Efficient pointwise estimation based on discrete data in ergodic nonparametric diffusions

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A truncated sequential procedure is constructed for estimating the drift coefficient at a given state point based on discrete data of ergodic diffusion process. A nonasymptotic upper bound is obtained for a pointwise absolute error risk. The optimal convergence rate and a sharp constant in the bounds are found for the asymptotic pointwise minimax risk. As a consequence, the efficiency is obtained of the proposed sequential procedure.

Keywords: discrete data; drift coefficient estimation; efficient procedure; ergodic diffusion process; minimax; nonparametric sequential estimation

1. Introduction

In this paper, we consider the following diffusion model:

$$\mathrm{d}y_t = S(y_t)\,\mathrm{d}t + \sigma(y_t)\,\mathrm{d}W_t, \qquad 0 \le t \le T, \tag{1.1}$$

where $(W_t)_{t\ge 0}$ is a scalar standard Wiener process, $S(\cdot)$ and $\sigma(\cdot)$ are unknown functions. This model appears in a number of applied problems of stochastic control, filtering, spectral analysis, identification of dynamic system, financial mathematics and others (see [1,3,23,27,28] and others for details).

The problem is to estimate the function $S(\cdot)$ at a point x_0 basing on the discrete time observations

$$(y_{t_j})_{1\le j \le N}, \qquad t_j = j\delta, \tag{1.2}$$

where $N = [T/\delta]$ and the frequency $\delta = \delta_T \in (0, 1)$ is a function of T that will be specified later.

The estimation problem of the function S was studied in a number of papers in the case of complete observations, that is when a continuous trajectory $(y_t)_{0\le t \le T}$ was observed. In the parametric case, this problem was considered apparently for the first time in the paper [2] for diffusion model of the axis of the equator precession. In that paper, a nonasymptotic distribution of the maximum likelihood estimator was found for a special Ornstein–Uhlenbeck process.

It should be noted that investigating nonasymptotic properties of parametric estimators in the models like to (1.1) comes to the analysis of nonlinear functionals of observations. At the most cases, this analysis is unproductive in nonasymptotic setting. In order to overcome the technique difficulties, the sequential analysis methods were used in [28] and [29] for estimating a scalar parameter. In [25], these methods were extended to estimating a multi-dimensional parameter as well. Moreover, in [26] truncated sequential procedures were developed that economizes the observation time.

In [7] and [8], a sequential approach was proposed for the pointwise nonparametric estimation in the ergodic models (1.1). Later in [10], the efficiency was studied of the proposed sequential procedures.

A sufficiently complete survey one can find in [27] on the nonparametric estimation in the ergodic model (1.1) when non sequential approaches are used.

In the cited papers, estimation problems were studied based on complete observations $(y_t)_{0 \le t \le T}$. In practice, usually one has at disposal discrete time observations even for continuous time models.

A natural question arises about proprieties and the behavior of estimates based on discrete time observations for such models. These problems were studied for several models. We cite some of them.

In [24], asymptotically normal estimators are constructed for both parameters (θ, σ), for a parametric ergodic one-dimensional diffusion model observed at discrete times $i\delta_n, 0 \le i \le n$, with drift $b(x, \theta)$ and diffusion coefficient $a(x, \sigma)$, when the observation frequency $\delta_n \to 0$ and $n\delta_n \to \infty$. Estimating is based on the property of discrete observations to be locally Gaussian. The author claims that asymptotic efficiency can be obtained under additional condition $n\delta_n^p \to 0$, where $p > 1$.

Parametric estimations were studied in the papers [17,30] and [21] for drift and diffusion coefficients in multidimensional diffusion processes when the observation frequency δ_n is as follows: $\delta_n \to 0, n\delta_n \to \infty$. In [17] the LAN property is proved in the ergodic case. The proof is based on the transformation of the log-likelihood ratio by the Malliavin calculus. Asymptotic normality is studied in [30] for the joint distribution of the maximum likelihood estimator of parameters in drift and diffusion coefficients. In [21] the tightness of estimators is proved without ergodicity or even recurrence assumptions.

Nonparametric estimation setting for models of kind (1.1) was considered firstly for estimating the unknown diffusion coefficient $\sigma^2(\cdot)$ based on discrete time observations on a fixed interval $[0, T]$, when the observation frequency goes to zero (see, e.g., [6,15,19,20] and the references therein).

Later, in [18] kernel estimates of drift and diffusion coefficients were studied for reflecting ergodic processes (1.1) taking the values into the interval $[0, 1]$ in the case of fixed observation frequency; the asymptotics are taken as the sample size goes to infinity. Minimax optimal convergence rates are found for estimators of the drift and the diffusion coefficients. Upper and lower bounds for L_2-risk are given as well.

So far as concerning the estimation in ergodic case, it should be noted that a sequential procedure was proposed in [19] for nonparametric estimating the drift coefficient of the process (1.1) in the integral metric. Some upper and lower asymptotic bounds were found for the $\mathbf{L}_p$-risks. Later, in the paper [5] a nonasymptotic oracle inequality was proved for the drift coefficient estimation problem in a special empiric quadratic risk based on discrete time observations. In the

asymptotic setting, when the observation frequency goes to zero and the length of the observation time interval tends to infinity, the constructed estimators reach the minimax optimal convergence rates.

This paper deals with the drift coefficient efficient nonparametric estimation at a given state point based on discrete time observations (1.2) in the absolute error risk. The unknown diffusion coefficient is a nuisance parameter. We find the minimax optimal convergence rate and we study the lower bound normalized by this convergence rate in the case when the frequency $\delta_T \to 0$ and $T\delta_T \to 0$ as $T \to \infty$.

Our approach is based on the sequential analysis developed in the papers [7,8], and [10] for the nonparametric estimation. This approach makes possible to replace the denominator by a constant in a sequential Nadaraya–Watson estimator.

Let us recall that in the case of complete observations (i.e., when a whole trajectory is observed) the sequential estimate efficiency was proved by making use of a uniform concentration inequality (see [11]), besides an indicator kernel estimator and a weak Hölder space of functions S were used.

As it turns out later in [14], the efficient kernel estimate in the above given sense provides to construct a selection model adaptive procedure that appears efficient in the quadratic $\mathbf{L}_2$-metric.

Therefore, in order to realize this program (i.e., from efficient pointwise estimators to an efficient $\mathbf{L}_2$-estimator) in the case of discrete time observations, one needs to obtain a suitable concentration inequality, that is done in [12]. It should be noted that in order to obtain nonasymptotic concentration inequality we make use of nonasymptotic bounds uniform over functions S and σ for the convergence rate in the ergodic theorem for the process (1.1). The latter result is proved in [13] and it is based on a new approach using Lyapounov's functions and the coupling method.

Further in this paper, we make use of the concentration inequality in order to find the explicit constant in the upper bound for weak Hölder's risk normalized by the optimal convergence rate and we prove that this upper bound is best over all possible estimators. It means the procedure is efficient.

The paper is organized as follows. In Section 2, we describe the functional classes. In Section 3 the sequential procedure is constructed. In Section 4, we obtain a nonasymptotic upper bound for the absolute error pointwise risk of the sequential procedure. In Section 5, we show that the proposed procedure is asymptotically efficient for the pointwise risk. All proofs are given in Section 6. In the Appendix, we give all necessary technical results.

2. Functional class

We consider the pointwise estimation problem for the function $S(\cdot)$ at a fixed point $x_0 \in \mathbb{R}$ for the model (1.1) with unknown diffusion coefficient σ. It is clear that to obtain a good estimate for the function $S(\cdot)$ at the point x_0 it is necessary to impose some conditions on the function $\vartheta = (S, \sigma)$ which provide that the observed process $(y_t)_{0 \le t \le T}$ returns to any vicinity of the point x_0 infinitely many times.

In this section, we describe a weak Hölder functional class which guarantees the ergodicity property for this model. First, for some $\mathbf{x}_* \ge |x_0| + 1$, $M > 0$ and $L > 1$ we denote by $\Sigma_{L,M}$ the

class of functions S from $\mathbf{C}^1(\mathbb{R})$ such that

$$\sup_{|x| \le \mathbf{x}_*} \left(\left| S(x) \right| + \left| \dot{S}(x) \right| \right) \le M$$

and

$$-L \le \inf_{|x| \ge \mathbf{x}_*} \dot{S}(x) \le \sup_{|x| \ge \mathbf{x}_*} \dot{S}(x) \le -L^{-1}.$$

Moreover, for some fixed parameters $0 < \sigma_{\min} \le \sigma_{\max} < \infty$ we denote by $\mathcal{V}$ the class of the functions σ from $\mathbf{C}^2(\mathbb{R})$ such that

$$\inf_{x \in \mathbb{R}} \left| \sigma(x) \right| \ge \sigma_{\min} \quad \text{and} \quad \sup_{x \in \mathbb{R}} \max \left(\left| \sigma(x) \right|, \left| \dot{\sigma}(x) \right|, \left| \ddot{\sigma}(x) \right| \right) \le \sigma_{\max}. \tag{2.1}$$

In this paper, we make use of the weak Hölder functions introduced in [9].

Definition 2.1. *We say that a function S satisfies the weak Hölder condition at the point $x_0 \in \mathbb{R}$ with the parameters $h, \varepsilon > 0$ and exponent $\beta = 1 + \alpha$, $\alpha \in (0, 1)$, if $S \in \mathbf{C}^1(\mathbb{R})$ and its derivative satisfies the following inequality*

$$\left| \int_{-1}^{1} z \int_{0}^{1} \left(\dot{S}(x_0 + uzh) - \dot{S}(x_0) \right) du \, dz \right| \le \varepsilon h^{\alpha}. \tag{2.2}$$

We will denote the set of all such functions by $\mathcal{H}_{x_0}^w(\varepsilon, \beta, h)$.

Note that

$$\int_{-1}^{1} z \int_{0}^{1} \left(\dot{S}(x_0 + uzh) - \dot{S}(x_0) \right) du \, dz = \Omega_{x_0, h}(S), \tag{2.3}$$

where $\Omega_{x_0, h}(S) = \int_{-1}^{1} (S(x_0 + hz) - S(x_0)) \, dz$. Therefore, the condition (2.2) for the functions from $\mathcal{H}_{x_0}^w(\varepsilon, \beta, h)$ is equivalent to the following one

$$\sup_{S \in \mathcal{H}_{x_0}^w(\varepsilon, \beta, h)} \left| \Omega_{x_0, h}(S) \right| \le \varepsilon h^{\beta}. \tag{2.4}$$

Let us denote by $\mathcal{H}_{x_0, M}^w(\varepsilon, \beta, h)$ the set of all functions D from $\mathcal{H}_{x_0}^w(\varepsilon, \beta, h)$ such that $\sup_{x \in \mathbb{R}}(|D(x)| + |\dot{D}(x)|) \le M/2$ and $D(x) = 0$ for $|x| \ge x_*$.

Let S_0 be a function from $\Sigma_{L, M/2}$ such that

$$\lim_{h \to 0} h^{-\beta} \Omega_{x_0, h}(S_0) = 0. \tag{2.5}$$

We define a vicinity $\mathcal{U}_M(x_0, \beta)$ of the function S_0 as follows

$$\mathcal{U}_M(x_0, \beta) = S_0 + \mathcal{H}_{x_0, M}^w(\varepsilon, \beta, h), \tag{2.6}$$

where $h = T^{-1/(2\beta+1)}$ and

$$\varepsilon = \varepsilon_T = \frac{1}{(\ln T)^{1+\gamma}} \tag{2.7}$$

for some $0 < \gamma < 1$. Obviously that $\mathcal{U}_M(x_0, \beta) \subset \Sigma_{L,M}$. Finally, we set

$$\Theta_\beta = \mathcal{U}_M(x_0, \beta) \times \mathcal{V}. \tag{2.8}$$

It should be noted that, for any $\vartheta \in \Theta_\beta$, there exists an invariant density which is defined as

$$q_\vartheta(x) = \left(\int_\mathbb{R} \sigma^{-2}(z) e^{\widetilde{S}(z)} \, dz \right)^{-1} \sigma^{-2}(x) e^{\widetilde{S}(x)}, \tag{2.9}$$

where $\widetilde{S}(x) = 2 \int_0^x \sigma^{-2}(v) S(v) \, dv$ (see, e.g., [16], Chapter 4.18, Theorem 2). It is easy to see that this density is uniformly bounded in the class (2.8), that is,

$$q^* = \sup_{x \in \mathbb{R}} \sup_{\vartheta \in \Theta_\beta} q_\vartheta(x) < +\infty \tag{2.10}$$

and bounded away from zero on the interval $[x_0 - 1, x_0 + 1]$, that is,

$$q_* = \inf_{x_0 - 1 \leq x \leq x_0 + 1} \inf_{\vartheta \in \Theta_\beta} q_\vartheta(x) > 0. \tag{2.11}$$

For any $\mathbb{R} \to \mathbb{R}$ function f from $\mathbf{L}_1(\mathbb{R})$, we set

$$\mathbf{m}_\vartheta(f) = \int_\mathbb{R} f(x) q_\vartheta(x) \, dx. \tag{2.12}$$

Assume that the frequency δ in the observations (1.2) is of the following asymptotic form (as $T \to \infty$)

$$\delta = \delta_T = O\left(\frac{\varepsilon_T}{T} \right), \tag{2.13}$$

where the function ε_T is introduced in (2.7).

Now, for any estimate (i.e., any $(y_t)_{0 \leq t \leq T}$ measurable function) $\widetilde{S}_T(x_0)$ of $S(x_0)$, we define the pointwise risk as follows

$$\mathcal{R}_\vartheta(\widetilde{S}_T) = \mathbf{E}_\vartheta \left| \widetilde{S}_T(x_0) - S(x_0) \right|. \tag{2.14}$$

Remark 2.1. It should be noted that in the definition of the weak Hölder class $\mathcal{H}_{x_0}^w(\varepsilon, \beta, h)$ in (2.2) the weak Hölder norm ε is chosen such that it goes to zero as $T \to \infty$ (see (2.7)) in contrast with the initial definition of this class given in [10], where the norm was fixed. But there an additional limit passage with $\varepsilon \to 0$ was done in the theorem about upper bound (see, Theorem 5.2 in [10]). Therefore, in this paper we choose $\varepsilon \to 0$ in the definition of the weak Hölder class instead of the additional limit passage with $\varepsilon \to 0$ in the theorem on the upper bound. Technically it is nearly the same, but the sense of the upper bound is clearer without the additional limit with $\varepsilon \to 0$.

3. Sequential procedure

In order to construct an efficient pointwise estimator of S, we begin with estimating the ergodic density $q = q_\vartheta$ at the point x_0 from first N_0 observations. We choose

$$N_0 = N^{\gamma_0} \quad \text{and} \quad 2/3 < \gamma_0 < 1. \tag{3.1}$$

We will make use of the following kernel estimator

$$\widehat{q}_T(x_0) = \frac{1}{2(N_0 - 1)\varsigma} \sum_{j=0}^{N_0 - 1} Q\left(\frac{y_{t_j} - x_0}{\varsigma}\right), \tag{3.2}$$

where $Q(y) = \mathbf{1}_{(|y| \leq 1)}$ and $\varsigma = \varsigma_T$ is a function of T such that

$$\varsigma_T = \mathrm{o}(T^{-\gamma_0/2}) \qquad \text{as } T \to \infty.$$

For $T \geq 3$, we set

$$\widetilde{q}_T(x_0) = \begin{cases} (\upsilon_T)^{1/2}, & \text{if } \widehat{q}_T(x_0) < (\upsilon_T)^{1/2}; \\ \widehat{q}_T(x_0), & \text{if } (\upsilon_T)^{1/2} \leq \widehat{q}_T(x_0) \leq (\upsilon_T)^{-1/2}; \\ (\upsilon_T)^{-1/2}, & \text{if } \widehat{q}_T(x_0) > (\upsilon_T)^{-1/2}, \end{cases} \tag{3.3}$$

where

$$\upsilon_T = \frac{1}{(\ln T)^{a_0}} \quad \text{and} \quad a_0 = \frac{\sqrt{\gamma + 1} - 1}{10}.$$

The properties of the estimates $\widehat{q}_T(x_0)$ and $\widetilde{q}_T(x_0)$ are studied in the Appendix.

Let us define the following stopping time

$$\varpi = \varpi_T = \inf\left\{j \geq N_0 \colon \sum_{i=N_0}^{j} \phi_i \geq H_T\right\}, \tag{3.4}$$

where H_T is a threshold, $\phi_i = \chi_{h,x_0}(y_{t_{i-1}})\mathbf{1}_{\{i \leq N\}} + \mathbf{1}_{\{i > N\}}$, $\chi_{h,x_0}(y) = Q((y - x_0)/h)$ and h is a positive bandwidth. We put $\varpi = \infty$ if the set $\{\cdot\}$ is empty. Obviously, that in the our case $\varpi < \infty$ a.s. since $\sum_{i \geq N_0} \phi_i = +\infty$ a.s.

Now we have to choose the threshold H_T. Note that in order to construct an efficient estimator one should use all, that is, N observations. Therefore, the threshold H_T should provide the asymptotic relations $\varpi_T \approx N$ and

$$\sum_{i=N_0}^{N} \phi_i = \sum_{i=N_0}^{N} \chi_{h,x_0}(y_{t_{i-1}}) \qquad \text{as } T \to \infty.$$

In order to obtain these relations, note that, due to the ergodic theorem,

$$\sum_{i=N_0}^{N} \chi_{h,x_0}(y_{t_{i-1}}) \approx 2h(N - N_0)q_\vartheta(x_0).$$

Hence, replacing in the right-hand side term the ergodic density with its corrected estimate yields the following definition of the threshold

$$H = H_T = h(N - N_0)\big(2\widetilde{q}_T(x_0) - \upsilon_T\big). \tag{3.5}$$

Note that in [7] it has been shown that the such form of the threshold H_T provides the optimal convergence rate. It is clear that

$$\varpi \leq N + H_T < N + h(N - N_0)/\sqrt{\upsilon_T},$$

that is, the stopping time ϖ is bounded. Now on the set $\Gamma_T = \{\varpi \leq N\}$ we define the correction coefficient $\varkappa = \varkappa_T$ as

$$\varkappa_T = \frac{H_T - \sum_{j=N_0}^{\varpi-1} \chi_{h,x_0}(y_{t_{j-1}})}{\chi_{h,x_0}(y_{t_{\varpi-1}})},$$

that is, on the set Γ_T

$$\sum_{j=N_0}^{\varpi-1} \chi_{h,x_0}(y_{t_{j-1}}) + \varkappa \chi_{h,x_0}(y_{t_{\varpi-1}}) = H_T.$$

Moreover, on the Γ_T^c we set $\varkappa_T = 1$. Using this definition, we introduce the weight sequence

$$\widetilde{\varkappa}_j = \mathbf{1}_{\{j < \varpi\}} + \varkappa \mathbf{1}_{\{j = \varpi\}}, \qquad j \geq 1. \tag{3.6}$$

One can check directly that, for any $j \geq 1$, the coefficients $\widetilde{\varkappa}_j$ are $\mathcal{F}_{t_{j-1}}$ measurable, where $\mathcal{F}_{t_j} = \sigma(y_{t_k}, 0 \leq k \leq j)$. Now we define the sequential estimator for $S(x_0)$ as

$$S_{h,T}^*(x_0) = \frac{1}{\delta H_T}\left(\sum_{j=N_0}^{N} \widetilde{\varkappa}_j \chi_{h,x_0}(y_{t_{j-1}})\Delta y_{t_j}\right)\mathbf{1}_{\Gamma_T}. \tag{3.7}$$

In the next section, we study nonasymptotic properties of this procedure.

Remark 3.1. Note that the correction coefficient of type (3.6) was used first in the paper [4] in order to construct an unbiased estimator of a scalar parameter in autoregressive processes AR(1). Here, we make use of the same idea for a nonparametric procedure.

Remark 3.2. In fact, our procedure uses only the observations belonging to the interval $[x_0 - h, x_0 + h]$, that results in the sample size asymptotically equals to $2Nh\widehat{q}_T(x_0)$. This is related with the choice of the estimator kernel that is an indicator function. It is easy to verify (see [8]) that this kernel minimizes the variance of stochastic term in the kernel estimator. Ultimately, the last result provides efficiency of the procedure.

4. Nonasymptotic estimation

In this section, an upper bound for the absolute error risk will be given in the case, when $S \in \Sigma_{L,M}, \sigma$ is differentiable and $\sigma_{\min} \leq |\sigma(x)| \leq \sigma_{\max}$ for any x. We will denote this case as $\vartheta \in \Sigma_{L,M} \times [\sigma_{\min}, \sigma_{\max}]$.

As we will see later in studying the estimator (3.7), the approximation term plays the crucial role. In the our case, this term is of the following form

$$\Upsilon_{1,T} = \frac{1}{\delta H_T} \sum_{j=N_0}^{N} \widetilde{\varkappa}_j \chi_{h,x_0}(y_{t_{j-1}}) \varrho_j, \tag{4.1}$$

where $\varrho_j = \int_{t_{j-1}}^{t_j} (S(y_u) - S(y_{t_{j-1}})) \, du$. One can show the following result.

Proposition 4.1. *For any $T \geq 3$,*

$$\sup_{\vartheta \in \Sigma_{L,M} \times [0, \sigma_{\max}]} \mathbf{E}_\vartheta \Upsilon_{1,T}^2 \leq \widetilde{L}^2 L_1 \delta, \tag{4.2}$$

where $\widetilde{L} = \max(L, M)$ and $L_1 = 2(\sigma_{\max}^2 + 2\delta(M^2 + L^3 D_* + L^2 x_*^2))$.

Further, we set

$$\Upsilon_{2,T} = \frac{1}{\delta H_T} \sum_{j=N_0}^{N} \widetilde{\varkappa}_j \chi_{h,x_0}(y_{t_{j-1}}) \varrho_j^*, \tag{4.3}$$

where $\varrho_j^* = \int_{t_{j-1}}^{t_j} (\sigma(y_u) - \sigma(y_{t_{j-1}})) \, dW_u$.

Proposition 4.2. *For any $T \geq 3$ for which $0 < \delta \leq 1$, one has*

$$\sup_{\vartheta \in \Sigma_{L,M} \times [0, \sigma_{\max}]} \mathbf{E}_\vartheta (\Upsilon_{2,T})^2 \leq \frac{\sigma_{\max}^2 L_1}{h(N - N_0)\sqrt{\upsilon_T}}. \tag{4.4}$$

Proofs of Propositions 4.1 and 4.2 are given in the Appendix.

Now we introduce the approximative term, that is,

$$B_T = \frac{1}{H_T} \sum_{j=N_0}^{N} \widetilde{\varkappa}_j f_h(y_{t_{j-1}}) \tag{4.5}$$

with $f_h(y) = \chi_{h,x_0}(y)(S(y) - S(x_0))$. Taking into account this formula, we can represent the error of estimator (3.7) on the set Γ_T as

$$S_{h,T}^*(x_0) - S(x_0) = \Upsilon_{1,T} + B_T + \mathbf{M}_T, \tag{4.6}$$

where

$$\mathbf{M}_T = \frac{1}{\delta H_T} \sum_{j=N_0}^{N} \widetilde{\varkappa}_j \chi_{h,x_0}(y_{t_{j-1}}) \eta_j$$

with $\eta_j = \int_{t_{j-1}}^{t_j} \sigma(y_u) \, dW_u$. Obviously, for any function S from $\Sigma_{L,M}$, the term B_T can be bounded as

$$|B_T| \leq h \max_{|x-x_0| \leq h} |\dot{S}(x)| \leq Mh. \tag{4.7}$$

Proposition 4.3. *For any $T \geq 3$, one has*

$$\sup_{\vartheta \in \Sigma_{L,M} \times [0,\sigma_{\max}]} \mathbf{E}_\vartheta \mathbf{M}_T^2 \leq \frac{\sigma_{\max}^2}{\delta h(N - N_0)\sqrt{\upsilon_T}}. \tag{4.8}$$

Hence, we obtain the following upper bound.

Theorem 4.4. *For any $h > 0$ and $T \geq 3$ for which $0 < \delta \leq 1$, one has*

$$\sup_{\vartheta \in \Sigma_{L,M} \times [\sigma_{\min},\sigma_{\max}]} \mathbf{E}_\vartheta \left| S_{h,T}^*(x_0) - S(x_0) \right| \leq U^*(\delta, h, T) + M\Pi_T^*, \tag{4.9}$$

where

$$U^*(\delta, h, T) = \widetilde{L}\sqrt{\delta L_1} + Mh + \frac{\sigma_{\max}}{\sqrt{\delta h(N - N_0)}\upsilon_T^{1/4}}$$

and

$$\Pi_T^* = \sup_{\vartheta \in \Sigma_{L,M} \times [\sigma_{\min},\sigma_{\max}]} \mathbf{P}_\vartheta \left(\Gamma_T^c\right).$$

Let us study now the last term in (4.9).

Proposition 4.5. *Assume that the parameter δ is of the asymptotic form* (2.13) *and $h \geq T^{-1/2}$. Then, for any $a > 0$,*

$$\lim_{T \to \infty} T^a \Pi_T^* = 0. \tag{4.10}$$

Proof of this proposition is given in the Appendix.

Remark 4.1. It should be noted that the main destination of the bound (4.9) is to obtain a sharp oracle inequality for a model selection procedure for the process (1.1) observed at discrete times. Recall (see, [14]), that an efficient model selection procedure for diffusion processes is based on estimators admitting on some set Γ_T a nonasymptotic representation of kind (4.6) satisfying the conditions (4.7) and (4.8) at estimation state points (x_j). In addition, the condition (4.10) must hold true for the set Γ_T. Moreover, the stochastic terms of kernel estimators, $\mathbf{M}_T = \mathbf{M}_T(x_j)$,

must be independent random variables for different points x_j. The last condition is provided by sequential approach since, for sequential kernel estimator, the term $\mathbf{M}_T$ is a Gaussian random variable on the set Γ_T.

5. Asymptotic efficiency

First of all, we study a lower bound for the risk (2.14). To this end, we set

$$\varsigma_\vartheta^*(x_0) = \frac{2q_\vartheta(x_0)}{\sigma^2(x_0)}. \tag{5.1}$$

This parameter provides a sharp asymptotic lower bound for the pointwise risk normalized by the minimax rate $\varphi_T = T^{\beta/(2\beta+1)}$.

Theorem 5.1. *The risk defined in* (2.14) *admits the following lower bound*

$$\varliminf_{T\to\infty} \varphi_T \inf_{\widetilde{S}_T} \sup_{\vartheta\in\Theta_\beta} \sqrt{\varsigma_\vartheta^*(x_0)} \mathcal{R}_\vartheta(\widetilde{S}_T) \geq \mathbf{E}|\xi|, \tag{5.2}$$

where infimum is taken over all possible estimators $\widetilde{S}_T$, ξ is a (0, 1) *Gaussian random variable.*

Theorem 5.2. *The kernel estimator $S_{h,T}^*$ defined in* (3.7) *with $h = T^{-1/(2\beta+1)}$ satisfies the following asymptotic inequality*

$$\varlimsup_{T\to\infty} \varphi_T \sup_{\vartheta\in\Theta_\beta} \sqrt{\varsigma_\vartheta^*(x_0)} \mathcal{R}_\vartheta(S_{h,T}^*) \leq \mathbf{E}|\xi|, \tag{5.3}$$

where ξ is a (0, 1) *Gaussian random variable.*

Remark 5.1. It should be noted that one cannot use the bound (4.9) in order to obtain the sharp asymptotic upper bound (5.3) because the upper bound (4.9) is obtained for a wider function class, that is, for $\vartheta \in \Sigma_{L,M} \times [\sigma_{\min}, \sigma_{\max}]$ and hence, it is not the best.

Notice that the Theorems 5.1 and 5.2 imply the following efficiency property.

Theorem 5.3. *The sequential procedure* (3.7) *with $h = T^{-1/(2\beta+1)}$ is asymptotically efficient in the following sense*:

$$\varlimsup_{T\to\infty} \varphi_T \sup_{\vartheta\in\Theta_\beta} \sqrt{\varsigma_\vartheta^*(x_0)} \mathcal{R}_\vartheta(S_{h,T}^*) = \varliminf_{T\to\infty} \varphi_T \inf_{\widetilde{S}_T} \sup_{\vartheta\in\Theta_\beta} \sqrt{\varsigma_\vartheta^*(x_0)} \mathcal{R}_\vartheta(\widetilde{S}_T),$$

where infimum is taken over all possible estimators $\widetilde{S}_T$.

Remark 5.2. The constant (5.1) provides the sharp asymptotic lower bound for the minimax pointwise risk. The calculation of this constant is possible by making use of the weak Hölder class. This functional class was introduced in [9] for regression models. For the first time, the constant (5.1) was obtained in the paper [10] at the pointwise estimation problem of the drift based on continuous time observations of the process (1.1) with the unit diffusion. Later this constant was used in the paper [14] to obtain the Pinsker constant for a quadratic risk in the adaptive estimation problem of the drift in the model (1.1) based on continuous time observations.

Remark 5.3. Note also that in this paper the efficient procedure is constructed when the regularity is known of the function to be estimated. In the case of unknown regularity, we shall use an approach based on the model selection similarly to that in the paper [14] which deals with continuous time observations. The announced result will be published in the next paper which is in the work.

Remark 5.4. In the paper, we studied only the case of Hölderian smoothness $1 + \alpha$ with $\alpha \in (0, 1)$. Efficiency is provided by the indicator kernel which minimizes the asymptotic variance of the stochastic term (see, e.g., [8]). If in the pointwise estimation problem the unknown function possesses a greater smoothness then, as known, one needs to make use of a kernel that should be orthogonal to all polynomials of orders less than the integral part of the smoothness order. It is clear that the kernel estimator is not efficient. Therefore, one needs to uses an other estimator, in particular, a local polynomial estimator but again with an indicator kernel. This is a subject of our future investigation in the pointwise setting for diffusion processes. It will based on ideas and results of $1 + \alpha$-smoothness case but, due to extreme complication, the new case cannot be considered in this paper.

6. Proofs

6.1. Lower bound

In this section, the Theorem 5.1 will be proved. Let us introduce the model (1.1) with $\sigma = 1$, that is,

$$\mathrm{d}y_t = S(y_t)\,\mathrm{d}t + \mathrm{d}W_t. \tag{6.1}$$

Now we define the risk corresponding to this model as follows

$$\mathcal{R}_S^*(\widetilde{S}_T) = \mathbf{E}_S \big| \widetilde{S}_T(x_0) - S(x_0) \big|, \tag{6.2}$$

where $\mathbf{E}_S$ denotes the expectation with respect to the distribution $\mathbf{P}_S$ of the process (6.1) in the space of continuous functions $\mathbf{C}[0, T]$. It is clear that

$$\sup_{\vartheta \in \Theta_\beta} \sqrt{\varsigma_\vartheta^*(x_0)} \mathcal{R}_\vartheta(\widetilde{S}_T) \geq \sup_{S \in \mathcal{U}_M(x_0, \beta)} \sqrt{2 q_S(x_0)} \mathcal{R}_S^*(\widetilde{S}_T), \tag{6.3}$$

where q_S is the invariant density for the process (6.1) which equals to q_ϑ with $\sigma = 1$. Let now g be a continuously differentiable probability density on the interval $[-1, 1]$. Then, for any $u \in \mathbb{R}$ and $0 < \nu < 1/4$, we set

$$S_{u,\nu}(x) = S_0(x) + \frac{u}{\varphi_T} V_\nu\left(\frac{x - x_0}{h}\right),$$

where $h = T^{-1/(2\beta+1)}$ and

$$V_\nu(x) = \frac{1}{\nu} \int_{-\infty}^{\infty} (\mathbf{1}_{(|u| \leq 1 - 2\nu)} + 2\mathbf{1}_{(1 - 2\nu \leq |u| \leq 1 - \nu)}) g\left(\frac{u - x}{\nu}\right) du.$$

It is easy to see directly that, for any $0 < \nu < 1/4$,

$$V_\nu(0) = 1 \quad \text{and} \quad \int_{-1}^{1} V_\nu(x) \, dx = 2.$$

Therefore, denoting $D_u(x) = S_{u,\nu}(x) - S_0(x)$, we obtain, for any $u \in \mathbb{R}$,

$$\Omega_{x_0,h}(D_u) = \int_{-1}^{1} \left(D_u(x_0 + hz) - D_u(x_0)\right) dz = 0.$$

Moreover, note that, for any fixed $b > 0$,

$$\sup_{|u| \leq b} \left|\dot{D}_u(x)\right| = \sup_{|u| \leq b} \left(|u| \varphi_T^{-1} h^{-1} \left|\dot{V}_\nu\left(\frac{x - x_0}{h}\right)\right|\right) \leq b T^{-\alpha/(2\beta+1)} \nu^{-2} \dot{g}^*,$$

where $\dot{g}^* = \sup_x |\dot{g}(x)|$. Therefore, $\sup_{x \in \mathbb{R}} \sup_{|u| \leq b} |\dot{D}_u(x)| \leq M/2$ for sufficiently large T and, in view of the equality (2.3), the functions $(S_{u,\nu})_{|u| \leq b}$ belong to the class $\mathcal{U}_M(x_0, \beta)$ for sufficiently large T. It implies that, for any $b > 0$ and for sufficiently large T, we can estimate from below the right-hand term in the inequality (6.3) as

$$\sup_{S \in \mathcal{U}_M(x_0,\beta)} \sqrt{2q_S(x_0)} \mathcal{R}_S^*(\widetilde{S}_T) \geq \sup_{|u| \leq b} \sqrt{2q_{S_{u,\nu}}(x_0)} \mathcal{R}_{S_{u,\nu}}^*(\widetilde{S}_T)$$

$$= \sup_{|u| \leq b} \sqrt{2q_{S_0}(x_0)} \mathcal{R}_{S_{u,\nu}}^*(\widetilde{S}_T) + Q_T,$$

where $Q_T = \sup_{|u| \leq b} \sqrt{2q_{S_{u,\nu}}(x_0)} \mathcal{R}_{S_{u,\nu}}^*(\widetilde{S}_T) - \sup_{|u| \leq b} \sqrt{2q_{S_0}(x_0)} \mathcal{R}_{S_{u,\nu}}^*(\widetilde{S}_T)$.

It is easy to see that

$$|Q_T| \leq \sup_{|u| \leq b} \left|\sqrt{2q_{S_{u,\nu}}(x_0)} - \sqrt{2q_{S_0}(x_0)}\right| \mathcal{R}_{S_{u,\nu}}^*(\widetilde{S}_T)$$

$$\leq \frac{1}{\sqrt{2q_*}} \sup_{|u| \leq b} \left|q_{S_{u,\nu}}(x_0) - q_{S_0}(x_0)\right| \mathcal{R}_{S_{u,\nu}}^*(\widetilde{S}_T).$$

Taking into account here that

$$\lim_{T\to\infty} \sup_{|u|\le b} \left|q_{S_{u,v}}(x_0) - q_{S_0}(x_0)\right| = 0,$$

we obtain the inequality (5.2) by making use of the Theorem 4.1 from [10]. Thus, we obtain the Theorem 5.1.

6.2. Upper bound

We begin with stating the following result for the term (4.5).

Proposition 6.1. *The function B_T defined in* (4.5) *satisfies the following asymptotic property*

$$\limsup_{T\to\infty} \varphi_T \sup_{\vartheta\in\Theta_\beta} \mathbf{E}_\vartheta |B_T| = 0. \tag{6.4}$$

The result is proved in the Appendix.

Now we prove Theorem 5.2. To this end, we set

$$\widetilde{\phi}(u) = \sum_{j=N_0}^{+\infty} \phi_j \mathbf{1}_{\{t_{j-1} < u \le t_j\}},$$

where the random variables $(\phi_i)_{i\ge 1}$ are defined in (3.4). Using this function, we introduce the stopping time

$$\tau = \tau_T = \inf\left\{t \ge T_0 \colon \int_{T_0}^t \widetilde{\phi}(u)\,du \ge \delta H_T\right\},$$

where $T_0 = t_{N_0} = \delta N_0$. As usually, we put $\tau = \infty$ if the set $\{\cdot\}$ is empty. Obviously that

$$\tau \le T + \delta H_T \le T + \delta h(N - N_0)/\sqrt{\upsilon_T}.$$

Due to the equality $\int_{T_0}^\infty \widetilde{\phi}(u)\,du = \infty$, one obtains immediately that the random variable

$$\xi_T = \frac{1}{\sqrt{\delta H_T}} \int_{T_0}^\tau \widetilde{\phi}(u)\,dW_u \tag{6.5}$$

is Gaussian $\mathcal{N}(0,1)$ (see, e.g., [28], Chapter 17). Now, using this property, we can rewrite the deviation (4.6) on set Γ_T as

$$S_{h,T}^*(x_0) - S(x_0) = B_T^* + \mathbf{M}_T^{(1)} + \sigma(x_0)\mathbf{M}_T^{(2)} + \frac{\sigma(x_0)}{\sqrt{\delta H_T}}\xi_T, \tag{6.6}$$

where $B_T^* = \Upsilon_{1,T} + \Upsilon_{2,T} + B_T$,

$$\mathbf{M}_T^{(1)} = \frac{1}{\delta H_T} \sum_{j=N_0}^N \widetilde{\varkappa}_j \chi_{h,x_0}(y_{t_{j-1}})\big(\sigma(y_{t_{j-1}}) - \sigma(x_0)\big)\Delta W_{t_j}$$

and

$$\mathbf{M}_T^{(2)} = \frac{1}{\delta H_T}\left(\sum_{j=N_0}^N \widetilde{\varkappa}_j \phi_j \Delta W_{t_j} - \int_{T_0}^\tau \widetilde{\phi}(u)\,\mathrm{d}W_u\right).$$

First, we note that the definition of the sequence $(\widetilde{\varkappa}_j)_{j\geq 1}$ in (3.6) implies

$$\sum_{j=N_0}^N \widetilde{\varkappa}_j \chi_{h,x_0}(y_{t_{j-1}}) \leq H_T \qquad \text{a.s.} \tag{6.7}$$

Therefore, through the condition (2.1)

$$\mathbf{E}_\vartheta\left(\mathbf{M}_T^{(1)}\right)^2 = \mathbf{E}_\vartheta\left(\frac{1}{\delta H_T^2}\sum_{j=N_0}^N \widetilde{\varkappa}_j^2 \chi_{h,x_0}(y_{t_{j-1}})\big(\sigma(y_{t_{j-1}}) - \sigma(x_0)\big)^2\right)$$

$$\leq \mathbf{E}_\vartheta \frac{h^2\sigma_{\max}^2}{\delta H_T}.$$

Taking into account here that, for $T \geq 3$,

$$H_T \geq h(N - N_0)(2\sqrt{\upsilon_T} - \upsilon_T) \geq h(N - N_0)\sqrt{\upsilon_T}, \tag{6.8}$$

we obtain

$$\lim_{T\to\infty} \varphi_T \sup_{\vartheta\in\Theta_\beta} \mathbf{E}_\vartheta\left|\mathbf{M}_T^{(1)}\right| = 0.$$

Now we study the term $\mathbf{M}_T^{(2)}$. To this end, note that $t_{\varpi-1} < \tau \leq t_\varpi$. Therefore, we can represent this term as

$$\mathbf{M}_T^{(2)} = \frac{1}{\delta H_T}\big(\varkappa\phi_\varpi \Delta W_{t_\varpi} - \phi_\varpi(W_\tau - W_{t_{\varpi-1}})\big).$$

Moreover, taking into account that the stopping times ϖ and τ are bounded, one gets

$$\mathbf{E}(\Delta W_{t_\varpi})^2 = \delta \quad\text{and}\quad \mathbf{E}(W_\tau - W_{t_{\varpi-1}})^2 = \mathbf{E}(\tau - t_{\varpi-1}) \leq \delta.$$

Therefore, from here and (6.8), we get

$$\mathbf{E}_\vartheta\left(\mathbf{M}_T^{(2)}\right)^2 \leq 2\mathbf{E}_\vartheta \frac{1}{\delta H_T^2} \leq \frac{2}{\delta h^2 \upsilon_T (N - N_0)^2}$$

and

$$\lim_{T\to\infty} \varphi_T \sup_{\vartheta\in\Theta_\beta} \mathbf{E}_\vartheta\left|\mathbf{M}_T^{(2)}\right| = 0.$$

Due to (4.2) and (4.4), it is easy to see that

$$\lim_{T \to \infty} \varphi_T \sup_{\vartheta \in \Theta_\beta} \mathbf{E}_\vartheta |\Upsilon_{i,T}| = 0, \qquad i = 1, 2.$$

To put an end to the proof of this theorem, we present the last term on the right-hand side of (6.6) as

$$\frac{\sigma(x_0)}{\sqrt{\delta H_T}} \xi_T = \frac{\sigma(x_0)}{\sqrt{\delta h(N - N_0)}} \left(\frac{1}{\sqrt{2q_\vartheta(x_0)}} \xi_T + K_T \xi_T \right),$$

where

$$K_T = \frac{1}{\sqrt{2\widetilde{q}_T(x_0) - \upsilon_T}} - \frac{1}{\sqrt{2q_\vartheta(x_0)}},$$

and we have to show that

$$\lim_{T \to \infty} \sup_{\vartheta \in \Theta_\beta} \mathbf{E}_\vartheta |K_T| |\xi_T| = 0. \tag{6.9}$$

It is easy to see that, for any $T > 0$, the random variable ξ_T is $(0, 1)$-Gaussian conditionally with respect to $\mathcal{F}_{T_0}$. Therefore,

$$\mathbf{E}_\vartheta |K_T| |\xi_T| = \sqrt{\frac{2}{\pi}} \mathbf{E}_\vartheta |K_T|.$$

Taking into account here Lemma A.5, we come to the equality (6.9). Hence Theorem 5.2.

7. Conclusion

In the paper, we studied the estimation problem of the function S when its smoothness is known. In the case of unknown smoothness, in order to construct an adaptive estimate based on discrete time observations (1.2) in the model (1.1) we shall use the approach developed in [7] for continuous time observations. The approach make use of Lepskii's procedure and sequential estimating. Note that Lepskii's procedure works here just thanks to sequential estimating since, for the sequential estimate of the function S, the stochastic term in the deviation (6.6) is a Gaussian random variable. This provides correct estimating the tail distribution of a kernel estimate and adapting for the pointwise risk. Moreover, for adaptive estimating in the case of quadratic risk, we shall apply the selection model developed in [14] to sequential kernel estimates (3.7). Note once more, that Gaussianity of the stochastic term in (6.6) is a cornerstone result for obtaining a sharp oracle inequality. It permits to find Pinsker's constant like to [14] and then to study the proposed procedure efficiency.

These both programs will be realized in the next paper.

Appendix

A.1. Geometric ergodicity

First of all, we recall that in [13] we have proved the following result.

Theorem A.1. *For any $\varepsilon > 0$, there exist constants $R = R(\varepsilon) > 0$ and $\kappa = \kappa(\varepsilon) > 0$ such that*

$$\sup_{u \geq 0} e^{\kappa u} \sup_{\|g\|_* \leq 1} \sup_{x \in \mathbb{R}} \sup_{\vartheta \in \Sigma_{L,M} \times \mathcal{V}} \frac{|\mathbf{E}_{\vartheta,x} g(y_u) - \mathbf{m}_\vartheta(g)|}{(1 + x^2)^\varepsilon} \leq R,$$

where $\mathbf{E}_{\vartheta,x}(\cdot) = \mathbf{E}_\vartheta(\cdot | y_0 = x)$, $\|g\|_* = \sup_x |g(x)|$.

A.2. Concentration inequality

For any $\mathbb{R} \to \mathbb{R}$ function f belonging to $\mathbf{L}_1(\mathbb{R})$, we set

$$\mathbf{D}_n(f) = \sum_{k=1}^{n} \big(f(y_{t_k}) - \mathbf{m}_\vartheta(f)\big). \tag{A.1}$$

Now we assume that the frequency δ in the observations (1.2) is of the following form

$$\delta = \delta_T = \frac{1}{(T+1)l_T}, \tag{A.2}$$

where the function l_T is such that,

$$\lim_{T \to \infty} \frac{l_T}{T^{1/2}} = 0 \quad \text{and} \quad \lim_{T \to \infty} \frac{l_T}{\ln T} = +\infty, \tag{A.3}$$

in particular, the function $l_T = (\ln T)^{1+\gamma}$ from (2.7) is of this kind. Moreover, let $\varkappa^* = \varkappa_T^*$ be a positive function satisfying the following properties

$$\lim_{T \to \infty} \varkappa_T^* = 0 \quad \text{and} \quad \lim_{T \to \infty} \frac{l_T(\varkappa_T^*)^5}{\ln T} = +\infty. \tag{A.4}$$

Theorem A.2 ([12]). *Assume that the frequency δ satisfies (A.2)–(A.3). Then, for any $a > 0$,*

$$\lim_{T \to \infty} T^a \sup_{h \geq T^{-1/2}} \sup_{\vartheta \in \Theta_\beta} \mathbf{P}_\vartheta\big(|\mathbf{D}_N(\chi_{h,x_0})| \geq \varkappa_T^* T\big) = 0. \tag{A.5}$$

A.3. Proof of Proposition 4.1

First, we note that by the Bunyakovskii–Cauchy–Schwarz inequality

$$\mathbf{E}_\vartheta \left(\varrho_j^2 | \mathcal{F}_{t_{j-1}} \right) \leq \delta \widetilde{L}^2 \int_{t_{j-1}}^{t_j} \mathbf{E}_\vartheta \left((y_u - y_{t_{j-1}})^2 | \mathcal{F}_{t_{j-1}} \right) du,$$

where $\widetilde{L} = \max(L, M)$. Note now that, for $t_{j-1} \leq u \leq t_j$,

$$\mathbf{E}_\vartheta \left((y_u - y_{t_{j-1}})^2 | \mathcal{F}_{t_{j-1}} \right) \leq 2\delta \left(\int_{t_{j-1}}^u \mathbf{E}_\vartheta \left(S^2(y_v) | \mathcal{F}_{t_{j-1}} \right) dv + \sigma_{\max}^2 \right)$$

$$\leq 2\delta \left(2 \int_{t_{j-1}}^u \left(M^2 + L^2 \mathbf{E}_\vartheta \left(y_v^2 | \mathcal{F}_{t_{j-1}} \right) \right) dv + \sigma_{\max}^2 \right).$$

Due to Proposition A.6, we can estimate the last conditional expectation as

$$\sup_{\vartheta \in \Sigma_{L,M} \times [0,\sigma_{\max}]} \sup_{t_{j-1} \leq u \leq t_j} \mathbf{E}_\vartheta \left(y_u^2 | \mathcal{F}_{t_{j-1}} \right) \leq D_* L + y_{t_{j-1}}^2.$$

Therefore, taking into account that $\chi_{h,x_0}(y_{t_{j-1}}) y_{t_{j-1}}^2 \leq x_*^2$, we obtain

$$\sup_{t_{j-1} \leq u \leq t_j} \sup_{\vartheta \in \Sigma_{L,M} \times [0,\sigma_{\max}]} \chi_{h,x_0}(y_{t_{j-1}}) \mathbf{E}_\vartheta \left((y_u - y_{t_{j-1}})^2 | \mathcal{F}_{t_{j-1}} \right) \leq L_1 \delta. \tag{A.6}$$

Therefore,

$$\sup_{j \geq 1} \sup_{\vartheta \in \Sigma_{L,M} \times [0,\sigma_{\max}]} \chi_{h,x_0}(y_{t_{j-1}}) \mathbf{E}_\vartheta \left(\varrho_j^2 | \mathcal{F}_{t_{j-1}} \right) \leq \widetilde{L}^2 L_1 \delta^3.$$

Making use of the inequality (6.7) yields the following upper bound, through the Bunyakovskii–Cauchy–Schwarz inequality,

$$\mathbf{E}_\vartheta \Upsilon_{1,T}^2 \leq \mathbf{E}_\vartheta \frac{1}{\delta^2 H_T} \sum_{j=N_0}^N \widetilde{\varkappa}_j \chi_{h,x_0}(y_{t_{j-1}}) \varrho_j^2$$

$$= \mathbf{E}_\vartheta \frac{1}{\delta^2 H_T} \sum_{j=N_0}^N \widetilde{\varkappa}_j \chi_{h,x_0}(y_{t_{j-1}}) \mathbf{E}_\vartheta \left(\varrho_j^2 | \mathcal{F}_{t_{j-1}} \right) \leq \widetilde{L}^2 L_1 \delta.$$

Hence, Proposition 4.1.

A.4. Proof of Proposition 4.2

Note that by the condition (2.14)

$$\mathbf{E}_\vartheta \left((\varrho_j^*)^2 | \mathcal{F}_{t_{j-1}} \right) = \int_{t_{j-1}}^{t_j} \mathbf{E}_\vartheta \left((\sigma(y_u) - \sigma(y_{t_{j-1}}))^2 | \mathcal{F}_{t_{j-1}} \right) du$$

$$\leq \sigma_{\max}^2 \int_{t_{j-1}}^{t_j} \mathbf{E}_\vartheta \left((y_u - y_{t_{j-1}})^2 | \mathcal{F}_{t_{j-1}} \right) du.$$

Therefore, using the inequality (A.6) one has

$$\sup_{j \geq 1} \sup_{\vartheta \in \Sigma_{L,M} \times [0,\sigma_{\max}]} \chi_{h,x_0}(y_{t_{j-1}}) \mathbf{E}_\vartheta \left((\varrho_j^*)^2 | \mathcal{F}_{t_{j-1}} \right) \leq \sigma_{\max}^2 L_1 \delta^2.$$

From here and (6.7) and, taking into account that $0 < \widetilde{\varkappa}_j \leq 1$, we obtain

$$\mathbf{E}_\vartheta \Upsilon_{2,T}^2 = \mathbf{E}_\vartheta \frac{1}{\delta^2 H_T^2} \sum_{j=N_0}^{N} \widetilde{\varkappa}_j^2 \chi_{h,x_0}(y_{t_{j-1}}) \mathbf{E}_\vartheta \left((\varrho_j^*)^2 | \mathcal{F}_{t_{j-1}} \right)$$

$$\leq \sigma_{\max}^2 L_1 \mathbf{E}_\vartheta \frac{1}{H_T}.$$

Now the inequality (6.8) yields (4.4). Hence, Proposition 4.2.

A.5. Proof of Proposition 4.3

Taking into account the inequalities (6.7) and (6.8), we obtain that, for any $T \geq 3$,

$$\mathbf{E}_\vartheta \mathbf{M}_T^2 = \mathbf{E}_\vartheta \frac{1}{\delta^2 H_T^2} \sum_{j=N_0}^{N} \chi_{h,x_0}(y_{t_{j-1}}) \widetilde{\varkappa}_j^2 \mathbf{E}_\vartheta \left(\eta_j^2 | \mathcal{F}_{t_{j-1}} \right)$$

$$\leq \mathbf{E}_\vartheta \frac{1}{\delta^2 H_T^2} \sum_{j=N_0}^{N} \widetilde{\varkappa}_j \chi_{h,x_0}(y_{t_{j-1}}) \int_{t_{j-1}}^{t_j} \sigma^2(y_u) \, \mathrm{d}u$$

$$\leq \frac{\sigma_{\max}^2}{\delta h (N - N_0) \sqrt{\upsilon_T}}.$$

Hence, Proposition 4.3.

A.6. Proof of Proposition 6.1

We start with setting

$$r_T = \frac{(2\widetilde{q}_T - \upsilon_T)h}{\mathbf{m}_\vartheta(\chi_{h,x_0})} \quad \text{and} \quad N_1 = N_0 + r_T(N - N_0).$$

Note that $N_1 - N_0 \leq (q_* \sqrt{\upsilon_T})^{-1} N := N_1^*$, for sufficiently large T. Moreover, we set

$$G_T = \frac{1}{H_T} \sum_{j=N_0}^{N_1} f_h(y_{t_{j-1}}) \quad \text{and} \quad \widehat{G}_T = G_T - B_T.$$

Using (2.12), we can represent the term G_T as

$$G_T = \frac{N_1 - N_0}{H_T} \mathbf{m}_\vartheta(f_h) + \frac{1}{H_T} \sum_{j=N_0}^{N_1} \widetilde{f}_h(y_{t_{j-1}}) := G_1(T) + G_2(T),$$

where $\widetilde{f}_h(y) = f_h(y) - \mathbf{m}_\vartheta(f_h)$. Taking into account that $\mathbf{m}_\vartheta(\chi_{h,x_0}) \geq 2hq_*$, we obtain

$$\left|G_1(T)\right| = \frac{r_T(N - N_0)h}{H_T}\left|\mathbf{m}_\vartheta^*(h)\right| \leq \frac{1}{2q_*}\left|\mathbf{m}_\vartheta^*(h)\right|, \qquad \mathbf{m}_\vartheta^*(h) = \frac{\mathbf{m}_\vartheta(f_h)}{h}.$$

Let us represent the last term as

$$\mathbf{m}_\vartheta^*(h) = q_\vartheta(x_0)\Omega_{x_0,h}(S) + \widetilde{\mathbf{m}}_\vartheta(h),$$

where $\widetilde{\mathbf{m}}_\vartheta(h) = \int_{-1}^1 (S(x_0 + hz) - S(x_0))(q_\vartheta(x_0 + hz) - q_\vartheta(x_0))\,dz$. Further, by the definition (2.6), one has

$$\Omega_{x_0,h}(S) = \Omega_{x_0,h}(S_0) + \Omega_{x_0,h}(D),$$

for some function D from $\mathcal{H}_{x_0}^w(\varepsilon, \beta, h)$. Therefore, the properties (2.4)–(2.5) and (2.7) yield

$$\lim_{h \to 0} \varphi_T \sup_{S \in \mathcal{U}_M(x_0,\beta)} \left|\Omega_{x_0,h}(S)\right| = 0.$$

Obviously, that

$$\limsup_{h \to 0} h^{-2} \sup_{\vartheta \in \Theta_\beta} \left|\widetilde{\mathbf{m}}_\vartheta(h)\right| < \infty.$$

Hence,

$$\limsup_{T \to \infty} \varphi_T \sup_{\vartheta \in \Theta_\beta} \mathbf{E}_\vartheta\left|G_1(T)\right| = 0.$$

Now we note that,

$$\mathbf{E}_\vartheta G_2^2(T) = \mathbf{E}_\vartheta \frac{1}{H_T^2}\left(\sum_{j=N_0}^{N_1-1} \Psi_j + \widetilde{f}_h^2(y_{t_{N_1-1}})\right),$$

where $\Psi_j = \widetilde{f}_h^2(y_{t_{j-1}}) + 2\widetilde{f}_h(y_{t_{j-1}})\sum_{l=j+1}^{N_1} \mathbf{E}_\vartheta(\widetilde{f}_h(y_{t_{l-1}})|\mathcal{F}_{t_{j-1}})$ and $\mathcal{F}_t = \sigma\{y_s, 0 \leq s \leq t\}$. Taking into account that $(y_t)_{t \geq 0}$ is a homogeneous Markov process and that $|\widetilde{f}_h(y)| \leq 2Mh$, we estimate from above the last conditional expectation, through the Theorem A.1 (for $\varepsilon = 1/2$), as

$$\left|\mathbf{E}_\vartheta\left(\widetilde{f}_h(y_{t_{l-1}})|\mathcal{F}_{t_{j-1}}\right)\right| = \left|\mathbf{E}_{\vartheta,y_{t_{j-1}}} \widetilde{f}_h(y_{t_{l-j}})\right| \leq 2MhR\left(1 + y_{t_{j-1}}^2\right)^{1/2}\mathrm{e}^{-\kappa t_{l-j}}$$

$$\leq 2MhR\left(1 + |y_{t_{j-1}}|\right)\mathrm{e}^{-\kappa\delta(l-j)}.$$

Therefore,

$$|\Psi_j| \leq 4M^2h^2\left(1 + \frac{2R(1 + |y_{t_{j-1}}|)}{e^{\kappa\delta} - 1}\right).$$

From here, bounding $e^{\kappa\delta} - 1$ by $\kappa\delta$, we get

$$\mathbf{E}_\vartheta G_2^2(T) \leq 8M^2h^2\mathbf{E}_\vartheta \frac{1}{H_T^2} \sum_{j=N_0}^{N_1} \left(1 + \frac{R}{\kappa\delta}(1 + |y_{t_{j-1}}|)\right)$$

$$\leq 8M^2h^2\left(1 + \frac{R}{\kappa\delta}\right)\mathbf{E}_\vartheta \frac{(N_1 - N_0)}{H_T^2}$$

$$+ 8M^2h^2\frac{R}{\kappa\delta}\mathbf{E}_\vartheta \frac{1}{H_T^2} \sum_{j=N_0}^{N_1} \left(\mathbf{E}_\vartheta\left(y_{t_{j-1}}^2|\mathcal{F}_{t_{N_0-1}}\right)\right)^{1/2}.$$

By making use of Proposition A.6, one obtains

$$\mathbf{E}_\vartheta G_2^2(T) \leq 8M^2h^2\left(1 + \frac{R}{\kappa\delta}\right)\mathbf{E}_\vartheta \frac{N_1 - N_0}{H_T^2}\left(1 + \sqrt{D_*L} + |y_{t_{N_0-1}}|\right).$$

Now from (6.8) it follows that

$$\frac{N_1 - N_0}{H_T^2} = \frac{1}{H_T\mathbf{m}_\vartheta(\chi_{h,x_0})} \leq \frac{1}{2h^2\sqrt{\upsilon_T}(N - N_0)q_*}.$$

Thus,

$$\sup_{\vartheta\in\Theta_\beta} \mathbf{E}_\vartheta G_2^2(T) \leq \frac{G^*}{\delta\sqrt{\upsilon_T}(N - N_0)},$$

where $G^* = 4M^2(\kappa + R)(1 + 2\sqrt{D_*L} + |y_0|)/(\kappa q_*)$. From this equality, we obtain immediately

$$\lim_{T\to\infty} \varphi_T \sup_{\vartheta\in\Theta_\beta} \mathbf{E}_\vartheta\left|G_2(T)\right| = 0.$$

Let us estimate the term $\widehat{G}_T$. Taking into account the lower bound (6.8), we get

$$|\widehat{G}_T| \leq \frac{Mh}{H_T}\left|\sum_{j=N_0}^{N_1} \chi_{h,x_0}(y_{t_{j-1}}) - H_T\right| + \frac{2Mh}{H_T}$$

$$\leq \frac{M}{\sqrt{\upsilon_T}(N - N_0)} \sum_{j=N_0}^{N_1} \left|\widetilde{\chi}_h(y_{t_{j-1}})\right| + \frac{2M}{\sqrt{\upsilon_T}(N - N_0)},$$

where $\widetilde{\chi}_h(y) = \chi_{h,x_0}(y) - \mathbf{m}_\vartheta(\chi_{h,x_0})$. By making use of the Theorem A.1 with $\varepsilon = 1/2$ one gets

$$\sum_{j \geq N_0} \mathbf{E}_{\vartheta,y_0} \big| \widetilde{\chi}_h(y_{t_{j-1}}) \big| \leq \frac{R \mathrm{e}^{-\kappa\delta(N_0-1)}}{1 - \mathrm{e}^{-\kappa\delta}} \big(1 + \sqrt{D_* L} + |y_0| \big).$$

This inequality implies directly

$$\lim_{T \to \infty} \varphi_T \sup_{\vartheta \in \Theta_\beta} \mathbf{E}_\vartheta |\widehat{G}_T| = 0.$$

Hence, Proposition 6.1.

A.7. Properties of the estimate (3.3)

Lemma A.3. *Assume that the parameter δ is of the form (2.13). Then, for any $a > 0$,*

$$\lim_{T \to \infty} T^a \sup_{\vartheta \in \Theta_\beta} \mathbf{P}_\vartheta \big(\big| \widehat{q}_T(x_0) - q_\vartheta(x_0) \big| > \upsilon_T \big) = 0.$$

Proof. Denoting $\psi_\varsigma(y) = (1/\varsigma)Q((y-x_0)/\varsigma)$ one has

$$\widehat{q}_T(x_0) - q_\vartheta(x_0) = \frac{1}{2} \int_{-1}^{1} \big(q_\vartheta(x_0 + \varsigma z) - q_\vartheta(x_0) \big) \, \mathrm{d}z$$

$$+ \frac{1}{2(N_0 - 1)} \mathbf{D}_{N_0-1}(\psi_\varsigma).$$

Therefore

$$\mathbf{P}_\vartheta \big(\big| \widehat{q}_T(x_0) - q_\vartheta(x_0) \big| > \upsilon_T \big)$$

$$\leq \mathbf{P}_\vartheta \left(\left| \int_{-1}^{1} \big(q_\vartheta(x_0 + \varsigma z) - q_\vartheta(x_0) \big) \, \mathrm{d}z \right| > \upsilon_T \right) + \mathbf{P}_\vartheta \left(\frac{1}{N_0 - 1} \mathbf{D}_{N_0-1}(\psi_\varsigma) > \upsilon_T \right).$$

The first term on the right-hand side equals to zero for sufficiently large T since

$$\left| \int_{-1}^{1} \big(q_\vartheta(x_0 + \varsigma z) - q_\vartheta(x_0) \big) \, \mathrm{d}z \right| \leq \varsigma^2 \ddot{q}^* < \upsilon_T,$$

for sufficiently large T, where $\ddot{q}^* = \sup_x \sup_\vartheta |\ddot{q}_\vartheta(x)| < \infty$. Applying Theorem A.2, to the second term on the right-hand side of the same inequality yields the Lemma A.3. $\qquad\square$

Lemma A.4. *Assume that the parameter δ is of the form (2.13). Then, for any $a > 0$,*

$$\lim_{T \to \infty} T^a \sup_{\vartheta \in \Theta_\beta} \mathbf{P}_\vartheta \big(\big| \widetilde{q}_T(x_0) - q_\vartheta(x_0) \big| > \upsilon_T \big) = 0.$$

Proof. Note, that for sufficiently large T (for that $\ln T \geq \max(q^{*2}, 1/q_*^2)$),

$$\left|\widetilde{q}_T(x_0) - q_\vartheta(x_0)\right| \leq \left|\widehat{q}_T(x_0) - q_\vartheta(x_0)\right|.$$

The lemma follows immediately from Lemma A.3. $\qquad\square$

Lemma A.5. *Assume that the parameter δ is of the form* (2.13). *Then,*

$$\limsup_{T \to \infty} \frac{1}{v_T^{1/2}} \sup_{\vartheta \in \Theta_\beta} \mathbf{E}_\vartheta \left| \frac{1}{\widetilde{q}_T(x_0) - v_T} - \frac{1}{q_\vartheta(x_0)} \right| \leq \frac{4}{q^*} < \infty. \tag{A.7}$$

Proof. Indeed, for sufficiently large T for which

$$v_T \leq \min\left(1/(q^*)^2, 1/4\right),$$

we obtain

$$\mathbf{E}_\vartheta \left| \frac{1}{\widetilde{q}_T(x_0) - v_T} - \frac{1}{q_\vartheta(x_0)} \right| \leq \frac{2v_T^{1/2}}{q_*} + \frac{2}{q_* v_T^{1/2}} \mathbf{E}_\vartheta \left| \widetilde{q}_T(x_0) - q_\vartheta(x_0) \right|$$

$$\leq \frac{4v_T^{1/2}}{q_*} + \frac{2}{q_* v_T} \mathbf{P}_\vartheta \left(\left| \widetilde{q}_T(x_0) - q_\vartheta(x_0) \right| > v_T \right).$$

Now Lemma A.4 implies the equality (A.7). Hence, Lemma A.5. $\qquad\square$

A.8. Moment inequality for the process (1.1)

We state the moment bound from [13].

Proposition A.6. *Let $(y_t)_{t \geq 0}$ be a solution of the equation* (1.1). *Then, for any $z \in \mathbb{R}$ and $m \geq 1$,*

$$\sup_{u \geq 0} \sup_{\vartheta \in \Sigma_{L,M} \times [0, \sigma_{\max}]} \mathbf{E}_{\vartheta, z}(y_u)^{2m} \leq (2m - 1)!! \left(D_* L + z^2 \right)^m, \tag{A.8}$$

where $\mathbf{E}_{\vartheta, z}(\cdot) = \mathbf{E}_\vartheta(\cdot | y_0 = z)$ and $D_* = (M + Lx_* + 2x_*)^2 (L + M) + \sigma_{\max}^2$.

Proof. To obtain this inequality, we make use of the method proposed in [22], page 20, for linear stochastic equations. First of all, note that thanks to Theorem 4.7 from [28], for any $T > 0$, there exists some $\varepsilon > 0$ such that, for each $\vartheta \in \Theta_\beta$ and $z \in \mathbb{R}$,

$$\sup_{0 \leq t \leq T} \mathbf{E}_{\vartheta, z} e^{\varepsilon y_t^2} < \infty. \tag{A.9}$$

Let us denote $D_\vartheta(y) = 2yS(y) + \sigma^2(y) + \check{L}y^2$ and $\check{L} = L^{-1}$. Taking into account that $0 < \check{L} < 1$ and $x_* \geq 1$, we obtain that, for $|y| \leq x_*$,

$$\left| D_\vartheta(y) \right| \leq x_*^2 (2M + 1) + \sigma_{\max}^2.$$

Let now $|y| \geq x_*$. Denoting by y_* the projection of y onto the interval $[-x_*, x_*]$ we obtain that

$$2yS(y) = 2yS(y_*) + 2y_*\big(S(y) - S(y_*)\big) + 2(y - y_*)\big(S(y) - S(y_*)\big)$$
$$\leq 2|y|M + 2Lx_*|y - y_*| - 2\check{L}|y - y_*|^2$$
$$\leq 2(M + Lx_* + 2x_*)|y| - 2\check{L}y^2.$$

Therefore,

$$\sup_{\vartheta \in \Sigma_{L,M} \times [0, \sigma_{\max}]} \sup_{y \in \mathbb{R}} D_\vartheta(y) \leq D_*.$$

By the Itô formula, we obtain

$$dy_u^{2m} = -m\check{L}y_u^{2m} dt + my_u^{2(m-1)}\big(D_\vartheta(y_u) + 2(m-1)\sigma^2(y_u)\big) dt$$
$$+ 2my_u^{2m-1}\sigma(y_u) dW_t.$$

Moreover, the property (A.9) yields that, for any $m \geq 1$,

$$\mathbf{E}_\vartheta \int_0^t e^{-m\check{L}(t-s)} y_s^{2m-1}\sigma(y_s) dW_s = 0.$$

Therefore, $\mathbf{E}_\vartheta y_t^{2m} \leq z^{2m} + m(2m-1)D_* \int_0^t e^{-m\check{L}(t-s)} \mathbf{E}_\vartheta y_s^{2(m-1)} ds$. Now the induction implies directly the bound (A.8). Hence, Proposition A.6. $\qquad\square$

Proposition A.7. *Let* $(y_t)_{t \geq 0}$ *be a solution of the equation* (1.1). *Then, for any* $z \in \mathbb{R}$ *and* $m \geq 1$, *and for any stopping time* τ *taking values in* $[0, T]$, *one has*

$$\sup_{\vartheta \in \Sigma_{L,M} \times [0, \sigma_{\max}]} \mathbf{E}_{\vartheta, z}(y_\tau)^{2m} \leq B^*(m, z)T \qquad (A.10)$$

and

$$\sup_{\vartheta \in \Sigma_{L,M} \times [0, \sigma_{\max}]} \mathbf{E}_{\vartheta, z} \sup_{0 \leq u \leq T}(y_u)^{2m} \leq B_1^*(m, z)T, \qquad (A.11)$$

where $B^*(m, z) = (2m - 1)!!(D_* L + z^2)^m (D_* + 2(m - 1)\sigma_{\max}^2)$ *and* $B_1^*(m, z) = 1 + mB^*(m + 1, z)$.

The proof of this proposition follows immediately from Proposition 1.1.5 in [22].

A.9. Proof of Proposition 4.5

It is clear, that to show (3.7) it suffices to check that, for any $a > 0$,

$$\lim_{T \to \infty} T^a \sup_{\vartheta \in \Sigma_{L,M} \times [\sigma_{\min}, \sigma_{\max}]} \mathbf{P}_\vartheta\big(\Gamma_T^c\big) = 0. \qquad (A.12)$$

Indeed, by the definition of ϖ

$$\mathbf{P}_\vartheta\left(\Gamma_T^c\right) = \mathbf{P}_\vartheta\left(\sum_{j=N_0}^N \chi_{h,x_0}(y_{t_{j-1}}) < H_T\right)$$

$$= \mathbf{P}_\vartheta\left(\mathbf{D}_{N_0,N-1}(\chi_{h,x_0}) < \left(2\widetilde{q}_T - \upsilon_T - \mathbf{m}_\vartheta^*(\chi_{h,x_0})\right)(N - N_0)h\right),$$

where $\mathbf{D}_{k,n}(f) = \mathbf{D}_n(f) - \mathbf{D}_k(f)$ and

$$\mathbf{m}_\vartheta^*(\chi_{h,x_0}) = \frac{\mathbf{m}_\vartheta(\chi_{h,x_0})}{h} = \int_{-1}^1 q_\vartheta(x_0 + hz)\,\mathrm{d}z.$$

Taking into account the definition of υ_T in (3.3) we obtain that, for sufficiently large T,

$$\sup_{\vartheta \in \Theta_\beta} \int_{-1}^1 \left|q_\vartheta(x_0 + hz) - q_\vartheta(x_0)\right|\,\mathrm{d}z \leq \upsilon_T/4.$$

Therefore, for such T,

$$\mathbf{P}_\vartheta(\varpi > N) \leq \mathbf{P}_\vartheta\left(\left|\widetilde{q}_T(x_0) - q_\vartheta(x_0)\right| > \upsilon_T/8\right)$$
$$+ \mathbf{P}_\vartheta\left(\left|\mathbf{D}_{N_0,N-1}(\chi_{h,x_0})\right| > Nh\upsilon_T/2\right).$$

Now we estimate the last term as

$$\mathbf{P}_\vartheta\left(\left|\mathbf{D}_{N_0,N-1}(\chi_{h,x_0})\right| > Nh\upsilon_T/2\right) \leq \mathbf{P}_\vartheta\left(\left|\mathbf{D}_{N-1}(\chi_{h,x_0})\right| > Nh\upsilon_T/4\right)$$
$$+ \mathbf{P}_\vartheta\left(\left|\mathbf{D}_{N_0}(\chi_{h,x_0})\right| > Nh\upsilon_T/4\right).$$

By applying Lemma A.3 and the inequality (A.5), we obtain (A.12).

Acknowledgements

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