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On asymptotic constants in the theory of extremes for Gaussian processes

A.B. DIEKER[1] and B. YAKIR[2]

[1] *H. Milton Stewart School of Industrial and Systems Engineering, Georgia Institute of Technology, Atlanta, GA 30332, USA. E-mail: ton.dieker@isye.gatech.edu*
[2] *Department of Statistics, Hebrew University of Jerusalem, Mount Scopus, Jerusalem 91905, Israel. E-mail: msby@mscc.huji.ac.il*

This paper gives a new representation of Pickands' constants, which arise in the study of extremes for a variety of Gaussian processes. Using this representation, we resolve the long-standing problem of devising a reliable algorithm for estimating these constants. A detailed error analysis illustrates the strength of our approach.

Keywords: extremes; Gaussian processes; Monte Carlo simulation; Pickands' constants

1. Introduction

Gaussian processes and fields have emerged as a versatile yet relatively tractable class of models for random phenomena. Gaussian processes have been applied fruitfully to risk theory, statistics, machine learning, and biology, while Gaussian fields have been applied to neuroimaging, astrophysics, oceanography, as well as to other fields. Extremes and level sets are particularly important in these applications (Azaïs and Wschebor [7]). New applications and theoretical developments continually revive the interest in Gaussian processes, see, for instance, Meka [27].

Although the understanding of Gaussian processes and fields has advanced steadily over the past decades, a variety of results related to extremes (tail asymptotics, extreme value theorems, laws of iterated logarithm) are only "explicit" up to certain constants. These constants are referred to as Pickands' constants after their discoverer (Pickands, III [29]). It is believed that these constants may never be calculated (Adler [1]).

These constants have remained so elusive that devising an estimation algorithm with certain performance guarantees has remained outside the scope of current methodology (Dębicki and Mandjes [18]). The current paper resolves this open problem for the classical Pickands' constants. Our main tool is a new representation for Pickands' constant, which expresses the constant as the expected value of a random variable with low variance and therefore it is suitable for simulation. Our approach also gives rise to a number of new questions, which could lead to further improvement of our simulation algorithm or its underlying theoretical foundation. We expect that our methodology carries through for all of Pickands' constants, not only for the classical ones discussed here.

Several different representations of Pickands' constants are known, typically arising from various methodologies for studying extremes of Gaussian processes. Hüsler [23] uses triangular arrays to interpret Pickands' constant as a clustering index. Albin and Choi [3] have recently

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rediscovered Hüsler's representation. For sufficiently smooth Gaussian processes, various level-crossing tools can be exploited (Azaïs and Wschebor [6], Kobelkov [25]). Yet another representation is found when a sojourn approach is taken (Berman [9]). Aldous [4] explains various connections heuristically and also gives intuition behind other fundamental results in extreme-value theory. We also mention Chapter 12 in Leadbetter *et al.* [26], who use methods different from those of Pickands but arrive at the same representation.

The approach advocated in the current paper is inspired by a method which has been applied successfully in various statistical settings, see Siegmund *et al.* [33] and references therein. This method relies on a certain change-of-measure argument, which results in asymptotic expressions with a term of the form $\mathbb{E}(M/S)$, where M and S are supremum-type and sum-type (or integral-type) functionals, respectively. This methodology can also be applied directly to study extremes of Gaussian processes, in which case it yields a new method for establishing tail asymptotics. This will be pursued elsewhere.

Throughout this paper, we let $B = \{B_t\colon t \in \mathbb{R}\}$ be a standard fractional Brownian motion with Hurst index $\alpha/2 \in (0, 1]$, that is, a centered Gaussian process for which

$$\mathbb{C}\mathrm{ov}(B_s, B_t) = \tfrac{1}{2}\bigl[|s|^\alpha + |t|^\alpha - |t-s|^\alpha\bigr].$$

Note that has stationary increments and variance function $\mathbb{V}\mathrm{ar}(B_t) = |t|^\alpha$. The process $\{Z_t\}$ defined through $Z_t = \sqrt{2}B_t - |t|^\alpha$ plays a key role in this paper. This stochastic process plays a fundamental role in the stochastic calculus for fractional Brownian motion (Bender and Parczewski [8]). The "classical" definition of Pickands' constant $\mathcal{H}_\alpha$ is

$$\mathcal{H}_\alpha = \lim_{T\to\infty} \frac{1}{T}\mathbb{E}\Bigl[\sup_{t\in[0,T]} \mathrm{e}^{Z_t}\Bigr]. \tag{1}$$

Current understanding of $\mathcal{H}_\alpha$ and related constants is quite limited. It is known that $\mathcal{H}_1 = 1$ and that $\mathcal{H}_2 = 1/\sqrt{\pi}$ (Bickel and Rosenblatt [10], Piterbarg [30]), and that $\mathcal{H}_\alpha$ is continuous as a function of α (Dębicki [16]). Most existing work focuses on obtaining sharp bounds for these constants (Aldous [4], Dębicki [15], Dębicki and Kisowski [17], Dębicki *et al.* [21], Shao [32], Harper [22]). Previous work on estimating Pickands' constant through simulation has yielded contradictory results (Burnecki and Michna [13], Michna [28]).

The next theorem forms the basis for our approach to estimate $\mathcal{H}_\alpha$. Note that the theorem expresses $\mathcal{H}_\alpha$ in the form $\mathbb{E}(M/S)$. A different but related representation is given in Proposition 2 below, and we give yet another representation in Proposition 4.

Theorem 1. *We have*

$$\mathcal{H}_\alpha = \mathbb{E}\left[\frac{\sup_{t\in\mathbb{R}} \mathrm{e}^{Z_t}}{\int_{-\infty}^{\infty} \mathrm{e}^{Z_t}\,\mathrm{d}t}\right].$$

The representation $\mathbb{E}(M/S)$ is well-suited for estimating Pickands' constant by simulation. Although both M and S are finite random variables with infinite mean, we provide theoretical evidence that their ratio has low variance and our empirical results show that this representation is suitable for simulation.

This paper is organized as follows. Section 2 establishes two results which together yield Theorem 1. In Section 3, we state an auxiliary result that plays a key role in several of the proofs in this paper. Section 4 gives an error analysis when $\mathbb{E}(M/S)$ is approximated by a related quantity that can be simulated on a computer. In Section 5, we carry out simulation experiments to estimate Pickands' constant. Some proofs are deferred to Appendix A, and a table with our simulation results is included as Appendix B.

2. Representations

This section is devoted to connections between Pickands' classical representation and our new representation, thus establishing Theorem 1. We also informally argue why our new representation is superior from the point of view of estimation. This is explored further in the next section.

The following well-known change-of-measure lemma forms the basis for our results.

Lemma 1. *Fix $t \in \mathbb{R}$, and set $Z^{(t)} = \{\sqrt{2}B_s - |s-t|^{2H}\colon\ s \in \mathbb{R}\}$. For an arbitrary measurable functional F on $\mathbb{R}^{\mathbb{R}}$, we have*

$$\mathbb{E}\mathrm{e}^{Z_t} F(Z) = \mathbb{E}F\bigl(|t|^{2H} + Z^{(t)}\bigr).$$

When the functional F is moreover translation-invariant (invariant under addition of a constant function), we have

$$\mathbb{E}\mathrm{e}^{Z_t} F(Z) = \mathbb{E}F(\theta_t Z),$$

where the shift θ_t is defined through $(\theta_t Z)_s = Z_{s-t}$.

Proof. Set $\mathbb{Q}(A) = \mathbb{E}[\mathrm{e}^{Z_t} 1_A]$, and write $\mathbb{E}^{\mathbb{Q}}$ for the expectation operator with respect to $\mathbb{Q}$. Select an integer k and $s_1 < s_2 < \cdots < s_k$. We show that $(Z_{s_1}, \ldots, Z_{s_k})$ under $\mathbb{Q}$ has the same distribution as $(|t|^{2H} + Z^{(t)}_{s_1}, \ldots, |t|^{2H} + Z^{(t)}_{s_k})$ under $\mathbb{P}$, by comparing generating functions: for any $\beta_1, \ldots, \beta_k \in \mathbb{R}$,

$$\begin{aligned}
\log \mathbb{E}^{\mathbb{Q}} \exp\Biggl(\sum_i \beta_i Z_{s_i}\Biggr) &= -|t|^{2H} - \sum_i \beta_i |s_i|^{2H} + \mathbb{V}\mathrm{ar}\Biggl[B_t + \sum_i \beta_i B_{s_i}\Biggr] \\
&= \sum_i 2\beta_i \,\mathbb{C}\mathrm{ov}(B_t, B_{s_i}) - \sum_i \beta_i |s_i|^{2H} + \mathbb{V}\mathrm{ar}\Biggl[\sum_i \beta_i B_{s_i}\Biggr] \\
&= \sum_i \beta_i \bigl[|t|^{2H} - |s_i - t|^{2H}\bigr] + \mathbb{V}\mathrm{ar}\Biggl[\sum_i \beta_i B_{s_i}\Biggr] \\
&= \sum_i \beta_i |t|^{2H} + \mathbb{E}\Biggl[\sum_i \beta_i Z^{(t)}_{s_i}\Biggr] + \frac{1}{2}\,\mathbb{V}\mathrm{ar}\Biggl[\sum_i \beta_i Z^{(t)}_{s_i}\Biggr] \\
&= \mathbb{E}\Biggl[\sum_i \beta_i\bigl(|t|^{2H} + Z^{(t)}_{s_i}\bigr)\Biggr] + \frac{1}{2}\,\mathbb{V}\mathrm{ar}\Biggl[\sum_i \beta_i\bigl(|t|^{2H} + Z^{(t)}_{s_i}\bigr)\Biggr].
\end{aligned}$$

The first claim of the lemma then immediately follows from the Cramér–Wold device.

Alternatively, one could carefully define a space on which the distribution of Z becomes a Gaussian measure and then note that the claim follows from the Cameron–Martin formula; see Bogachev [12], Proposition 2.4.2 and Dieker [19] for key ingredients for this approach.

When the functional F is translation-invariant, we conclude that

$$\mathbb{E}^{\mathbb{Q}} F(Z) = \mathbb{E} F\bigl(|t|^{2H} + Z^{(t)}\bigr) = \mathbb{E} F\bigl(Z^{(t)} - \sqrt{2} B_t\bigr) = \mathbb{E} F(\theta_t Z),$$

and this proves the second claim in the lemma. □

The next corollary readily implies subadditivity of $\mathbb{E}[\sup_{0\le t\le T} \mathrm{e}^{Z_t}]$ as a function of T, a well-known fact that immediately yields the existence of the limit in (1). Evidently, we must work under the usual separability conditions, which ensure that the supremum functional is measurable.

Corollary 1. *For any $a < b$, we have*

$$\mathbb{E}\Bigl[\sup_{a\le t\le b} \mathrm{e}^{Z_t}\Bigr] = \mathbb{E}\Bigl[\sup_{0\le t\le b-a} \mathrm{e}^{Z_t}\Bigr], \qquad \mathbb{E}\biggl[\int_a^b \mathrm{e}^{Z_t}\,\mathrm{d}t\biggr] = \mathbb{E}\biggl[\int_0^{b-a} \mathrm{e}^{Z_t}\,\mathrm{d}t\biggr].$$

Proof. Applying Lemma 1 for $t = a$ to the translation-invariant functionals F given by $F(z) = \sup_{a\le s\le b} \mathrm{e}^{z_s - z_a}$ and $F(z) = \int_a^b \mathrm{e}^{z_t - z_a}\,\mathrm{d}t$ yields the claims. (The second claim is also immediate from $\mathbb{E}\mathrm{e}^{Z_t} = 1$.) □

Corollary 2. *For $T > 0$, we have*

$$\frac{1}{T}\mathbb{E}\Bigl[\sup_{0\le t\le T} \mathrm{e}^{Z_t}\Bigr] = \int_0^1 \mathbb{E}\biggl[\frac{\sup_{-uT\le s\le (1-u)T} \mathrm{e}^{Z_s}}{\int_{-uT}^{(1-u)T} \mathrm{e}^{Z_s}\,\mathrm{d}s}\biggr]\,\mathrm{d}u. \tag{2}$$

Proof. Applying Lemma 1 to the translation-invariant functional F given by

$$F(z) = \frac{\sup_{t\in[0,T]} \mathrm{e}^{z_t}}{\int_0^T \mathrm{e}^{z_u}\,\mathrm{d}u}$$

yields that, for any $T > 0$,

$$\begin{aligned}
\frac{1}{T}\mathbb{E}\Bigl[\sup_{0\le t\le T} \mathrm{e}^{Z_t}\Bigr] &= \frac{1}{T}\int_0^T \mathbb{E}\biggl[\mathrm{e}^{Z_t} \times \frac{\sup_{0\le s\le T} \mathrm{e}^{Z_s}}{\int_0^T \mathrm{e}^{Z_s}\,\mathrm{d}s}\biggr]\,\mathrm{d}t \\
&= \frac{1}{T}\int_0^T \mathbb{E}\biggl[\frac{\sup_{-t\le s\le T-t} \mathrm{e}^{Z_s}}{\int_{-t}^{T-t} \mathrm{e}^{Z_s}\,\mathrm{d}s}\biggr]\,\mathrm{d}t,
\end{aligned}$$

and the statement of the lemma follows after a change of variable. □

The left-hand side of the identity (2) converges to $\mathcal{H}_\alpha$ by definition. The next proposition shows that the right-hand side of (2) converges to our new representation, thereby proving Theorem 1. The proof of the proposition itself is deferred to Appendix A.

Proposition 1. *For any $u \in (0, 1)$, we have*

$$\lim_{T\to\infty} \mathbb{E}\left[\frac{\sup_{-uT\le s\le(1-u)T} \mathrm{e}^{Z_s}}{\int_{-uT}^{(1-u)T} \mathrm{e}^{Z_s}\,\mathrm{d}s}\right] = \mathbb{E}\left[\frac{\sup_{t\in\mathbb{R}} \mathrm{e}^{Z_t}}{\int_{-\infty}^{\infty} \mathrm{e}^{Z_t}\,\mathrm{d}t}\right] < \infty.$$

Moreover,

$$\mathcal{H}_\alpha = \lim_{T\to\infty} \int_0^1 \mathbb{E}\left[\frac{\sup_{-uT\le s\le(1-u)T} \mathrm{e}^{Z_s}}{\int_{-uT}^{(1-u)T} \mathrm{e}^{Z_s}\,\mathrm{d}s}\right] \mathrm{d}u = \mathbb{E}\left[\frac{\sup_{t\in\mathbb{R}} \mathrm{e}^{Z_t}}{\int_{-\infty}^{\infty} \mathrm{e}^{Z_t}\,\mathrm{d}t}\right].$$

Apart from establishing Theorem 1, this proposition gives two ways of approximating $\mathcal{H}_\alpha$. The speed at which the prelimits tend to $\mathcal{H}_\alpha$ is different for these two representations. For the second “integral” representation, which is the classical representation in view of Corollary 2, the speed of convergence to $\mathcal{H}_\alpha$ can be expected to be slow. Indeed, it is known to be of order $1/\sqrt{T}$ in the Brownian motion case (e.g., Dębicki and Kisowski [17]). This is in stark contrast with the speed of convergence in the first representation (e.g., for $u = 1/2$), as analyzed in the next section. Our study shows that the slow convergence speed in the classical definition is due to values of u close to the endpoints of the integration interval [0, 1] in the right-hand side of (2).

It is instructive to compare our new representation of $\mathcal{H}_\alpha$ with the classical representation of Pickands' constant through a discussion of variances. Note that $\mathbb{E}\mathrm{e}^{Z_s} = 1$, $\mathbb{V}\mathrm{ar}\,\mathrm{e}^{Z_s} = \mathrm{e}^{\mathbb{V}\mathrm{ar}(Z_s)} - \mathrm{e}^{-\mathbb{V}\mathrm{ar}(Z_s)}$, so that the variance blows up as s grows large. As a result, one can expect that $\sup_{0\le t\le T} \mathrm{e}^{Z_t}$ has high variance for large T. Moreover, significant contributions to its expectation come from values of t close to T. These two observations explain why it is hard to reliably estimate Pickands' constant from the classical definition.

Our new representation does not have these drawbacks. Let us focus on the special case $\alpha = 2$, for which it is known that $\mathcal{H}_2 = 1/\sqrt{\pi}$. Writing N for a standard normal random variable, we obtain that

$$\mathcal{H}_2 = \mathbb{E}\left[\frac{\sup_{t\in\mathbb{R}} \mathrm{e}^{\sqrt{2}tN-t^2}}{\int_{t\in\mathbb{R}} \mathrm{e}^{\sqrt{2}tN-t^2}\,\mathrm{d}t}\right] = \mathbb{E}\left[\frac{\sup_{t\in\mathbb{R}} \mathrm{e}^{-(t-N/\sqrt{2})^2}}{\int_{t\in\mathbb{R}} \mathrm{e}^{-(t-N/\sqrt{2})^2}\,\mathrm{d}t}\right] = \frac{1}{\int_{\mathbb{R}} \mathrm{e}^{-t^2}\,\mathrm{d}t} = \frac{1}{\sqrt{\pi}}.$$

It follows from this calculation that M/S has zero variance for $\alpha = 2$, so we can expect it to have very low variance for values of α close to 2.

We next present an alternative representation for $\mathcal{H}_\alpha$ in the spirit of Theorem 1. The proof of Corollary 2 shows that for any locally finite measure μ,

$$\frac{1}{T}\mathbb{E}\Big[\sup_{0\le t\le T} \mathrm{e}^{Z_t}\Big] = \int_0^1 \mathbb{E}\left[\frac{\sup_{-uT\le s\le(1-u)T} \mathrm{e}^{Z_s}}{\int_{-uT}^{(1-u)T} \mathrm{e}^{Z_s}\mu(\mathrm{d}s)}\right]\mu^{(T)}(\mathrm{d}u), \tag{3}$$

where $\mu^{(T)}(\mathrm{d}u) = \mu(T\,\mathrm{d}u)/T$. Of particular interest is the case where μ is the counting measure on $\eta\mathbb{Z}$. Then $\mu^{(T)}$ converges weakly to Leb/η, where Leb stands for Lebesgue measure. In view of this observation, the following analog of Proposition 1 is natural. The proof is given in Appendix A.

Proposition 2. *For any $\eta > 0$, we have*

$$\mathcal{H}_\alpha = \mathbb{E}\left[\frac{\sup_{t\in\mathbb{R}} e^{Z_t}}{\eta\sum_{k\in\mathbb{Z}} e^{Z_{k\eta}}}\right].$$

This identity is particularly noteworthy since the integral in the denominator in the representation of Theorem 1 can apparently be replaced with an approximating sum. For $\alpha = 2$, this means that for any $\eta > 0$,

$$\int_{\mathbb{R}} \frac{dy}{\sum_{k\in\mathbb{Z}} e^{ky\eta^2 - k^2\eta^2}} = 2.$$

We have not been able to verify this intriguing equality directly, but numerical experiments suggest that this identity indeed holds.

We conclude this section with two further related results. For $\eta > 0$, define the "discretized" Pickands constant through

$$\mathcal{H}_\alpha^\eta = \lim_{T\to\infty} \frac{1}{T}\mathbb{E}\Big[\sup_{k\in\mathbb{Z}:0\le k\eta\le T} e^{Z_{k\eta}}\Big].$$

The proof of the next proposition requires discrete analogs of Corollary 2 and Proposition 1, with suprema taken over a grid and integrals replaced by sums (for the first equality). The proof is omitted since it follows the proofs of these results verbatim.

Proposition 3. *For any $\eta > 0$, we have*

$$\mathcal{H}_\alpha^\eta = \mathbb{E}\left[\frac{\sup_{k\in\mathbb{Z}} e^{Z_{k\eta}}}{\eta\sum_{k\in\mathbb{Z}} e^{Z_{k\eta}}}\right] = \mathbb{E}\left[\frac{\sup_{k\in\mathbb{Z}} e^{Z_{k\eta}}}{\int_{-\infty}^{\infty} e^{Z_t}\, dt}\right].$$

The second representation for $\mathcal{H}_\alpha^\eta$ in this proposition immediately shows that $\mathcal{H}_\alpha = \lim_{\eta\downarrow 0} \mathcal{H}_\alpha^\eta$ by the monotone convergence theorem and sample path continuity.

A different application of Lemma 1 yields further representations for $\mathcal{H}_\alpha$ and $\mathcal{H}_\alpha^\eta$. Let F_t be the indicator of the event that the supremum of its (sample path) argument occurs at t. Since $E[F_t(Z)F_s(Z)] = 0$ for all $s \ne t$, we have

$$\mathbb{E}\Big[\sup_{k\in\mathbb{Z}:0\le k\eta\le T} e^{Z_{k\eta}}\Big] = \sum_{\ell=0}^{\lfloor T/\eta\rfloor} \mathbb{E}\big[e^{Z_{\ell\eta}} F_{\ell\eta}(Z)\big] = \sum_{\ell=0}^{\lfloor T/\eta\rfloor} \mathbb{P}\Big(\sup_{k\in\mathbb{Z}:-\ell\le k\le T/\eta-\ell} Z_{k\eta} = 0\Big),$$

where we use Lemma 1 to obtain the last equality. This can be written as

$$\frac{1}{T}\mathbb{E}\Big[\sup_{k\in\mathbb{Z}:0\le k\eta\le T} e^{Z_{k\eta}}\Big] = \int_0^1 \mathbb{P}\Big(\sup_{k\in\mathbb{Z}:-uT\le k\eta\le(1-u)T} Z_{k\eta} = 0\Big)\mu^{(T)}(du),$$

where, as before, $\mu^{(T)}(\mathrm{d}u) = \mu(T\,\mathrm{d}u)/T$ and μ is the counting measure on $\eta\mathbb{Z}$. Note the similarity with (3). Taking the limit as $T \to \infty$ requires verifications similar to those in the proof of Proposition 2; the details are given in Appendix A. The resulting representation is a two-sided version of the Hüsler–Albin–Choi representation (Albin and Choi [3], Hüsler [23]), and appears to be new.

Proposition 4. *For $\eta > 0$, we have*

$$\mathcal{H}_\alpha^\eta = \eta^{-1}\mathbb{P}\Bigl(\sup_{k\in\mathbb{Z}} Z_{k\eta} = 0\Bigr)$$

and therefore

$$\mathcal{H}_\alpha = \lim_{\eta\downarrow 0}\eta^{-1}\mathbb{P}\Bigl(\sup_{k\in\mathbb{Z}} Z_{k\eta} = 0\Bigr).$$

From the point of view of simulation, one difficulty with this representation is that one would have to estimate small probabilities when η is small. Unless one develops special techniques, it would require many simulation replications to reliably estimate these probabilities. As discussed below, such a task is computationally extremely intensive.

3. An auxiliary bound

This section presents a simple auxiliary bound which plays a key role in the next section. To formulate it, let Z_t^η be the following approximation of Z_t on a grid with mesh $\eta > 0$:

$$Z_t^\eta = \begin{cases} Z_{\eta\lfloor t/\eta\rfloor}, & \text{for } t > 0, \\ Z_{\eta\lceil t/\eta\rceil}, & \text{otherwise,} \end{cases}$$

and define B_t^η similarly in terms of B_t.

Let J be a fixed compact closed interval, assumed to be fixed throughout this section. We write

$$\Delta(\eta) = \sup_{t\in J}\bigl(Z_t - Z_t^\eta\bigr), \qquad \delta(\eta) = \sqrt{2}\sup_{t\in J}\bigl(B_t - B_t^\eta\bigr).$$

Define $M_J = \sup_{u\in J} \mathrm{e}^{Z_u}$ and $S_J^\eta = \int_J \mathrm{e}^{Z_u^\eta}\,\mathrm{d}u$. Note that

$$\frac{M_J}{S_J^\eta} \le \mathrm{e}^{\Delta(\eta)}\frac{M_J^\eta}{S_J^\eta} \le \frac{1}{\eta}\mathrm{e}^{\Delta(\eta)}.$$

Given an event E, we have for $\tau > \mathrm{e}^{\mathbb{E}\Delta(\eta)}$,

$$\begin{aligned}
&\mathbb{E}\bigl(M_J/S_J^\eta; E\bigr) \\
&\quad \le \mathbb{E}\bigl(M_J/S_J^\eta; M_J/S_J^\eta > \tau/\eta\bigr) + \frac{\tau}{\eta}\mathbb{P}(E)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\eta}\int_\tau^\infty \mathbb{P}\bigl(M_J/S_J^\eta > y/\eta\bigr)\,\mathrm{d}y + \frac{\tau}{\eta}\mathbb{P}\bigl(M_J/S_J^\eta > \tau/\eta\bigr) + \frac{\tau}{\eta}\mathbb{P}(E) \\
&\le \frac{1}{\eta}\int_\tau^\infty \mathbb{P}\bigl(\mathrm{e}^{\Delta(\eta)} > y\bigr)\,\mathrm{d}y + \frac{\tau}{\eta}\mathbb{P}\bigl(\mathrm{e}^{\Delta(\eta)} > \tau\bigr) + \frac{\tau}{\eta}\mathbb{P}(E) \\
&\le \frac{1}{\eta}\int_\tau^\infty \exp\biggl(-\frac{(\log(y)-\mathbb{E}\Delta(\eta))^2}{4\eta^\alpha}\biggr)\,\mathrm{d}y + \frac{\tau}{\eta}\exp\biggl(-\frac{(\log(\tau)-\mathbb{E}\Delta(\eta))^2}{4\eta^\alpha}\biggr) + \frac{\tau}{\eta}\mathbb{P}(E),
\end{aligned}
$$

where the last inequality uses Borell's inequality, for example, Adler and Taylor [2], Theorem 2.1.1. We can bound this further by bounding $\mathbb{E}\Delta(\eta)$. After setting

$$
\kappa(\eta) = \sup_{t\in J}\bigl(\mathbb{V}\mathrm{ar}(Z_t) - \mathbb{V}\mathrm{ar}\bigl(Z_t^\eta\bigr)\bigr),
$$

we obtain that $\Delta(\eta) \le \kappa(\eta) + \delta(\eta)$. We next want to apply Theorem 1.3.3 of Adler and Taylor [2] to bound $\mathbb{E}\delta(\eta)$, but the statement of this theorem contains an unspecified constant. Our numerical experiments require that all constants be explicit, and therefore we directly work with the bound derived in the proof of this theorem. Choose $r = 1/(2\eta^{\alpha/2})$, and set $N_j = |J|r^{j/H}$. The proof of this theorem shows that

$$
\mathbb{E}\delta(\eta) \le \sqrt{\frac{2\pi}{\log(2)}}\sum_{j=2}^\infty 2^{3/2}r^{-j+1}\sqrt{\log\bigl(2^{j+1}N_j^2\bigr)} =: \mathcal{E}(\eta),
$$

which is readily evaluated numerically.

As a result, whenever $\tau > \mathrm{e}^{\mathcal{E}(\eta)+\kappa(\eta)}$, we have

$$
\begin{aligned}
\mathbb{E}\bigl(M_J/S_J^\eta; E\bigr) \le{}& \frac{1}{\eta}\int_\tau^\infty \exp\biggl(-\frac{(\log(y)-\kappa(\eta)-\mathcal{E}(\eta))^2}{4\eta^\alpha}\biggr)\,\mathrm{d}y \\
&+ \frac{\tau}{\eta}\exp\biggl(-\frac{(\log(\tau)-\kappa(\eta)-\mathcal{E}(\eta))^2}{4\eta^\alpha}\biggr) + \frac{\tau}{\eta}\mathbb{P}(E).
\end{aligned}
$$

To apply this bound, one needs to select τ appropriately. Note that we may let τ depend on the interval J.

4. Estimation

This section studies the effect of truncation and discretization of Z on $\mathcal{H}_\alpha$. The bounds we develop are used in the next section, where we perform a simulation study in order to estimate $\mathcal{H}_\alpha$.

In addition to $\mathcal{H}_\alpha$ and $\mathcal{H}_\alpha^\eta$, the following quantities play a key role throughout the remainder of this paper:

$$
\mathcal{H}_\alpha(T) = \mathbb{E}\biggl[\frac{\sup_{-T\le t\le T}\mathrm{e}^{Z_t}}{\int_{-T}^T \mathrm{e}^{Z_t}\,\mathrm{d}t}\biggr], \qquad \mathcal{H}_\alpha^\eta(T) = \mathbb{E}\biggl[\frac{\sup_{-T/\eta\le k\le T/\eta}\mathrm{e}^{Z_{k\eta}}}{\eta\sum_{-T/\eta\le k\le T/\eta}\mathrm{e}^{Z_{k\eta}}}\biggr],
$$

where it is implicit that t is a continuous-time parameter and k only takes integer values. Throughout, we assume that the truncation horizon $T > 0$ and mesh size η are fixed. We also assume for convenience that T is an integer multiple of η.

We now introduce some convenient abbreviations. For fixed $0 < a_1 < a_2 < \cdots$, we write $J_0 = (-a_1, a_1)$ and $J_j = J_j^+ = [a_j, a_{j+1})$, $J_{-j} = J_j^- = (-a_{j+1}, -a_j]$ with $j \ge 1$. Throughout this section, we use $a_1 = T$. Write $M_j = \sup_{t\in J_j} \mathrm{e}^{Z_t}$, $S_j = \int_{J_j} \mathrm{e}^{Z_t}\,\mathrm{d}t$, $M_j^\eta = \sup_{k:k\eta\in J_j} \mathrm{e}^{Z_{k\eta}}$, and $S_j^\eta = \eta\sum_{k:k\eta\in J_j} \mathrm{e}^{Z_{k\eta}}$, and set $M = \sup_{j\in\mathbb{Z}} M_j$, $S = \sum_{j\in\mathbb{Z}} S_j$, and $S^\eta = \sum_{j\in\mathbb{Z}} S_j^\eta$. The length of an interval J_j is denoted by $|J_j|$.

The first step in our error analysis is a detailed comparison of $\mathcal{H}_\alpha = \mathbb{E}(M/S^\eta)$ and $\mathbb{E}(M_0/S_0^\eta)$, which entails truncation of the horizon over which the supremum and sum are taken. As a second step, we compare $\mathbb{E}(M_0/S_0^\eta)$ to $\mathcal{H}_\alpha^\eta(T) = \mathbb{E}(M_0^\eta/S_0^\eta)$, which entails approximating the maximum on a discrete mesh.

4.1. Truncation

This subsection derives upper and lower bounds on $\mathbb{E}(M/S^\eta)$ in terms of $\mathbb{E}(M_0/S_0^\eta)$. For convenience we derive our error bounds for $a_j = T(1+\gamma)^{j-1}$ for $j \ge 1$, for some $\gamma > 0$. Presumably sharper error bounds can be given when the choice of the a_j is optimized.

4.1.1. *An upper bound*

We derive an upper bound on $\mathbb{E}(M/S^\eta)$ in terms of $\mathbb{E}(M_0/S_0^\eta)$. Since $S \ge S_j$ for any $j \in \mathbb{Z}$, we have

$$\begin{aligned}
\mathbb{E}\bigl(M/S^\eta\bigr) &= \mathbb{E}\biggl[\frac{M_0}{S^\eta}; M = M_0\biggr] + \sum_{j\ne 0}\mathbb{E}\biggl[\frac{M_j}{S^\eta}; M = M_j\biggr] \\
&\le \mathbb{E}\bigl(M_0/S_0^\eta\bigr) + \sum_{j\ne 0}\mathbb{E}\biggl[\frac{M_j}{S_j^\eta}; M_j > 1\biggr] \\
&\le \mathbb{E}\bigl(M_0/S_0^\eta\bigr) + 2\sum_{j\ge 1}\mathbb{E}\biggl[\frac{M_j}{S_j^\eta}; \sqrt{2}\sup_{s\in J_j} B_s > \min_{s\in J_j}|s|^\alpha\biggr].
\end{aligned} \tag{4}$$

Set $E_j = \{\sqrt{2}\max_{s\in J_j} B_s > \min_{s\in J_j}|s|^\alpha\}$. To further bound (4), we use the bounds developed in Section 3. Thus, the next step is to bound $\mathbb{P}(E_j)$ from above. We write τ_j for τ used in the jth term. Using the facts that B has stationary increments and is self-similar, we find that by Theorem 2.8 in Adler [1],

$$\begin{aligned}
\mathbb{E}\Bigl(\max_{s\in J_j} B_s\Bigr) &= \mathbb{E}\Bigl(\max_{0\le s\le |J_j|} B_s\Bigr) = |J_j|^{\alpha/2}\mathbb{E}\Bigl(\max_{0\le s\le 1} B_s\Bigr) \\
&\le 2|J_j|^{\alpha/2}\mathbb{E}\Bigl(\max_{0\le s\le 1} sN\Bigr) = |J_j|^{\alpha/2},
\end{aligned} \tag{5}$$

where N stands for a standard normal random variable. We derive a bound on $\mathbb{P}(E_j)$ in a slightly more general form for later use. It follows from Borell's inequality that, for $0 < a < b$, $c \in \mathbb{R}$,

$$\mathbb{P}\Big(\sqrt{2} \max_{s\in[a,b]} B_s > c + a^\alpha\Big) \le \exp\bigg(-\frac{[c + a^\alpha - \sqrt{2}(b-a)^{\alpha/2}]^2}{4b^\alpha}\bigg), \tag{6}$$

provided $c + a^\alpha > \sqrt{2}(b-a)^{\alpha/2}$. Specialized to $\mathbb{P}(E_j)$, we obtain that for $j \ge 1$, $a_j^\alpha > \sqrt{2}(a_{j+1} - a_j)^{\alpha/2}$,

$$\mathbb{P}(E_j) \le \exp\bigg\{-\frac{[a_j^\alpha - \sqrt{2}(a_{j+1} - a_j)^{\alpha/2}]^2}{4a_{j+1}^\alpha}\bigg\} = \exp\bigg\{-\frac{(a_j^{\alpha/2} - \gamma^{\alpha/2}\sqrt{2})^2}{4(1+\gamma)^\alpha}\bigg\},$$

provided $T > \gamma 2^{1/\alpha}$.

Thus, the error is upper bounded by $\exp(-c'T^\alpha)$ for some constant c' as $T \to \infty$. As a result, the error decreases to zero much faster than any polynomial, unlike the classical representation for which the error can be expected to be polynomial as previously discussed. This is one of the key advantages of our new representation.

4.1.2. *A lower bound*

We derive a lower bound on $\mathbb{E}(M/S^\eta)$ in terms of $\mathbb{E}(M_0/S_0^\eta)$ as follows:

$$\begin{aligned}
\mathbb{E}\big(M/S^\eta\big) &\ge \mathbb{E}\bigg[\frac{M_0}{S_0^\eta} \cdot \frac{S_0^\eta}{S_0^\eta + \sum_{j\ne 0} S_j^\eta}; \varepsilon S_0^\eta \ge \sum_{j\ne 0} S_j^\eta\bigg] \\
&\ge \frac{1}{1+\varepsilon}\mathbb{E}\bigg[\frac{M_0}{S_0^\eta}; \varepsilon S_0^\eta \ge \sum_{j\ne 0} S_j^\eta\bigg] \\
&= \frac{1}{1+\varepsilon}\mathbb{E}\big(M_0/S_0^\eta\big) - \frac{1}{1+\varepsilon}\mathbb{E}\bigg[\frac{M_0}{S_0^\eta}; \varepsilon S_0^\eta < \sum_{j\ne 0} S_j^\eta\bigg].
\end{aligned}$$

Set $E = \{\varepsilon S_0^\eta < \sum_{j\ne 0} S_j^\eta\}$. To apply the technique from Section 3, we seek an upper bound on $\mathbb{P}(E)$. Let $0 < \delta < T$, to be determined later. Since $S_0^\eta \ge \eta$, we obtain

$$\mathbb{P}(E) \le \mathbb{P}\bigg(\sum_{j\ne 0} S_j^\eta > \varepsilon\eta\bigg) \le 2\sum_{j\ge 1} \mathbb{P}\big(S_j^\eta > \varepsilon\eta q_j\big)$$

for any probability distribution $\{q_j\colon\ j \ne 0\}$. We find it convenient to take $q_j = \psi(1+\psi)^{-|j|}/2$ for some $\psi > 0$ and $j \ne 0$. An upper bound on S_j^η for $j \ge 1$ is

$$S_j^\eta \le (a_{j+1} - a_j)\mathrm{e}^{-a_j^\alpha}\mathrm{e}^{\sqrt{2}\max_{s\in J_j} B_s} = \gamma a_j \mathrm{e}^{-a_j^\alpha}\mathrm{e}^{\sqrt{2}\max_{s\in J_j} B_s}.$$

For $j \ge 1$, we therefore have

$$\begin{aligned}\mathbb{P}\bigl(S_j^\eta > \varepsilon\eta q_j\bigr) &\le \mathbb{P}\bigl(\mathrm{e}^{\sqrt{2}\max_{s\in J_j} B_s} > \varepsilon\eta \mathrm{e}^{a_j^\alpha} q_j/(\gamma a_j)\bigr)\\ &= \mathbb{P}\Bigl(\sqrt{2}\max_{s\in J_j} B_s > a_j^\alpha + \log\bigl[\varepsilon\eta q_j/(\gamma a_j)\bigr]\Bigr)\\ &\le \exp\biggl(-\frac{(\log[\varepsilon\eta q_j/(\gamma a_j)] + a_j^\alpha - \sqrt{2}\gamma^{\alpha/2}a_j^{\alpha/2})^2}{4(1+\gamma)^\alpha a_j^\alpha}\biggr),\end{aligned}$$

provided T is large enough so that the expression inside the square is nonnegative. The last inequality follows from (6).

4.2. Approximating the supremum on a mesh

We now find upper and lower bounds on $\mathbb{E}(M_0/S_0^\eta)$ in terms of $\mathbb{E}(M_0^\eta/S_0^\eta)$.

For the upper bound, we note that

$$\mathbb{E}\bigl(M_0/S_0^\eta\bigr) \le \mathrm{e}^\varepsilon \mathbb{E}\bigl(M_0^\eta/S_0^\eta\bigr) + \mathbb{E}\bigl(M_0/S_0^\eta; \Delta_0(\eta) > \varepsilon\bigr).$$

We use the technique from Section 3 to bound $\mathbb{E}(M_0^\eta/S_0^\eta; \Delta_0(\eta) > \varepsilon)$, which requires a bound on $\mathbb{P}(\Delta_0(\eta) > \varepsilon)$. Writing $\kappa_0(\eta) = \max(\eta^\alpha, T^\alpha - (T-\eta)^\alpha)$, we use the self-similarity in conjunction with Borell's inequality and (5) to deduce that

$$\begin{aligned}\mathbb{P}\bigl(\Delta_0(\eta) > \varepsilon\bigr) &\le \mathbb{P}\Bigl(\sup_{t\in(-T,T)} \sqrt{2}\bigl(B_t - B_t^\eta\bigr) > \varepsilon - \kappa_0(\eta)\Bigr)\\ &\le \frac{2T}{\eta}\mathbb{P}\Bigl(\sqrt{2}\sup_{t\in(0,1)} \eta^{\alpha/2} B_t > \varepsilon - \kappa_0(\eta)\Bigr)\\ &\le \frac{2T}{\eta}\mathbb{P}\biggl(\sup_{t\in[0,1]} B_t > \frac{\varepsilon - \kappa_0(\eta)}{\sqrt{2}\eta^{\alpha/2}}\biggr)\\ &\le \frac{2T}{\eta}\exp\biggl(-\frac{1}{2}\biggl[\frac{\varepsilon - \kappa_0(\eta)}{\sqrt{2}\eta^{\alpha/2}} - 1\biggr]^2\biggr),\end{aligned}$$

provided $\varepsilon > \kappa_0(\eta)$.

A lower bound on $\mathbb{E}(M_0/S_0^\eta)$ in terms of $\mathbb{E}(M_0^\eta/S_0^\eta)$ follows trivially:

$$\mathbb{E}\bigl(M_0/S_0^\eta\bigr) \ge \mathbb{E}\bigl(M_0^\eta/S_0^\eta\bigr).$$

4.3. Conclusions

We summarize the bounds we have obtained. For any $\varepsilon > 0$, we have derived the following upper bound:

$$\mathcal{H}_\alpha \le e^\varepsilon \mathbb{E}\big(M_0^\eta / S_0^\eta\big) + \mathbb{E}\left[\frac{M_0}{S_0^\eta}; \Delta_0(\eta) > \varepsilon\right] + 2\sum_{j\ge 1} \mathbb{E}\left[\frac{M_j}{S_j^\eta}; \sqrt{2}\sup_{s\in J_j} B_s > \min_{s\in J_j} |s|^\alpha\right], \tag{7}$$

where the second and third terms are bounded further using Section 3. Note that this requires selecting a τ for each of the terms; we will come back to this in the next section.

For any $\varepsilon > 0$, we have derived the following lower bound:

$$\mathcal{H}_\alpha \ge \frac{1}{1+\varepsilon}\mathbb{E}\big(M_0^\eta / S_0^\eta\big) - \frac{1}{1+\varepsilon}\mathbb{E}\left[\frac{M_0}{S_0^\eta}; \varepsilon S_0^\eta < \sum_{j\ne 0} S_j^\eta\right], \tag{8}$$

and we again use Section 3. We note that we may choose a different ε for the upper bound and the lower bound, which we find useful in the next section.

5. Numerical experiments

This section consists of two parts. The first part studies $\mathcal{H}_\alpha^\eta(T)$ for suitable choices of the simulation horizon T and the discretization mesh η, and uses the previous section to estimate bounds on $\mathcal{H}_\alpha$. In the second part of this section, we present a heuristic method for obtaining sharper estimates for $\mathcal{H}_\alpha$.

Simulation of fractional Brownian motion is highly nontrivial, but there exists a vast body of literature on the topic. The fastest available algorithms simulate the process on an equispaced grid, by simulation of the (stationary) increment process, which often called fractional Gaussian noise. We use the method of Davies and Harte [14] for simulating n points of a fractional Gaussian noise. This method requires that n be a power of two. In this approach, the covariance matrix is embedded in a so-called circulant matrix, for which the eigenvalues can easily be computed. The algorithm relies on the Fast Fourier Transform (FFT) for maximum efficiency; the computational effort is of order $n \log n$ for a sample size of length n. For more details on simulation of fractional Brownian motion, we refer to Dieker [20].

5.1. Confidence intervals

Our next aim is to give a point estimate for $\mathcal{H}_\alpha^\eta(T)$ and use the upper and lower bounds from the previous section to obtain an interval estimate for Pickands' constant $\mathcal{H}_\alpha$.

The truncation and discretization errors both critically depend on α, but we choose T and η to be fixed throughout our experiments in order to use a simulation technique known as *common random numbers*. This means that the same stream of (pseudo)random numbers is used for all values of α. By choosing T and η independent of α, the realizations of fractional Brownian motion in the nth simulation replication are perfectly dependent for different values of α. As a result, our estimate of $\mathcal{H}_\alpha$ as a function of α is smoothened without any statistical sacrifice.

Since T and η are fixed, our estimates for $\mathcal{H}_\alpha$ are likely to be far off from $\mathcal{H}_\alpha^\eta(T)$ for small α. In that regime our algorithm becomes unreliable, since the truncation horizon would have to grow so large that it requires ever more computing power to produce an estimate. Any method that relies on truncating the simulation horizon suffers from this problem, and it seems unlikely that truncation can be avoided. There is some understanding of the asymptotic behavior of $\mathcal{H}_\alpha$ as $\alpha \downarrow 0$ (Shao [32], Harper [22]) so this regime is arguably less interesting from a simulation point of view. Since we cannot trust the simulation output for small α, we focus our experiments on $\alpha \geq 7/10$.

Somewhat arbitrarily, we chose to calibrate errors using $\alpha = 1$, so that our estimates of $\mathcal{H}_\alpha(T)$ are close to $\mathcal{H}_\alpha$ for $\alpha \geq 1$. The closer one sets the calibration point to 0, the higher one has to choose T (and thus more computing power). We estimate $\mathcal{H}_\alpha^\eta(T)$ using 1500 simulation replications, which takes about three days on a modern computer for each value of α. We carry out the simulation for $\alpha = 14/20, 15/20, \ldots, 40/20$, and interpolate linearly between the simulated points. A high-performance computing environment is used to run the experiments in parallel.

We choose the parameters so that the simulated error bounds from the previous section yield an error of approximately 3% for $\alpha = 1$. The most crucial parameter in the error analysis is ε. We note that a different ε can be used for the lower and upper bound, and that ε may depend on α, so we take advantage of this extra flexibility to carefully select ε. For the upper bound in (7) we use $\varepsilon = 0.005 + 0.025 \cdot (2 - \alpha)$, and for the lower bound in (8) we use $\varepsilon = (0.005 + 0.025 \cdot (2 - \alpha))/3$. We use $T = 128$ and $\eta = 1/2^{18}$.

We next discuss how we have chosen the other parameters in the error analysis from Section 4. These have been somewhat optimized. Equations (7) and (8) produce bounds on $\mathcal{H}_\alpha$ in terms of $\mathcal{H}_\alpha^\eta(T)$ in view of Section 3, but this requires selecting some τ for each term for which Section 3 is applied. We use $\tau_j = 1.3 \cdot (1.005)^{j-1}$ for the j-term in the infinite sum, and $\tau = 1.4$ for any of the other terms. We set $\gamma = 0.025$ for the growth rate of a_j, and we use $\psi = 0.3$ for the decay rate of q_j. For these parameter values, all event-independent terms in Section 3 are negligible. Finally, we replace $\mathcal{H}_\alpha^\eta(T)$ in the resulting bounds with its estimate.

In Figure 1, we plot our estimates of $\mathcal{H}_\alpha^\eta(T)$ as a function of α (blue, solid), along with their 95% confidence interval (green, dotted) and our bounds for $\mathcal{H}_\alpha$ (red, dash-dotted). The numerical values are given in Appendix B. Note that the errors we find for $\alpha < 1$ are so large that our error bounds are essentially useless. We do believe that the simulated values are reliable approximations to $\mathcal{H}_\alpha$, but the bounds from our error analysis are too loose.

A well-known conjecture states that $\mathcal{H}_\alpha = 1/\Gamma(1/\alpha)$ (Dębicki and Mandjes [18]), but (to our knowledge) it lacks any foundation other than that $\lim_{\alpha\downarrow 0} \mathcal{H}_\alpha = \lim_{\alpha\downarrow 0} 1/\Gamma(1/\alpha) = 0$, $\mathcal{H}_1 = 1/\Gamma(1)$, and $\mathcal{H}_2 = 1/\Gamma(1/2)$. A referee communicated to us that this conjecture is due to K. Breitung. Our simulation gives strong evidence that this conjecture is not correct: the function $1/\Gamma(1/\alpha)$ is the magenta, dashed curve in Figure 1, and we see that the confidence interval and error bounds are well above the curve for α in the range 1.6–1.8. Note that we cannot *exclude* that this conjecture holds, since our error bounds are based on Monte Carlo experiments. However, this formula arguably serves as a reasonable approximation for $\alpha \geq 1$.

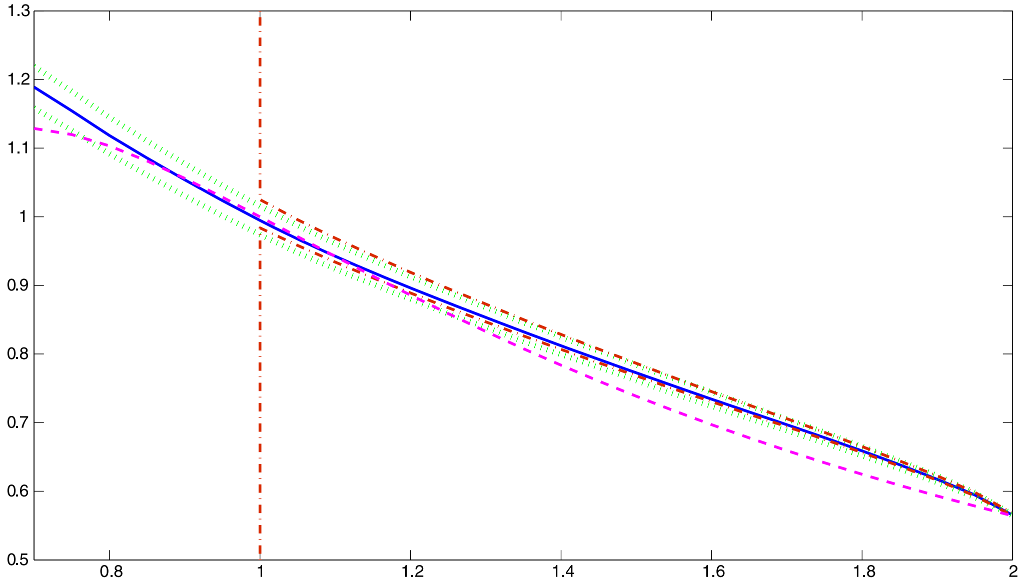

Figure 1. Point estimates (blue, solid) and interval estimates (green, dotted) for $\mathcal{H}_\alpha^\eta(T)$ as a function of α. Our error analysis shows that $|\mathcal{H}_\alpha^\eta(T) - \mathcal{H}_\alpha|$ is at most 0.03 for $\alpha \geq 1$ (red, dash-dotted). We also plot $1/\Gamma(1/\alpha)$ (magenta, dashed).

5.2. A regression-based approach

In the previous subsection, we approximated $\mathcal{H}_\alpha$ by $\mathcal{H}_\alpha^\eta(T)$. The main contribution to the error is the discretization step, so we now focus on a refined approximation based on the behavior of $\mathcal{H}_\alpha^\eta(T)$ as $\eta \downarrow 0$.

This approach relies on the rate at which $\mathcal{H}_\alpha^\eta(T)$ converges to $\mathcal{H}_\alpha(T)$. We state this as a conjecture, it is outside the scope of the current paper to (attempt to) prove it.

Conjecture 1. *For fixed $T > 0$, we have $\lim_{\eta\downarrow 0} \eta^{-\alpha/2}[\mathcal{H}_\alpha(T) - \mathcal{H}_\alpha^\eta(T)] \in (0, \infty)$. We also have $\lim_{\eta\downarrow 0} \eta^{-\alpha/2}[\mathcal{H}_\alpha - \mathcal{H}_\alpha^\eta] \in (0, \infty)$.*

We motivate this conjecture as follows. We focus on the last part of the conjecture for brevity. Since $\mathcal{H}_\alpha = \mathbb{E}[M/S^\eta]$ by Proposition 2, we obtain that

$$\eta^{-\alpha/2}\left[\mathcal{H}_\alpha - \mathcal{H}_\alpha^\eta\right] = \mathbb{E}\left[\eta^{-\alpha/2}\left(e^{m-m^\eta} - 1\right) \times \frac{M^\eta}{S^\eta}\right],$$

where $m^\eta = \log M^\eta$ and $m = \log M$. The right-hand side equals approximately

$$\mathbb{E}\left[\eta^{-\alpha/2}\left(m - m^\eta\right) \times \frac{M^\eta}{S^\eta}\right].$$

This expectation involves a product of two random variables. The random variable M^η/S^η converges almost surely to the finite random variable M/S as $\eta \downarrow 0$. Although we are not aware of any existing results on the behavior of $\eta^{-\alpha/2}(m - m^\eta)$ or its expectation, we expect that the random variable $\eta^{-\alpha/2}(m - m^\eta)$ converges in distribution. Indeed, this is suggested by prior work on related problems, see Asmussen *et al.* [5] for the case $\alpha = 1$ and Hüsler *et al.* [24], Seleznjev [31] for general results on interpolation approximations for Gaussian processes (which is different but related). The rate of convergence of m^η to m (or for finite-horizon analogs) seems to be of general fundamental interest, but falls outside the scope of this paper.

Conjecture 1 implies that for some $c = c(T) > 0$, for small η, we have approximately

$$\mathcal{H}_\alpha^\eta(T) = \mathcal{H}_\alpha(T) - c\eta^{\alpha/2}.$$

This allows us to perform an ordinary linear regression to simultaneously estimate c and $\mathcal{H}_\alpha(T)$ from (noisy) estimates of $\mathcal{H}_\alpha^\eta(T)$ for different (small) values of η and fixed α. One could use the same simulated fractional Brownian motion trace for different values of η, but it is also possible to use independent simulation experiments for different values of η. The latter approach is computationally less efficient, but it has the advantage that classical regression theory becomes available for constructing confidence intervals of $\mathcal{H}_\alpha(T)$.

Even though we do not have a formal justification for this approach, we have carried out regressions with the same simulated trace for different values of η. The results are reported in Figure 2. The simulation experiments are exactly the same as those underlying Figure 1, and in particular we have used the same parameter values. The red, dashed curves are estimates for $\mathcal{H}_\alpha^\eta(T)$ for $\eta = 2^{-14}, 2^{-13}, 2^{-12}, 2^{-11}$. Using the regression approach, we estimate $\mathcal{H}_\alpha^\eta(T)$ for $\eta = 2^{-18}$ and compare it with our simulation estimates for the same value of η (blue, solid). The two resulting curves are indistinguishable in Figure 2, and the difference is of order 10^{-3}. We have also plotted our regression-based estimate of $\mathcal{H}_\alpha(T)$ (green, dash-dotted).

It is instructive to look at the resulting estimate for $\mathcal{H}_1(T)$, since we know that $\mathcal{H}_1 = 1$. Our estimate for $\mathcal{H}_1(T)$ is 0.9962650, which is indeed closer to its true value. As the number of simulation replications increases, we expect much more improvement.

Appendix A: Proofs

Proof of Proposition 1. Our proof of (9) uses several ideas that are similar to those in Sections 3 and 4, so our exposition here is concise. We fix some η for which $1/\eta$ is a (large) integer; its exact value is irrelevant. Recall that the quantities $J_j, M_j, S_j, M_j^\eta, S_j^\eta$ from Section 4 have been introduced with respect to parameters $0 < a_1 < a_2 < \cdots$. Here we use different choices: $a_1 = \lceil 2^{1/(2\alpha)} \rceil$, $a_j = a_{j-1} + 1$ for $j \ge 2$.

Abusing notation slightly, we write $M_{[-uT,(1-u)T]} = \sup_{s\in[-uT,(1-u)T]} \mathrm{e}^{Z_s}$ and $S_{[-uT,(1-u)T]} = \int_{-uT}^{(1-u)T} \mathrm{e}^{Z_s}\,\mathrm{d}s$. Since $M_{[-uT,(1-u)T]} \to M$ and $S_{[-uT,(1-u)T]} \to S$ almost surely as $T \to \infty$ for $u \in (0,1)$, both claims follow after showing that

$$\lim_{A\to\infty} \sup_{T>0} \sup_{u\in(0,1)} \mathbb{E}\left[\frac{M_{[-uT,(1-u)T]}}{S_{[-uT,(1-u)T]}}; \frac{M_{[-uT,(1-u)T]}}{S_{[-uT,(1-u)T]}} > A\right] = 0. \tag{9}$$

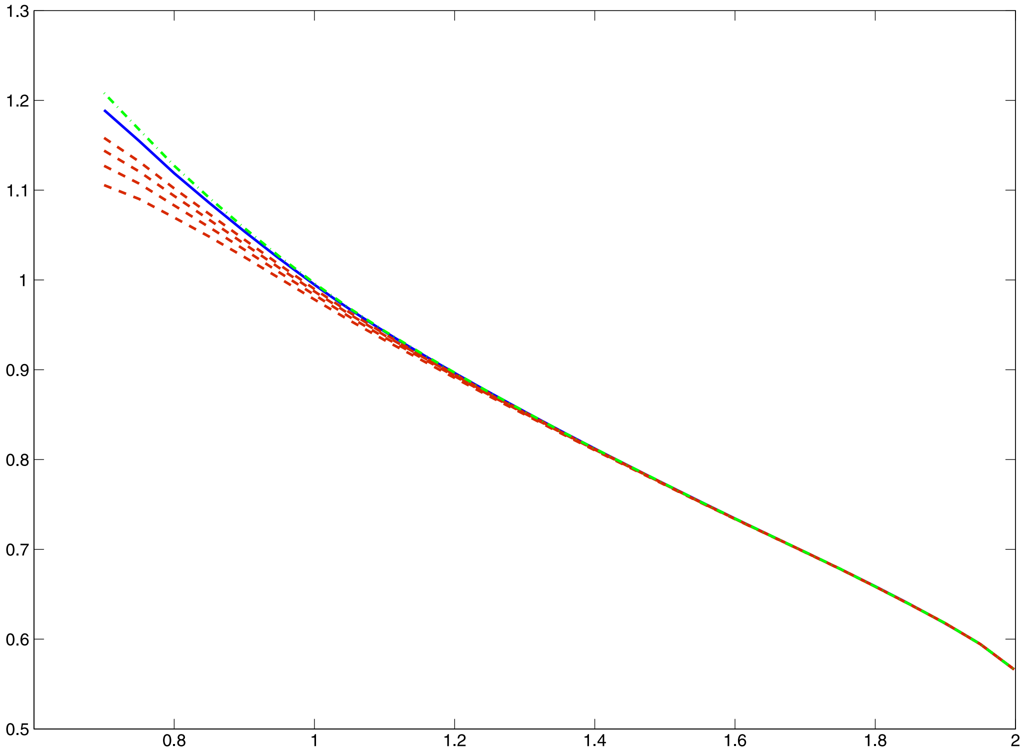

Figure 2. Estimation of $\mathcal{H}_\alpha^\eta(T)$ for different values of η.

Write $\kappa_j = \kappa_j(\eta) = \sup_{t\in J_j}[\mathbb{V}\mathrm{ar}(Z_t) - \mathbb{V}\mathrm{ar}(Z_t^\eta)]$. First, suppose that $-uT$ and $(1-u)T$ lie in $\{\dots, -a_2, -a_1, a_1, a_2, \dots\}$. On the event $\{M_{[-uT,(1-u)T]} = M_j\}$ for some $j \in \mathbb{Z}$, we have

$$\begin{aligned} \frac{M_{[-uT,(1-u)T]}}{S_{[-uT,(1-u)T]}} &\le \frac{M_j}{S_j} \le \mathrm{e}^{2\sqrt{2}\sup_{s\in J_j}|B_s - B_s^\eta| + \kappa_j} \frac{M_j^\eta}{S_j^\eta} \\ &\le \frac{1}{\eta}\mathrm{e}^{2\sqrt{2}\sup_{s\in J_j}|B_s - B_s^\eta| + \kappa_j}. \end{aligned} \tag{10}$$

Note that this bound remains valid if $-uT$ and $(1-u)T$ fail to lie in $\{\dots, -a_2, -a_1, a_1, a_2, \dots\}$.

Since $\mathbb{E}[\mathrm{e}^{2\sqrt{2}\sup_{s\in J_0}|B_s - B_s^\eta|}] < \infty$ by Borell's inequality, (9) follows after we establish that

$$\lim_{A\to\infty}\sum_{j=1}^{\infty}\mathbb{E}\big[\mathrm{e}^{2\sqrt{2}\sup_{s\in J_j}|B_s - B_s^\eta| + \kappa_j}; \mathrm{e}^{2\sqrt{2}\sup_{s\in J_j}|B_s - B_s^\eta| + \kappa_j} > A, M_j > 1\big] = 0.$$

To this end, we observe that for $j \geq 1$

$$\begin{aligned}
&\mathbb{E}\big[\mathrm{e}^{2\sqrt{2}\sup_{s\in J_j}|B_s-B_s^\eta|+\kappa_j}; M_j > 1\big] \\
&\quad\leq \sqrt{\mathbb{E}\big[\mathrm{e}^{4\sqrt{2}\sup_{s\in J_j}|B_s-B_s^\eta|+2\kappa_j}\big]\mathbb{P}(M_j > 1)} \\
&\quad\leq \sqrt{\mathrm{e}^{2\kappa_j}\mathbb{E}\big[\mathrm{e}^{4\sqrt{2}\sup_{s\in[0,1]}|B_s-B_s^\eta|}\big]\mathbb{P}\Big(\sup_{t\in J_j}\sqrt{2}B_t > a_j^\alpha\Big)} \\
&\quad\leq C\mathrm{e}^{\kappa_j}\exp\biggl(-\frac{(a_j^\alpha-\sqrt{2}\mathbb{E}[\sup_{t\in J_j}B_t])^2}{8(a_j+1)^\alpha}\biggr) \\
&\quad\leq C\mathrm{e}^{\kappa_j}\exp\biggl(-\frac{(a_j^\alpha-\sqrt{2}])^2}{8(a_j+1)^\alpha}\biggr),
\end{aligned}$$

where C denotes some constant and we have used (5) to obtain the last inequality. Note that $a_j^\alpha > \sqrt{2}$ for our choice of a_j. The resulting expression is summable, which establishes the required inequality by the monotone convergence theorem. □

Proof of Proposition 2. Our starting point is (3) and the accompanying remarks. By Theorem 1.5.5 in Billingsley [11], it suffices to show that $\mathrm{Leb}(E) = 0$, where E consists of all $u \in [0, 1]$ for which

$$\lim_{T\to\infty}\mathbb{E}\biggl[\frac{M_{[-u_T T,(1-u_T)T]}}{S^\eta_{[-u_T T,(1-u_T)T]}}\biggr] = \mathbb{E}\biggl[\frac{M}{S^\eta}\biggr]$$

fails to hold for some $\{u_T\}$ with $u_T \to u$. With minor modifications to the bound (10) since, we work with S^η instead of S, the proof of Proposition 1 shows that

$$\lim_{A\to\infty}\sup_{T>0}\sup_{u\in(0,1)}\mathbb{E}\biggl[\frac{M_{[-uT,(1-u)T]}}{S^\eta_{[-uT,(1-u)T]}}; \frac{M_{[-uT,(1-u)T]}}{S^\eta_{[-uT,(1-u)T]}} > A\biggr] = 0.$$

This implies that $E \subseteq \{0, 1\}$, so its Lebesgue measure is zero. □

Proof of Proposition 4. As in the proof of Proposition 2, it suffices to show that, whenever $u_T \to u \in (0, 1)$,

$$\lim_{T\to\infty}\mathbb{P}\Bigl(\sup_{k\in\mathbb{Z}:-u_T T\leq k\eta\leq(1-u_T)T} Z_{k\eta} = 0\Bigr) = \mathbb{P}\Bigl(\sup_{k\in\mathbb{Z}} Z_{k\eta} = 0\Bigr).$$

A sandwich argument readily establishes that

$$\lim_{T\to\infty}\sup_{k\in\mathbb{Z}:-u_T T\leq k\eta\leq(1-u_T)T} Z_{k\eta} = \sup_{k\in\mathbb{Z}} Z_{k\eta}.$$

The claim follows since almost sure convergence implies convergence in distribution. □

Appendix B: Simulated values

This appendix lists our estimates for $\mathcal{H}_\alpha^\eta(T)$ in tabular form for $\eta = 1/2^{18}$ and $T = 128$, along with the sample standard deviation. We also list the lower and upper bounds on $\mathcal{H}_\alpha$, where we note that these are estimated values since they depend on $\mathcal{H}_\alpha^\eta(T)$. We cannot report these bounds for $\alpha < 1$, since our choice of parameter values causes the methodology to break down. Our methods can be applied with different parameter values to obtain bounds in this regime, but this requires more computing time and is not pursued in this paper. These numerical results are summarized in Table B.1.

Table B.1. Our numerical results

α	Estimate $\mathcal{H}_\alpha^\eta(T)$	Sample stddev M_0/S_0^η	Lower bound $\mathcal{H}_\alpha$	Upper bound $\mathcal{H}_\alpha$
0.700	1.1888337	0.5998979	–	–
0.750	1.1543904	0.5614484	–	–
0.800	1.1184290	0.5257466	–	–
0.850	1.0855732	0.4919238	–	–
0.900	1.0539127	0.4625016	–	–
0.950	1.0235620	0.4360272	–	–
1.000	0.9946978	0.4116689	0.9837218	1.0250320
1.050	0.9674279	0.3892142	0.9582444	0.9956451
1.100	0.9424383	0.3665194	0.9338777	0.9687150
1.150	0.9191131	0.3442997	0.9111406	0.9435593
1.200	0.8963231	0.3239746	0.8889154	0.9190136
1.250	0.8743162	0.3048379	0.8674489	0.8953298
1.300	0.8532731	0.2864521	0.8469212	0.8726894
1.350	0.8322652	0.2698805	0.8264114	0.8501401
1.400	0.8121016	0.2540026	0.8067235	0.8285072
1.450	0.7922732	0.2390896	0.7873523	0.8072685
1.500	0.7727308	0.2248372	0.7682494	0.7863726
1.550	0.7531251	0.2112524	0.7490677	0.7654634
1.600	0.7342039	0.1970492	0.7305511	0.7453000
1.650	0.7155531	0.1821118	0.7122884	0.7254599
1.700	0.6970209	0.1665167	0.6941287	0.7057883
1.750	0.6782065	0.1503939	0.6756727	0.6858794
1.800	0.6585134	0.1339708	0.6563256	0.6651316
1.850	0.6384329	0.1156335	0.6365762	0.6440437
1.900	0.6176244	0.0953090	0.6160842	0.6222740
1.950	0.5944161	0.0698590	0.5931803	0.5981428
1.998	0.5663460	0.0146697	0.5653943	0.5692133

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