f and g to be functions. Then the worthy martingale measure M can be extended into an isonormal Gaussian process $(M(h); h \in \mathcal{H}_T)$ with respect to which we can do Malliavin calculus. With this, we can define $\mathbb{D}^{1,2}(\mathcal{H})$ and the important space $\mathbb{L}^{1,2}(\mathcal{H}) := L^2([0, T]; \mathbb{D}^{1,2}(\mathcal{H}))$, with norm

$$\|f\|_{\mathbb{L}^{1,2}(\mathcal{H})}^2 = \mathbb{E}[\|f\|_{\mathcal{H}_T}^2] + \mathbb{E}[\|Df\|_{\mathcal{H}_T \otimes \mathcal{H}_T}^2] < \infty.$$

On this space the divergence operator $\delta: L^2(\Omega; \mathcal{H}_T) \rightarrow L^2(\Omega)$ is well-defined and continuous; see [14], equation (1.47). For simple processes f , which are all in $\mathbb{L}^{1,2}(\mathcal{H})$ the anticipating Walsh integral is given by

$$\int_0^t f(t, x) M(\delta t, dx) = \sum_{j=1}^d Z_j (M_{t \wedge b_j}(A_j) - M_{t \wedge a_j}(A_j)) + \sum_{j=1}^d \langle DZ_j, 1_{[a_j, b_j]} 1_{A_j} \rangle_{\mathcal{H}_T}.$$

Then one can extend this integral to integrands in the space $\mathbb{L}^{1,2}(\mathcal{H})$, where the Malliavin differentiable simple processes are dense. This extension is done using the following isometry:

$$\begin{aligned} \mathbb{E}[(u \cdot M)^2] &= \mathbb{E}[\|u\|_{\mathcal{H}_T}^2] + \mathbb{E}[\|Du\|_{\mathcal{H}_T \otimes \mathcal{H}_T}^2] \\ &= \mathbb{E}[\|u\|_{\mathcal{H}_T}^2] \\ &\quad + \mathbb{E}\left[\int_0^t \int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} D_{s_2, y_2 - z_2} u(s_1, y_1) \right. \\ &\quad \left. \times D_{s_1, y_1 - z_1} u(s_2, y_2) dy_1 dy_2 \Gamma(dz_1) \Gamma(dz_2) ds_1 ds_2 \right]. \end{aligned} \tag{6.4}$$

Another way to think about this integral is by using the equivalence of the anticipating Walsh integral and an infinite sum of anticipating integrals with respect to Brownian motion. In fact, we have for all $u \in \mathbb{L}^{1,2}(\mathcal{H})$

$$\int_0^t \int_{\mathbb{R}^d} u(s, y) M(\delta s, dy) = \sum_{k \in \mathbb{N}} \int_0^t \langle u(s, \cdot), e_k \rangle_{\mathcal{H}} \delta B^k(s),$$

where $(e_k)_{k \in \mathbb{N}}$ is a CONS of \mathcal{H} , B is a Brownian motion on \mathcal{H} and $B^k := \langle B, e_k \rangle_{\mathcal{H}}$ are independent real-valued Brownian motions, for which anticipating calculus is well known. Therefore, one could also take the last equality as the definition of the anticipating Walsh integral.

6.2. The random-field X-integral

Now we start with the definition of the X-integral using a random-field approach. In contrast to the definition of the X-integral in Section 3, we have to start here with defining the stochastic integral on elementary processes first and then extend it to simple processes and further. This is due to the fact that a similar integration by parts procedure done in Section 3 is not easily applicable since one would get nontrivial boundary terms which are hard to interpret.

Take \mathcal{H} and \mathcal{H}_T to be the Hilbert spaces defined in the previous subsection. The martingale measure that we will use in the following will be the one using the homogeneous noise in (6.3), which appeared in [5]. We will however not use it in its full generality, but we only look at functions f for which the following norm:

$$\|f\|_0^2 = \int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(s, y) f(s, y - z) dy \Gamma(dz) ds < \infty. \tag{6.5}$$

This is a norm – and in fact it is generated by an inner product $\langle \cdot, \cdot \rangle_0$ – as one can see by going to its spectral representation. We will however not do this, since we cannot deal with distributions as integrators g in the kernel (3.4). Therefore, we define the Hilbert space \mathcal{H}_T as the completion of the simple functions with respect to the inner product $\langle \cdot, \cdot \rangle_0$. This is then a function space which does not contain distributions. We also look at random functions in $L^2(\Omega; \mathcal{H}_T)$ equipped with the obvious norm and inner product. The following assumptions are made.

Assumption 6.1. Fix $T > 0$ and $t \in [0, T]$, and let $M = (M_t(A); t \in [0, T], A \in \mathcal{B}_b(\mathbb{R}^d))$ be a worthy martingale measure with covariation measure as in (6.5). Furthermore, $g : [0, T]^2 \times (\mathbb{R}^d)^2 \rightarrow \mathbb{R}$ is a deterministic function for all $0 \leq s < t \leq T$ and all $x, y \in \mathbb{R}^d$ and $(\sigma(t, x); (t, x) \in [0, T] \times \mathbb{R}^d)$ is a Malliavin differentiable random field such that $(s, y) \mapsto g(t, s, x, y) \times \sigma(s, y)$ is integrable with respect to M for all $(t, x) \in [0, T] \times \mathbb{R}^d$, that is, this random function is in $\mathbb{L}^{1,2}(\mathcal{H})$. Assume for all $(s, y) \in [0, T] \times \mathbb{R}^d$ that the function $(t, x) \mapsto g(t, s, x, y)$ has bounded variation on $[u, v] \times \mathbb{R}^d$ for all $0 \leq s < u < v \leq t$.

Now we turn to the definition of the integral. To this end, let first $0 \leq a < b \leq T, A \in \mathcal{B}_b(\mathbb{R}^d)$ be of the form $A = \times_{j=1}^d [a_j, b_j]$ and set $Y(t, x, \omega) = 1_{(a,b]}(t)1_A(x)Z(\omega)$, where Z is a bounded Malliavin differentiable random variable, not necessarily measurable with respect to \mathcal{F}_a . Note that since the spatial argument of the integrator process X only appears in g , and since X is linear in the kernel g , one can derive using (2.1)

$$\begin{aligned} X(t, A) &= \int_0^t \int_{\mathbb{R}^d} g(t, s, A, y) \sigma(s, y) M(ds, dy) \\ &= \int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} 1_A(z) g(t, s, dz, y) \sigma(s, y) M(ds, dy). \end{aligned} \tag{6.6}$$

In the calculation that follows, we aim at arriving at a similar kernel as in (3.4). In order to keep the notation tidy, we do this calculation for the case when $g(s, s; \cdot, \cdot)$ is a well-defined object, thus yielding a kernel as in (3.5). By following the same arguments as given below, one can however also arrive at (3.4). Using the obvious definition for the stochastic integral for simple processes and (6.6), one derives for the simple process Y

$$\begin{aligned} &\int_0^t \int_{\mathbb{R}^d} Y(s, y) X(ds, dy) \\ &:= Z(X(t \wedge b, A) - X(t \wedge a, A)) \\ &= Z \int_0^t \int_{\mathbb{R}^d} \left(1_{[a,b]}(s) \int_{\mathbb{R}^d} 1_A(z) g(s, s; dz, y) \right. \\ &\quad + 1_{[0,b]}(s) \left(\int_{\mathbb{R}^d} 1_A(z) g(b, s; dz, y) - \int_{\mathbb{R}^d} 1_A(z) g(s, s; dz, y) \right) \\ &\quad - 1_{[0,a]}(s) \left(\int_{\mathbb{R}^d} 1_A(z) g(a, s; dz, y) \right. \\ &\quad \left. \left. - \int_{\mathbb{R}^d} 1_A(z) g(s, s; dz, y) \right) \right) \sigma(s, y) M(\delta s, dy) \\ &= Z \int_0^t \int_{\mathbb{R}^d} \left(1_{[a,b]}(s) \int_{\mathbb{R}^d} 1_A(z) g(s, s; dz, y) + 1_{[0,a]}(s) \int_a^b \int_{\mathbb{R}^d} 1_A(z) g(du, s; dz, y) \right. \\ &\quad \left. + 1_{[a,b]}(s) \int_s^b \int_{\mathbb{R}^d} 1_A(z) g(du, s; dz, y) \right) \sigma(s, y) M(\delta s, dy). \end{aligned}$$

Note that the last two integral terms one the right-hand side of the previous equation are equal to

$$\int_{a \vee s}^{b \wedge t} \int_{\mathbb{R}^d} 1_A(z)g(du, s; dz, y) = \int_s^t \int_{\mathbb{R}^d} 1_{[a,b]}(u)1_A(z)g(du, s; dz, y).$$

Plugging this into the equation above yields

$$\begin{aligned} & \int_0^t \int_{\mathbb{R}^d} Y(s, y)X(ds, dy) \\ &= Z \int_0^t \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} 1_{[a,b]}(s)1_A(z)g(s, s; dz, y) \right. \\ & \quad \left. + \int_s^t 1_{[a,b]}(u)1_A(z)g(du, s; dz, y) \right) \sigma(s, y)M(\delta s, dy). \end{aligned} \tag{6.7}$$

Similar to (3.4), we define an integration kernel

$$\begin{aligned} & \mathcal{K}_g(h)(t, s, y) \\ &:= \int_{\mathbb{R}^d} h(s, z)g(t, s; dz, y) + \int_s^t \int_{\mathbb{R}^d} (h(u, z) - h(s, z))g(du, s; dz, y). \end{aligned} \tag{6.8}$$

Note that this term is well-defined if g is a function of bounded variation spatial argument z and of bounded variation on subintervals bounded away from s in the temporal argument t . As in Section 3, one can rewrite this kernel in some situations. If $g(s, s; z, y)$ exists almost everywhere (with respect to $dy \Gamma(dz) ds$), then one has

$$\mathcal{K}_g(h)(t, s, y) = \int_{\mathbb{R}^d} h(s, z)g(s, s; dz, y) + \int_s^t \int_{\mathbb{R}^d} h(u, z)g(du, s; dz, y).$$

If the function g has a partial derivative with respect to each coordinate, then

$$\begin{aligned} \mathcal{K}_g(h)(t, s, y) &= \int_{\mathbb{R}^d} h(s, z) \frac{\partial^d}{\partial z_1 \dots \partial z_d} g(t, s; z, y) dz \\ & \quad + \int_s^t \int_{\mathbb{R}^d} (h(u, z) - h(s, z)) \frac{\partial^{d+1}}{\partial u \partial z_1 \dots \partial z_d} g(u, s; z, y) dz du. \end{aligned}$$

Using the kernel in (6.8), we obtain by pulling the random variable inside the stochastic integral

$$\begin{aligned} & \int_0^t \int_{\mathbb{R}^d} Y(s, y)X(ds, dy) = Z \int_0^t \int_{\mathbb{R}^d} \mathcal{K}_g(1_{[a,b]}1_A)(t, s, y)\sigma(s, y)M(\delta s, dy) \\ &= \int_0^t \int_{\mathbb{R}^d} \mathcal{K}_g(1_{[a,b]}1_A Z)(t, s, y)\sigma(s, y)M(\delta s, dy) \\ & \quad + \langle DZ, \mathcal{K}_g(1_{[a,b]}1_A)(t, \cdot, *)\sigma(\cdot, *) \rangle_{\mathcal{H}_T}. \end{aligned}$$

The second term of the previous equality can be manipulated as follows in order to get an expression which explicitly involves the original integrand Y

$$\begin{aligned} & \langle DZ, \mathcal{K}_g(1_{[a,b]}1_A)(t, \cdot, *)\sigma(\cdot, *) \rangle_{\mathcal{H}_T} \\ &= \int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} D_{s,y-z} Z \mathcal{K}_g(1_{[a,b]}1_A)(t, s, y) \sigma(s, y) dy \Gamma(dz) ds \\ &= \int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} D_{s,y-z} \mathcal{K}_g(Y)(t, s, y) \sigma(s, y) dy \Gamma(dz) ds. \end{aligned} \tag{6.9}$$

In total, we have for the random-field X -integral

$$\begin{aligned} \int_0^t \int_{\mathbb{R}^d} Y(s, y) X(ds, dy) &= \int_0^t \int_{\mathbb{R}^d} \mathcal{K}_g(Y)(t, s, y) \sigma(s, y) M(ds, dy) \\ &+ \int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} D_{s,y-z} \mathcal{K}_g(Y)(t, s, y) \sigma(s, y) dy \Gamma(dz) ds. \end{aligned} \tag{6.10}$$

First, we see the similarities to the definition in (3.6), where here we add the spatial component which in turn forces us to include a possible spatial correlation into the definition of the integral. Note that if we consider the uncorrelated case, then $\Gamma = \delta_0$ and one gets an expression without z . Now we extend the definition of the integral by using the isometry in (6.4) so that the integral (6.10) is defined for all $Y \in \mathbb{L}^{1,2}(\mathcal{H})$ for which similar integrability conditions as in Definition 3.2 hold.

Definition 6.2. Let X be the random field defined in (6.1) together with Assumption 6.1. We say that the random field $Y = (Y(t, x); (t, x) \in [0, T] \times \mathbb{R}^d) \in \mathcal{P}_0$ belongs to the domain of the stochastic integral with respect to the random field X , if:

- (i) the process $(Y(s, z))_{z \in \mathbb{R}^d}$ is integrable with respect to $g(t, s; dz, y)$ almost surely and (t, s, y) -almost everywhere,
- (ii) the process $(Y(u, z) - Y(s, z))_{u \in (s,t] \times \mathbb{R}^d}$ is integrable with respect to $g(du, s; dz, y)$ almost surely and (s, y) -almost everywhere,
- (iii) $(s, y) \mapsto \mathcal{K}_g(Y)(t, s, y) \sigma(s, y) 1_{[0,t]}(s)$ is in the domain of the martingale measure M , that is, $\mathcal{K}_g(Y)(t, \cdot, *) \sigma(\cdot, *) 1_{[0,t]}(\cdot) \in \mathcal{P}_0$ and
- (iv) $\mathcal{K}_g(Y)(t, s, y)$ is Malliavin differentiable with respect to $D_{s,y-z}$ for all $s \in [0, t]$ and $y, z \in \mathbb{R}^d$ and the random field

$$(s, y, z) \mapsto \text{tr}_{\mathcal{H}_1} D_{s,y-z} (\mathcal{K}_g(Y)(t, s, y)) \sigma(s, y)$$

$\lambda|_{[0,T]} \otimes \lambda|_{\mathbb{R}^d} \otimes \Gamma$ -integrable on $[0, t] \times \mathbb{R}^d \times \mathbb{R}^d$ almost surely.

We denote this by $Y \in \mathcal{I}^X([0, t] \times \mathbb{R}^d)$ and the integral $\int_0^t \int_{\mathbb{R}^d} Y(s, y) X(ds, dy)$ is defined by (6.10).

The main reason why we have introduced this integral is to be able to integrate processes $(Y(t, x, y); t \in [0, T], x, y \in \mathbb{R}^d)$ with respect to t and y and obtain a random field which has

a pointwise (in x) interpretation as a real-valued random variable, rather than as an element in some abstract Hilbert space. This could serve to deduce properties of ambit fields such as path continuity, or existence of densities at each point.

We could easily derive similar properties for the random-field X -integral as in Section 4.1, but we restrict ourselves to showing that the random field integral can be rewritten as a Hilbert-valued stochastic integral when we interpret the process X as a stochastic process with values in some Hilbert space, as it was shown in [6,18].

Proposition 6.3. *Let $Y \in \mathcal{I}^X([0, t] \times \mathbb{R}^d)$. Then $Y \in \mathcal{I}^X(0, t)$ with $\mathcal{H}_1 = \mathcal{H}$ and $\mathcal{H}_3 = \mathbb{R}$ and*

$$\int_0^t \int_{\mathbb{R}^d} Y(s, y) X(ds, dy) = \int_0^t Y(s) dX(s),$$

where X is interpreted as a random field on the left-hand side and as an \mathcal{H}_2 -valued Volterra process on the right-hand one.

Proof. At first we argue that if $Y \in \mathcal{I}^X([0, t] \times \mathbb{R}^d)$, then $Y \in \mathcal{I}^X(0, t)$. In fact, from the remark at the end of the previous subsection it follows that if Y is in the domain of the martingale measure integral, then it is also in the domain of the integral with respect to a \mathcal{H} -valued Brownian motion. Furthermore, the properties Definition 6.2(ii)–(iv) are exactly Definition 3.2(i)–(iii) written out in the special case of $\mathcal{H}_1 = \mathcal{H}$. This yields the first assertion.

Second, we show that the stochastic integrals with respect to the martingale measure and the \mathcal{H} -valued Brownian motion are equivalent. This follows by a straightforward adaption of the proof of [6], Proposition 2.6, because by Definition 6.2(iii) we conclude that

$$\mathcal{K}_g(Y)(t, \cdot, *) \sigma(\cdot, *) 1_{[0,t]}(\cdot) \in L^2(\Omega; \mathcal{H}_T)$$

and

$$\int_0^t \int_{\mathbb{R}^d} \mathcal{K}_g(Y)(t, s, y) \sigma(s, y) M(\delta s, dy) = \sum_{k \in \mathbb{N}} \int_0^t \langle \mathcal{K}_g(Y)(t, s, *) \sigma(s, *) , e_k \rangle_{\mathcal{H}} dB^k(s),$$

where B is an \mathcal{H} -valued Wiener process and $B^k := \langle B, e_k \rangle_{\mathcal{H}}$. This yields the equality of the stochastic integrals in the X -integral and the random-field X -integral.

For the pathwise integrals, we start with the elementary processes given by $Y(t, x, \omega) = 1_{[a,b]}(t) 1_A(x) Z(\omega)$ and use the representation in (6.9) to obtain

$$\begin{aligned} & \int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} D_{s,y-z} \mathcal{K}_g(Y)(t, s, y) \sigma(s, y) dy \Gamma(dz) ds \\ &= \int_0^t \langle D_s Z, \mathcal{K}_g(1_{[a,b]} 1_A)(t, s, *) \sigma(s, *) \rangle_{\mathcal{H}} ds \\ &= \sum_{k \in \mathbb{N}} \int_0^t \langle D_s Z, e_k \rangle_{\mathcal{H}} \langle \mathcal{K}_g(1_{[a,b]} 1_A)(t, s, *) \sigma(s, *) , e_k \rangle_{\mathcal{H}} ds \end{aligned}$$

$$\begin{aligned}
 &= \sum_{k \in \mathbb{N}} \int_0^t \langle D_{s,k} \mathcal{K}_g(Y)(t, s, *) \sigma(s, *) , e_k \rangle_{\mathcal{H}} ds \\
 &= \text{tr}_{\mathcal{H}} \int_0^t D_s \mathcal{K}_g(Y)(t, s) \sigma(s) ds.
 \end{aligned}$$

Then we extend the last equality to all processes $Y \in \mathcal{P}_0$ which satisfy Definition 6.2(iv). This, together with the equality above, implies the assertion. \square

Now we treat a first example for the random field X -integral, another one will follow in Section 7. Here, we focus on the nonlinear stochastic heat equation with null initial condition

$$\begin{aligned}
 \left(\frac{\partial}{\partial t}(t, x) - \Delta \right) u(t, x) &= b(u(t, x)) + \sigma(u(t, x)) \dot{F}(t, x) \quad (t, x \in (0, T] \times \mathbb{R}^d) \\
 u(0, x) &= 0, \quad x \in \mathbb{R}^d.
 \end{aligned}$$

The fundamental solution to the associated PDE (the heat equation) is given in any spatial dimensions by

$$g(t, x) = \frac{1}{(4\pi t)^{d/2}} \exp\left(\frac{-|x|^2}{4t}\right).$$

Let in the following without much loss of generality $d = 1$. Note that the random-field solution to the SPDE above is given by

$$\begin{aligned}
 u(t, x) &= \int_0^t \int_{\mathbb{R}^d} g(t-s, x-y) \sigma(u(s, y)) M(ds, dy) \\
 &\quad + \int_0^t \int_{\mathbb{R}^d} g(t-s, x-y) b(u(s, y)) dy ds,
 \end{aligned}$$

almost surely for all $(t, x) \in [0, T] \times \mathbb{R}^d$, with M being the martingale measure corresponding to the noise \dot{F} . We see that the stochastic integral has the form of a Volterra process and, therefore, we can apply the integration theory developed in this section. Since the fundamental solution g is differentiable in both arguments and stationary in time and space, we can calculate $\mathcal{K}_g(Y)(t, s, y)$ to be

$$\begin{aligned}
 &\mathcal{K}_g(Y)(t, s, y) \\
 &= \int_{\mathbb{R}^d} Y(s, y+z) g(t-s, dz) + \int_0^{t-s} \int_{\mathbb{R}^d} Y(u+s, z+y) g(du, dz) \\
 &= \int_{\mathbb{R}^d} Y(s, y+z) \partial_z g(t-s, z) dz + \int_0^{t-s} \int_{\mathbb{R}^d} Y(u+s, z+y) \partial_{(u,z)}^2 g(u, z) dz du,
 \end{aligned}$$

where ∂_z denotes the partial derivative with respect to the spatial argument, and $\partial_{(u,z)}^2$ is the mixed spatial and temporal argument. Note the heat kernel estimates for the derivatives

$$|\partial_x \partial_t^a g(t, s, x, y)| \leq c(x - y)(t - s)^{-(3+2a)/2} \exp\left(-c \frac{(x - y)^2}{t - s}\right)$$

for $a \in \{0, 1\}$. This implies that all Y which are almost surely polynomially bounded are integrable with respect to g . This implies (i) and (ii) in Definition 6.2. In order for (iii) to be satisfied, one needs that $\mathcal{K}_g(Y)(t, \cdot, *)\sigma(\cdot, *) \in \mathcal{P}_0$. This depends on the concrete form of σ and Γ . In order for the pathwise integral to well-defined in $L^2(\Omega)$ which implies (iv), one has to assume that Y is Malliavin differentiable and that the integrand of the pathwise integral is in $L^2([0, T] \times \Omega; L^1(\mathbb{R}^d))$.

7. Generalization of the random-field X -integral

In this section, we follow two objectives. First, we want to present another explicit example for the X -integral defined in Section 3, and thus show how nonregular fundamental solutions to partial differential equations enter into this framework. Second, we want to show a first idea how to generalize the random-field X -integral in Section 6 in a way similar to the generalizations of the Walsh integral by Dalang in [5], going from functions to distributions as integrands. More specifically, we investigate under which conditions on Y the kernel $\mathcal{K}_g(Y)(t, s)$ is well-defined if g is chosen to be the fundamental solution to a partial differential operator which is singular in the spatial argument.

For this, we focus on the specific example of a wave equation in different spatial dimensions d given by

$$\left(\frac{\partial^2}{\partial t^2} - \Delta_d\right)u(t, x) = \delta_{0,0}, \tag{7.1}$$

with null initial conditions, where Δ_d is the d -dimensional Laplace operator and $\delta_{0,0}$ is the Dirac delta distribution in time and space. The solution to this equation is the fundamental solution to the wave equation, which differs with the spatial dimension. In the dimensions $d = 1$ and $d = 2$ the fundamental solution is a function, which we can treat with similar methods as in the example with the heat kernel in Section 6.

For $d = 3$, the fundamental solution is given by $g(t, s) = c\rho_{t-s}^3/(t - s)$, where $c > 0$ is a constant and ρ_t^3 is the surface measure on the sphere in three dimensions with radius t . Therefore, the mass of this measure is equal to ct^2 . Since this fundamental solution is not a regular function anymore, the random-field X -integral of Section 6 is no longer well-defined. So we use the Hilbert-valued integral from Section 3 with the Hilbert space \mathcal{H} introduced in Section 6.1.

Proposition 7.1. *Suppose that Assumption 3.1 and the integrability conditions for σ in Definition 3.2 hold. Assume that Y be a random element with values in the linear functionals on \mathcal{H} such that Y is differentiable and Y' does not have a singularity of order greater or equal than 2 at zero, that is, $Y' \in \mathcal{O}(v^{-2})$ at zero. Then, for the special choice of $g(t, s) = c\rho_{t-s}^3/(t - s)$, $Y \in \mathcal{I}^X(0, t)$.*

Proof. By performing integration by parts, we see that

$$\begin{aligned} c^{-1}\mathcal{K}_g(Y)(t, s) &= Y(s)\frac{\rho_{t-s}^3}{t-s} + \int_0^{t-s} (Y(v+s) - Y(s))\frac{d}{dv}\left(\frac{\rho_v^3}{v}\right)dv \\ &= Y(s)\frac{\rho_{t-s}^3}{t-s} + (Y(v+s) - Y(s))\frac{\rho_v^3}{v}\Big|_{v=0}^{t-s} - \int_0^{t-s} Y'(v)\frac{\rho_v^3}{v}dv \\ &= Y(t)\frac{\rho_{t-s}^3}{t-s} - \int_0^{t-s} Y'(v)\frac{\rho_v^3}{v}dv, \end{aligned}$$

where the integral in the last line is understood as a Bochner integral. Since the total mass of the measure ρ_v^3 grows quadratically in v , the last equality tells us that under the conditions on Y , $\mathcal{K}_g(Y)(t, s)$ is well-defined as a linear operator on \mathcal{H} . In this case, $\mathcal{K}_g(Y)(t, s)$ exists as a linear operator on \mathcal{H} defined by

$$\mathcal{K}_g(Y)(t, s)\sigma(s) = Y(t)\frac{\rho_{t-s}^3(\sigma(s))}{t-s} - \int_0^{t-s} Y'(v)\frac{\rho_v^3(\sigma(s))}{v}dv,$$

where

$$\rho_v^3(\sigma(s)) = \int_{\partial B_3(0, v)} \sigma(s, y)\rho_v^3(dy) = v^2 \int_{\partial B_3(0, 1)} \sigma(s, vy)\rho_1^3(dy). \tag{7.2}$$

Under these conditions the X -integral is given by

$$\begin{aligned} \int_0^t Y(s) dX(s) &= \int_0^t \left(Y(t)\frac{\rho_{t-s}^3}{t-s} - \int_0^{t-s} Y'(v)\frac{\rho_v^3}{v}dv \right) \sigma(s)\delta B(s) \\ &\quad + \text{tr}_{\mathcal{H}} \int_0^t \left(D_s Y(t)\frac{\rho_{t-s}^3}{t-s} - \int_0^{t-s} D_s Y'(v)\frac{\rho_v^3}{v}dv \right) \sigma(s) ds, \end{aligned}$$

and this holds for any σ for which (7.2) is finite and for which the integrability conditions from Definition 3.2 hold. □

The same procedure can be done in even higher dimensions d . The fundamental solution to the wave equation becomes in these cases

$$g(t) = \begin{cases} \frac{2\pi^{d/2}}{\Gamma(d/2)} 1_{r>0} \left(\frac{1}{t} \frac{\partial}{\partial t}\right)^{(d-3)/2} \frac{\rho_r^d}{t}, & \text{if } d \text{ is odd,} \\ \frac{2\pi^{d/2}}{\Gamma(d/2)} 1_{r>0} \left(\frac{1}{t} \frac{\partial}{\partial t}\right)^{(d-2)/2} (t^2 - |x|^2)_+^{-1/2}, & \text{if } d \text{ is even.} \end{cases}$$

This can also be treated by integration by parts. Let us show this for the case $d = 4$, where all the higher-order cases follow in the same manner. For $d = 4$, we deal with the regular distribution

$(t^2 - |\cdot|^2)_+^{-1/2}$ given for each test function f by

$$(t^2 - |\cdot|^2)_+^{-1/2} f = \int_{B_4(0,t)} f(y) \frac{dy}{\sqrt{t^2 - |y|^2}} = t^2 \int_{B_4(0,1)} f(ty) \frac{dy}{\sqrt{1 - |y|^2}}.$$

Proposition 7.2. *Suppose that Assumption 3.1 and the integrability conditions for σ in Definition 3.2 hold. Assume that Y be a random element with values in the linear functionals on \mathcal{H} such that Y is differentiable and $Y \in o(v^{-1})$, $Y' \in o(v^{-2})$ and $Y'' \in o(v^{-3})$ at zero. Then, for the special choice of $g(t, s) = c\rho_{t-s}^4/(t - s)$, $Y \in \mathcal{I}^X(0, t)$.*

Proof. We compute as in the previous proposition

$$\begin{aligned} c_4^{-1} \mathcal{K}_g(Y)(t, s) &= Y(s)g(t - s) + \int_0^{t-s} (Y(v + s) - Y(s)) \frac{d}{dv} \left(\frac{1}{v} \frac{d}{dv} \right) (v^2 - |\cdot|^2)_+^{-1/2} dv \\ &= Y(t)g(t - s) - \frac{Y'(t - s)}{t - s} ((t - s)^2 - |\cdot|^2)_+^{-1/2} \\ &\quad + \int_0^{t-s} \frac{Y''(v)v - Y'(v)}{v^2} (v - |\cdot|^2)_+^{-1/2} dv. \end{aligned} \tag{7.3}$$

In order for this to be well-defined, we need the following condition for an integration by parts procedure:

$$\lim_{v \downarrow 0} \frac{Y'(v)}{v} (v^2 - |\cdot|^2)_+^{-1/2} = 0.$$

Under the conditions on Y , all the terms in (7.3) are well-defined. □

Now we want to go a step further and show that even the random-field integral exists. We show this in the case $d = 3$, but the method can be applied in higher dimensions, also. So we assume in the following that $Y(s, *)$ vanishes at infinity for all $s \in [0, T]$ and that it is differentiable in s and in x . Then we can apply an integration by parts procedure and obtain

$$\int_{\mathbb{R}^3} Y(s, z) g(t, s; dz, y) = \int_{\mathbb{R}^3} g(t, s; z, y) \partial_z^3 Y(s, z) dz,$$

where $\partial_z^3 = \partial^3 / \partial z_1 \partial z_2 \partial z_3$. Under these conditions, we can rewrite the integral with respect to the martingale measure and use its isometry to obtain

$$\begin{aligned} &\mathbb{E} \left[\left(\int_0^t \int_{\mathbb{R}^d} \mathcal{K}_g(Y)(t, s, y) \sigma(s, y) M(ds, dy) \right)^2 \right] \\ &= \mathbb{E} \left[\int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathcal{K}_g(Y)(t, s, y - z) \mathcal{K}_g(Y)(t, s, y) \sigma(s, y - z) \sigma(s, y) dy \Gamma(dz) ds \right]. \end{aligned} \tag{7.4}$$

Note that by integration by parts with respect to the temporal and spatial argument

$$\begin{aligned}
 & \mathcal{K}_g(Y)(t, s, y - z)\mathcal{K}_g(Y)(t, s, y) \\
 &= \left(\int_{\mathbb{R}^3} \partial_{x^1}^3 Y(t, x^1)g(t - s; x^1 - y + z) dx^1 \right. \\
 & \quad \left. + \int_0^{t-s} \int_{\mathbb{R}^3} \partial_{v,x^1}^4 Y(v, x^1)g(v; dx^1 - y + z) dx^1 dv \right) \\
 & \quad \times \left(\int_{\mathbb{R}^3} \partial_{x^2}^3 Y(t, x^2)g(t - s; x^2 - y) dx^2 + \int_0^{t-s} \int_{\mathbb{R}^3} \partial_{v,x^2}^4 Y(v, x^2)g(v; x^2 - y) dx^2 dv \right).
 \end{aligned} \tag{7.5}$$

Factoring out this product and substituting this into (7.4) yields to four stochastic integrals of which we will explicitly only treat the first one. So we have

$$\begin{aligned}
 & \mathbb{E} \left[\int_0^t \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \left(\int_{\mathbb{R}^3} \partial_{x^1}^3 Y(t, x^1)g(t - s, x^1 - y + z) dx^1 \right) \right. \\
 & \quad \left. \times \left(\int_{\mathbb{R}^3} \partial_{x^2}^3 Y(t, x^2)g(t - s, x^2 - y) dx^2 \right) \sigma(s, y - z)\sigma(s, y) dy \Gamma(dz) ds \right] \\
 & \leq \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \sup_{(r,y) \in [0,T] \times \mathbb{R}^3} \mathbb{E} [|\partial_{x^1}^3 Y(t, x^1)\partial_{x^2}^3 Y(t, x^2)| | \sigma(r, y) |^2] \\
 & \quad \times \int_0^t \int_{\mathbb{R}^3} |\exp(i\xi(x^1 + x^2))| | \mathcal{F}\Lambda(t - s)(\xi) |^2 \mu(d\xi) ds dx^1 dx^2 \\
 & \leq \int_{\mathbb{R}^3} (\mathbb{E} [|\partial_x^3 Y(t, x) |^{2p}])^{1/p} dx \\
 & \quad \times \sup_{(r,y) \in [0,T] \times \mathbb{R}^3} \mathbb{E} [|\sigma(r, y) |^{2q}]^{1/q} \int_0^t \int_{\mathbb{R}^3} |\mathcal{F}\Lambda(t - s)(\xi) |^2 \mu(d\xi) ds,
 \end{aligned}$$

where in the last line we used Hölder’s inequality with $p^{-1} + q^{-1} = 1$. The other three terms in (7.5) and the pathwise integral involving the Malliavin derivative can be treated in a similar way, so that at the end the random field integral $\int_0^t \int_{\mathbb{R}^3} Y(s, y)X(ds, dy)$ exists when g is the fundamental solution to the three-dimensional wave equation if σ has uniform (in time and space) $2q$ -moments, if

$$\int_0^t \int_{\mathbb{R}^3} | \mathcal{F}\Lambda(t - s)(\xi) |^2 \mu(d\xi) ds < \infty,$$

for \mathcal{F} being the Fourier transform, and if

$$\int_{\mathbb{R}^3} (\mathbb{E} [|\partial_x^3 Y(t, x) |^{2p}])^{1/(2p)} + \int_0^{t-s} \int_{\mathbb{R}^3} (\mathbb{E} [|\partial_{u,x}^4 Y(u, x) |^{2p}])^{1/p} dx du < \infty.$$

Here, we see a certain trade-off between the integrability conditions of Y and σ , if we let one of the two to be more irregular, the other one has to be more regular. A similar condition with the

Malliavin derivatives of Y has to hold so that the pathwise integral is well-defined. All this means that Y has to be in some Sobolev-type space where the (weak) derivatives of the process and of the Malliavin derivatives of the process exist and an integrability condition has to hold. These conditions are far from optimal, but serve as a first step toward a general existence theorem for the random-field X -integral when g is a distribution.

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