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Sharp ellipticity conditions for ballistic behavior of random walks in random environment

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We sharpen ellipticity criteria for random walks in i.i.d. random environments introduced by Campos and Ramírez which ensure ballistic behavior. Furthermore, we construct new examples of random environments for which the walk satisfies the polynomial ballisticity criteria of Berger, Drewitz and Ramírez. As a corollary, we can exhibit a new range of values for the parameters of Dirichlet random environments in dimension $d=2$ under which the corresponding random walk is ballistic.

Keywords: Dirichlet distribution; max-flow min-cut theorem; random walk in random environment; reinforced random walks

1. Introduction

We continue the study initiated in [3] sharpening the ellipticity criteria which ensure ballistic behavior of random walks in random environment. Furthermore, we apply our results to exhibit a new class of ballistic random walks in Dirichlet random environments in dimension $d=2$.

For $x \in \mathbb{R}^d$, denote by $|x|_1$ and $|x|_2$ its L^1 and L^2 norm, respectively. Call $U := \{e \in \mathbb{Z}^d\colon |e|_1 = 1\} = \{e_1, \ldots, e_{2d}\}$ the canonical vectors with the convention that $e_{d+i} = -e_i$ for $1 \le i \le d$. We set $\mathcal{P} := \{p(e)\colon p(e) \ge 0, \sum_{e\in U} p(e) = 1\}$.

An environment is an element $\omega := \{\omega(x)\colon x \in \mathbb{Z}^d\}$ of the environment space $\Omega := \mathcal{P}^{\mathbb{Z}^d}$. We denote the components of $\omega(x)$ by $\omega(x, e)$.

The random walk in the environment ω starting from x is the Markov chain $\{X_n\colon n \ge 0\}$ in $\mathbb{Z}^d$ with law $P_{x,\omega}$ defined by the condition $P_{x,\omega}(X_0 = x) = 1$ and the transition probabilities

$$P_{x,\omega}(X_{n+1} = x + e | X_n = x) = \omega(x, e)$$

for each $x \in \mathbb{Z}^d$ and $e \in U$.

Let $\mathbb{P}$ be a probability measure defined on the environment space Ω endowed with its Borel σ-algebra, such that $\{\omega(x)\colon x \in \mathbb{Z}^d\}$ is i.i.d. under $\mathbb{P}$. We call $P_{x,\omega}$ the quenched law of the random walk in random environment (RWRE) starting from x, and $P_x := \int P_{x,\omega}\, \mathrm{d}\mathbb{P}(\omega)$ the averaged or annealed law of the RWRE starting from x.

1350-7265

The law $\mathbb{P}$ is said to be elliptic if for every $x \in \mathbb{Z}^d$ and $e \in U$, $\mathbb{P}(\omega(x,e) > 0) = 1$. We say that $\mathbb{P}$ is uniformly elliptic if there exists a constant $\gamma > 0$ such that for every $x \in \mathbb{Z}^d$ and $e \in U$, $\mathbb{P}(\omega(x,e) \geq \gamma) = 1$.

Given $l \in \mathbb{S}^{d-1}$ we say that the RWRE is transient in direction l if

$$P_0(A_l) = 1,$$

with

$$A_l := \Big\{ \lim_{n\to\infty} X_n \cdot l = \infty \Big\}.$$

Furthermore, it is ballistic in direction l if P_0-a.s.

$$\liminf_{n\to\infty} \frac{X_n \cdot l}{n} > 0.$$

Given $\Lambda \subset \mathbb{Z}^d$, we denote its outer boundary by

$$\partial\Lambda := \big\{ x \notin \Lambda \colon\ |x-y|_1 = 1 \text{ for some } y \in \Lambda \big\}.$$

We denote any nearest neighbour path with n steps joining two points $x, y \in \mathbb{Z}^d$ by $(x_1, x_2, \ldots, x_n)$, where $x_1 = x$ and $x_n = y$.

1.1. Polynomial condition, ellipticity condition

In [1], Berger, Drewitz and Ramírez introduced a polynomial ballisticity condition within the uniformly elliptic context, which was later extended to the elliptic case by Campos and Ramírez in [3]. This condition will be of interest for our results. It is effective, in the sense that it can a priori be verified explicitly for a given environment.

To define it, we need for each $L, \tilde{L} > 0$ and $l \in \mathbb{S}^{d-1}$ to consider the box

$$B_{l,L,\tilde{L}} := R\big((-L, L) \times (-\tilde{L}, \tilde{L})^{d-1}\big) \cap \mathbb{Z}^d,$$

where R is a rotation of $\mathbb{R}^d$ that verifies $R(e_1) = l$.

For each subset $A \subset \mathbb{Z}^d$ we denote the first exit time from the set A as

$$T_A := \min\{n \geq 0\colon\ X_n \notin A\}.$$

Define also the half space

$$H_l := \big\{ l' \in \mathbb{R}^d \colon\ l' \cdot l \geq 0 \big\}.$$

We now choose $\alpha > 0$ such that

$$\eta_\alpha := \max_{e \in H_l \cap \mathbb{Z}^d} \mathbb{E}\left(\frac{1}{\omega(0,e)^\alpha} \right) < \infty$$

and let

$$c_0 := \tfrac{2}{3}3^{120d^4+3000d((1/\alpha)\log\eta_\alpha)^2}. \tag{1.1}$$

Definition 1. *Given $M \geq 1$, we say that condition $(P)_M$ in direction l is satisfied (also written as $(P)_M|l)$ if there exists $L \geq c_0$ and $\tilde{L} \leq 70L^3$ such that one has the following upper bound for the probability that the walk does not exit the box $B_{l,L,\tilde{L}}$ through its front side*:

$$P_0(X_{T_{B_{l,L,\tilde{L}}}} \cdot l < L) \leq \frac{1}{L^M}.$$

This condition has proven to be useful in the uniformly elliptic case. Indeed, $(P)_M$ for $M \geq 15d+5$ implies ballisticity (see [1]).

For non-uniformly elliptic environments in dimensions $d \geq 2$, there exist elliptic random walks which are transient in a given direction but not ballistic in that direction (see, e.g., Sabot–Tournier [10], Bouchet [2]). In [3], Campos and Ramírez introduced ellipticity criteria on the law of the environment which ensure ballisticity if condition $(P)_M$ is satisfied for $M \geq 15d+5$ In this article we will sharpen this ellipticity criteria.

Remark 1. In definition (1.6) of [3], an incorrect value of the constant c_0 is given, different from the definition in (1.1). Nevertheless, it is straightforward to check that the argument of Section 3.1 of [3] showing that $(P)_M$ implies $(T)_{\gamma_L}$ does not change.

Let us first recall the ellipticity condition of [3]. For all $M \geq 1$, the polynomial condition $(P)_M$ implies the existence of an asymptotic direction (see, e.g., Simenhaus [11]): there exists $\hat{v} \in \mathbb{S}^{d-1}$ such that P_0-a.s.,

$$\lim_{n\to\infty} \frac{X_n}{|X_n|_2} = \hat{v}.$$

We call $\hat{v}$ the asymptotic direction.

Definition 2. *Let $\beta > 0$. We say that the law of the environment satisfies the ellipticity condition $(E')_\beta$ if there exists an $\{\alpha(e)\colon e \in U\} \in (0,\infty)^{2d}$ such that*

$$\kappa\big(\{\alpha(e)\colon e \in U\}\big) := 2\sum_{e'} \alpha(e') - \max_{e\in U}\big(\alpha(e) + \alpha(-e)\big) > \beta \tag{1.2}$$

and for every $e \in U$

$$\mathbb{E}\left(\prod_e \omega(0,e)^{-\alpha(e)}\right) < \infty. \tag{1.3}$$

Furthermore, when $\hat{v}$ exists, we say that the ellipticity condition $(E')_\beta$ is satisfied towards the asymptotic direction if there exists an $\{\alpha(e)\colon e \in U\}$ satisfying (1.2) and (1.3) and such that there exists $\alpha_1 > 0$ that satisfies $\alpha(e) = \alpha_1$ for $e \in H_{\hat{v}} \cap U$ while $\alpha(e) \leq \alpha_1$ for $e \in U \setminus H_{\hat{v}}$.

Remark 2. In [3], (1.3) is replaced by $\mathbb{E}(\prod_{e'\neq e}\omega(0,e')^{-\alpha(e')})<\infty$. Those two conditions are in fact equivalent. The direct implication is straightforward. And since $1\leq\sum_{e\in U}\mathbb{1}_{\{\omega(0,e)\geq 1/(2d)\}}$, we get

$$\mathbb{E}\left(\prod_{e}\omega(0,e)^{-\alpha(e)}\right)\leq\sum_{e\in U}(2d)^{\alpha(e)}\mathbb{E}\left(\prod_{e'\neq e}\omega\big(0,e'\big)^{-\alpha(e')}\right)<\infty.$$

This gives the reverse implication.

Remark 3. Knowing the existence of $\hat{v}$ does not mean that we know its value. In most cases, $\hat{v}$ is found to be inaccessible. A notable exception is the result of Tournier [15] that gives the value of $\hat{v}$ in the case of random walks in Dirichlet environments.

1.2. Ballisticity results

Our main results are a generalization of Theorems 1.2 and 1.3 of [3] where we remove the "towards the asymptotic direction" condition of Theorems 1.2 and 1.3 of [3].

Let $\tau_1^{\hat{v}}$ be the first renewal time in the direction $\hat{v}$, its precise definition is recalled in the next section. We prove the following tail estimate on renewal times, which improves Proposition 5.1 of [3].

Theorem 1. *Let $l\in\mathbb{S}^{d-1}$, $\beta>0$ and $M\geq 15d+5$. Assume that $(P)_M|l$ is satisfied and that $(E')_\beta$ holds (cf. (1.2), (1.3)). Then*

$$\limsup_{u\to\infty}(\log u)^{-1}\log P_0\big(\tau_1^{\hat{v}}>u\big)\leq-\beta.$$

The condition (E'_β) is sharp in a sense that is made precise in Remark 4 below. Together with previous results of Sznitman, Zerner, Seppäläinen and Rassoul-Agha, cf. [8,12,14,18], it implies the following.

Theorem 2 (Law of large numbers). *Consider a random walk in an i.i.d. environment in dimensions $d\geq 2$. Let $l\in\mathbb{S}^{d-1}$ and $M\geq 15d+5$. Assume that the random walk satisfies condition $(P)_M|l$ and the ellipticity condition $(E')_1$. Then the random walk is ballistic in direction l and there is a $v\in\mathbb{R}^d$, $v\neq 0$ such that*

$$\lim_{n\to\infty}\frac{X_n}{n}=v,\qquad P_0\text{-}a.s.$$

Theorem 3 (Central limit theorems). *Consider a random walk in an i.i.d. environment in dimensions $d\geq 2$. Let $l\in\mathbb{S}^{d-1}$ and $M\geq 15d+5$. Assume that the random walk satisfies condition $(P)_M|l$.*

(a) (*Annealed central limit theorem.*) *If $(E')_2$ is satisfied, then*

$$\varepsilon^{1/2}\big(X_{[\varepsilon^{-1}n]}-\big[\varepsilon^{-1}n\big]v\big)$$

converges in law under P_0 as $\varepsilon \to 0$ to a Brownian motion with non-degenerate covariance matrix.

(b) (*Quenched central limit theorem.*) *If $(E')_{176d}$ is satisfied, then $\mathbb{P}$-a.s. we have that*

$$\varepsilon^{1/2}\big(X_{[\varepsilon^{-1}n]} - \big[\varepsilon^{-1}n\big]v\big)$$

converges in law under $P_{0,\omega}$ as $\varepsilon \to 0$ to a Brownian motion with non-degenerate covariance matrix.

Removing the "towards the asymptotic direction" is a real improvement: in Section 1.3.2, we will give some examples of environments (in the class of Dirichlet environments) that satisfy (E'_β) but not towards the asymptotic direction. For those environments, our new theorems allow to prove a LLN or CLT. Furthermore, our final goal would be to get a ballisticity condition that depends only locally on the environment (i.e., a condition that depends only on the law of the environment at one point). Condition $(E')_\beta$ is local, whereas $(E')_\beta$ towards the asymptotic dimension is not: removing the "towards the asymptotic direction" is then a first step in this direction. Ideally, we would also need to get rid of condition $(P)_M|l$, that is not local either. This is a much more difficult problem, not solved even in the uniformly elliptic case.

Remark 4. The condition of Theorem 1 is sharp under the following assumption on the tail behavior of the environment at one site: there exists some $(\beta_e)_{e\in U}$, $\beta_e \ge 0$, and a positive constant $C > 1$ such that for all $e \in U$

$$C^{-1}\Big(\prod_{e'\in U, e'\neq e} t_{e'}^{\beta_{e'}}\Big) \le \mathbb{P}\big(\omega(0,e') \le t_{e'}, \forall e' \in U, e' \neq e\big) \le C\Big(\prod_{e'\in U, e'\neq e} t_{e'}^{\beta_{e'}}\Big) \tag{1.4}$$

for all $(t_{e'})_{e'\in U\setminus\{e\}}$, $0 \le t_{e'} \le 1$. Indeed, in this case we easily see that $(E')_\beta$ is satisfied if and only if $\beta < 2\sum_{e'}\beta_{e'} - \max_{e\in U}(\beta_e + \beta_{-e})$. On the other hand, if $\beta \ge 2\sum_{e'}\beta_{e'} - \max_{e\in U}(\beta_e + \beta_{-e})$ then $\mathbb{E}((\tau_1^{\hat{v}})^\beta) = \infty$. Indeed, consider a direction e_0 which realizes the maximum in $\max_{e\in U}(\beta_e + \beta_{-e})$ and set $K = \{0, e_0\}$. We denote by $\partial_+ K$ the set of edges that exit the set K composed of the edges $\{(0,e)\}_{e\neq e_0}$ and $\{(e_0,e)\}_{e\neq -e_0}$. For small $t > 0$, under the condition that $\omega(x, y-x) \le t$ for all $(x,y) \in \partial_+ K$ we have $P_{0,\omega}(T_K \ge n) \ge (1-(2d-1)t)^n$. Hence,

$$\begin{aligned} P_0(T_K \ge n) &\ge \big(1-(2d-1)/n\big)^n \mathbb{P}\big(\omega(x,y-x) \le 1/n, \forall (x,y) \in \partial_+ K\big) \\ &\ge \big(1-(2d-1)/n\big)^n C^{-1} n^{-(\sum_{e'}\beta_{e'} - (\beta_{e_0}+\beta_{-e_0}))} \end{aligned}$$

which implies that $E_0(T_K^\beta) = \infty$. Since T_K is clearly a lower bound for the first renewal time it gives the result.

Dirichlet environment (cf. the next section) is a typical example of environment that satisfies condition (1.4).

Remark 5. Theorem 1.1 of [3] states that for i.i.d. environments in dimensions $d \ge 2$ satisfying the ellipticity condition $(E')_0$, the polynomial condition $(P)_M|l$ (for $l \in \mathbb{S}^{d-1}$ and $M \ge 15d+5$) is equivalent to Sznitman's condition $(T')|l$ (see, e.g., [13] for the definition). We can therefore replace $(P)_M|l$ by $(T')|l$ in the statements of Theorems 2 and 3.

1.3. New examples of random walks satisfying the polynomial condition

In this article, we also introduce new examples of RWRE in environments which are not uniformly elliptic and which satisfy the polynomial condition $(P)_M$ for $M \geq 15d + 5$. In Section 1.3.1, we prove the polynomial condition for a subset of marginal nestling random walks, including a particular two-dimensional environment introduced by Campos and Ramírez in [3]. In Section 1.3.2, we prove the polynomial condition for a class of random walks in Dirichlet random environments which do not necessarily satisfy Kalikow's condition. In both cases, we present the case of environments for which our new Theorems 2 and 3 prove necessary to study the behaviour of the walks.

1.3.1. *Example within the class of marginal nestling random walks*

Following Sznitman [12], we say that a law $\mathbb{P}$ on Ω is *marginal nestling* if the convex hull K_o of the support of the law of

$$d(0, \omega) := \sum_{e \in U} \omega(0, e)e$$

is such that $0 \in \partial K_o$. We will prove in Section 4 that a certain subset of the marginal nestling laws satisfies the polynomial condition.

Theorem 4. *Consider an elliptic law $\mathbb{P}$ under which $\{\omega(x) \colon x \in \mathbb{Z}^d\}$ are i.i.d. Assume that there exists an $r > 1$ such that $\omega(0, e_1) = r\omega(0, e_{1+d})$. Then the polynomial condition $(P)_M|e_1$ is satisfied for some $M \geq 15d + 5$.*

Remark 6. This theorem is valid for all i.i.d. elliptic environments satisfying $\omega(0, e_1) = r\omega(0, e_{1+d})$, including uniformly elliptic environments. However, the environments are marginal nestling only in the non-uniformly elliptic case.

The above result includes an example suggested in [3], by Campos and Ramírez, of an environment which satisfies the polynomial condition and for which the random walk is directionally transient but not ballistic. They showed that on this environment, $(E')_\alpha$ is satisfied for α smaller but arbitrarily close to 1, and that the walk is transient but not ballistic in a given direction. The proof that this environment satisfies the polynomial condition was left for a future work.

Let us define the environment introduced in [3]. Let φ be any random variable taking values on the interval $(0, 1/4)$ and such that the expected value of $\varphi^{-1/2}$ is infinite, while for every $\varepsilon > 0$, the expected value of $\varphi^{-(1/2-\varepsilon)}$ is finite. Let X be a Bernoulli random variable of parameter $1/2$. We now define $\omega(0, e_1) = 2\varphi$, $\omega(0, -e_1) = \varphi$, $\omega(0, e_2) = X\varphi + (1 - X)(1 - 4\varphi)$ and $\omega(0, -e_2) = X(1 - 4\varphi) + (1 - X)\varphi$.

For every $\varepsilon > 0$, this environment satisfies $(E')_{1-\varepsilon}$: traps can appear because the random walk can get caught on two edges of the type $(x, e_2), (x + e_2, -e_2)$. Furthermore, it is transient in direction e_1 but not ballistic in that direction.

1.3.2. *Examples within the class of Dirichlet random environments*

Random Walks in Dirichlet Environment (RWDE) are interesting because of the analytical simplifications they offer, and because of their link with reinforced random walks. Indeed, the annealed law of a RWDE corresponds to the law of a linearly directed-edge reinforced random walk [4,7].

Given a family of positive weights $(\beta_1, \ldots, \beta_{2d})$, a random i.i.d. Dirichlet environment is a law on Ω constructed by choosing independently at each site $x \in \mathbb{Z}^d$ the values of $(\omega(x, e_i))_{i \in [\![1,2d]\!]}$ according to a Dirichlet law with parameters $(\beta_1, \ldots, \beta_{2d})$. That is, at each site we choose independently a law with density

$$\frac{\Gamma(\sum_{i=1}^{2d} \beta_i)}{\prod_{i=1}^{2d} \Gamma(\beta_i)} \left(\prod_{i=1}^{2d} x_i^{\beta_i - 1} \right) \mathrm{d}x_1 \cdots \mathrm{d}x_{2d-1}$$

on the simplex $\{(x_1, \ldots, x_{2d}) \in]0,1]^{2d}, \sum_{i=1}^{2d} x_i = 1\}$. Here Γ denotes the Gamma function $\Gamma(\beta) = \int_0^\infty t^{\beta-1} \mathrm{e}^{-t}\, \mathrm{d}t$, and $\mathrm{d}x_1 \cdots \mathrm{d}x_{2d-1}$ represents the image of the Lebesgue measure on $\mathbb{R}^{2d-1}$ by the application $(x_1, \ldots, x_{2d-1}) \to (x_1, \ldots, x_{2d-1}, 1 - x_1 - \cdots - x_{2d-1})$. Obviously, the law does not depend on the specific role of x_{2d}.

Remark 7. Given a Dirichlet law of parameters $(\beta_1, \ldots, \beta_{2d})$, the ellipticity condition $(E')_\beta$ is satisfied if and only if

$$\kappa\big((\beta_1, \ldots, \beta_{2d})\big) = 2\left(\sum_{i=1}^{2d} \beta_i \right) - \max_{i=1,\ldots,d} (\beta_i + \beta_{i+d}) > \beta.$$

As stated in Remark 4, this ellipticity condition is optimal to get Theorem 1 in the case of Dirichlet environments. Remark that for Dirichlet environments, for all $\beta > 0$, $(E')_\beta$ is much sharper that $(E')_\beta$ towards the asymptotic direction. Indeed, the result of Tournier [15] gives us the explicit value of $\hat{v}$ in the case of Dirichlet laws: $\hat{v} = \frac{\sum_{i=1}^{2d} \beta_i e_i}{\| \sum_{i=1}^{2d} \beta_i e_i \|}$. Without loss of generality, we can assume that $\beta_i \geq \beta_{i+d}$ for $1 \leq i \leq d$. This implies that $e_i \cdot \hat{v} \geq 0$ for $1 \leq i \leq d$. If we define $\tilde{\beta}_i := \min_{1 \leq j \leq d} \beta_j$ and $\tilde{\beta}_{i+d} := \min(\tilde{\beta}_i, \beta_{i+d})$ for $1 \leq i \leq d$, we can see that $(E')_\beta$ is satisfied towards the asymptotic direction if and only if $\kappa(\{\tilde{\beta}_i\colon 1 \leq i \leq 2d\}) > \beta$.

In the case of RWDE, it has been proved that Kalikow's condition, and thus the (T') condition, is satisfied whenever

$$\sum_{i=1}^{d} |\beta_i - \beta_{i+d}| > 1 \tag{1.5}$$

(see Enriquez and Sabot in [5] and Tournier in [16]). The characterization of Kalikow's condition in terms of the parameters of a RWDE remains an open question. On the other hand, we believe that for RWDE condition (T') is satisfied if and only if $\max_{1 \leq i \leq d} |\beta_i - \beta_{i+d}| > 0$. Nevertheless, in this article we are able to prove the following result.

Theorem 5. *Let $\beta_1, \beta_2, \dots, \beta_d, \beta_{d+2}, \dots, \beta_{2d}$ be fixed positive numbers. Then, there exists an $\varepsilon \in (0,1)$ depending on these numbers such that if β_{1+d} is chosen so that $\beta_{1+d} \le \varepsilon$, the Random Walk in Dirichlet Environment with parameters $(\beta_1, \dots, \beta_{2d})$ satisfies condition $(P)_M|e_1$ for $M \ge 15d + 5$.*

Theorem 5 gives as a corollary new examples of RWDE which are ballistic in dimension $d = 2$ since they do not correspond to ranges of the parameters satisfying condition (1.5) of Tournier [16] and Sabot and Enriquez [5] (see the following remark for the case $d \ge 3$). Indeed, by Theorem 2, if

$$2\sum_{i=1}^{2d} \beta_i - \max_{1\le i\le d} (\beta_i + \beta_{i+d}) > 1$$

and one of the parameters $\{\beta_i\colon 1 \le i \le d\}$ is small enough, the walk is ballistic.

Remark 8. In dimension $d \ge 3$, in [2,9], precise conditions on the existence of an invariant measure viewed from the particle absolutely continuous with respect to the law have been given; this allows to characterize completely the parameters for which there is ballisticity, but it fails to give information on the (T') condition and on the tails of renewal times. It also fails to give a CLT.

Theorem 3 then gives us annealed CLTs for Dirichlet laws when the parameters $(\beta_1, \dots, \beta_{2d})$ satisfy $2\sum_{i=1}^{2d} \beta_i - \max_{1\le i\le d}(\beta_i + \beta_{i+d}) > 2$ along with condition (1.5) or the hypothesis of Theorem 5.

Remark 9. For the Dirichlet laws in dimension $d = 2$ with parameters $(\beta_1, \dots, \beta_4)$ satisfying $2\sum_{i=1}^{4} \beta_i - \max_{1\le i\le 2}(\beta_i + \beta_{i+d}) > 1$, with one of the parameters $\{\beta_i\colon 1 \le i \le 4\}$ small enough, but for which $(E')_1$ is not satisfied toward the asymptotic direction, our Theorem 2 gives the ballisticity when the results of [3] would not have been enough.

For the Dirichlet laws in dimension $d \ge 2$ with parameters $(\beta_1, \dots, \beta_{2d})$ satisfying $2\sum_{i=1}^{2d} \beta_i - \max_{1\le i\le d}(\beta_i + \beta_{i+d}) > 2$, with condition (1.5) or the hypothesis of Theorem 5, but for which $(E')_2$ is not satisfied toward the asymptotic direction, our Theorem 3 gives the annealed CLT when the results of [3] would not have been enough.

This illustrates the relevance of having removed the "toward the asymptotic direction" hypothesis in Theorem 1.

2. First tools for the proofs

In this section, we will introduce some tools that will prove necessary for the proof of Theorem 1.

2.1. Regeneration times

The proofs in [3] are based on finding bounds on the regeneration times. We thus begin by giving the definition and some results about the regeneration times with respect to a fixed direction l. In the following, we suppose that the walk is transient in direction l.

We define $\{\theta_n\colon n \geq 1\}$ as the canonical time shift on $(\mathbb{Z}^d)^{\mathbb{N}}$. For $l \in \mathbb{S}^{d-1}$ and $u \geq 0$, we define the time

$$T_u^l := \min\{n \geq 0\colon X_n \cdot l \geq u\}.$$

Set

$$a > 2\sqrt{d} \tag{2.1}$$

and

$$D^l := \min\{n \geq 0\colon X_n \cdot l < X_0 \cdot l\}.$$

We define

$$\begin{aligned} S_0 &:= 0, \qquad M_0 := X_0 \cdot l, \\ S_1 &:= T^l_{M_0+a}, \qquad R_1 := D^l \circ \theta_{S_1} + S_1, \\ M_1 &:= \max\{X_n \cdot l\colon 0 \leq n \leq R_1\}, \end{aligned}$$

and recursively for $k \geq 1$,

$$\begin{aligned} S_{k+1} &:= T^l_{M_k+a}, \qquad R_{k+1} := D^l \circ \theta_{S_{k+1}} + S_{k+1}, \\ M_{k+1} &:= \max\{X_n \cdot l\colon 0 \leq n \leq R_{k+1}\}. \end{aligned}$$

The first regeneration time is then defined as

$$\tau_1 := \min\{k \geq 1\colon S_k < \infty, R_k = \infty\}.$$

We can now define recursively in n the $(n+1)$th regeneration time τ_{n+1} as $\tau_1(X_\cdot) + \tau_n(X_{\tau_1+\cdot} - X_{\tau_1})$. We will occasionally write $\tau_1^l, \tau_2^l, \dots$ to emphasize the dependence on the chosen direction.

Remark 10. The condition (2.1) on a is only necessary to prove the non-degeneracy of the covariance matrix of part (a) of Theorem 3.

It is a standard fact (see, e.g., Sznitman and Zerner [14]) to show that under the assumption of transience in direction l, the sequence $((\tau_1, X_{(\tau_1+\cdot)\wedge\tau_2} - X_{\tau_1}), (\tau_2 - \tau_1, X_{(\tau_2+\cdot)\wedge\tau_3} - X_{\tau_2}), \dots)$ is independent and (except for its first term) i.i.d. Its law is the same as the law of τ_1 with respect to the conditional probability measure $P_0(\cdot|D^l = \infty)$.

Those regeneration times are particularly useful to us because of the two following theorems.

Theorem 6 (Sznitman and Zerner [14], Zerner [18], Sznitman [12]). *Consider a RWRE in an elliptic i.i.d. environment. Let $l \in \mathbb{S}^{d-1}$ and assume that there is a neighbourhood V of l such that for every $l' \in V$ the random walk is transient in the direction l'. Then there is a deterministic v such that P_0-a.s.*

$$\lim_{n\to\infty} \frac{X_n}{n} = v.$$

Furthermore, the following are satisfied.

(a) *If* $E_0(\tau_1) < \infty$, *the walk is ballistic and* $v \neq 0$.
(b) *If* $E_0(\tau_1^2) < \infty$,

$$\varepsilon^{1/2}\big(X_{[\varepsilon^{-1}n]} - \big[\varepsilon^{-1}n\big]v\big)$$

converges in law under P_0 *to a Brownian motion with non-degenerate covariance matrix.*

Theorem 7 (Rassoul-Agha and Seppäläinen [8]). *Consider a RWRE in an elliptic i.i.d. environment. Take* $l \in \mathbb{S}^{d-1}$ *and let* τ_1 *be the corresponding regeneration time. Assume that*

$$E_0\big(\tau_1^p\big) < \infty,$$

for some $p > 176d$. *Then* $\mathbb{P}$-*a.s. we have that*

$$\varepsilon^{1/2}\big(X_{[\varepsilon^{-1}n]} - \big[\varepsilon^{-1}n\big]v\big)$$

converges in law under $P_{0,\omega}$ *to a Brownian motion with non-degenerate covariance matrix.*

2.2. Atypical quenched exit estimate

The proof of Theorem 1 is based on an atypical quenched exit estimate proved in [3]. We will also need this result, and thus recall it in this section. Let us first introduce some notations.

Without loss of generality, we can assume that e_1 is contained in the open half-space defined by the asymptotic direction so that

$$\hat{v} \cdot e_1 > 0.$$

We define the hyperplane:

$$H := \big\{x \in \mathbb{R}^d \colon x \cdot e_1 = 0\big\}.$$

Let $P := P_{\hat{v}}$ be the projection on the asymptotic direction along the hyperplane H defined for $z \in \mathbb{Z}^d$ by

$$P(z) := \left(\frac{z \cdot e_1}{\hat{v} \cdot e_1}\right)\hat{v},$$

and $Q := Q_l$ be the projection of z on H along $\hat{v}$ so that

$$Q(z) \colon = z - P(z).$$

Now, for $x \in \mathbb{Z}^d$, $\beta > 0$, $\rho > 0$ and $L > 0$, we define the tilted boxes with respect to the asymptotic direction $\hat{v}$ by:

$$B_{\beta,L}(x) := \big\{y \in \mathbb{Z}^d \text{ s.t. } -L^\beta < (y-x) \cdot e_1 < L \text{ and } \big\|Q(y-x)\big\|_\infty < \rho L^\beta\big\} \tag{2.2}$$

and their front boundary by

$$\partial^+ B_{\beta,L}(x) := \big\{y \in \partial B_{\beta,L}(x) \text{ s.t. } (y-x) \cdot e_1 = L\big\}.$$

We have the following.

Proposition 8 (Atypical Quenched Exit Estimate, Proposition 4.1 of [3]). *Assume there exists $\alpha > 0$ such that $\eta_\alpha := \max_{e\in U} \mathbb{E}((\frac{1}{\omega(0,e)})^\alpha) < \infty$. Take $M \geq 15d + 5$ such that $(P)_M|l$ is satisfied. Let $\beta_0 \in (1/2, 1)$, $\beta \in (\frac{\beta_0+1}{2}, 1)$ and $\zeta \in (0, \beta_0)$. Then, for each $\gamma > 0$ we have that*

$$\limsup_{L\to\infty} L^{-g(\beta_0,\beta,\zeta)} \log \mathbb{P}\big(P_{0,\omega}\big(X_{T_{B_{\beta,L}(0)}} \in \partial^+ B_{\beta,L}(0)\big) \leq \mathrm{e}^{-\gamma L^\beta}\big) < 0,$$

where

$$g(\beta_0, \beta, \zeta) := \min\big\{\beta + \zeta, 3\beta - 2 + (d-1)(\beta - \beta_0)\big\}.$$

2.3. Some results on flows

The main tool that enables us to improve the results of [3] is the use of flows and max-flow min-cut theorems. We need some definitions and properties that we will detail in this section. In the following, we consider a finite directed graph $G = (V, E)$, where V is the set of vertices and E is the set of edges. For all $e \in E$, we denote by $\underline{e}$ and $\overline{e}$ the vertices that are the tail and head of the edge e (the edge e goes from $\underline{e}$ to $\overline{e}$).

Definition 3. *We consider a finite directed graph $G = (V, E)$. A flow from a set $A \subset V$ to a set $Z \subset V$ is a non-negative function $\theta : E \to \mathbb{R}_+$ such that*:

- $\forall x \in (A \cup Z)^c$, $\operatorname{div}\theta(x) = 0$.
- $\forall x \in A$, $\operatorname{div}\theta(x) \geq 0$.
- $\forall x \in Z$, $\operatorname{div}\theta(x) \leq 0$.

Where the divergence operator is $\operatorname{div} : \mathbb{R}^E \to \mathbb{R}^V$ *such that for all $x \in V$,*

$$\operatorname{div}\theta(x) = \sum_{e\in E, \underline{e}=x} \theta(e) - \sum_{e\in E, \overline{e}=x} \theta(e).$$

A unit flow from A to Z is a flow such that $\sum_{x\in A} \operatorname{div}\theta(x) = 1$. (Then we have also $\sum_{x\in Z} \operatorname{div}\theta(x) = -1$.)

We will need the following generalized version of the max-flow min-cut theorem.

Proposition 9 (Proposition 1 of [9]). *Let $G = (V, E)$ be a finite directed graph. Let $(c(e))_{e\in E}$ be a set of non-negative reals (called capacities). Let x_0 be a vertex and $(p_x)_{x\in V}$ be a set of non-negative reals. There exists a non-negative function $\theta : E \to \mathbb{R}_+$ such that*

$$\operatorname{div}\theta = \sum_{x\in V} p_x(\delta_{x_0} - \delta_x) \tag{2.3}$$

and

$$\forall e \in E, \qquad \theta(e) \leq c(e) \tag{2.4}$$

if and only if for all subset $K \subset V$ containing x_0 we have

$$c(\partial_+ K) \geq \sum_{x \in K^c} p_x, \tag{2.5}$$

where $\partial_+ K = \{e \in E, \underline{e} \in K, \overline{e} \in K^c\}$ and $c(\partial_+ K) = \sum_{e \in \partial_+ K} c(e)$. The same is true if we restrict the condition (2.5) to the subsets K such that any $y \in K$ can be reached from x_0 following a directed path in K.

We will give here an idea of the proof, that explains why we call this result a generalized version of the classical max-flow min-cut theorem. The complete proof can be found in [9].

Idea of the proof. If θ satisfies (2.3) and (2.4), then

$$\sum_{e, \underline{e} \in K, \overline{e} \in K^c} \theta(e) - \sum_{e, \overline{e} \in K, \underline{e} \in K^c} \theta(e) = \sum_{x \in K} \operatorname{div} \theta(x) = \sum_{x \in K^c} p_x.$$

It implies (2.5) by (2.4) and positivity of θ.

The reversed implication is an easy consequence of the classical max-flow min-cut theorem on finite directed graphs (see, e.g., [6] Section 3.1). If $(c(e))_{e \in E}$ satisfies (2.5), we consider the new graph $\tilde{G} = (V \cup \{\delta\}, \tilde{E})$, where

$$\tilde{E} = E \cup \{(x, \delta), x \in V\}.$$

We define a new set of capacities $(\tilde{c}(e))_{e \in \tilde{E}}$ where $c(e) = \tilde{c}(e)$ for $e \in E$ and $\tilde{c}((x, \delta)) = p_x$. The strategy is to apply the max-flow min-cut theorem with capacities $\tilde{c}$ and with source x_0 and sink δ. It gives a flow $\tilde{\theta}$ on $\tilde{G}$ between x_0 and δ with strength $\sum_{x \in V} p_x$ and such that $\tilde{\theta} \leq \tilde{c}$. The function θ obtained by restriction of $\tilde{\theta}$ to E satisfies (2.4) and (2.3). $\square$

For the proof of Theorem 1, we will consider the oriented graph $(\mathbb{Z}^d, E_{\mathbb{Z}^d})$ where $E_{\mathbb{Z}^d} := \{(x, y) \in (\mathbb{Z}^d)^2 \text{ s.t. } |x - y|_1 = 1\}$. This graph is not finite, but we will only consider flows with compact support ($\theta(e) = 0$ for all e except in a finite subset of $E_{\mathbb{Z}^d}$). We can then proceed as if the graph were finite, and use the previous definition and proposition.

3. Proof of Theorem 1

We will prove Theorem 1 using the atypical quenched exit estimate Proposition 8. Let us give a rough idea of the proof. We first show that the event $\{\tau_1 > u\}$ is concentrated on the event that the random walk does not exit a box of side $(C \log u)^{1/\beta}$, for an appropriate choice of C, before time u. Now on this last event, necessarily, the walk must visit some point of this box at least $N_u := u/(C \log u)^{d/\beta}$ times. But due to Proposition 8 and the strong Markov property, the probability that this point is visited N_u times is less than $(1 - \frac{1}{u^{1-\varepsilon}})^{u/(C \log u)^{d/\beta}}$, for some $\varepsilon > 0$ which depends on the choice of C. This last quantity tends quickly to 0, and then the dominant term bounding $P_0(\tau_1 > u)$ will be the probability to exit the box of side $(C\log u)^{1/\beta}$ before time u.

Let $l \in \mathbb{S}^{d-1}$, $\beta > 0$ and $M \geq 15d + 5$. Assume that $(P)_M|l$ is satisfied and that $(E')_\beta$ holds.

Let us take a rotation $\hat{R}$ such that $\hat{R}(e_1) = \hat{v}$. We fix $\beta' \in (\frac{5}{6}, 1)$, $M > 0$ and for simplicity we will write τ_1 instead of $\tau_1^{\hat{v}}$.

For $u > 1$, take

$$L = L(u) := \left(\frac{1}{4M\sqrt{d}}\right)^{1/\beta'} (\log u)^{1/\beta'},$$

$$C_L := \hat{R}\left(\left[-\frac{L}{2(\hat{v}\cdot e_1)}, \frac{L}{2(\hat{v}\cdot e_1)}\right]^d\right) \cap \mathbb{Z}^d.$$

Following the proof of Proposition 5.1 in [3], we write

$$P_0(\tau_1 > u) \leq P_0(\tau_1 > u, T_{C_{L(u)}} \leq \tau_1) + \mathbb{E}\big(F_1^c, P_{0,\omega}(T_{C_{L(u)}} > u)\big) + \mathbb{P}(F_1),$$

with

$$F_1 := \left\{\omega \in \Omega\colon\ t_\omega(C_{L(u)}) > \frac{u}{(\log u)^{1/\beta'}}\right\}$$

and

$$t_\omega(A) := \min\left\{n \geq 0\colon\ \sup_x P_{x,\omega}(T_A > n) \leq \frac{1}{2}\right\}.$$

As in [3], the term $P_0(\tau_1 > u, T_{C_{L(u)}} \leq \tau_1)$ is bounded thanks to condition $(P)_M|l$, and the term $\mathbb{E}(F_1^c, P_{0,\omega}(T_{C_{L(u)}} > u))$ is bounded thanks to the strong Markov property. This part of the original proof is not modified, so we will not give more details here. It gives the existence for every $\gamma \in (\beta', 1)$ of a constant $c > 0$ such that:

$$P_0(\tau_1 > u) \leq \frac{\mathrm{e}^{-cL(u)^\gamma}}{c} + \left(\frac{1}{2}\right)^{\lfloor(\log u)^{1/\beta'}\rfloor} + \mathbb{P}(F_1).$$

It only remains to show that we can find a constant $C > 0$ such that $\mathbb{P}(F_1) \leq Cu^{-\beta}$ for u big enough.

For each $\omega \in \Omega$, still as in [3], there exists $x_0 \in C_{L(u)}$ such that

$$P_{x_0,\omega}(\tilde{H}_{x_0} > T_{C_{L(u)}}) \leq \frac{2|C_{L(u)}|}{t_\omega(C_{L(u)})},$$

where for $y \in \mathbb{Z}^d$, $\tilde{H}_y = \min\{n \geq 1 : X_n = y\}$. It gives

$$\mathbb{P}(F_1) \leq \mathbb{P}\left(\omega \in \Omega \text{ s.t. } \exists x_0 \in C_{L(u)} \text{ s.t. } P_{x_0,\omega}(\tilde{H}_{x_0} > T_{C_{L(u)}}) \leq \frac{2(\log u)^{1/\beta'}}{u}|C_{L(u)}|\right).$$

We define for each point $x \in C_{L(u)}$ a point y_x, closest from $x + 2\frac{L^{\beta'}}{\hat{v}\cdot e_1}\hat{v}$. To bound $\mathbb{P}(F_1)$, we will need paths that go from x to y_x with probability big enough and the atypical quenched exit estimate (Proposition 8).

Define:

$$N := \frac{|\hat{v}| \log u}{2M\sqrt{d}(\hat{v}\cdot e_1)}.$$

It is straightforward that

$$N - 1 \le |y_x - x|_1 \le N + 1.$$

The following of the proof will be developed in three parts: first, we will construct unit flows $\theta_{i,x}$ going from $\{x, x+e_i\}$ to $\{y_x, y_x + e_i\}$, for all $x \in C_{L(u)}$. Then we will construct paths with those flows, and use the atypical quenched exit estimate to bound $\mathbb{P}(F_1)$ in the case that those paths are big enough. We will conclude by bounding the probability that the paths are not big enough.

3.1. Construction of the flows $\theta_{i,x}$

We consider the oriented graph $(\mathbb{Z}^d, E_{\mathbb{Z}^d})$ where $E_{\mathbb{Z}^d} := \{(x, y) \in (\mathbb{Z}^d)^2 \text{ s.t. } |x - y|_1 = 1\}$. We want to construct unit flows $\theta_{i,x}$ going from $\{x, x+e_i\}$ to $\{y_x, y_x+e_i\}$, for all $x \in C_{L(u)}$. But there are additional constraints, as we will need them to construct paths that have a probability big enough. The aim of this section is to prove the following proposition.

Proposition 10. *For all $x \in C_{L(u)}$, for all $\alpha_1, \dots, \alpha_{2d}$ positive constants, there exists $2d$ unit flows $\theta_{i,x}$: $E_{\mathbb{Z}^d} \to \mathbb{R}_+$, respectively, going from $\{x, x+e_i\}$ to $\{y_x, y_x+e_i\}$, such that*:

$$\forall e \in E_{\mathbb{Z}^d}, \qquad \theta_{i,x}(e) \le \frac{\alpha(e)}{\kappa_i}, \tag{3.1}$$

where $\kappa_i := 2\sum_{j=1}^{2d} \alpha_j - (\alpha_i + \alpha_{i+d})$, and $\alpha(e) := \alpha_j$ for e of the type (z, e_j).

Furthermore, we can construct $\theta_{i,x}$ with a finite support, and in a way that allows to find γ and $S \subset E_{\mathbb{Z}^d}$, $|S|$ independent of u, such that $\theta_{i,x}(e)\kappa_i \le \gamma < \alpha(e)$ for all $e \in S^c$.

We will construct the $\theta_{i,x}$ to prove their existences. For this, we need three steps. Let $B(x, R)$ be the box of $\mathbb{Z}^d$ of center x and radius R, and $B_i(x, R)$ be the same box, where the vertices x and $x + e_i$ are merged (and we suppress the edge between them). We note $E_{B(x,R)} := \{(x, y) \in E_{\mathbb{Z}^d} \cap (B(x, R))^2\}$ and $E_{B_i(x,R)} := \{(x, y) \in E_{\mathbb{Z}^d} \cap (B_i(x, R))^2\}$ the corresponding sets of edges. We will construct a unit flow in the graph $(B_i(x, R), E_{B_i(x,R)})$ from $\{x, x+e_i\}$ to $B_i(x, R)^c$, a unit flow in the graph $(B_i(y_x, R), E_{B_i(y_x,R)})$ from $B_i(y_x, R)^c$ to $\{y_x, y_x+e_i\}$, and then connect them. At each step, we will ensure that condition (3.1) is fulfilled.

First step. Construction of a unit flow from $\{x, x+e_i\}$ to $B_i(x, R)^c$:

Lemma 1. *Set $x \in C_{L(u)}$, and $\alpha_1, \dots, \alpha_{2d}$ positive constants. If $R \geq \frac{\max_i \kappa_i}{\min_j \alpha_j}$, there exists $2d$ unit flows $\theta_{i,x} : E_{B_i(x,R)} \to \mathbb{R}_+$ such that*:

$$\operatorname{div} \theta_{i,x} = \sum_{z \in \partial B_i(x,R)} \frac{1}{|\partial B_i(x,R)|} (\delta_x - \delta_z)$$

and

$$\forall e \in E_{B_i(x,R)}, \qquad \theta_{i,x}(e) \leq \frac{\alpha(e)}{\kappa_i},$$

where $\partial B_i(x,R) = \{z \in B_i(x,R)$ that has a neighbour in $B_i(x,R)^c\}$.

The divergence condition ensures that the flow will be a unit flow, that it goes from x, and that it leaves $B_i(x,R)$ uniformly on the boundary of the box.

Proof of Lemma 1. The result is a simple application of Proposition 9. We fix $x \in C_{L(u)}$ and i between 1 and $2d$. Define $p_z = \frac{1}{|\partial B_i(x,R)|}$ if $z \in \partial B_i(x,R)$, $p_z = 0$ if $z \notin \partial B_i(x,R)$.

To prove the result, we only have to check that $\forall K \subset B_i(x,R)$ containing x, $\sum_{e \in \partial_+ K} \frac{\alpha(e)}{\kappa_i} \geq \sum_{z \notin K} p_z$, where $\partial_+ K = \{e \in E_{B_i(x,R)}$ s.t. $\underline{e} \in K$ and $\overline{e} \notin K\}$.

We have two cases to examine:

- If $K \cap \partial B_i(x,R) = \varnothing$, $\sum_{z \notin K} p_z = 1$. We then need $\sum_{e \in \partial_+ K} \alpha(e) \geq \kappa_i$. For $K = \{x\}$, $\sum_{e \in \partial_+ K} \alpha(e) = \kappa_i$ as we merged x and $x + e_i$. For bigger K, we consider for all $j \neq i$ the paths $(x + ne_j)_{n \in \mathbb{N}}$ and for all $j \neq i + d$ the paths $(x + e_i + ne_j)_{n \in \mathbb{N}}$. They intersect the boundary of K in $2d + 1$ different points, and the exit directions give us the corresponding α_j, that sum to κ_i. It gives that $\sum_{e \in \partial_+ K} \alpha(e) \geq \kappa_i$.
- If $K \cap \partial B_i(x,R) \neq \varnothing$, $\sum_{z \notin K} p_z < 1$. As K contains a path from x to $\partial B_i(x,R)$, $\sum_{e \in \partial_+ K} \frac{\alpha(e)}{\kappa_i} \geq \frac{R \min_j(\alpha_j + \alpha_{j+d})}{\kappa_i}$. It is bigger than 1 thanks to the hypothesis on R. It gives the result. □

Second step. By the same way, we construct a flow $\theta_{i,x} : E_{B_i(y_x,R)} \to \mathbb{R}_+$ such that

$$\operatorname{div} \theta_{i,x} = \sum_{z \in \partial B_i(y_x,R)} \frac{1}{|\partial B_i(y_x,R)|} (\delta_z - \delta_{y_x})$$

and

$$\forall e \in E_{B_i(y_x,R)}, \qquad \theta_{i,x}(e) \leq \frac{\alpha(e)}{\kappa_i}.$$

Third step. We will join the flows on $E_{B_i(x,R)}$ and $E_{B_i(y_x,R)}$ with simple paths, to get a flow on $E_{\mathbb{Z}^d}$. Take $R \geq \frac{\max_i \kappa_i}{\min_j \alpha_j}$, and make sure that $\frac{1}{|\partial B(x,R)|} < \frac{\alpha(e)}{\kappa_i}$ for all $e \in E_{\mathbb{Z}^d}$ (always possible by taking R big enough, R depends only on the α_i and the dimension).

We can find $|\partial B(x,R)|$ simple paths $\pi_j \subset E_{\mathbb{Z}^d}$ satisfying:

- $\forall j$, π_j connects a point of $\partial B(x, R)$ to a point of $\partial B(y_x, R)$.
- $\forall j$, π_j stays outside of $B(x, R)$ and $B(y_x, R)$, except from the departure and arrival points.
- If two paths intersect, they perform jumps in different direction after the intersection (no edge is used by two paths). If (x, e_i) is in a path, then $(x + e_i, -e_i)$ is not in any path.
- The number of steps of each path is close to N: there exists constants K_1 and K_2 independent of u such that the length of π_j is smaller than $K_1 N + K_2$.

(E.g., we can use the paths $\pi^{(i,j)}$ page 45 of [3], and make them exit the ball $B(x, R)$ instead of $\{x, x + e_i\}$).

For all i, $\partial B_i(x, R) = \partial B(x, R)$ and $\partial B_i(y_x, R) = \partial B(y_x, R)$ as soon as $R > 1$. By construction, $-\operatorname{div} \theta_{i,x}(z_1) = \operatorname{div} \theta_{i,x}(z_2) = \frac{1}{|\partial B(x,R)|}$ for any $z_1 \in \partial B_i(x, R)$ and $z_2 \in \partial B_i(y_x, R)$. We can then join the flows of the first two steps by defining a flow $\theta_{i,x}(e) = \frac{1}{|\partial B(y_x,R)|}$ for all $e \in \pi_j$ (and 0 on all the other edges of $E_{\mathbb{Z}^d}$).

We have thus constructed a unit flow $\theta_{i,x}$ on $E_{\mathbb{Z}^d}$, from $\{x, x + e_i\}$ to $\{y_x, y_x + e_i\}$, satisfying (3.1) ((3.1) is satisfied on $E_{B_i(x,R)}$ and $E_{B_i(y_x,R)}$ as $R \geq \frac{\max_i \kappa_i}{\min_j \alpha_j}$ thanks to Lemma 1, and outside those balls as $\frac{1}{|\partial B(x,R)|} < \frac{\alpha(e)}{\kappa_i}$ for all $e \in E_{\mathbb{Z}^d}$). It concludes the proof of the first part of Proposition 10.

As $\theta_{i,x}(e) = 0$ out of the finite set $E_{B_i(x,R)} \cup E_{B_i(y_x,R)} \cup \{e \in \pi_j, 1 \leq j \leq |\partial B(x, R)|\}$, the flow has a finite support. And as we made sure that $\frac{1}{|\partial B(x,R)|} < \frac{\alpha(e)}{\kappa_i}$, we can take $S = B(x, R) \cup B(y_x, R)$ and $\gamma = \frac{\kappa_i}{|\partial B(x,R)|}$ to conclude the proof.

3.2. Bounds for $\mathbb{P}(F_1)$

We apply Proposition 10 for the $\alpha_1, \ldots, \alpha_{2d}$ of the definition of $(E')_\beta$ (see (1.2) and (1.3)). It gives flows $\theta_{i,x}$ on $E_{\mathbb{Z}^d}$, constructed as in the previous section.

We can decompose a given $\theta_{i,x}$ (for i and x fixed) in a finite set of weighted paths, each path starting from x or $x + e_i$ and arriving to y_x or $y_x + e_i$. It suffices to choose a path σ where the flow is always positive, to give it a weight $p_\sigma := \min_{e \in \sigma} \theta_{i,x}(e) > 0$ and to iterate with the new flow $\theta(e) := \theta_{i,x}(e) - p_\sigma \mathbb{1}_{e \in \sigma}$.

The weight p_σ of a path σ then satisfies: for all $e \in E_{\mathbb{Z}^d}$, $\theta_{i,x}(e) = \sum_{\sigma \text{ containing } e} p_\sigma$. As $\theta_{i,x}$ is a unit flow, we get $\sum_{\sigma \text{ path of } \theta_{i,x}} p_\sigma = 1$. We will use those weights in the next section, to prove that those paths are "big enough" with high probability.

We now introduce:

$$F_{2,i} = \left\{ \omega \in \Omega \text{ s.t. } \forall x \in C_{L(u)}, \exists \sigma \text{ path of } \theta_{i,x}, \omega_\sigma := \prod_{e \in \sigma} \omega_e > u^{(1/M)-1} \right\},$$

where we recall that M is the parameter for condition $(P)_M$, and

$$F_2 = \bigcap_{i=1}^{2d} F_{2,i}.$$

Define

$$F_3 := F_2 \cap \left\{ \omega \in \Omega \text{ s.t. } \exists x_0 \in C_{L(u)} \text{ s.t. } P_{x_0,\omega}(\tilde{H}_{x_0} > T_{C_{L(u)}}) \le \frac{2(\log u)^{1/\beta'}}{u} |C_{L(u)}| \right\}.$$

We get immediately:

$$\mathbb{P}(F_1) \le \mathbb{P}(F_3) + \mathbb{P}\big(F_2^c\big).$$

It gives two new terms to bound. We start by bounding $\mathbb{P}(F_3)$. For this we will use the same method as in [3]: on the event F_3, for all $1 \le i \le 2d$ we can use a path σ of $\theta_{i,x}$ to join x or $x + e_i$ to y_x or $y_x + e_i$. It gives:

$$\begin{aligned}\omega(x_0, e_i) u^{(1/M)-1} \min_{z \in \{y_{x_0}, y_{x_0}+e_i\}} P_{z,\omega}(T_{C_{L(u)}} < H_{x_0}) &\le P_{x_0,\omega}(T_{C_{L(u)}} < \tilde{H}_{x_0}) \\ &\le \frac{2(\log u)^{1/\beta'}}{u} |C_{L(u)}|,\end{aligned}$$

where the factor $\omega(x_0, e_i)$ corresponds to the probability of jumping from x to $x + e_i$, in the case where the path σ starts from $x + e_i$.

As $\sum_{i=1}^{2d} \omega(x_0, e_i) = 1$, it gives

$$u^{(1/M)-1} \min_{z \in V(y_{x_0})} P_{z,\omega}(T_{C_{L(u)}} < H_{x_0}) \le \frac{4d(\log u)^{1/\beta'}}{u} |C_{L(u)}|,$$

where $V(y_{x_0}) := \{y_{x_0}, (y_{x_0} + e_i)_{i=1,\dots,2d}\}$.

In particular, on F_3, we can see that for u large enough $V(y_{x_0}) \subset C_{L(u)}$. As a result, on F_3, we have for u large enough

$$\begin{aligned}\min_{z \in V(y_{x_0})} P_{z,\omega}(X_{T_{z+U_{\beta',L}}} \cdot e_1 > z \cdot e_1) &\le \min_{z \in V(y_{x_0})} P_{z,\omega}(T_{C_{L(u)}} < H_{x_0}) \\ &\le \frac{1}{u^{1/(2M)}} = \mathrm{e}^{-2\sqrt{d} L(u)^{\beta'}},\end{aligned}$$

where

$$U_{\beta',L} := \big\{ x \in \mathbb{Z}^d \colon -L^{\beta'} < x \cdot e_1 < L \big\}.$$

From this and using the translation invariance of the measure $\mathbb{P}$, we conclude that:

$$\begin{aligned}&\mathbb{P}\left(\exists x_0 \in C_{L(u)} \text{ s.t. } P_{x_0,\omega}(\tilde{H}_{x_0} > T_{C_{L(u)}}) \le \frac{4d(\log u)^{1/\beta'}}{u} |C_{L(u)}|, F_2 \right) \\ &\quad \le \mathbb{P}\Big(\exists x_0 \in C_{L(u)} \text{ s.t. } \min_{z \in V(y_{x_0})} P_{z,\omega}(X_{T_{z+U_{\beta',L}}} \cdot e_1 > z \cdot e_1) \le \mathrm{e}^{-2\sqrt{d} L(u)^{\beta'}} \Big) \\ &\quad \le (2d+1) |C_{L(u)}| \mathbb{P}\big(P_{0,\omega}(X_{T_{U_{\beta',L(u)}}} \cdot e_1 > 0) \le \mathrm{e}^{-2\sqrt{d} L(u)^{\beta'}} \big) \\ &\quad \le (2d+1) |C_{L(u)}| \mathbb{P}\big(P_{0,\omega}(X_{T_{B_{\beta',L(u)}}} \cdot e_1 > 0) \le \mathrm{e}^{-2\sqrt{d} L(u)^{\beta'}} \big),\end{aligned}$$

where the tilted box $B_{\beta',L(u)}$ is defined as in (2.2).

We conclude with the atypical quenched exit estimate (Proposition 8): there exists a constant $c>0$ such that for each $\beta_0 \in (\frac{1}{2},1)$ one has:

$$\mathbb{P}(F_3) \le \frac{1}{c}\mathrm{e}^{-cL(u)^{g(\beta_0,\beta',\zeta)}},$$

where $g(\beta_0,\beta',\zeta)$ is defined as in Proposition 8.

Note that for each $\beta' \in (\frac{5}{6},1)$ there exists a $\beta_0 \in (\frac{1}{2},\beta)$ such that for every $\zeta \in (0,\frac{1}{2})$ one has $g(\beta_0,\beta',\zeta) > \beta'$. Therefore, replacing L by its value, we proved that there exists $c>0$ such that:

$$\mathbb{P}(F_3) \le cu^{-\beta}.$$

3.3. Bound for $\mathbb{P}(F_2^c)$

To conclude the bound for $\mathbb{P}(F_1)$ and the proof of Theorem 1, it only remains to control $\mathbb{P}(F_2^c)$. It is in this section that we will use the conditions that were imposed on $\theta_{i,x}$ during the construction of the flows, as well as condition $(E')_\beta$

$$\begin{aligned}\mathbb{P}\big(F_2^c\big) &\le \sum_{i=1}^{2d}\mathbb{P}\big(F_{2,i}^c\big)\\ &\le \sum_{i=1}^{2d}\sum_{x\in C_{L(u)}}\mathbb{P}\big(\forall\sigma \text{ path of } \theta_{i,x}, \omega_\sigma \le u^{(1/M)-1}\big).\end{aligned}$$

As $\theta_{i,x}$ is a unit flow, if $\forall\sigma$ path of $\theta_{i,x}$, $\omega_\sigma \le u^{(1/M)-1}$ then:

$$\sum_{\sigma \text{ path of } \theta_{i,x}} p_\sigma\omega_\sigma \le u^{(1/M)-1}\sum_{\sigma \text{ path of } \theta_{i,x}} p_\sigma = u^{(1/M)-1}.$$

Jensen's inequality then gives:

$$\prod_{e\in E_{\mathbb{Z}^d}}\omega_e^{\theta_{i,x}(e)} = \prod_{\sigma \text{ path of } \theta_{i,x}}\omega_\sigma^{p_\sigma} \le u^{(1/M)-1}.$$

It allows to write:

$$\begin{aligned}\mathbb{P}\big(F_2^c\big) &\le \sum_{i=1}^{2d}\sum_{x\in C_{L(u)}}\mathbb{P}\Big(\prod_{e\in E_{\mathbb{Z}^d}}\omega_e^{\theta_{i,x}(e)} \le u^{(1/M)-1}\Big)\\ &\le \sum_{i=1}^{2d}\sum_{x\in C_{L(u)}}\frac{\mathbb{E}(\prod_{e\in E_{\mathbb{Z}^d}}\omega_e^{-\kappa_i\theta_{i,x}(e)})}{u^{-\kappa_i((1/M)-1)}}.\end{aligned}$$

We will use the integrability given by the flows to bound the expectations. The independence of the environment gives (for i and x fixed):

$$\mathbb{E}\Big(\prod_e \omega_e^{-\kappa_i\theta_{i,x}(e)}\Big) = \prod_{z\in\mathbb{Z}^d}\mathbb{E}\Big(\prod_{e \text{ s.t. } \underline{e}=z}\omega_e^{-\kappa_i\theta_{i,x}(e)}\Big)$$
$$= \prod_{z\in S}\mathbb{E}\Big(\prod_{e \text{ s.t. } \underline{e}=z}\omega_e^{-\kappa_i\theta_{i,x}(e)}\Big)\prod_{z\notin S}\mathbb{E}\Big(\prod_{e \text{ s.t. } \underline{e}=z}\omega_e^{-\kappa_i\theta_{i,x}(e)}\Big),$$

where we recall that $S = B(x,R)\cup B(y_x,R)$.

As $\theta_{i,x}$ satisfies (3.1), the ellipticity condition $(E')_\beta$ gives that each of the expectations $\mathbb{E}(\prod_{e \text{ s.t. } \underline{e}=z}\omega_e^{-\kappa_i\theta_{i,x}(e)})$ are finite.

By construction $|S|$ is finite and does not depend on u: $\prod_{z\in S}\mathbb{E}(\prod_{e \text{ s.t. } \underline{e}=z}\omega_e^{-\kappa_i\theta_{i,x}(e)})$ is a finite constant independent on u.

It remains to deal with the case of $z\notin S$. As we chose R to get $\theta_{i,x}(e)\kappa_i < \gamma$ for the edges outside S, and thanks to the bounds on the number of edges with positive flow (there is a finite number of paths, and each path has a bounded length), we have:

$$\prod_{z\notin S}\mathbb{E}\Big(\prod_{e \text{ s.t. } \underline{e}=z}\omega_e^{-\kappa_i\theta_{i,x}(e)}\Big) \le \mathbb{E}\Big(\prod_{e \text{ s.t. } \underline{e}=0}\omega_e^{-\gamma}\Big)^{c_1N+c_2},$$

where c_1 and c_2 are positive constants, independent of u. Then, putting all of those bounds together,

$$\mathbb{P}\big(F_2^c\big) \le \sum_{i=1}^{2d}\sum_{x\in C_{L(u)}} C_1C_2^{C_3N}u^{\kappa_i((1/M)-1)} \le \sum_{i=1}^{2d}\sum_{x\in C_{L(u)}} C_4u^{((C_5+\kappa_i)/M)-\kappa_i}$$
$$\le C_6(\log u)^{C_7}u^{(C_8/M)-\min_i\kappa_i},$$

where all the constants C_i are positive and do not depend on u. As Remark 5 tells us that we can choose M as large as we want, we can get $\frac{C_8}{M}$ as small as we want.

Then we can find a constant $C>0$ such that $\mathbb{P}(F_2^c)\le Cu^{-\beta}$ for u big enough. It concludes the proof.

4. New examples of random walks satisfying the polynomial condition

4.1. Proof of Theorem 4

Consider the box $B_{e_1,L,\tilde{L}}$ for $\tilde{L} = 70L^3$. We want to find some integer $L > c_0$ such that

$$P_0(X_{T_{B_{e_1,L,\tilde{L}}}}\cdot e_1 < L) \le \frac{1}{L^M},$$

for some $M \geq 15d + 5$. We first decompose this probability according to whether the exit point of the random walk from the box $B_{e_1,L,\tilde{L}}$ is on the bottom or on one of the sides of the box, so that,

$$\begin{aligned} &P_0(X_{T_{B_{e_1,L,\tilde{L}}}} \cdot e_1 < L) \\ &\quad = P_0(X_{T_{B_{e_1,L,\tilde{L}}}} \cdot e_1 = -L) + \sum_{i=2}^{d} \bigl(P_0(X_{T_{B_{e_1,L,\tilde{L}}}} \cdot e_i = \tilde{L}) + P_0(X_{T_{B_{e_1,L,\tilde{L}}}} \cdot e_i = -\tilde{L})\bigr). \end{aligned}$$

We will first bound the probability to exit through the sides. We do the computations for $P_0(X_{T_{B_{e_1,L,\tilde{L}}}} \cdot e_2 = \tilde{L})$ but the other term can be dealt with in the same way. Suppose that $X_{T_{B_{e_1,L,\tilde{L}}}} \cdot e_2 = \tilde{L}$, and define $n_0, \dots, n_{\tilde{L}-1}$ the finite hitting times of new levels in direction e_2 as follows:

$$n_k := \min\{n \geq 0 \text{ s.t. } X_n \cdot e_2 \geq k\}.$$

To simplify notation define $\varphi(x) := \omega(x, e_{1+d})$. We now choose a constant $1 > \delta > 0$, and we will call "good point" any $x \in \mathbb{Z}^2$ such that $\varphi(x) > \delta$. We define $p := \mathbb{P}(\varphi(x) > \delta)$. Note that p does not depend on x since the environment is i.i.d., so that it depends only on δ and the law of φ.

We now introduce the event that a great number of the X_{n_k} are good points:

$$C_1 := \left\{ X_{T_{B_{e_1,L,\tilde{L}}}} \cdot e_2 = \tilde{L} \text{ and at least } \frac{p}{2}\tilde{L} \text{ of the } X_{n_k}, 1 \leq k \leq \tilde{L} - 1, \text{ are good} \right\}.$$

We get immediately

$$P_0(X_{T_{B_{e_1,L,\tilde{L}}}} \cdot e_2 = \tilde{L}) = P_0(C_1) + P_0\bigl(\{X_{T_{B_{e_1,L,\tilde{L}}}} \cdot e_2 = \tilde{L}\} \cap \bigl(C_1^c\bigr)\bigr).$$

By construction of the X_{n_k} and independence of the environment, and with Z an independent random variable following a binomial law of parameters p and $\tilde{L}$, we can bound the second term of the sum:

$$\begin{aligned} P_0\bigl(\{X_{T_{B_{e_1,L,\tilde{L}}}} \cdot e_2 = \tilde{L}\} \cap \bigl(C_1^c\bigr)\bigr) &\leq P\left(Z \leq \frac{p}{2}\tilde{L}\right) \\ &\leq \exp\left(-2\frac{(p\tilde{L} - p\tilde{L}/2)^2}{\tilde{L}}\right) \\ &= \exp\left(-\frac{p^2\tilde{L}}{2}\right), \end{aligned}$$

where the last inequality is Hoeffding's inequality.

It only remains to bound $P_0(C_1)$. For that, we introduce the following new event

$$C_2 := C_1 \cap \left\{ X_{n_k+1} - X_{n_k} = e_1 \text{ for at least } \frac{\delta p}{4}\tilde{L} \text{ of the good } X_{n_k} \right\},$$

that states that the walk goes often in direction e_1 just after reaching a X_{n_k} that is a good point.

We can then write

$$P_0(C_1) = P_0(C_2) + P_0\big(C_1 \cap \big(C_2^c\big)\big).$$

To bound $P_0(C_1 \cap (C_2^c))$, we use the uniform bound "$\varphi(x) > \delta$" for good points that gives us that $\omega(x, e_1) > r\delta$ on those points. Getting Z' an independent random variable following a binomial law of parameters $r\delta$ and $\lfloor \frac{p}{2}\tilde{L}\rfloor$, it gives:

$$P_0\big(C_1 \cap \big(C_2^c\big)\big) \le P\left(Z' \le \frac{\delta p}{4}\tilde{L}\right) \le \exp\left(-p\delta^2\tilde{L}\left(r - \frac{1}{2}\right)^2\right).$$

It only remains to bound $P_0(C_2)$. Set n^+ (resp., n^-) the total number of jumps in direction e_1 (resp., $-e_1$) before exiting the box $B_{e_1,L,\tilde{L}}$. We will need a third new event

$$C_3 := \left\{n^+ \ge \frac{1+r}{2}n^-\right\},$$

that allows us to write

$$P_0(C_2) = P_0(C_2 \cap C_3) + P_0\big(C_2 \cap \big(C_3^c\big)\big).$$

First, notice that for L big enough, $C_2 \cap C_3 = \varnothing$. Indeed, C_1 implies that we exit the box $B_{e_1,L,\tilde{L}}$ by the side "$x \cdot e_2 = \tilde{L}$". Now, since the vertical displacement of the walk before exiting the box $B_{e_1,L,\tilde{L}}$ is $n^+ - n^-$, on the event C_3 we know that this displacement is at least equal to $\frac{r-1}{r+1}n^+$. Therefore, since on C_2 the walk makes at least $\frac{\delta p}{4}\tilde{L} = \frac{35\delta p}{2}L^3$ moves in the direction e_1, on $C_2 \cap C_3$ its vertical displacement before exiting the box is at least $\frac{35\delta p(r-1)}{2(r+1)}L^3$. Since on $C_2 \cap C_3$ the walk exits the box by the "$x \cdot e_2 = \tilde{L}$" side, we see that for L larger than $L_1 := \sqrt{\frac{2(1+r)}{35\delta p(r-1)}}$ the event $C_2 \cap C_3$ is empty.

We now want to bound $P_0(C_2 \cap (C_3^c))$

$$\begin{aligned} P_0\big(C_2 \cap \big(C_3^c\big)\big) &\le P_0\left(n^+ \ge \frac{\delta p}{4}\tilde{L} \text{ and } n^+ < \frac{1+r}{2}n^-\right) \\ &\le P_0\left(n^+ + n^- \ge \frac{\delta p}{4}\tilde{L} \text{ and } \big(n^+ + n^-\big)\frac{2}{3+r} < n^-\right). \end{aligned}$$

Now note that whenever we go through a vertical edge from a point x, the law of the environment tells us that it is an edge (x, e_1) with probability $\frac{r}{1+r}$, and $(x, -e_1)$ with probability $\frac{1}{1+r}$. Then, defining Z'' as a random variable following a binomial law of parameters $\frac{1}{1+r}$ and $\lfloor \frac{p\delta}{4}\tilde{L}\rfloor$, we have the bound:

$$\begin{aligned} P_0\big(C_2 \cap \big(C_3^c\big)\big) &\le P\left(Z'' \ge \frac{2}{3+r}\frac{p\delta}{4}\tilde{L}\right) \\ &\le \exp\left(-2p\delta\tilde{L}\left(\frac{1}{3+r} - \frac{2}{p\delta\tilde{L}(1+r)}\right)^2\right), \end{aligned}$$

where we need $1 \leq \frac{(r+1)p\delta\tilde{L}}{2(3+r)}$ to apply Hoeffding's inequality in the last inequality. We can find L_2 such that this is true for $L \geq L_2$.

Choose $M \geq 15d + 5$. By putting all of our previous bounds together, we finally get, for all $L \geq L_2$,

$$\begin{aligned} &P_0(X_{T_{B_{e_1,L,\tilde{L}}}} \cdot e_2 = \tilde{L}) \\ &\quad \leq \exp\left(-\frac{p^2\tilde{L}}{2}\right) + \exp\left(-p\delta^2\tilde{L}\left(r - \frac{1}{2}\right)^2\right) + \exp\left(-2p\delta\tilde{L}\left(\frac{1}{3+r} - \frac{2}{p\delta\tilde{L}(1+r)}\right)^2\right), \end{aligned}$$

where we recall that $\tilde{L} = 70L^3$, $\delta > 0$ and $p = \mathbb{P}(\varphi(x) > \delta)$. Then, for any choice of δ, we can find $L_3 \geq \max(c_0, L_1, L_2)$ such that for all $L \geq L_3$,

$$\sum_{i=2}^{d} \big(P_0(X_{T_{B_{e_1,L,\tilde{L}}}} \cdot e_i = \tilde{L}) + P_0(X_{T_{B_{e_1,L,\tilde{L}}}} \cdot e_i = -\tilde{L})\big) \leq \frac{1}{2L^M}.$$

We now only need to bound $P_0(X_{T_{B_{e_1,L,\tilde{L}}}} \cdot e_1 = -L)$ to prove $(P)_M|e_1$. We will use again the notations n^+ (resp., n^-) for the total number of jumps in direction e_1 (resp., $-e_1$) before exiting the box $B_{e_1,L,\tilde{L}}$. Suppose that $X_{T_{B_{e_1,L,\tilde{L}}}} \cdot e_1 = -L$. Then necessarily $n^+ < n^-$, which gives $n^+ < \frac{n^+ + n^-}{2}$. As n^+ conditioned to $n^+ + n^-$ follows a binomial law of parameters $\frac{r}{1+r} > \frac{1}{2}$ and $n^+ + n^-$, Hoeffding's inequality gives the bound:

$$P_0\left(n^+ < \frac{n^+ + n^-}{2} \,\middle|\, n^+ + n^-\right) \leq \exp\left(-2\big(n^+ + n^-\big)\left(\frac{r}{1+r} - \frac{1}{2}\right)^2\right).$$

But $X_{T_{B_{e_1,L,\tilde{L}}}} \cdot e_1 = -L$ also gives that necessarily, $n^- \geq L$. Then

$$\begin{aligned} P_0(X_{T_{B_{e_1,L,\tilde{L}}}} \cdot e_1 = -L) &\leq P_0\left(n^+ < \frac{n^+ + n^-}{2} \text{ and } n^- \geq L\right) \\ &\leq \sum_{m=L}^{\infty} \exp\left(-2m\left(\frac{r}{1+r} - \frac{1}{2}\right)^2\right). \end{aligned}$$

Therefore, we can find $L_4 \geq L_3$ such that for all $L \geq L_4$,

$$P_0(X_{T_{B_{e_1,L,\tilde{L}}}} \cdot e_1 = -L) \leq \frac{1}{2L^M},$$

from where we conclude that for all $L \geq L_4$,

$$P_0(X_{T_{B_{e_1,L,\tilde{L}}}} \cdot e_1 < L) \leq \frac{1}{L^M}.$$

4.2. Proof of Theorem 5

It is classical to represent Dirichlet distributions with independent gamma random variables: if $\gamma_1, \dots, \gamma_N$ are independent gamma random variables with parameters $\beta_1, \dots, \beta_N$, then $\frac{\gamma_1}{\sum \gamma_i}, \dots, \frac{\gamma_N}{\sum \gamma_i}$ is a Dirichlet random variable with parameters $(\beta_1, \dots, \beta_N)$. We get a restriction property as an easy consequence of this representation (see [17], pages 179–182): if $(\xi_1, \dots, \xi_N)$ is a Dirichlet random variable with parameters $(\beta_1, \dots, \beta_N)$, for any J non-empty subset of $\{1, \dots, N\}$, the random variable $(\frac{\xi_j}{\sum_{i \in J} \xi_i})_{j \in J}$ follows a Dirichlet law with parameters $(\beta_j)_{j \in J}$ and is independent of $\sum_{i \in J} \xi_i$. This property will be useful in the following.

We consider the box $B_{e_1, L, \tilde{L}}$ for $\tilde{L} = 70L^3$, and want to find some $L > c_0$ such that $P_0(X_{T_{B_{e_1,L,\tilde{L}}}} \cdot e_1 < L) \le \frac{1}{L^M}$ to prove $(P)_M | e_1$.

Let $l_i := \{x \in \mathbb{Z}^d \text{ s.t. } x \cdot e_1 = i\}$ and $t_i := \min\{n \ge 0\colon\ X_n \in l_i, X_{n+1} \notin l_i\}$. We first consider the events that, when the walk arrives on l_i for the first time, it gets out of it by an edge in direction e_1 (the alternative being getting out by an edge in direction $-e_1$):

$$G_{1,i} := \{X_{t_i+1} - X_{t_i} = e_1\}.$$

At the point X_{t_i}, we know that the walk will go either to $X_{t_i} + e_1$ or to $X_{t_i} - e_1$. Thanks to the restriction property of the Dirichlet laws, we know that $\frac{\omega(X_{t_i}, e_1)}{\omega(X_{t_i}, e_1) + \omega(X_{t_i}, -e_1)}$ follows a beta law of parameters $(\beta_1, \beta_1 + \beta_{1+d})$ and is independent from the previous trajectory of the walk on l_i. Indeed, we already know by construction that it is independent from the environment on the other points of l_i, and the restriction property gives the independence from $\frac{\omega(X_{t_i}, e_j)}{1 - \omega(X_{t_i}, e_1) - \omega(X_{t_i}, -e_1)}$ for all $j \neq 1, -1$, which corresponds to the law of the previous trajectory from X_{t_i} on l_i. Then

$$P_0(G_{1,i}) = \frac{\beta_1}{\beta_1 + \beta_{1+d}}.$$

Now define

$$G_1 := \bigcap_{i=1}^{L} G_{1,i},$$

and note that

$$P_0\bigl(G_1^c\bigr) \le L \frac{\beta_{1+d}}{\beta_1 + \beta_{1+d}}.$$

We can now write

$$\begin{aligned} &P_0(X_{T_{B_{e_1,L,\tilde{L}}}} \cdot e_1 < L) \\ &\quad \le P_0\bigl(\{X_{T_{B_{e_1,L,\tilde{L}}}} \cdot e_1 < L\} \cap G_1\bigr) + L \frac{\beta_{1+d}}{\beta_1 + \beta_{1+d}}, \end{aligned}$$

and we only need to bound the first term of this sum. If G_1 is satisfied, the walk cannot get out of the box $B_{e_1,L,\tilde{L}}$ by the "lower boundary" $\{x \in \mathbb{Z}^d \text{ s.t. } x \cdot e_1 = -L\}$. Then the walk has to get out by one of the $2d-2$ "side boundaries":

$$P_0\big(\{X_{T_{B_{e_1,L,\tilde{L}}}} \cdot e_1 < L\} \cap G_1\big) = P_0\left(\bigcup_{j=2}^{d} \{X_{T_{B_{e_1,L,\tilde{L}}}} \cdot e_j = \pm\tilde{L}\} \cap G_1\right).$$

On the event $\bigcup_{j=1}^{d}\{X_{T_{B_{e_1,L,\tilde{L}}}} \cdot e_j = \pm\tilde{L}\}$ define $n_0, \dots, n_{\tilde{L}-1}$ as the finite hitting times of new levels in any direction perpendicular to e_1 as follows:

$$n_k := \min\Big\{n \geq 0 \text{ s.t. } \max_{2\leq j\leq d} |X_n \cdot e_j| \geq k\Big\}.$$

Let now $p = \frac{\beta_1}{1+\sum_{i\neq 1+d}\beta_i}$ and consider the event

$$G_3 := G_1 \cap \left\{X_{n_k+1} - X_{n_k} = e_1 \text{ for at least } \frac{p}{2}\tilde{L} \text{ of the points } X_{n_k}\right\}.$$

Suppose $\beta_{1+d} \leq 1$, then $p \leq \mathbb{E}(\omega(0,e_1))$. Consider now a random variable Z with a binomial law of parameters p and $\tilde{L}$. Using Hoeffding's inequality, we see that

$$P\big(G_3^c\big) \leq P\left(Z \leq \frac{p\tilde{L}}{2}\right) \leq \exp\left(-\frac{p^2}{2}\tilde{L}\right).$$

But clearly $G_1 \cap G_3 = \varnothing$ for $L \geq L_0 := \sqrt{\frac{1}{35p}}$. Therefore, we have in this case

$$P_0\left(\bigcup_{j=2}^{d}\{X_{T_{B_{e_1,L,\tilde{L}}}} \cdot e_j = \pm\tilde{L}\} \cap G_1\right) \leq \exp\left(-\frac{p^2}{2}\tilde{L}\right).$$

Putting the previous bounds together, we finally get for all $L \geq L_0$:

$$P_0(X_{T_{B_{e_1,L,\tilde{L}}}} \cdot e_1 < L) \leq L\frac{\beta_{1+d}}{\beta_1 + \beta_{1+d}} + \exp\left(-\frac{p^2}{2}\tilde{L}\right).$$

Let now L_1 be such that for all $L \geq L_1$

$$\exp\left(-\frac{p^2}{2}\tilde{L}\right) \leq \frac{1}{2L^M}.$$

The constant c_0 (cf. (1.1)) is increasing in β_{1+d}. Therefore, in the region where $\beta_{1+d} \leq \beta_1$, it does not depend on β_{1+d}. Call this value c_0'. On the other hand, by construction, L_0 and L_1 do not depend on β_{1+d}. Take now $L_2 := \max\{c_0', L_0, L_1\}$.

We can then choose β_{1+d} (necessarily $\le \beta_1$) so that

$$L_2 \frac{\beta_{1+d}}{\beta_1 + \beta_{1+d}} \le \frac{1}{2L_2^M}.$$

We then conclude that for this choice of β_{1+d} there exists an $L \ge c_0$ such that

$$P_0(X_{T_{B_{e_1,L_2,\tilde{L}}}} \cdot e_1 < L) \le \frac{1}{L^M}.$$

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