Bernoulli **22**(2), 2016, 1113–1130
DOI: 10.3150/14-BEJ688

Excursion probability of Gaussian random fields on sphere

DAN CHENG[1] and YIMIN XIAO[2]

[1]*Department of Statistics, North Carolina State University, 2311 Stinson Drive, Campus Box 8203, Raleigh, NC 27695, USA. E-mail: cheng@stt.msu.edu*
[2]*Department of Statistics and Probability, Michigan State University, 619 Red Cedar Road, C-413 Wells Hall, East Lansing, MI 48824, USA. E-mail: xiao@stt.msu.edu; url: http://www.stt.msu.edu/~xiaoyimi*

Let $X = \{X(x)\colon x \in \mathbb{S}^N\}$ be a real-valued, centered Gaussian random field indexed on the N-dimensional unit sphere $\mathbb{S}^N$. Approximations to the excursion probability $\mathbb{P}\{\sup_{x\in\mathbb{S}^N} X(x) \ge u\}$, as $u \to \infty$, are obtained for two cases: (i) X is locally isotropic and its sample functions are non-smooth and; (ii) X is isotropic and its sample functions are twice differentiable. For case (i), the excursion probability can be studied by applying the results in Piterbarg (*Asymptotic Methods in the Theory of Gaussian Processes and Fields* (1996) Amer. Math. Soc.), Mikhaleva and Piterbarg (*Theory Probab. Appl.* **41** (1997) 367–379) and Chan and Lai (*Ann. Probab.* **34** (2006) 80–121). It is shown that the asymptotics of $\mathbb{P}\{\sup_{x\in\mathbb{S}^N} X(x) \ge u\}$ is similar to Pickands' approximation on the Euclidean space which involves Pickands' constant. For case (ii), we apply the expected Euler characteristic method to obtain a more precise approximation such that the error is super-exponentially small.

Keywords: Euler characteristic; excursion probability; Gaussian random fields on sphere; Pickands' constant

1. Introduction

Even though the characterizations of isotropic covariance functions and variograms on spheres were given long time ago by Schoenberg [35] and Gangolli [11], respectively, and random fields on the sphere were studied by Obukhov [28], Yaglom [44] and Jones [19], it is the applications in atmospherical sciences, geophysics, solar physics, medical imaging and environmental sciences (see, e.g., Genovese *et al.* [12], Oh and Li [29], Stein [37], Cabella and Marinucci [6], Tebaldi and Sansó [42], Hansen *et al.* [14]) that have stimulated the recent rapid development in statistics of random fields on the sphere. Various new random field models have been constructed and new probabilistic and statistical methods have been developed. For example, Jun and Stein [21,22], Huang, Zhang and Robeson [16], Jun [20], Hitczenko and Stein [15], Ma [24], Du, Ma and Li [9] and Gneiting [13] have constructed several classes of real or vector-valued random fields on spheres; Istas [17,18] has constructed spherical fractional Brownian motion (SFBM), which has fractal sample functions, and studied its Karhunen–Loève expansion and other properties. Lang and Schwab [23] characterized sample Hölder continuity and sample differentiability of isotropic Gaussian random fields on the two-dimensional sphere $\mathbb{S}^2$ in terms of their angular power spectra. We refer to the recent book by Marinucci and Peccati [25] for a systematic account on theory and statistical inferences of random fields on the sphere $\mathbb{S}^N$, with a view towards applications to cosmology.

1350-7265

In this paper, we consider a real-valued, centered (locally) isotropic Gaussian random field $X = \{X(x) \colon x \in \mathbb{S}^N\}$, indexed on the N-dimensional unit sphere $\mathbb{S}^N$, and investigate the asymptotic properties of the excursion probability $\mathbb{P}\{\sup_{x\in\mathbb{S}^N} X(x) \ge u\}$ as $u \to \infty$. Such excursion probabilities are important in probability theory, statistics and their applications. In particular, we mention that the above excursion probability has appeared in Sun [38], Park and Sun [30] for determining the P-value in studying exploratory projection pursuit and, as illustrated by Sun [40], is useful for constructing simultaneous confidence region for a function $f : \mathbb{S}^N \to \mathbb{R}$. In his studies of projection-based depth functions, Zuo [45] has shown that Gaussian random fields on sphere appear as scaling limit of sample projection median (see Theorems 3.2 and 3.3 in Zuo [45]) and the excursion probability of the limiting Gaussian field is useful for constructing confidence regions for the true projection median (see Remark 3.2 in Zuo [45]). For further information on extreme value theory of Gaussian random fields on Euclidean spaces or manifolds and statistical applications, we refer to Adler and Taylor [2], Adler, Taylor and Worsley [3] and Marinucci and Peccati [25].

For studying the excursion probability of $X = \{X(x) \colon x \in \mathbb{S}^N\}$, we will distinguish two cases: (i) the sample function of X, denoted as $X(\cdot)$, is non-smooth and, (ii) $X(\cdot) \in C^2$ a.s., and to apply very different methods. In the non-smooth case, the asymptotics of the excursion probability $\mathbb{P}\{\sup_{x\in\mathbb{S}^N} X(x) \ge u\}$ as $u \to \infty$ can be studied by applying the results in Piterbarg [32], Mikhaleva and Piterbarg [27] or Chan and Lai [7], which are extensions of the seminal result of Pickands [31] under various local stationarity conditions. We will make use of Theorem 2.1 in [7] to prove Theorem 2.4 in Section 2, and the method can also be applied to other Gaussian fields on sphere with more complicated local covariance structures, see Section 2.2 for the example of standardized spherical fractional Brownian motion. For the smooth case, we consider isotropic Gaussian fields on sphere. Thanks to the special representation of covariance function (Theorem 3.1), we are able to apply the general theory of Adler and Taylor [2] to compute the Lipschitz–Killing curvatures induced by the field and hence derive the approximation to the excursion probability, see Theorem 3.7 and Corollary 3.9 below. Such an approximation is more precise than that in Theorem 2.4 for the non-smooth case and the error is super-exponentially small.

We should mention that Mikhaleva and Piterbarg [27] have established asymptotic results for the excursion probability of Gaussian fields on a finite-dimensional smooth manifold in $\mathbb{R}^{N+1}$. Their theorems can be applied to obtain results similar to Theorem 2.4 below for a Gaussian field X on the sphere $\mathbb{S}^N$, provided X is the restriction on $\mathbb{S}^N$ of a Gaussian field defined on $\mathbb{R}^{N+1}$. This approach is very useful, but may not be able to deal with all locally isotropic Gaussian random fields on $\mathbb{S}^N$. For instance, Huang, Zhang and Robeson [16] have recently shown that the restriction of some commonly used stationary isotropic covariance functions on $\mathbb{R}^{N+1}$ may not be a valid covariance functions on the sphere (when the Euclidean metric is replaced by the spherical metric). Similarly, another method proposed by Ma ([24], Theorem 4) to obtain valid covariance functions on $\mathbb{S}^N$ from those on $\mathbb{R}^{N+1}$ is only able to produce a proper subset of all covariance functions on $\mathbb{S}^N$ (cf. Ma [24], page 775). Their works have motivated us to deal with Gaussian fields on sphere directly to establish asymptotic results for $\mathbb{P}\{\sup_{x\in\mathbb{S}^N} X(x) \ge u\}$.

Motivated by Mikhaleva and Piterbarg [27], as well as pointed out by an anonymous referee, it would be interesting to study the excursion probability for Gaussian fields over Riemannian manifolds (beyond sphere), whose covariance functions satisfy (2.1) with $d(x, y)$ being the geodesic distance of x and y. This is beyond the scope of the present paper, but we believe that a Pickands-type approximation similar to Theorem 2.4 still holds. As pointed out by an anonymous referee, the problem in the smooth case may be more challenging because there is no analog of the Gegenbauer polynomials to characterize the covariance functions of Gaussian fields over general Riemannian manifolds.

We end the Introduction with some notation. Let $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ denote, respectively, the Euclidean norm and the inner product in $\mathbb{R}^{N+1}$ (or in $\mathbb{R}^N$, which will be clear from the context). Denote by $d(\cdot,\cdot)$ the spherical distance on $\mathbb{S}^N$, that is, $d(x,y)=\arccos\langle x,y\rangle$, $\forall x,y\in\mathbb{S}^N$. For two functions $f(t)$ and $g(t)$, we say $f(t)\sim g(t)$ as $t\to t_0\in[-\infty,+\infty]$ if $\lim_{t\to t_0} f(t)/g(t)=1$.

2. Non-smooth Gaussian fields on sphere

We start with case (i) where the sample functions of $X=\{X(x)\colon\ x\in\mathbb{S}^N\}$ may be non-smooth. This case is easier and we show that the asymptotics of the excursion probability $\mathbb{P}\{\sup_{x\in\mathbb{S}^N} X(x)\ge u\}$, as $u\to\infty$, can be derived from the results in Piterbarg [32], Mikhaleva and Piterbarg [27] and Chan and Lai [7].

2.1. Locally isotropic Gaussian fields on sphere

Let $X=\{X(x)\colon\ x\in\mathbb{S}^N\}$ be a centered Gaussian field with covariance function C satisfying

$$C(x,y)=1-cd^{\alpha}(x,y)\bigl(1+\mathrm{o}(1)\bigr)\qquad \text{as } d(x,y)\to 0, \tag{2.1}$$

for some constants $c>0$ and $\alpha\in(0,2]$. When $X(\cdot)$ is smooth, we have $\alpha=2$.

Covariance functions satisfying (2.1) behave isotropically in a local sense, hence the corresponding random fields fall under the general category of locally isotropic random fields. Similarly to Gaussian fields defined on the Euclidean space (cf. Adler [1]), one can show that, when $\alpha\in(0,2)$, the sample function of X is not differentiable and the fractal dimensions of its trajectories are determined by α. See Andreev and Lang [4], Hansen *et al.* [14] and Lang and Schwab [23] for related regularity results.

There are many examples of covariances of isotropic Gaussian fields on $\mathbb{S}^N$ that satisfy (2.1). A well-known example is $C(x,y)=\mathrm{e}^{-cd^{\alpha}(x,y)}$, where $c>0$ and $\alpha\in(0,1]$ (cf. e.g., Huang, Zhang and Robeson [16], page 725). In their studies on germ-grain (or random ball) models on the sphere $\mathbb{S}^N$, Estrade and Istas ([10], Remark 2.5 and Lemma 3.1) discovered an isotropic Gaussian field W^{β} on $\mathbb{S}^N$ with $0<\beta<1/2$, whose covariance function satisfies (2.1) for $\alpha=2\beta\in(0,1]$. (From here one can show that, even though W^{β} and the spherical fractional Brownian motion $B_{\beta}(x)$ introduced by Istas [17] are different, they share some local properties (e.g., they have the same Hölder continuity and fractal dimensions). In Remark 2.5 below, we will compare the excursion probabilities of W^{β} and the standardized SFBM.) Moreover, as in Yadrenko [43] and Ma [24], one can apply the identity

$$\|x-y\|=2\sin\biggl(\frac{d(x,y)}{2}\biggr)\qquad \forall x,y\in\mathbb{S}^N$$

to construct covariance functions that satisfy (2.1) from isotropic covariance functions $K(\cdot)$ on $\mathbb{R}^N$ which satisfy $K(x) = 1 - c_1\|x\|^\alpha(1 + \mathrm{o}(1))$ as $\|x\| \to 0$. In particular, the following covariance function C given by Soubeyrand, Enjalbert and Sache [36]

$$C(x, y) = 1 - \left(\sin \frac{d(x, y)}{c^{1/\alpha}}\right)^\alpha \mathbb{1}_{\{d(x,y) \le \pi c^{1/\alpha}\}}, \tag{2.2}$$

where $c > 0$ and $\alpha \in (0, 2)$ are constants, satisfies (2.1). See Huang, Zhang and Robeson [16] and Gneiting [13] for further comments on (2.2) and more examples.

For $x = (x_1, \ldots, x_{N+1}) \in \mathbb{S}^N$, its corresponding spherical coordinate $\theta = (\theta_1, \ldots, \theta_N)$ is defined as follows.

$$\begin{aligned}
x_1 &= \cos\theta_1, \\
x_2 &= \sin\theta_1 \cos\theta_2, \\
x_3 &= \sin\theta_1 \sin\theta_2 \cos\theta_3, \\
&\vdots \\
x_N &= \sin\theta_1 \sin\theta_2 \cdots \sin\theta_{N-1} \cos\theta_N, \\
x_{N+1} &= \sin\theta_1 \sin\theta_2 \cdots \sin\theta_{N-1} \sin\theta_N,
\end{aligned} \tag{2.3}$$

where $0 \le \theta_i \le \pi$ for $1 \le i \le N - 1$ and $0 \le \theta_N < 2\pi$.

We define the Gaussian field $\widetilde{X} = \{\widetilde{X}(\theta)\colon\ \theta \in [0, \pi]^{N-1} \times [0, 2\pi)\}$ by $\widetilde{X}(\theta) := X(x)$ and denote by $\widetilde{C}$ the covariance function of $\widetilde{X}$ accordingly. The following elementary lemma characterizes the local behavior of the spherical distance. It provides a useful tool for establishing the relation between local behaviors of covariance functions C and $\widetilde{C}$. Since we cannot find such a result in the literature, for readers' convenience, we provide here a short proof.

Lemma 2.1. *Let $x, y \in \mathbb{S}^N$ and let x be fixed. Then as $d(y, x) \to 0$,*

$$d^2(y, x) \sim (\varphi_1 - \theta_1)^2 + \big(\sin^2\theta_1\big)(\varphi_2 - \theta_2)^2 + \cdots + \left(\prod_{i=1}^{N-1} \sin^2\theta_i\right)(\varphi_N - \theta_N)^2. \tag{2.4}$$

Here and in the sequel, $\theta = (\theta_1, \ldots, \theta_N)$ and $\varphi = (\varphi_1, \ldots, \varphi_N)$ are the spherical coordinates of x and y, respectively.

Proof. For $x, y \in \mathbb{S}^N$, we see that $d(y, x) \sim \|y - x\|$ as $d(y, x) \to 0$, and

$$\begin{aligned}
\|y - x\|^2 = {}& 2 - 2\cos(\varphi_1 - \theta_1) + 2(\sin\varphi_1 \sin\theta_1)\big[1 - \cos(\varphi_2 - \theta_2)\big] \\
&+ \cdots + 2\left(\prod_{i=1}^{N-1} \sin\varphi_i \sin\theta_i\right)\big[1 - \cos(\varphi_N - \theta_N)\big].
\end{aligned}$$

It follows from the spherical coordinates that $d(y, x) \to 0$ is equivalent to $\|\varphi - \theta\| \to 0$. (There is an exception for θ with $\theta_N = 0$, since for those φ such that $d(y, x) \to 0$ and φ_N tending to 2π, $\|\varphi - \theta\|$ does not tend to 0. In such case, we may treat θ_N as 2π instead of 0 and this does not affect the result thanks to the periodicity.) Therefore, as $d(y, x) \to 0$, (2.4) follows from Taylor's expansion. □

Next, we recall from Chan and Lai [7] some results on the excursion probability of Gaussian fields over the Euclidean space. Let $0 < \alpha \le 2$ and let $\{W_t(s)\colon s \in [0, \infty)^N\}$ ($t \in \mathbb{R}^N$) be a family of Gaussian fields such that

$$\begin{aligned}
\mathbb{E}\big(W_t(s)\big) &= -\|s\|^\alpha r_t\big(s/\|s\|\big), \\
\operatorname{Cov}\big(W_t(s), W_t(v)\big) &= \|s\|^\alpha r_t\big(s/\|s\|\big) + \|v\|^\alpha r_t\big(v/\|v\|\big) \\
&\quad - \|s - v\|^\alpha r_t\big((s - v)/\|s - v\|\big),
\end{aligned} \tag{2.5}$$

where $r_t(\cdot) : \mathbb{S}^{N-1} \to \mathbb{R}_+$ is a continuous function which satisfies

$$\sup_{v \in \mathbb{S}^{N-1}} \big|r_t(v) - r_s(v)\big| \to 0 \qquad \text{as } s \to t. \tag{2.6}$$

Define

$$H_\alpha^r(t) = \lim_{K \to \infty} K^{-N} \int_0^\infty \mathrm{e}^u \mathbb{P}\Big\{\sup_{s \in [0, K]^N} W_t(s) \ge u\Big\} \,\mathrm{d}u. \tag{2.7}$$

Denote by H_α the usual Pickands' constant, that is

$$H_\alpha = \lim_{K \to \infty} K^{-N} \int_0^\infty \mathrm{e}^u \mathbb{P}\Big\{\sup_{s \in [0, K]^N} Z(s) \ge u\Big\} \,\mathrm{d}u,$$

where $\{Z(s)\colon s \in [0, \infty)^N\}$ is a Gaussian field such that

$$\mathbb{E}\big(Z(s)\big) = -\|s\|^\alpha, \qquad \operatorname{Cov}\big(Z(s), Z(v)\big) = \|s\|^\alpha + \|v\|^\alpha - \|s - v\|^\alpha.$$

It is clear that $H_\alpha^r(t)$ becomes H_α when $r_t \equiv 1$.

Let $D \subset \mathbb{R}^N$ be a bounded N-dimensional Jordan measurable set, that is, the boundary of D has N-dimensional Lebesgue measure 0. Let $Y = \{Y(t), t \in \mathbb{R}^N\}$ be a real-valued, centered Gaussian field such that its covariance function C_Y satisfies

$$C_Y(t, t + s) = 1 - \|s\|^\alpha r_t\big(s/\|s\|\big)\big(1 + \mathrm{o}(1)\big) \qquad \text{as } \|s\| \to 0, \tag{2.8}$$

for some constant $\alpha \in (0, 2]$, uniformly over $t \in \bar{D}$, the closure of D.

We will make use of the following theorem of Chan and Lai [7]. One can also apply similar results in Piterbarg [32], Mikhaleva and Piterbarg [27], which are formulated under somewhat different local stationarity conditions. Having the functions $r_t(\cdot)$ in (2.8) makes the following theorem slightly easier to apply.

Theorem 2.2 (Chan and Lai [7], Theorem 2.1). *Let $D \subset \mathbb{R}^N$ be a bounded N-dimensional Jordan measurable set. Suppose the Gaussian field $\{Y(t)\colon\ t \in \mathbb{R}^N\}$ satisfies condition (2.8), in which $r_t(\cdot)\colon \mathbb{S}^{N-1} \to \mathbb{R}_+$ is a continuous function such that the convergence (2.6) is uniform in $\bar{D}$ and $\sup_{t\in\bar{D}, v\in\mathbb{S}^{N-1}} r_t(v) < \infty$. Then as $u \to \infty$,*

$$\mathbb{P}\Big\{\sup_{t\in D} Y(t) \geq u\Big\} \sim u^{2N/\alpha}\Psi(u)\int_D H_\alpha^r(t)\,\mathrm{d}t.$$

Here and in the sequel, $\Psi(u) = (\sqrt{2\pi}u)^{-1}\mathrm{e}^{-u^2/2}$.

The lemma below establishes the relation between $H_\alpha^r(t)$ and H_α for a special class of functions $r_t(\cdot)$.

Lemma 2.3. *Let $\{W_t(s)\colon\ s \in [0,\infty)^N\}$ $(t \in \mathbb{R}^N)$ be a family of Gaussian fields satisfying (2.5) with $r_t(v) = \|M_t v\|^\alpha$ for all $v \in \mathbb{S}^{N-1}$, where, for every $t \in \mathbb{R}^N$, M_t is a non-degenerate $N \times N$ matrix. Then $H_\alpha^r(t) = |\det M_t| H_\alpha$ for each $t \in \mathbb{R}^N$.*

Proof. Let $t \in \mathbb{R}^N$ be fixed and consider the centered Gaussian field $\overline{W}_t = \{\overline{W}_t(s), s \in [0,\infty)^N\}$ defined by $\overline{W}_t(s) = W_t(M_t^{-1}s)$. Then by (2.5), $\overline{W}_t$ satisfies

$$\mathbb{E}\big(\overline{W}_t(s)\big) = -\|s\|^\alpha, \qquad \mathrm{Cov}\big(\overline{W}_t(s), \overline{W}_t(v)\big) = \|s\|^\alpha + \|v\|^\alpha - \|s-v\|^\alpha. \tag{2.9}$$

Let $B_K = [0,K]^N$ and $M_t B_K = \{s \in \mathbb{R}^N\colon\ \exists v \in B_K$ such that $s = M_t v\}$. Then $\mathrm{Vol}(M_t B_K) = |\det M_t|\,\mathrm{Vol}(B_K)$ and $\sup_{s\in B_K} W_t(s) = \sup_{s\in M_t B_K} \overline{W}_t(s)$, it follows from (2.7) that

$$\begin{aligned} H_\alpha^r(t) &= \lim_{K\to\infty}\frac{1}{\mathrm{Vol}(B_K)}\int_0^\infty \mathrm{e}^u\mathbb{P}\Big\{\sup_{s\in B_K} W_t(s) \geq u\Big\}\,\mathrm{d}u \\ &= \lim_{K\to\infty}\frac{\mathrm{Vol}(M_t B_K)}{\mathrm{Vol}(B_K)}\frac{1}{\mathrm{Vol}(M_t B_K)}\int_0^\infty \mathrm{e}^u\mathbb{P}\Big\{\sup_{s\in M_t B_K} \overline{W}_t(s) \geq u\Big\}\,\mathrm{d}u \\ &= |\det M_t|\lim_{K\to\infty}\frac{1}{\mathrm{Vol}(M_t B_K)}\int_0^\infty \mathrm{e}^u\mathbb{P}\Big\{\sup_{s\in M_t B_K} \overline{W}_t(s) \geq u\Big\}\,\mathrm{d}u. \end{aligned} \tag{2.10}$$

Because of (2.9), we can modify the proofs in [34] to show that

$$H_\alpha = \lim_{K\to\infty}\frac{1}{\mathrm{Vol}(M_t B_K)}\int_0^\infty \mathrm{e}^u\mathbb{P}\Big\{\sup_{s\in M_t B_K} \overline{W}_t(s) \geq u\Big\}\,\mathrm{d}u. \tag{2.11}$$

Comparing (2.10) and (2.11) gives the result. □

For any $T \subset \mathbb{S}^N$, we denote by $D \subset [0,\pi]^{N-1} \times [0,2\pi)$ the set corresponding to T under the spherical coordinates (2.3). We say that T is an N-dimensional Jordan measurable set on $\mathbb{S}^N$ if D is an N-dimensional Jordan measurable set in $\mathbb{R}^N$. Now we can prove our main result of this section.

Theorem 2.4. *Let $\{X(x)\colon x \in \mathbb{S}^N\}$ be a centered Gaussian random field satisfying condition (2.1) and let $T \subset \mathbb{S}^N$ be an N-dimensional Jordan measurable set on $\mathbb{S}^N$. Then as $u \to \infty$,*

$$\mathbb{P}\Big\{\sup_{x\in T} X(x) \geq u\Big\} \sim c^{N/\alpha} \operatorname{Area}(T) H_\alpha u^{2N/\alpha} \Psi(u),$$

where $\operatorname{Area}(T)$ *denotes the spherical area of T and $c > 0$ is the constant in* (2.1).

Proof. For any $\theta \in [0, \pi]^{N-1} \times [0, 2\pi)$, let $M_\theta = c^{1/\alpha} \operatorname{diag}(1, \sin\theta_1, \dots, \prod_{i=1}^{N-1} \sin\theta_i)$ be the $N \times N$ diagonal matrix. By Lemma 2.1, condition (2.1) implies

$$\widetilde{C}(\theta, \theta + \xi) = 1 - \|\xi\|^\alpha r_\theta\big(\xi/\|\xi\|\big)\big(1 + \mathrm{o}(1)\big) \qquad \text{as } \|\xi\| \to 0,$$

where $r_\theta(\tau) = \|M_\theta \tau\|^\alpha$, $\forall \tau \in \mathbb{S}^{N-1}$. Then by Theorem 2.2, as $u \to \infty$,

$$\mathbb{P}\Big\{\sup_{x\in T} X(x) \geq u\Big\} = \mathbb{P}\Big\{\sup_{\theta\in D} \widetilde{X}(\theta) \geq u\Big\} \sim u^{2N/\alpha} \Psi(u) \int_D H_\alpha^r(\theta)\, \mathrm{d}\theta. \tag{2.12}$$

It follows from Lemma 2.3 that for any $\theta \in [0, \pi]^{N-1} \times [0, 2\pi)$ such that M_θ is non-degenerate (i.e., $\prod_{i=1}^{N-1} \sin\theta_i \neq 0$),

$$H_\alpha^r(\theta) = c^{N/\alpha} \left(\prod_{i=1}^{N-1} \sin^{N-i} \theta_i\right) H_\alpha.$$

Note that $(\prod_{i=1}^{N-1} \sin^{N-i} \theta_i)\, \mathrm{d}\theta$ is the spherical area element and M_θ is non-degenerate for almost every $\theta \in D$, we obtain

$$\int_D H_\alpha^r(\theta)\, \mathrm{d}\theta = c^{N/\alpha} \operatorname{Area}(T) H_\alpha.$$

Plugging this into (2.12) gives the desired result. □

2.2. Standardized spherical fractional Brownian motion

Theorem 2.4 provides a nice approximation to the excursion probability for locally isotropic Gaussian random fields on $\mathbb{S}^N$ whose covariance functions satisfy (2.1). When the local behavior of the covariance function becomes more complicated, Theorem 2.4 may not be applicable anymore. However, we can still apply Lemma 2.1 to find the corresponding local behavior of covariance function under spherical coordinates and then apply Theorem 2.2 to obtain the asymptotics for the excursion probability. In the following, we use spherical fractional Brownian motion on sphere as an illustrating example.

Let o be a fixed point on $\mathbb{S}^N$. The spherical fractional Brownian motion (SFBM) $B_\beta = \{B_\beta(x)\colon x \in \mathbb{S}^N\}$ is defined by Istas [17] as a centered real-valued Gaussian random field such that $B_\beta(o) = 0$ and

$$\mathbb{E}\big(B_\beta(x) - B_\beta(y)\big)^2 = d^{2\beta}(x, y) \qquad \forall x, y \in \mathbb{S}^N,$$

where $\beta \in (0, 1/2]$. It follows immediately that

$$\mathrm{Cov}\big(B_\beta(x), B_\beta(y)\big) = \tfrac{1}{2}\big(d^{2\beta}(x,o) + d^{2\beta}(y,o) - d^{2\beta}(x,y)\big).$$

Without loss of generality, we take $o = (1, 0, \ldots, 0) \in \mathbb{R}^{N+1}$, whose corresponding spherical coordinate is $(0, \ldots, 0) \in \mathbb{R}^N$. We consider the standardized SFBM $X = \{X(x): x \in \mathbb{S}^N \setminus \{o\}\}$ defined by

$$X(x) = \frac{B_\beta(x)}{d^\beta(x,o)} \qquad \forall x \in \mathbb{S}^N \setminus \{o\}. \tag{2.13}$$

Then the covariance of X is

$$C(x,y) = \mathrm{Cov}\big(X(x), X(y)\big) = \frac{d^{2\beta}(x,o) + d^{2\beta}(y,o) - d^{2\beta}(x,y)}{2d^\beta(x,o)d^\beta(y,o)}.$$

Note that, under the spherical coordinates, $d(x,o) = \theta_1$ and $d(y,o) = \varphi_1$, together with Lemma 2.1, we obtain that the covariance function of the corresponding Gaussian field $\widetilde{X}$ satisfies

$$\begin{aligned}\widetilde{C}(\theta,\varphi) &= \mathrm{Cov}\big(\widetilde{X}(\theta), \widetilde{X}(\varphi)\big)\\ &= 1 - \big(1 + \mathrm{o}(1)\big)\\ &\quad\times \frac{1}{2\theta_1^{2\beta}}\left[(\varphi_1 - \theta_1)^2 + \big(\sin^2\theta_1\big)(\varphi_2 - \theta_2)^2 + \cdots + \left(\prod_{i=1}^{N-1}\sin^2\theta_i\right)(\varphi_N - \theta_N)^2\right]^\beta\end{aligned}$$

as $d(x,y) \to 0$. Let

$$\begin{aligned}M_\theta &= \frac{1}{2^{1/(2\beta)}\theta_1}\,\mathrm{diag}\left(1, \sin\theta_1, \ldots, \prod_{i=1}^{N-1}\sin\theta_i\right),\\ r_\theta(\tau) &= \|M_\theta\tau\|^{2\beta} \qquad \forall\tau \in \mathbb{S}^{N-1},\end{aligned}$$

and $\xi = \varphi - \theta$, then as $\|\xi\| \to 0$,

$$\widetilde{C}(\theta, \theta + \xi) = 1 - \|\xi\|^{2\beta} r_\theta\big(\xi/\|\xi\|\big)\big(1 + \mathrm{o}(1)\big).$$

Let $T \subset \mathbb{S}^N$ be an N-dimensional Jordan measurable set such that $o \notin \bar{T}$, and denote its corresponding domain under the spherical coordinates by D, which implies $\theta_1 \neq 0$ for any $\theta \in \bar{D}$. By Theorem 2.2, as $u \to \infty$,

$$\mathbb{P}\Big\{\sup_{x\in T} X(x) \ge u\Big\} = \mathbb{P}\Big\{\sup_{\theta\in D}\widetilde{X}(\theta) \ge u\Big\} \sim u^{N/\beta}\Psi(u)\int_D H_{2\beta}^r(\theta)\,\mathrm{d}\theta.$$

For any θ such that M_θ is non-degenerate (i.e., $\prod_{i=1}^{N-1}\sin\theta_i \neq 0$), Lemma 2.3 gives

$$H^r_{2\beta}(\theta) = \frac{1}{2^{N/(2\beta)}\theta_1^N}\left(\prod_{i=1}^{N-1}\sin^{N-i}\theta_i\right)H_{2\beta}.$$

Therefore, as $u \to \infty$,

$$\mathbb{P}\Big\{\sup_{x\in T} X(x) \geq u\Big\} \sim u^{N/\beta}\Psi(u)2^{-N/(2\beta)}H_{2\beta}\int_D \theta_1^{-N}\left(\prod_{i=1}^{N-1}\sin^{N-i}\theta_i\right)\mathrm{d}\theta. \tag{2.14}$$

Remark 2.5. Comparing the excursion probabilities in (2.14) for the standardized SFBM X and in Theorem 2.4 for the isotropic Gaussian field W^β, which is defined in Estrade and Istas [10], we see that the constant in (2.14) is more complicated.

3. Smooth isotropic Gaussian fields on sphere

In this section, we study the excursion probability of smooth isotropic Gaussian fields on sphere. Related to the results in this section, we mention that [8] have determined the height distribution and overshoot distribution of local maxima of smooth isotropic Gaussian random fields on sphere.

3.1. Preliminaries

Given $\lambda > 0$ and an integer $n \geq 0$, the *ultraspherical polynomial* (or *Gegenbauer polynomial*) of degree n, denoted by $P_n^\lambda(t)$, is defined by the expansion

$$\big(1 - 2rt + r^2\big)^{-\lambda} = \sum_{n=0}^{\infty} r^n P_n^\lambda(t), \qquad t \in [-1, 1].$$

For $\lambda = 0$, we follow Schoenberg [35] and define $P_n^0(t) = \cos(n \arccos t) = T_n(t)$, where T_n $(n \geq 0)$ are *the Chebyshev polynomials of the first kind* defined by the expansion

$$\frac{1 - rt}{1 - 2rt + r^2} = \sum_{n=0}^{\infty} r^n T_n(t), \qquad t \in [-1, 1].$$

For reference later on, we recall the following formulae on P_n^λ.

(i) For all $n \geq 0$, $P_n^0(1) = 1$, and if $\lambda > 0$ (cf. Szegő [41], page 80),

$$P_n^\lambda(1) = \binom{n + 2\lambda - 1}{n}. \tag{3.1}$$

(ii) For all $n \geq 0$,

$$\frac{\mathrm{d}}{\mathrm{d}t} P_n^0(t) = n P_{n-1}^1(t), \tag{3.2}$$

and if $\lambda > 0$ (cf. Szegő [41], page 81),

$$\frac{\mathrm{d}}{\mathrm{d}t} P_n^\lambda(t) = 2\lambda P_{n-1}^{\lambda+1}(t). \tag{3.3}$$

The following theorem by Schoenberg [35] characterizes the covariance function of an isotropic Gaussian field on sphere (see also Gneiting [13]).

Theorem 3.1. *Let $N \geq 1$, then a continuous function $C(\cdot,\cdot): \mathbb{S}^N \times \mathbb{S}^N \to \mathbb{R}$ is the covariance of an isotropic Gaussian field on $\mathbb{S}^N$ if and only if it has the form*

$$C(x, y) = \sum_{n=0}^{\infty} a_n P_n^\lambda\big(\langle x, y\rangle\big), \qquad x, y \in \mathbb{S}^N,$$

where $\lambda = (N-1)/2$, $a_n \geq 0$ and $\sum_{n=0}^\infty a_n P_n^\lambda(1) < \infty$.

Remark 3.2. Note that for the case of $N = 1$ and $\lambda = 0$, $\sum_{n=0}^\infty a_n P_n^0(1) < \infty$ is equivalent to $\sum_{n=0}^\infty a_n < \infty$; while for $N \geq 2$ and $\lambda = (N-1)/2$, (3.1) implies that $\sum_{n=0}^\infty a_n P_n^\lambda(1) < \infty$ is equivalent to $\sum_{n=0}^\infty n^{N-2} a_n < \infty$.

When $N = 2$ and $\lambda = 1/2$, P_n^λ $(n \geq 0)$ become the *Legendre polynomials*. For more results on isotropic Gaussian fields on $\mathbb{S}^2$, we refer to Marinucci and Peccati [25]. Regularity and smoothness properties of Gaussian field $\{X(x): x \in \mathbb{S}^2\}$ have recently been obtained by Lang and Schwab [23] in terms of the corresponding angular power spectrum.

The following statement (**A**1) is a smoothness condition for Gaussian fields on sphere. In Lemma 3.3 below, we show that it implies $X(\cdot) \in C^2(\mathbb{S}^N)$ a.s.

(**A**1). The covariance $C(\cdot,\cdot)$ of $\{X(x): x \in \mathbb{S}^N\}$ satisfies

$$C(x, y) = \sum_{n=0}^{\infty} a_n P_n^\lambda\big(\langle x, y\rangle\big), \qquad x, y \in \mathbb{S}^N,$$

where $\lambda = \frac{N-1}{2}$, $a_n \geq 0$ and $\sum_{n=1}^\infty n^{N+8} a_n < \infty$ if $N \geq 2$; $\sum_{n=1}^\infty n^{10} a_n < \infty$ if $N = 1$.

Lemma 3.3. *Let $\{X(x): x \in \mathbb{S}^N\}$ be an isotropic Gaussian field such that* (**A**1) *is fulfilled. Then $X(\cdot) \in C^2(\mathbb{S}^N)$ a.s.*

Proof. We first consider $N \geq 2$. By Theorem 3.1, each $P_n^\lambda(\langle t, s\rangle)$ is the covariance of an isotropic Gaussian field on $\mathbb{S}^N$ and hence the Cauchy–Schwarz inequality implies

$$\big|P_n^\lambda\big(\langle x, y\rangle\big)\big| \leq P_n^\lambda\big(\langle x, x\rangle\big) = P_n^\lambda(1) \qquad \forall x, y \in \mathbb{S}^N. \tag{3.4}$$

Combining (**A**1) with (3.1), (3.3) and (3.4), together with the fact $P_0^\lambda(t) \equiv 1$, we obtain that there exist positive constants M_1 and M_2 such that

$$\sup_{t\in[-1,1]} \sum_{n=0}^{\infty} a_n \left| \left(\frac{\mathrm{d}^5}{\mathrm{d}t^5} P_n^\lambda(t) \right) \right| \le M_1 \sum_{n=5}^{\infty} a_n P_{n-5}^{\lambda+5}(1) \le M_2 \sum_{n=1}^{\infty} n^{N+8} a_n < \infty.$$

This shows that $C(\cdot,\cdot) \in C^5(\mathbb{S}^N \times \mathbb{S}^N)$. The proof for $N = 1$ is similar once we apply both (3.2) and (3.3). Therefore, by arguments via charts (cf. Auffinger [5]) and the results in Potthoff [33] (though the results therein are for $X(\cdot) \in C^1$, they can be extended easily to the case of higher-order smoothness), we conclude that $X(\cdot) \in C^2(\mathbb{S}^N)$ a.s. □

By Schoenberg [35] or Gneiting [13], $C(\cdot,\cdot)$ is a covariance function on $\mathbb{S}^N$ for every $N \ge 1$ if and only if it has the form

$$C(x,y) = \sum_{n=0}^{\infty} b_n \langle x, y\rangle^n, \qquad x, y \in \mathbb{S}^N,$$

where $b_n \ge 0$ and $\sum_{n=0}^{\infty} b_n < \infty$. Then similarly to (**A**1), we may state the smoothness condition (**A**1′) below for this special class of Gaussian fields on sphere.

(**A**1′). The covariance $C(\cdot,\cdot)$ of $\{X(x)\colon x \in \mathbb{S}^N\}$ satisfies

$$C(x,y) = \sum_{n=0}^{\infty} b_n \langle x, y\rangle^n, \qquad x, y \in \mathbb{S}^N,$$

where $b_n \ge 0$ and $\sum_{n=0}^{\infty} n^5 b_n < \infty$.

We obtain below an analogue of Lemma 3.3. Since the proof is similar, it is omitted.

Lemma 3.4. *Let $\{X(x)\colon x \in \mathbb{S}^N\}$ be an isotropic Gaussian field such that* (**A**1′) *is fulfilled. Then $X(\cdot) \in C^2(\mathbb{S}^N)$ a.s.*

3.2. Excursion probability

Let $\chi(A_u(X, \mathbb{S}^N))$ be the Euler characteristic of excursion set $A_u(X,\mathbb{S}^N) = \{x \in \mathbb{S}^N\colon X(x) \ge u\}$ (cf. Adler and Taylor [2]). Denote by $H_j(x)$ the Hermite polynomial of order j, that is,

$$H_j(x) = (-1)^j \mathrm{e}^{x^2/2} \frac{\mathrm{d}^j}{\mathrm{d}x^j} \bigl(\mathrm{e}^{-x^2/2}\bigr).$$

Denote $\omega_j = \frac{2\pi^{(j+1)/2}}{\Gamma((j+1)/2)}$, the spherical area of the j-dimensional unit sphere $\mathbb{S}^j$.

Before stating our results, we need another regularity condition for the Gaussian field.

(**A**2). For each $x \in \mathbb{S}^N$, the joint distribution of $(X(x), \nabla X(x), \nabla^2 X(x))$ is non-degenerate.

Lemma 3.5. *Let $\{X(x)\colon x \in \mathbb{S}^N\}$ be a centered, unit-variance, isotropic Gaussian field satisfying* (**A**1) *and* (**A**2). *Then*

$$\mathbb{E}\{\chi(A_u(X,\mathbb{S}^N))\} = \sum_{j=0}^{N} (C')^{j/2} \mathcal{L}_j(\mathbb{S}^N)\rho_j(u),$$

where the constant C' is defined as

$$C' = \begin{cases} (N-1)\displaystyle\sum_{n=1}^{\infty} \binom{n+N-1}{N} a_n, & \text{if } N \geq 2, \\ \displaystyle\sum_{n=1}^{\infty} n^2 a_n, & \text{if } N = 1, \end{cases} \tag{3.5}$$

and where $\rho_0(u) = (2\pi)^{-1/2}\int_u^\infty \mathrm{e}^{-x^2/2}\,\mathrm{d}x$, $\rho_j(u) = (2\pi)^{-(j+1)/2} H_{j-1}(u)\mathrm{e}^{-u^2/2}$ for $j \geq 1$ and

$$\mathcal{L}_j(\mathbb{S}^N) = \begin{cases} 2\dbinom{N}{j}\dfrac{\omega_N}{\omega_{N-j}}, & \text{if } N-j \text{ is even}, \\ 0, & \text{otherwise} \end{cases} \tag{3.6}$$

(*for $j = 0, 1, \ldots, N$*) *are the Lipschitz–Killing curvatures of $\mathbb{S}^N$* (*cf.* (6.3.8) *in Adler and Taylor* [2]).

Remark 3.6. In Lemma 3.5, if condition (**A**1) is replaced by (**A**1$'$), then it can be seen from the proof below that the result still holds with C' being replaced by $C' = \sum_{n=1}^\infty nb_n$.

Proof of Lemma 3.5. By Theorem 12.4.1 in Adler and Taylor [2], we only need to show that the Lipschitz–Killing curvatures induced by X on $\mathbb{S}^N$ are $\mathcal{L}_j(X,\mathbb{S}^N) = (C')^{j/2}\mathcal{L}_j(\mathbb{S}^N)$ for $j = 0, 1, \ldots, N$.

The Riemannian structure induced by X on $\mathbb{S}^N$ is defined as

$$g_{x_0}^{X,\mathbb{S}^N}(\xi_{x_0},\sigma_{x_0}) := \mathbb{E}\{(\xi_{x_0}X)\cdot(\sigma_{x_0}X)\} = \xi_{x_0}\sigma_{x_0}C(x,y)|_{x=y=x_0} \qquad \forall x_0 \in \mathbb{S}^N,$$

where $\xi_{x_0}, \sigma_{x_0} \in T_{x_0}\mathbb{S}^N$, the tangent space of $\mathbb{S}^N$ at x_0 (cf. Adler and Taylor [2], page 305). We may choose two smooth curves on $\mathbb{S}^N$, say $\gamma(t)$, $\tau(s)$, $t, s \in [0,1]$, such that $\gamma(0) = \tau(0) = x_0$ and $\gamma'(0) = \xi_{x_0}$, $\tau'(0) = \sigma_{x_0}$. We first consider $N \geq 2$, then

$$\begin{aligned} \xi_{x_0}\sigma_{x_0}C(x,y)|_{x=y=x_0} &= \frac{\partial}{\partial t}\frac{\partial}{\partial s} C(\gamma(t),\tau(s))\Big|_{t=s=0} \\ &= \frac{\partial}{\partial t}\frac{\partial}{\partial s}\sum_{n=0}^{\infty} a_n P_n^\lambda(\langle\gamma(t),\tau(s)\rangle)\Big|_{t=s=0} \\ &= \frac{\partial}{\partial t}\sum_{n=1}^{\infty} a_n (N-1) P_{n-1}^{\lambda+1}(\langle\gamma(t),x_0\rangle)\langle\gamma(t),\sigma_{x_0}\rangle\Big|_{t=0} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{n=2}^{\infty} a_n (N-1)(N+2) P_{n-2}^{\lambda+2}\big(\langle x_0, x_0\rangle\big)\langle \xi_{x_0}, x_0\rangle \langle x_0, \sigma_{x_0}\rangle \\
&\quad + \sum_{n=1}^{\infty} a_n (N-1) P_{n-1}^{\lambda+1}\big(\langle x_0, x_0\rangle\big)\langle \xi_{x_0}, \sigma_{x_0}\rangle \\
&= \left(\sum_{n=1}^{\infty} a_n (N-1) P_{n-1}^{\lambda+1}(1)\right)\langle \xi_{x_0}, \sigma_{x_0}\rangle = C'\langle \xi_{x_0}, \sigma_{x_0}\rangle,
\end{aligned}
$$

where the third and fourth equalities follow from (3.3), while the fifth equality is due to the facts $\langle x_0, x_0\rangle = 1$ and $\langle \xi_{x_0}, x_0\rangle = \langle \sigma_{x_0}, x_0\rangle = 0$, since the vector x_0 is always orthogonal to its tangent space. The case $N = 1$ can be proved similarly once we apply (3.2) instead of (3.3).

Hence the induced metric is

$$
g_{x_0}^{X,\mathbb{S}^N}(\xi_{x_0}, \sigma_{x_0}) = C'\langle \xi_{x_0}, \sigma_{x_0}\rangle \qquad \forall x_0 \in \mathbb{S}^N.
$$

By the definition of Lipschitz–Killing curvatures, one has $\mathcal{L}_j(X, \mathbb{S}^N) = (C')^{j/2}\mathcal{L}_j(\mathbb{S}^N)$, where $\mathcal{L}_j(\mathbb{S}^N)$ are the original Lipschitz–Killing curvatures of $\mathbb{S}^N$ given by (3.6). We have finished the proof. □

Applying Lemma 3.5 and Theorem 14.3.3 in Adler and Taylor [2], we obtain immediately the following approximation for the excursion probability.

Theorem 3.7. *Suppose the conditions in Lemma 3.5 hold. Then, under the notation therein, there exists a constant $\alpha_0 > 0$ such that as $u \to \infty$,*

$$
\mathbb{P}\Big\{\sup_{x\in\mathbb{S}^N} X(x) \geq u\Big\} = \sum_{j=0}^{N} \big(C'\big)^{j/2}\mathcal{L}_j\big(\mathbb{S}^N\big)\rho_j(u) + o\big(e^{-\alpha_0 u^2 - u^2/2}\big). \tag{3.7}
$$

Remark 3.8. The following are some remarks.

- Under the conditions in Theorem 3.7, the covariance function C satisfies (2.1) with $\alpha = 2$. Since for $\alpha = 2$, Pickands' constant $H_2 = \pi^{-N/2}$, one can check that the approximation in Theorem 2.4 only provides the leading term of the approximation in Theorem 3.7. This also affects the errors in two approximations: the error in the former one is only $o(u^{N-1}e^{-u^2/2})$, while the error in the latter one is $o(e^{-\alpha_0 u^2 - u^2/2})$.
- By applying the tube method, Sun [39] gave a two-term approximation formula for the excursion probability of a class of differentiable Gaussian random field $\{X(x), x \in I\}$, where $I \subset \mathbb{R}^N$ is a bounded convex set. Her results can be applied to provide a two-term approximation for the excursion probability in (3.7) for some special cases. See Park and Sun [30], page 73.
- Recently Marinucci and Vadlamani [26] have computed the Lipschitz–Killing curvatures of excursion set and derived a very precise approximation for the excursion probability of a

class of nonlinear functionals of a smooth Gaussian random field on $\mathbb{S}^2$. In the linear case (i.e., $q = 1$) Theorem 21 of Marinucci and Vadlamani [26] is a special case of (3.7) with $N = 2$.

If the sphere $\mathbb{S}^N$ is replaced by a more general subset $T \subset \mathbb{S}^N$, by revising Lemma 3.5 and applying Theorem 14.3.3 in Adler and Taylor [2] again, we obtain the following corollary.

Corollary 3.9. *Suppose the conditions in Lemma 3.5 hold. Let $T \subset \mathbb{S}^N$ be a k-dimensional, locally convex, regular stratified manifold (cf. Adler and Taylor [2], page 198), then there exists $\alpha_0 > 0$ such that as $u \to \infty$,*

$$\mathbb{P}\Big\{\sup_{x\in T} X(x) \geq u\Big\} = \sum_{j=0}^{k} \big(C'\big)^{j/2} \mathcal{L}_j(T)\rho_j(u) + \mathrm{o}\big(\mathrm{e}^{-\alpha_0 u^2 - u^2/2}\big), \tag{3.8}$$

where $\mathcal{L}_j(T)$ are the Lipschitz–Killing curvatures of T (cf. Adler and Taylor [2], page 175), C' and $\rho_j(u)$ are as in Lemma 3.5.

The parameter set $T \subset \mathbb{S}^N$ in Corollary 3.9 is assumed to be nice enough. Roughly speaking, it looks like a convex set and can be decomposed into several smooth manifolds, see Adler and Taylor [2] for a rigorous definition. Also, the jth Lipschitz–Killing curvature $\mathcal{L}_j(T)$ can be viewed as the measure of the j-dimensional boundary of T. One may use Steiner's formula (Adler and Taylor [2], page 142) to compute the Lipschitz–Killing curvatures of T exactly. In particular, if T is a semisphere of dimension one, then $\mathcal{L}_0(T) = 1$ and $\mathcal{L}_1(T) = \pi$. If T is a semisphere of dimension two, then $\mathcal{L}_0(T) = 1$, $\mathcal{L}_1(T) = \pi$ and $\mathcal{L}_2(T) = 2\pi$. More generally, if T is a k-dimensional, locally convex, regular stratified manifold, then $\mathcal{L}_0(T)$ is the Euler characteristic, $\mathcal{L}_k(T)$ is the volume and $\mathcal{L}_{k-1}(T)$ is half of the surface area. For the other $\mathcal{L}_j(T)$, $1 \leq j \leq k-2$, we can apply Steiner's formula to find their values.

Lastly, to further illustrate the main results of this paper, we give more examples on approximating the excursion probability of Gaussian fields on spheres, including both smooth and non-smooth cases.

Example 3.1. The canonical Gaussian field on $\mathbb{S}^N$, denoted by X, has covariance function given by $C(x, y) = \langle x, y\rangle$ (cf. Adler and Taylor [2]). Since $C(x, y) = \cos d(x, y)$, it satisfies

$$C(x, y) = 1 - \tfrac{1}{2} d^2(x, y)\big(1 + \mathrm{o}(1)\big), \qquad \text{as } d(x, y) \to 0.$$

Applying Theorem 2.4 with $T = \mathbb{S}^N$, $c = 1/2$ and $\alpha = 2$, we obtain an approximation to the excursion probability:

$$\mathbb{P}\Big\{\sup_{x\in\mathbb{S}^N} X(x) \geq u\Big\} \sim 2^{-N/2}\,\mathrm{Area}\big(\mathbb{S}^N\big) H_2 u^N \Psi(u) = (2\pi)^{-(N+1)/2} \omega_N u^{N-1} \mathrm{e}^{-u^2/2}.$$

However, by applying Theorem 3.7 with $C' = 1$, we get a more precise approximation:

$$\mathbb{P}\Big\{\sup_{x\in\mathbb{S}^N} X(x) \ge u\Big\} = \sum_{j=0}^{N} \mathcal{L}_j\big(\mathbb{S}^N\big)\rho_j(u) + \mathrm{o}\big(\mathrm{e}^{-\alpha_0 u^2 - u^2/2}\big).$$

Example 3.2. Consider the Hamiltonian of the pure p-spin model on $\mathbb{S}^{N-1}$

$$H_{N,p}(x) = \frac{1}{N^{(p-1)/2}} \sum_{i_1,\dots,i_p=1}^{N} J_{i_1,\dots,i_p} x_{i_1}\cdots x_{i_p} \qquad \forall x = (x_1,\dots,x_N) \in \mathbb{S}^{N-1},$$

where $J_{i_1,\dots,i_p}$ are independent standard Gaussian random variables. Then $H_{N,p}$ and $H_{N,p'}$ are independent for any $p \ne p'$ and

$$\mathbb{E}\big\{H_{N,p}(x)H_{N,p}(y)\big\} = \frac{1}{N^{p-1}}\langle x, y\rangle^p.$$

Let $(b_p)_{p\ge 2}$ be a sequence of positive numbers such that $\sum_{p=2}^{\infty} 2^p b_p < \infty$ and define

$$X(x) = \sum_{p=2}^{\infty} b_p H_{N,p}(x).$$

Then X is a smooth Gaussian random field on $\mathbb{S}^{N-1}$ with covariance

$$C(x, y) = \sum_{p=2}^{\infty} \frac{b_p^2}{N^{p-1}}\langle x, y\rangle^p.$$

We can apply Theorem 3.7 or Corollary 3.9 to approximate the excursion probability.

Example 3.3. Consider the Gaussian field $\{X(x)\colon x \in \mathbb{S}^N\}$ with covariance structure $C(x, y) = 1 - \frac{2}{\pi} d(x, y)$ (cf. Zuo [45], Remark 3.3). Since $d(x, y) = \arccos\langle x, y\rangle$, we have

$$C(x, y) = \sum_{n=0}^{\infty} \frac{(2n)!}{4^n (n!)^2 (2n+1)}\langle x, y\rangle^{2n+1} := \sum_{n=0}^{\infty} b_n \langle x, y\rangle^n. \tag{3.9}$$

It is easy to check that $\sum_{n=0}^{\infty} n b_n = \infty$, $(\mathbf{A1}')$ is not satisfied and hence Theorem 3.7 is not applicable. Instead, we may use Theorem 2.2 to get an approximation to the excursion probability. This result allows one to construct confidence regions for the true projection median defined in Zuo ([45], Section 3) without using the bootstrapping techniques.

Acknowledgements

The authors thank Professor Enkelejd Hashorva, Dr. Lanpeng Ji for stimulating discussions and Dr. Xiaohui Liu for pointing out the connection between Gaussian random fields on sphere and projection depth functions in Zuo [45]. They thank the referees for their constructive comments which have led to improvements of the manuscript.

Research partially supported by NSF Grants DMS-13-09856 and DMS-13-07470.

References

[1] Adler, R.J. (2010). *The Geometry of Random Fields*, SIAM Classics ed. Philadelphia: SIAM.

[2] Adler, R.J. and Taylor, J.E. (2007). *Random Fields and Geometry*. *Springer Monographs in Mathematics*. New York: Springer. MR2319516

[3] Adler, R.J., Taylor, J.E. and Worsley, K.J. (2012). Applications of random fields and geometry: Foundations and case studies. In preparation.

[4] Andreev, R. and Lang, A. (2014). Kolmogorov–Chentsov theorem and differentiability of random fields on manifolds. *Potential Anal*. **41** 761–769. MR3264819

[5] Auffinger, A. (2011). Random matrices, complexity of spin glasses and heavy tailed processes. Ph.D. thesis, New York Univ., ProQuest LLC, Ann Arbor, MI. MR2912239

[6] Cabella, P. and Marinucci, D. (2009). Statistical challenges in the analysis of cosmic microwave background radiation. *Ann. Appl. Stat*. **3** 61–95. MR2668700

[7] Chan, H.P. and Lai, T.L. (2006). Maxima of asymptotically Gaussian random fields and moderate deviation approximations to boundary crossing probabilities of sums of random variables with multidimensional indices. *Ann. Probab*. **34** 80–121. MR2206343

[8] Cheng, D. and Schwartzman, A. (2015). Distribution of the height of local maxima of Gaussian random fields. *Extremes*. To appear. Available at arXiv:1307.5863.

[9] Du, J., Ma, C. and Li, Y. (2013). Isotropic variogram matrix functions on spheres. *Math. Geosci*. **45** 341–357. MR3107159

[10] Estrade, A. and Istas, J. (2010). Ball throwing on spheres. *Bernoulli* **16** 953–970. MR2759164

[11] Gangolli, R. (1967). Positive definite kernels on homogeneous spaces and certain stochastic processes related to Lévy's Brownian motion of several parameters. *Ann. Inst. H. Poincaré Sect. B* (*N.S.*) **3** 121–226. MR0215331

[12] Genovese, C.R., Miller, C.J., Nichol, R.C., Arjunwadkar, M. and Wasserman, L. (2004). Nonparametric inference for the cosmic microwave background. *Statist. Sci*. **19** 308–321. MR2146946

[13] Gneiting, T. (2013). Strictly and non-strictly positive definite functions on spheres. *Bernoulli* **19** 1327–1349. MR3102554

[14] Hansen, L.V., Thorarinsdottir, T.L., Gneiting, T. and Richards, D. (2015). Lévy particles: Modelling and simulating star-shaped random sets. *Adv. Appl. Probab.* To appear.

[15] Hitczenko, M. and Stein, M.L. (2012). Some theory for anisotropic processes on the sphere. *Stat. Methodol*. **9** 211–227. MR2863609

[16] Huang, C., Zhang, H. and Robeson, S.M. (2011). On the validity of commonly used covariance and variogram functions on the sphere. *Math. Geosci*. **43** 721–733. MR2824128

[17] Istas, J. (2005). Spherical and hyperbolic fractional Brownian motion. *Electron. Commun. Probab*. **10** 254–262 (electronic). MR2198600

[18] Istas, J. (2006). Karhunen–Loève expansion of spherical fractional Brownian motions. *Statist. Probab. Lett*. **76** 1578–1583. MR2245581

[19] Jones, R.H. (1963). Stochastic processes on a sphere. *Ann. Math. Statist*. **34** 213–218. MR0170378

[20] Jun, M. (2011). Non-stationary cross-covariance models for multivariate processes on a globe. *Scand. J. Stat*. **38** 726–747. MR2859747

[21] Jun, M. and Stein, M.L. (2007). An approach to producing space-time covariance functions on spheres. *Technometrics* **49** 468–479. MR2394558

[22] Jun, M. and Stein, M.L. (2008). Nonstationary covariance models for global data. *Ann. Appl. Stat.* **2** 1271–1289. MR2655659

[23] Lang, A. and Schwab, C. (2013). Isotropic Gaussian random fields on the sphere: Regularity, fast simulation, and stochastic partial differential equations. Preprint. Available at arXiv:1305.1170.

[24] Ma, C. (2012). Stationary and isotropic vector random fields on spheres. *Math. Geosci.* **44** 765–778. MR2956272

[25] Marinucci, D. and Peccati, G. (2011). *Random Fields on the Sphere: Representation, Limit Theorems and Cosmological Applications. London Mathematical Society Lecture Note Series* **389**. Cambridge: Cambridge Univ. Press. MR2840154

[26] Marinucci, D. and Vadlamani, S. (2013). High-frequency asymptotics for Lipschitz–Killing curvatures of excursion sets on the sphere. Preprint. Available at arXiv:1303.2456.

[27] Mikhaleva, T.L. and Piterbarg, V.I. (1997). On the distribution of the maximum of a Gaussian field with constant variance on a smooth manifold. *Theory Probab. Appl.* **41** 367–379.

[28] Obukhov, A.M. (1947). Statistically homogeneous fields on a sphere. *Uspekhi Math. Nauk* **2** 196–198.

[29] Oh, H.-S. and Li, T.-H. (2004). Estimation of global temperature fields from scattered observations by a spherical-wavelet-based spatially adaptive method. *J. R. Stat. Soc. Ser. B. Stat. Methodol.* **66** 221–238. MR2035768

[30] Park, M.G. and Sun, J. (1998). Tests in projection pursuit regression. *J. Statist. Plann. Inference* **75** 65–90. MR1671694

[31] Pickands, J. III (1969). Upcrossing probabilities for stationary Gaussian processes. *Trans. Amer. Math. Soc.* **145** 51–73. MR0250367

[32] Piterbarg, V.I. (1996). *Asymptotic Methods in the Theory of Gaussian Processes and Fields. Translations of Mathematical Monographs* **148**. Providence, RI: Amer. Math. Soc. MR1361884

[33] Potthoff, J. (2010). Sample properties of random fields III: Differentiability. *Commun. Stoch. Anal.* **4** 335–353. MR2668626

[34] Qualls, C. and Watanabe, H. (1973). Asymptotic properties of Gaussian random fields. *Trans. Amer. Math. Soc.* **177** 155–171. MR0322943

[35] Schoenberg, I.J. (1942). Positive definite functions on spheres. *Duke Math. J.* **9** 96–108. MR0005922

[36] Soubeyrand, S., Enjalbert, J. and Sache, I. (2008). Accounting for roughness of circular processes: Using Gaussian random processes to model the anisotropic spread of airborne plant disease. *Theoret. Population Bio.* **73** 92–103.

[37] Stein, M.L. (2007). Spatial variation of total column ozone on a global scale. *Ann. Appl. Stat.* **1** 191–210. MR2393847

[38] Sun, J. (1991). Significance levels in exploratory projection pursuit. *Biometrika* **78** 759–769. MR1147012

[39] Sun, J. (1993). Tail probabilities of the maxima of Gaussian random fields. *Ann. Probab.* **21** 34–71. MR1207215

[40] Sun, J. (2001). Multiple comparisons for a large number of parameters. *Biom. J.* **43** 627–643. MR1859377

[41] Szegő, G. (1975). *Orthogonal Polynomials*, 4th ed. Providence, RI: Amer. Math. Soc. MR0372517

[42] Tebaldi, C. and Sansó, B. (2009). Joint projections of temperature and precipitation change from multiple climate models: A hierarchical Bayesian approach. *J. Roy. Statist. Soc. Ser. A* **172** 83–106. MR2655606

[43] Yadrenko, A.M. (1983). *Spectral Theory of Random Fields*. New York: Optimization Software. MR0697386

[44] Yaglom, A.M. (1961). Second-order homogeneous random fields. In *Proc*. 4*th Berkeley Sympos*. *Math*. *Statist*. *and Prob*., *Vol*. *II* 593–622. Berkeley, CA: Univ. California Press. MR0146880
[45] Zuo, Y. (2003). Projection-based depth functions and associated medians. *Ann*. *Statist*. **31** 1460–1490. MR2012822

Received January 2014 and revised June 2014