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Small-time expansions for local jump-diffusion models with infinite jump activity

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We consider a Markov process X, which is the solution of a stochastic differential equation driven by a Lévy process Z and an independent Wiener process W. Under some regularity conditions, including non-degeneracy of the diffusive and jump components of the process as well as smoothness of the Lévy density of Z outside any neighborhood of the origin, we obtain a small-time second-order polynomial expansion for the tail distribution and the transition density of the process X. Our method of proof combines a recent regularizing technique for deriving the analog small-time expansions for a Lévy process with some new tail and density estimates for jump-diffusion processes with small jumps based on the theory of Malliavin calculus, flow of diffeomorphisms for SDEs, and time-reversibility. As an application, the leading term for out-of-the-money option prices in short maturity under a local jump-diffusion model is also derived.

Keywords: local jump-diffusion models; option pricing; small-time asymptotic expansion; transition densities; transition distributions

1. Introduction

The small-time asymptotic behavior of the transition densities of Markov processes $\{X_t(x)\}_{t\geq 0}$ with deterministic initial condition $X_0(x) = x$ has been studied for a long time, with a certain focus to consider either purely-continuous or purely-discontinuous processes. Starting from the problem of existence, there are several sets of sufficient conditions for the existence of the transition density of $X_t(x)$, denoted hereafter $p_t(\cdot; x)$. A stream in this direction is based on the machinery of Malliavin calculus, originally developed for continuous diffusions (see the monograph Nualart [24]) and, then, extended to Markov process with jumps (see the monograph Bichteler, Gravereaux and Jacod [6]). This approach can also yield estimates of the transition density $p_t(\cdot; x)$ in small time t. For purely-jump Markov processes, the key assumption is that the Lévy measure of the process admits a smooth Lévy density. The pioneer of this approach was Léandre [18], who obtained the first-order small-time asymptotic behavior of the transition density for fully supported Lévy densities. This result was extended in Ishikawa [16] to the case where the point y cannot be reached with only one jump from x but rather with finitely many jumps, while Picard [26] developed a method that can also be applied to Lévy measures with a non-zero singular component (see also Picard [27] and Ishikawa [17] for other related results).

The main result in Léandre [18] states that, for $y \neq 0$,

$$\lim_{t\to 0} \frac{1}{t} p_t(x+y; x) = g(x; y),$$

where $g(x; y)$ is the so-called Lévy density of the process X to be defined below (see (1.5)). Léandre's approach consisted of first separating the small jumps (say, those with sizes smaller than an $\varepsilon > 0$) and the large jumps of the underlying Lévy process, and then conditioning on the number of large jumps by time t. Malliavin's calculus was then applied to control the resulting density given that there is no large jump. For $\varepsilon > 0$ small enough, the term when there is only one large jump was proved to be equivalent, up to a remainder of order $\mathrm{o}(t)$, to the term resulting from a model in which there is no small-jump component at all. Finally, the terms when there is more than one large jump were shown to be of order $\mathrm{O}(t^2)$.

Higher-order expansions of the transition density of Markov processes with jumps have been considered quite recently and only for processes with finite jump activity (see, e.g., Yu [34]) or for Lévy processes with possibly infinite jump-activity. We focus on the literature of the latter case due to its close connection to the present work. Rüschendorf and Woerner [31] was the first work to consider higher-order expansions for the transition densities of Lévy processes using Léandre's approach. Concretely, the following expansion for the transition densities $\{p_t(y)\}_{t\geq 0}$ of a Lévy process $\{Z_t\}_{t\geq 0}$ was proposed therein:

$$p_t(y) := \frac{\mathrm{d}}{\mathrm{d}y}\mathbb{P}(Z_t \leq y) = \sum_{n=1}^{N-1} a_n(y)\frac{t^n}{n!} + \mathrm{O}\big(t^N\big) \qquad (y \neq 0, N \in \mathbb{N}). \tag{1.1}$$

As in Léandre [18], the idea was to justify that each higher-order term (say, the term corresponding to k large jumps) can be replaced, up to a remainder of order $\mathrm{O}(t^N)$, by the resulting density as if there were no small-jump component. However, this approach is able to produce the correct expressions for the higher-order coefficients $a_2(y), \dots$ only in the compound Poisson case (cf. Figueroa-López and Houdré [11]). The problem was subsequently resolved in Figueroa-López, Gong and Houdré [10] (see Section 6 therein as well as Figueroa-López and Houdré [11] for a preliminary related result), using a new approach, under the assumption that the Lévy density of the Lévy process $\{Z_t\}_{t\geq 0}$ is sufficiently smooth and bounded outside any neighborhood of the origin. There are two key ideas in Figueroa-López, Gong and Houdré [10], Figueroa-López and Houdré [11]. Firstly, instead of working directly with the transition densities, the following analog expansions for the tail probabilities were first obtained:

$$\mathbb{P}(Z_t \geq y) = \sum_{n=1}^{N-1} A_n(y)\frac{t^n}{n!} + t^N \mathcal{R}_t(y) \qquad (y > 0, N \in \mathbb{N}), \tag{1.2}$$

where $\sup_{0<t\leq t_0} |\mathcal{R}_t(y)| < \infty$, for some $t_0 > 0$. Secondly, by considering a smooth thresholding of the large jumps (so that the density of large jumps is smooth) and conditioning on the size of the first jump, it was possible to regularize the discontinuous functional $\mathbf{1}_{\{Z_t\geq x\}}$ and, subsequently, proceed to use an iterated Dynkin's formula (see Section 3.2 below for more information) to expand the resulting smooth moment functions $\mathbb{E}(f(Z_t))$ as a power series in t. Equation (1.1) was then obtained by differentiation of (1.2), after justifying that the functions $A_n(y)$ and the remainder $\mathcal{R}_t(y)$ were differentiable in y.

The results and techniques described in the previous paragraph open the door to the study of higher-order expansions for the transition densities of more general Markov models with infinite jump-activity. We take the analysis one step further and consider a jump-diffusion model with non-degenerate diffusion and jump components. Our analysis can also be applied to purely-discontinuous processes as in Léandre [18], but we prefer to consider a "mixture model" due to its relevance in financial applications where empirical evidence supports models containing both continuous and jump components (see Section 6 below for detailed references in this direction). More concretely, we consider the following stochastic differential equations (SDE) driven by a Wiener process $\{W_t\}_{t\geq 0}$ and an independent pure-jump Lévy process $\{Z_t\}_{t\geq 0}$:

$$\begin{aligned} X_t(x) = x + \int_0^t b\big(X_u(x)\big)\,\mathrm{d}u + \int_0^t \sigma\big(X_u(x)\big)\,\mathrm{d}W_u \\ + \sum_{u\in(0,t]:\ |\Delta Z_u|\geq 1} \gamma\big(X_{u^-}(x), \Delta Z_u\big) + \sum^{c}_{u\in(0,t]:\ 0<|\Delta Z_u|\leq 1} \gamma\big(X_{u^-}(x), \Delta Z_u\big). \end{aligned} \tag{1.3}$$

Here, $\Delta Z_u := Z_u - Z_{u^-} := Z_u - \lim_{s\nearrow t} Z_s$ denotes the jump of Z at time u, while $\sum^c$ denotes the compensated Poisson sum of the terms therein. The functions $b, \sigma : \mathbb{R} \to \mathbb{R}$, $\gamma : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are some suitable deterministic functions so that (1.3) is well-posed.

As it will be evident from our work, an important difficulty to deal with the model (1.3) arises from the more complex interplay of the jump and continuous components. In particular, conditioning on the first "big jump" of $\{X_s(x)\}_{s\leq t}$ leads us to consider the short-time expansions of the tail probability of a SDE with random initial value $\tilde{J}$, which creates important, albeit interesting, subtleties. More concretely, in the case of a Lévy process (i.e., when b, σ, and γ above are state-independent), conditioning on the first big jump naturally leads to analyzing the small-time expansion of the tail probability $\mathbb{P}(X_t^{\varepsilon}(x) + \tilde{J} \geq x + y)$, where $\{X_s^{\varepsilon}(x)\}$ stands for the "small jump" component of $\{X_s(x)\}$ (see the end of Section 2 for the terminology). This task is relatively simple to handle since the smooth density of $\tilde{J}$ "regularizes" the problem. By contrast, in the general local jump-diffusion model, conditioning on the first big jump leads to consider $\mathbb{P}(X_t^{\varepsilon}(x + \tilde{J}) \geq x + y)$, a problem that does not allow a direct application of Dynkin's formula. Instead, to obtain the second-order expansion of the latter tail probability, we need to rely on smooth approximations of the tail probability building on the theoretical machinery of the flow of diffeomorphisms for SDEs and time-reversibility.

Under certain regularity conditions on b, σ and γ, as well as the Lévy measure ν of Z, we show the following second-order expansion (as $t \to 0$) for the tail distribution of $\{X_t(x)\}_{t\geq 0}$:

$$\mathbb{P}\big(X_t(x) \geq x + y\big) = t A_1(x; y) + \frac{t^2}{2} A_2(x; y) + \mathrm{O}\big(t^3\big) \qquad \text{for } x \in \mathbb{R}, y > 0. \tag{1.4}$$

The assumptions required for (1.4) include boundedness and sufficient smoothness of the SDE's coefficients as well as non-degeneracy conditions on $|\partial_\zeta \gamma(x, \zeta)|$ and $|1 + \partial_x \gamma(x, \zeta)|$. As in

Léandre [18], the key assumption on the Lévy measure ν of Z is that this admits a density $h:\mathbb{R}\setminus\{0\}\to\mathbb{R}_+$ that is bounded and sufficiently smooth outside any neighborhood of the origin. In that case, the leading term $A_1(x;y)$ depends only on the jump component of the process as follows

$$A_1(x;y)=\nu\bigl(\bigl\{\zeta:\ \gamma(x,\zeta)\geq y\bigr\}\bigr)=\int_{\{\zeta:\gamma(x,\zeta)\geq y\}}h(\zeta)\,\mathrm{d}\zeta.$$

The second-order term $A_2(x;y)$ admits a more complex (but explicit) representation, which enables us, for instance, to precisely characterize the effects of the drift b and the diffusion σ of the process in the likelihood of a "large" positive move (say, a move of size more than y) during a short time period t (see Remark 4.2 below for further details).

Once the asymptotic expansion for tail distribution is obtained, we proceed to obtain a second-order expansion for the transition density function $p_t(y;x)$. As expected from taking formal differentiation of the tail expansion (1.4) with respect to y, the leading term of $p_t(x+y;x)$ is of the form $tg(x;y)$ for $y>0$, where $g(x;y)$ is the so-called Lévy density of the process $\{X_t(x)\}_{t\geq 0}$ defined by

$$g(x;y):=-\frac{\partial}{\partial y}\nu\bigl(\bigl\{\zeta:\ \gamma(x,\zeta)\geq y\bigr\}\bigr)\qquad(y>0),\tag{1.5}$$

while the second-order term takes the form $-\partial_y A_2(x;y)t^2/2$. One of the main subtleties here arises from attempting to control the density of $X_t(x)$ given that there is no "large" jump. To this end, we generalize the result in Léandre [18] to the case where there is a non-degenerate diffusion component. Again, Malliavin calculus is proved to be the key tool for this task.

Let us briefly make some remarks about the practical relevance of our results. Short-time asymptotics for the transition densities and distributions of Markov processes are important tools in many applications such as non-parametric estimation methods of the model under high-frequency sampling data and numerical approximations of functionals of the form $\Phi_t(x):=\mathbb{E}(\phi(X_T(x)))$. In many of these applications, a certain discretization of the continuous-time object under study is needed and, in that case, short-time asymptotics are important not only in developing such discrete-time approximations but also to determine the rate of convergence of the discrete-time proxies to their continuous-time counterparts.

As an instance of the applications referred to in the previous paragraph, a problem that has received a great deal of attention in the last few years is the study of small-time asymptotics for option prices and implied volatilities (see, e.g., Gatheral et al. [15], Feng, Forde and Fouque [8], Forde and Jacquier [13], Berestyki, Busca and Florent [5], Figueroa-López and Forde [9], Roper [30], Tankov [33], Gao and Lee [14], Muhle-Karbe and Nutz [23], Figueroa-López, Gong and Houdré [10]). As a byproduct of the asymptotics for the tail distributions (1.4), we derive here the leading term of the small-time expansion for the arbitrage-free prices of out-of-the-money European call options. Specifically, let $\{S_t\}_{t\geq 0}$ be the stock price process and denote $X_t=\log S_t$ for each $t\geq 0$. We assume that $\mathbb{P}$ is the option pricing measure and that under this measure the process $\{X_t\}_{t\geq 0}$ is of the form in (1.3). Then, we prove that

$$\lim_{t\to 0}\frac{1}{t}\mathbb{E}(S_t-K)_+=\int_{-\infty}^{\infty}\bigl(S_0\mathrm{e}^{\gamma(x,\zeta)}-K\bigr)_+h(\zeta)\,\mathrm{d}\zeta,\tag{1.6}$$

which extends the analog result for exponential Lévy model (cf. Roper [30] and Tankov [33]). A related paper is Levendorskii [20], where (1.6) was obtained for a wide class of multi-factor Lévy Markov models under certain technical conditions (see Theorem 2.1 therein), including the requirement that $\lim_{t\to 0} \mathbb{E}(S_t - K)_+/t$ exists in the "out-of-the-money region" and some stringent integrability conditions on the Lévy density h.

The paper is organized as follows. In Section 2, we introduced the model and the assumptions needed for our results. The probabilistic tools, such as the iterated Dynkin's formula as well as tail estimates for semimartingales with bounded jumps, are presented in Section 3. The main results of the paper are then stated in Sections 4 and 5, where the second-order expansion for the tail distributions and the transition densities are obtained, respectively. The application of the expansion for the tail distribution to option pricing in local jump-diffusion financial models is presented in Section 6. The proofs of our main results as well as some preliminaries of Malliavin calculus on Wiener–Poisson spaces are given in several appendices.

2. Setup, assumptions and notation

Throughout, $C_b^{\geq 1}$ (resp., C_b^{∞}) represents the class of continuous (resp., bounded) functions with bounded and continuous partial derivatives of arbitrary order $n \geq 1$. We let $Z := \{Z_t\}_{t\geq 0}$ be a pure-jump Lévy process with Lévy measure ν and $\{W_t\}_{t\geq 0}$ be a Wiener process independent of Z, both of which are defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, equipped with the natural filtration $(\mathcal{F}_t)_{t\geq 0}$ generated by W and Z and augmented by all the null sets in $\mathcal{F}$ so that it satisfies the *usual conditions* (see, e.g., Chapter I in Protter [29]). The jump measure of the process Z is denoted by $M(\mathrm{d}u, \mathrm{d}\zeta) := \#\{u > 0\colon (u, \Delta Z_u) \in du \times d\zeta\}$, where $\Delta Z_u := Z_u - Z_{u^-} := Z_u - \lim_{s\nearrow t} Z_s$ denotes the jump Z at time u. This is necessarily a Poisson random measure on $\mathbb{R}_+ \times \mathbb{R}\setminus\{0\}$ with mean measure $\mathbb{E}M(\mathrm{d}u, \mathrm{d}\zeta) = \mathrm{d}u\,\nu(\mathrm{d}\zeta)$. The corresponding compensated random measure is denoted $\bar{M}(\mathrm{d}u, \mathrm{d}\zeta) := M(\mathrm{d}u, \mathrm{d}\zeta) - \mathrm{d}u\,\nu(\mathrm{d}\zeta)$.

As stated in the Introduction, in this paper, we consider the following local jump-diffusion model:

$$
\begin{aligned}
X_t(x) = x &+ \int_0^t b\bigl(X_u(x)\bigr)\,\mathrm{d}u + \int_0^t \sigma\bigl(X_u(x)\bigr)\,\mathrm{d}W_u \\
&+ \int_0^t \int_{|\zeta|>1} \gamma\bigl(X_{u^-}(x), \zeta\bigr) M(\mathrm{d}u, \mathrm{d}\zeta) \\
&+ \int_0^t \int_{|\zeta|\leq 1} \gamma\bigl(X_{u^-}(x), \zeta\bigr) \bar{M}(\mathrm{d}u, \mathrm{d}\zeta),
\end{aligned}
\tag{2.1}
$$

where $b, \sigma : \mathbb{R} \to \mathbb{R}$ and $\gamma : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ are deterministic functions satisfying suitable conditions under which (2.1) admits a unique solution. Typical sufficient conditions for (2.1) to be well-posed include linear growth and Lipschitz continuity of the coefficients b, σ, and γ (see, e.g., Applebaum [3], Theorem 6.2.3, Oksendal and Sulem [25], Theorem 1.19).

Below, we will make use of the following assumptions about Z:

(C1) The Lévy measure ν of Z has a $C^{\infty}(\mathbb{R} \setminus \{0\})$ strictly positive density h such that, for every $\varepsilon > 0$ and $n \geq 0$,

$$\sup_{|\zeta|>\varepsilon} \left| h^{(n)}(\zeta) \right| < \infty. \tag{2.2}$$

Remark 2.1. Condition (2.2) is actually needed for the tail probabilities of $\{X_t(x)\}_{t\geq 0}$ to admit an expansion in integer powers of time. Indeed, even in the simplest pure Lévy case ($X_t(x) = Z_t + x$), it is possible to build examples where $\mathbb{P}(Z_t \geq y)$ converges to 0 at a fractional power of t in the absence of (2.2)(ii) (see Marchal [21]).

Throughout the paper, the jump coefficient γ is assumed to satisfy the following conditions:

(C2)(a) $\gamma(\cdot, \cdot) \in C_b^{\geq 1}(\mathbb{R} \times \mathbb{R})$ and $\gamma(x, 0) = 0$ for all $x \in \mathbb{R}$;
(C2)(b) There exists a constant $\delta > 0$ such that $|\partial_\zeta \gamma(x, \zeta)| \geq \delta$, for all $x, \zeta \in \mathbb{R}$.

Both of the previous conditions were also imposed in Léandre [18]. Note that (C2)(a) implies that, for any $\varepsilon > 0$, there exists $C_\varepsilon < \infty$ such that

$$\sup_x \left| \frac{\partial^i \gamma(x, \zeta)}{\partial x^i} \right| \leq C_\varepsilon |\zeta| \tag{2.3}$$

for all $|\zeta| \leq \varepsilon$ and $i \geq 0$. Condition (C2)(b) is imposed so that, for each $x \in \mathbb{R}$, the mapping $\zeta \to \gamma(x, \zeta)$ admits an inverse function $\gamma^{-1}(x, \zeta)$ with bounded derivatives. Note that (C2)(b) together with the continuity of $\partial\gamma(x, \zeta)/\partial\zeta$ implies that the mapping $\zeta \to \gamma(x, \zeta)$ is either strictly increasing or decreasing for all x.

We will also require the following boundedness and non-degeneracy conditions:

(C3) The functions $b(x)$ and $v(x) := \sigma^2(x)/2$ belong to $C_b^{\infty}(\mathbb{R})$.
(C4) There exists a constant $\delta > 0$ such that, for all $x, \zeta \in \mathbb{R}$,

$$\text{(i)} \quad \left| 1 + \frac{\partial \gamma(x, \zeta)}{\partial x} \right| \geq \delta, \qquad \text{(ii)} \quad \sigma(x) \geq \delta. \tag{2.4}$$

Remark 2.2. Boundedness conditions of the type (C3) above are not restrictive in practice. Indeed, on one hand, extremely large values of b and σ will not typically make sense in a particular financial or physical phenomenon in mind (e.g., a large volatility value σ could hardly be justified financially). On the other hand, a stochastic model with arbitrary (but sufficiently regular) functions b and v could be closely approximated by a model with C_b^{∞} functions b and v. The condition (2.4)(i), which was also imposed in Léandre [18], guarantees the a.s. existence of a flow $\Phi_{s,t}(x) : \mathbb{R} \to \mathbb{R}, x \to X_{s,t}(x)$ of diffeomorphisms for all $0 \leq s \leq t$ (cf. Léandre [18]), where here $\{X_{s,t}(x)\}_{t\geq s}$ is defined as in (2.1) but with initial condition $X_{s,s}(x) = x$. Finally, let us mentioned that condition (C4)(ii) is used only for the density expansion, but not the tail expansion.

As it is usually the case with Lévy processes, we shall decompose Z into a compound Poisson process and a process with bounded jumps. More specifically, let $\phi_\varepsilon \in C^\infty(\mathbb{R})$ be a truncation function such that $\mathbf{1}_{|\zeta|\geq\varepsilon} \leq \phi_\varepsilon(\zeta) \leq \mathbf{1}_{|\zeta|\geq\varepsilon/2}$ and let $Z(\varepsilon) := \{Z_t(\varepsilon)\}_{t\geq 0}$ and $Z'(\varepsilon) := \{Z'_t(\varepsilon)\}_{t\geq 0}$ be independent Lévy processes with respective Lévy densities

$$h_\varepsilon(\zeta) := \phi_\varepsilon(\zeta)h(\zeta) \quad \text{and} \quad \bar{h}_\varepsilon(\zeta) := \bigl(1-\phi_\varepsilon(\zeta)\bigr)h(\zeta). \tag{2.5}$$

Clearly, we have that

$$Z \stackrel{\mathfrak{D}}{=} Z'(\varepsilon) + Z(\varepsilon). \tag{2.6}$$

The process $Z'(\varepsilon)$, that we referred to as the small-jump component of Z, is a pure-jump Lévy process with jumps bounded by ε. In contrast, the process $Z(\varepsilon)$, hereafter referred to as the big-jump component of Z, is taken to be a compound Poisson process with intensity of jumps $\lambda_\varepsilon := \int \phi_\varepsilon(\zeta)h(\zeta)\,\mathrm{d}\zeta$ and jumps $\{J_i^\varepsilon\}_{i\geq 1}$ with probability density function

$$\breve{h}_\varepsilon(\zeta) := \frac{\phi_\varepsilon(\zeta)h(\zeta)}{\lambda_\varepsilon}. \tag{2.7}$$

Throughout the paper, $\{\tau_i\}_{i\geq 1}$ and $N := \{N_t^\varepsilon\}_{t\geq 0}$, respectively, denote the jump arrival times and the jump counting process of the compound Poisson process $Z(\varepsilon)$, and $J := J^\varepsilon$ represents a generic random variable with density $\breve{h}_\varepsilon(\zeta)$.

The next result will be needed in what follows. The different properties below follow from standard applications of the implicit function theorem, and the required smoothness and non-degeneracy conditions stated above. We refer the reader to Figueroa-López, Luo and Ouyang [12] for a detailed proof.

Lemma 2.1. *Under the conditions* (C1), (C2) *and* (C4), *the following statements hold*:

1. *Let $\tilde{\gamma}(z,\zeta) := \gamma(z,\zeta) + z$. Then, for each $z \in \mathbb{R}$, the mapping $\zeta \to \tilde{\gamma}(z,\zeta)$ (equiv. $\zeta \to \gamma(z,\zeta)$) is invertible and its inverse $\tilde{\gamma}^{-1}(z,\zeta)$ (resp., $\gamma^{-1}(z,\zeta)$) is $C_b^{\geq 1}(\mathbb{R}\times\mathbb{R})$.*
2. *Both $\tilde{\gamma}(z,J^\varepsilon)$ and $\gamma(z,J^\varepsilon)$ admit densities in $C_b^\infty(\mathbb{R}\times\mathbb{R})$, denoted by $\widetilde{\Gamma}(\zeta;z) := \widetilde{\Gamma}_\varepsilon(\zeta;z)$ and $\Gamma(\zeta;z) := \Gamma_\varepsilon(\zeta;z)$, respectively. Furthermore, they have the representation*:

$$\widetilde{\Gamma}_\varepsilon(\zeta;z) = \breve{h}_\varepsilon\bigl(\tilde{\gamma}^{-1}(z,\zeta)\bigr)\left|\frac{\partial\gamma}{\partial\zeta}\bigl(z,\tilde{\gamma}^{-1}(z,\zeta)\bigr)\right|^{-1}, \tag{2.8}$$

$$\Gamma_\varepsilon(\zeta;z) = \breve{h}_\varepsilon\bigl(\gamma^{-1}(z,\zeta)\bigr)\left|\frac{\partial\gamma}{\partial\zeta}\bigl(z,\gamma^{-1}(z,\zeta)\bigr)\right|^{-1}. \tag{2.9}$$

3. *The mappings $(z,\zeta) \to \mathbb{P}(\tilde{\gamma}(z,J^\varepsilon) \geq \zeta)$ and $(z,\zeta) \to \mathbb{P}(\gamma(z,J^\varepsilon) \geq \zeta)$ are $C_b^\infty(\mathbb{R}\times\mathbb{R})$.*
4. *The mapping $z \to u := z + \gamma(z,\zeta)$ admits an inverse, denoted hereafter $\bar{\gamma}(u,\zeta)$, that belongs to $C_b^{\geq 1}(\mathbb{R}\times\mathbb{R})$.*

We finish this section with the definition of some important processes. Let $\widetilde{M}$ and $M' := M'_\varepsilon$ denote the jump measure of the process $\widetilde{Z} := Z(\varepsilon) + Z'(\varepsilon)$ and $Z'(\varepsilon)$, respectively. For each

$\varepsilon \in (0,1)$, we construct a process $\{\widetilde{X}_s(\varepsilon, x)\}_{s\geq 0}$, defined as the solution of the SDE

$$\begin{aligned}\widetilde{X}_t(\varepsilon, x) = x &+ \int_0^t b\big(\widetilde{X}_u(\varepsilon, x)\big)\,\mathrm{d}u + \int_0^t \sigma\big(\widetilde{X}_u(\varepsilon, x)\big)\,\mathrm{d}\widetilde{W}_u \\ &+ \int_0^t \int_{|\zeta|>1} \gamma\big(\widetilde{X}_{u^-}(\varepsilon, x), \zeta\big)\widetilde{M}(\mathrm{d}u, \mathrm{d}\zeta) \\ &+ \int_0^t \int_{|\zeta|\leq 1} \gamma\big(\widetilde{X}_{u^-}(\varepsilon, x), \zeta\big)\overline{\widetilde{M}}(\mathrm{d}u, \mathrm{d}\zeta),\end{aligned}$$

where $\overline{\widetilde{M}}$ is the compensated measure of $\widetilde{M}$ and $\widetilde{W}$ is a Wiener process, which is independent of $\widetilde{Z}$. In terms of the jumps of the processes $Z(\varepsilon)$ and $Z'(\varepsilon)$, we can express $\widetilde{X}(\varepsilon, x)$ as

$$\begin{aligned}\widetilde{X}_t(\varepsilon, x) = x &+ \int_0^t b_\varepsilon\big(\widetilde{X}_u(\varepsilon, x)\big)\,\mathrm{d}u + \int_0^t \sigma\big(\widetilde{X}_u(\varepsilon, x)\big)\,\mathrm{d}\widetilde{W}_u \\ &+ \sum_{i=1}^{N_t^\varepsilon} \gamma\big(\widetilde{X}_{\tau_i^-}(\varepsilon, x), J_i^\varepsilon\big) + \int_0^t \int \gamma\big(\widetilde{X}_{u^-}(x), \zeta\big)\bar{M}'(\mathrm{d}u, \mathrm{d}\zeta),\end{aligned} \tag{2.10}$$

where $\bar{M}'$ is the compensated random measure $\bar{M}'(\mathrm{d}u, \mathrm{d}\zeta) := M'(\mathrm{d}u, \mathrm{d}\zeta) - \bar{h}_\varepsilon(\zeta)\,\mathrm{d}u\,\mathrm{d}\zeta$ and

$$b_\varepsilon(x) := b(x) - \int_{|\zeta|\leq 1} \gamma(x, \zeta) h_\varepsilon(\zeta)\,\mathrm{d}\zeta.$$

Since Z has the same distribution law as $\widetilde{Z} := Z(\varepsilon) + Z'(\varepsilon)$, the process $\{\widetilde{X}_t(\varepsilon, x)\}_{t\geq 0}$ has the same distribution as $\{X_t(x)\}_{t\geq 0}$. Hence, in order to obtain the short time asymptotics of $\mathbb{P}(X_t(x) \geq x + y)$, we can (and will) analyze the behavior of $\mathbb{P}(\widetilde{X}_t(\varepsilon, x) \geq x + y)$. For simplicity and with certain abuse of notation, we shall write from now on $X(x)$ instead of $\widetilde{X}(\varepsilon, x)$ and W instead of $\widetilde{W}$.

Next, we let $\{X_s(\varepsilon, \varnothing, x)\}_{s\geq 0}$ be the solution of the SDE:

$$\begin{aligned}X_s(\varepsilon, \varnothing, x) = x &+ \int_0^s b_\varepsilon\big(X_u(\varepsilon, \varnothing, x)\big)\,\mathrm{d}u + \int_0^s \sigma\big(X_u(\varepsilon, \varnothing, x)\big)\,\mathrm{d}W_u \\ &+ \int_0^s \int \gamma\big(X_{u^-}(\varepsilon, \varnothing, x), \zeta\big)\bar{M}'(\mathrm{d}u, \mathrm{d}\zeta).\end{aligned} \tag{2.11}$$

As seeing from the representation (2.10), the law of the process (2.11) can be interpreted as the law of $\{\widetilde{X}_s(\varepsilon, x)\}_{0\leq s\leq t} = \{X_s(x)\}_{0\leq s\leq t}$ conditioning on not having any "big" jumps during $[0, t]$. In other words, denoting the law of a process Y (resp., the conditional law of Y given an event B) by $\mathcal{L}(Y)$ (resp., $\mathcal{L}(Y|B)$), we have that, for each fixed $t > 0$,

$$\mathcal{L}\big(\big\{X_s(x)\big\}_{0\leq s\leq t} | N_t^\varepsilon = 0\big) = \mathcal{L}\big(\big\{X_s(\varepsilon, \varnothing, x)\big\}_{0\leq s\leq t}\big).$$

Similarly, for a collection of times $0 < s_1 < \cdots < s_n$, let $\{X_s(\varepsilon, \{s_1, \ldots, s_n\}, x)\}_{s\geq 0}$ be the solution of the SDE:

$$
\begin{aligned}
X_s\big(\varepsilon, \{s_1, \ldots, s_n\}, x\big) := x &+ \int_0^s b_\varepsilon\big(X_u\big(\varepsilon, \{s_1, \ldots, s_n\}, x\big)\big)\,\mathrm{d}u \\
&+ \int_0^s \sigma\big(X_u\big(\varepsilon, \{s_1, \ldots, s_n\}, x\big)\big)\,\mathrm{d}W_u \\
&+ \sum_{i:\, s_i \leq s} \gamma\big(X_{s_i^-}\big(\varepsilon, \{s_1, \ldots, s_n\}, x\big), J_i^\varepsilon\big) \\
&+ \int_0^s \int \gamma\big(X_{u^-}\big(\varepsilon, \{s_1, \ldots, s_n\}, x\big), \zeta\big)\bar{M}'(\mathrm{d}u, \mathrm{d}\zeta).
\end{aligned}
$$

From (2.10), it then follows that

$$
\mathcal{L}\big(\big\{X_s(x)\big\}_{0\leq s\leq t} \big| N_t^\varepsilon = n, \tau_1 = s_1, \ldots, \tau_n = s_n\big) = \mathcal{L}\big(\big\{X_s\big(\varepsilon, \{s_1, \ldots, s_n\}, x\big)\big\}_{0\leq s\leq t}\big).
$$

The previous two processes will be needed in order to implement Léandre's approach in which the tail distribution $\mathbb{P}(X_t(x) \geq x + y)$ is expanded in powers of time by conditioning on the number of jumps of $Z(\varepsilon)$ by time t.

3. Probabilistic tools

Throughout, $C_b^n(I)$ (resp., C_b^n) denotes the class of functions having continuous and bounded derivatives of order $0 \leq k \leq n$ in an open interval $I \subset \mathbb{R}$ (resp., in $\mathbb{R}$). Also, $\|g\|_\infty = \sup_y |g(y)|$.

3.1. Uniform tail probability estimates

The following general result will be important in the sequel.

Proposition 3.1. *Let M be a Poisson random measure on $\mathbb{R}_+ \times \mathbb{R}_0$ with mean measure $\mathbb{E}M(\mathrm{d}u, \mathrm{d}\zeta) = \nu(\mathrm{d}\zeta)\,\mathrm{d}t$ and $\bar{M}$ be its compensated random measure. Let $Y := Y^{(x)}$ be the solution of the SDE*

$$
Y_t = x + \int_0^t \bar{b}(Y_s)\,\mathrm{d}s + \int_0^t \bar{\sigma}(Y_s)\,\mathrm{d}W_s + \int_0^t \int \bar{\gamma}(Y_{s-}, \zeta)\bar{M}(\mathrm{d}s, \mathrm{d}\zeta).
$$

Assume that $\bar{b}(x)$ and $\bar{\sigma}(x)$ are uniformly bounded and $\bar{\gamma}(x, \zeta)$ is such that, for a constant $S \in (0, \infty)$, $\sup_y |\bar{\gamma}(y, \zeta)| \leq S(|\zeta| \wedge 1)$, for ν-a.e. ζ. In particular, the jumps of $\{Y_t\}_{t\geq 0}$ are bounded by S, and there exists a constant k such that the quadratic variation for the martingale part of Y is bounded by kt for any time t. Then there exists a constant $C(S, k)$ depending on S and k, such that, for any fixed $p > 0$ and all $0 \leq t \leq 1$,

$$
\mathbb{P}\Big\{\sup_{0\leq s\leq t} |Y_s - x| \geq 2pS\Big\} \leq C(S, k)t^p.
$$

Proof. Let

$$V_t = \int_0^t \bar{\sigma}(Y_s)\,\mathrm{d}W_s + \int_0^t \int \bar{\gamma}(Y_{s-}, z)\bar{M}(\mathrm{d}s, \mathrm{d}z)$$

be the martingale part of Y_t. It is clear that V_t is a martingale with its jumps bounded by S. Moreover, in light of the boundedness of $\bar{\sigma}$ and $\bar{\gamma}$, its quadratic variation satisfies $\langle V, V\rangle_t \le kt$, for some constant k. By equation (9) in Lepeltier and Marchal [19], we have

$$\mathbb{P}\Big\{\sup_{0\le s\le t} |V_s| \ge C\Big\} \le 2\exp\Big[-\lambda C + \frac{\lambda^2}{2}kt\big(1+\exp[\lambda S]\big)\Big] \qquad \text{for all } C, \lambda > 0. \tag{3.1}$$

Now take $C = 2pS$ and $\lambda = |\log t|/2S$, the claimed result follows for the martingale part V_t of Y_t. By equation (9) in Lepeltier and Marchal [19] and the fact that the drift term is bounded by $\|\bar{b}\|_\infty t$, we have for all $C, \lambda > 0$

$$\begin{aligned}\mathbb{P}\Big\{\sup_{0\le s\le t} |Y_s - x| \ge C\Big\} &\le \mathbb{P}\Big\{\sup_{0\le s\le t} |V_s| \ge C - t\|\bar{b}\|_\infty\Big\} \\ &\le 2\exp\Big[-\lambda\big(C - \|\bar{b}\|_\infty t\big) + \frac{\lambda^2}{2}kt\big(1+\exp[\lambda S]\big)\Big].\end{aligned} \tag{3.2}$$

Now take $C = 2pS$ and $\lambda = |\log t|/2S$, the claimed result follows. □

As a direct corollary of the previous proposition, we have the following estimate for the tail probability of the small-jump component $\{X_t(\varepsilon, \varnothing, x)\}_{t\ge}$ of X defined in (2.11). We also provide a related estimate for the tail probability of $\exp(|X_t(\varepsilon, \varnothing, x)|)$, which will be needed for the asymptotic result of option prices discussed in Section 6 below.

Lemma 3.1. *Fix any $\eta > 0$ and a positive integer N. Then, under the conditions* (C2)–(C3) *of Section* 2, *there exist an $\varepsilon := \varepsilon(N, \eta) > 0$ and $C := C(N, \eta) < \infty$ such that*

(1) *For all $t < 1$,*

$$\sup_{0<\varepsilon'<\varepsilon, x\in\mathbb{R}} \mathbb{P}\big(\big|X_t\big(\varepsilon', \varnothing, x\big) - x\big| \ge \eta\big) < Ct^N. \tag{3.3}$$

(2) *For all $t < 1$,*

$$\sup_{\varepsilon'<\varepsilon, x\in\mathbb{R}} \int_{\mathrm{e}^\eta}^\infty \mathbb{P}\big(\mathrm{e}^{|X_t(\varepsilon', \varnothing, x) - x|} \ge s\big)\,\mathrm{d}s < Ct^N.$$

Proof. The first statement is a special case of Proposition 3.1, which can be applied in light of the boundedness conditions (C3) as well as the condition (C2)(a). To prove the second statement, we keep the notation of the proof of Proposition 3.1 and note that, by (3.2), there exists a constant

$C > 0$ such that

$$\int_{e^\eta}^{\infty} \mathbb{P}\{|X_t(\varepsilon, \varnothing, x) - x| \geq \log s\} \, ds \leq C \int_{e^\eta}^{\infty} \exp\left[-\lambda \log s + \frac{\lambda^2}{2} kt\bigl(1 + \exp[\lambda\varepsilon]\bigr)\right] ds$$

$$= \frac{C e^\eta}{(\lambda - 1)e^{\lambda\eta}} \exp\left[\frac{\lambda^2}{2} kt\bigl(1 + \exp[\lambda\varepsilon]\bigr)\right].$$

Now it suffices to take $\lambda = |\log t|/2\varepsilon$ and $\varepsilon = \eta/2N$. □

3.2. Iterated Dynkin's formula

We now proceed to state a second-order iterated Dynkin's formula for the "small-jump component" of X, $\{X_t(\varepsilon, \varnothing, x)\}_{t\geq 0}$, defined in (2.11). To this end, let us first recall that the infinitesimal generator of $X(\varepsilon, \varnothing, x)$, hereafter denoted by L_ε, can be written as follows (cf. Oksendal and Sulem [25], Theorem 1.22):

$$\begin{aligned} L_\varepsilon f(y) &:= \mathcal{D}_\varepsilon f(y) + \mathcal{I}_\varepsilon f(y) \qquad \text{with} \\ \mathcal{D}_\varepsilon f(y) &:= \frac{\sigma^2(y)}{2} f''(y) + b_\varepsilon(y) f'(y), \\ \mathcal{I}_\varepsilon f(y) &:= \int \bigl(f\bigl(y + \gamma(y,\zeta)\bigr) - f(y) - \gamma(y,\zeta) f'(y)\bigr) \bar{h}_\varepsilon(\zeta) \, d\zeta. \end{aligned} \tag{3.4}$$

The following two alternative representations of $\mathcal{I}_\varepsilon f$ will be useful in the sequel:

$$\mathcal{I}_\varepsilon f(y) = \int \int_0^1 f''\bigl(y + \gamma(y,\zeta)\beta\bigr)(1-\beta) \, d\beta \bigl(\gamma(y,\zeta)\bigr)^2 \bar{h}_\varepsilon(\zeta) \, d\zeta \tag{3.5}$$

$$\begin{aligned} = \int \int_0^1 \bigl[& f''\bigl(y + \gamma(y,\zeta\beta)\bigr)\bigl(\partial_\zeta \gamma(y,\zeta\beta)\bigr)^2 + f'\bigl(y + \gamma(y,\zeta\beta)\bigr)\partial_\zeta^2 \gamma(y,\zeta\beta) \\ & - f'(y)\partial_\zeta^2 \gamma(y,\zeta\beta)\bigr](1-\beta) \, d\beta \, \zeta^2 \bar{h}_\varepsilon(\zeta) \, d\zeta. \end{aligned} \tag{3.6}$$

In particular, from the previous representations, it is evident that $\mathcal{I}_\varepsilon f$ is well-defined whenever $f \in C_b^2$, in view of (2.3), which follows from our condition (C2)(a).

The n-order iterated Dynkin's formula for the process $X(\varepsilon, \varnothing, x)$ takes the generic form

$$\mathbb{E} f\bigl(X_t(\varepsilon, \varnothing, x)\bigr) = \sum_{k=0}^{n-1} \frac{t^k}{k!} L_\varepsilon^k f(x) + \frac{t^n}{(n-1)!} \int_0^1 (1-\alpha)^{n-1} \mathbb{E}\bigl\{L_\varepsilon^n f\bigl(X_{\alpha t}(\varepsilon, \varnothing, x)\bigr)\bigr\} \, d\alpha, \tag{3.7}$$

where as usual $L_\varepsilon^0 f = f$ and $L_\varepsilon^n f = L_\varepsilon(L_\varepsilon^{n-1} f)$, $n \geq 1$. (3.7) can be proved for $n = 1$ using Itô's formula (see Oksendal and Sulem [25], Theorem 1.23) while, for a general order n, it can be proved by induction, provided that the iterated generators $L_\varepsilon^k f$ satisfy sufficient smoothness and boundedness conditions for any $k = 0, \ldots, n$. The next lemma explicitly states the second-order formula so that we can refer to it in the sequel. Its proof is standard and is omitted for the sake of brevity (see Figueroa-López, Luo and Ouyang [12] for the details).

Lemma 3.2. *For a fix $\varepsilon \in (0,1)$, let $K_{\varepsilon,m}$ denote a finite constant whose value only depends on $\int \zeta^2 \bar{h}_\varepsilon(\zeta)\,d\zeta$, $\|f^{(k)}\|_\infty$, $\|b^{(k)}\|_\infty$, and $\|v^{(k)}\|_\infty$ with $k = 0, \dots, m$. Then, under the conditions* (C1)–(C3) *of Section 2, the following assertions hold true*:

1. *For any function f in C_b^2, $\sup_y L_\varepsilon f(y) \le K_{\varepsilon,2}$, and the iterated Dynkin's formula (3.7) is satisfied with $n = 1$.*
2. *If, additionally, $f \in C_b^4$, then $\sup_y L_\varepsilon^2 f(y) \le K_{\varepsilon,4}$ and, furthermore, the iterated Dynkin's formula (3.7) is satisfied with $n = 2$.*

4. Second-order expansion for the tail distributions

We are ready to state our first main result; namely, we characterize the small-time behavior of the tail distribution of $\{X_t(x)\}_{t\ge 0}$:

$$\bar{F}_t(x,y) := \mathbb{P}\big(X_t(x) \ge x + y\big) \qquad (y > 0). \tag{4.1}$$

As in Léandre [18], the key idea is to take advantage of the decomposition (2.6), by conditioning on the number of "large" jumps occurring before time t. Concretely, recalling that $\{N_t^\varepsilon\}_{t\ge 0}$ and $\lambda_\varepsilon := \int \phi_\varepsilon(\zeta) h(\zeta)\,d\zeta$ represent the jump counting process and the jump intensity of the large-jump component process $\{Z_t(\varepsilon)\}_{t\ge 0}$ of Z, we have

$$\mathbb{P}\big(X_t(x) \ge x + y\big) = e^{-\lambda_\varepsilon t} \sum_{n=0}^{\infty} \mathbb{P}\big(X_t(x) \ge x + y | N_t^\varepsilon = n\big) \frac{(\lambda_\varepsilon t)^n}{n!}. \tag{4.2}$$

The first term in (4.2) (when $n = 0$) can be written as

$$\mathbb{P}\big(X_t(x) \ge x + y | N_t^\varepsilon = 0\big) = \mathbb{P}\big(X_t(\varepsilon, \varnothing, x) \ge x + y\big).$$

In light of (3.3), this term can be made $O(t^N)$ for an arbitrarily large $N \ge 1$, by taking ε small enough. In order to deal with the other terms in (4.2), we use the iterated Dynkin's formula introduced in Section 3.2. The following is the main result of this section (see Appendix A for the proof). Below, h_ε and $\bar{h}_\varepsilon$ denote the Lévy densities defined in (2.5), while $g(x; y)$ denotes the so-called Lévy density of the process $\{X_t(x)\}_{t\ge 0}$ defined by

$$g(x; y) := \begin{cases} -\dfrac{\partial}{\partial y} \displaystyle\int_{\{\zeta : \gamma(x,\zeta) \ge y\}} h(\zeta)\,d\zeta, & y > 0, \\ \dfrac{\partial}{\partial y} \displaystyle\int_{\{\zeta : \gamma(x,\zeta) \le y\}} h(\zeta)\,d\zeta, & y < 0. \end{cases} \tag{4.3}$$

for $y \ne 0$. In light of Lemma 2.1, g admits the representation:

$$g(x; y) = h\big(\gamma^{-1}(x,y)\big) \big|(\partial_\zeta \gamma)\big(x, \gamma^{-1}(x,y)\big)\big|^{-1},$$

where $\partial_\zeta \gamma$ is the partial derivative of the function $\gamma(x,\zeta)$ with respect to its second variable.

Theorem 4.1. *Let $x \in \mathbb{R}$ and $y > 0$. Then, under the conditions* (C1)–(C4) *of Section* 2, *we have*

$$\bar{F}_t(x, y) := \mathbb{P}\big(X_t(x) \geq x + y\big) = t A_1(x; y) + \frac{t^2}{2} A_2(x; y) + \mathrm{O}\big(t^3\big) \tag{4.4}$$

as $t \to 0$, where $A_1(x; y)$ and $A_2(x; y)$ admit the following representations (for $\varepsilon > 0$ small enough):

$$\begin{aligned} A_1(x; y) &:= \int_y^\infty g(x; \zeta)\,\mathrm{d}\zeta = \int_{\{\gamma(x,\zeta)\geq y\}} h(\zeta)\,\mathrm{d}\zeta, \\ A_2(x; y) &:= \mathcal{D}(x; y) + \mathcal{J}_1(x; y) + \mathcal{J}_2(x; y), \end{aligned}$$

with

$$\begin{aligned} \mathcal{D}(x; y) &= b_\varepsilon(x)\bigg(\frac{\partial}{\partial x}\int_y^\infty g(x; \zeta)\,\mathrm{d}\zeta + g(x; y)\bigg) + b_\varepsilon(x+y) g(x; y) \\ &\quad + \frac{\sigma^2(x)}{2}\bigg(\frac{\partial^2}{\partial x^2}\int_y^\infty g(x; \zeta)\,\mathrm{d}\zeta + 2\frac{\partial}{\partial x} g(x; y) - \frac{\partial}{\partial y} g(x; y)\bigg) \\ &\quad - \frac{\sigma(x+y)}{2}\bigg(\sigma(x+y)\frac{\partial}{\partial y} g(x; y) + 2\sigma'(x+y) g(x; y)\bigg), \\ \mathcal{J}_1(x; y) &= \int\bigg(\int_{y-\gamma(x,\bar{\zeta})}^\infty g\big(x + \gamma(x, \bar{\zeta}); \zeta\big)\,\mathrm{d}\zeta + \int_{\bar{\gamma}(x+y,\bar{\zeta})-x}^\infty g(x; \zeta)\,\mathrm{d}\zeta - 2\int_y^\infty g(x; \zeta)\,\mathrm{d}\zeta \\ &\quad - \gamma(x, \bar{\zeta})\partial_x \int_y^\infty g(x; \zeta)\,\mathrm{d}\zeta - \gamma(x, \bar{\zeta}) g(x; y) - \gamma(x+y, \bar{\zeta}) g(x; y)\bigg)\bar{h}_\varepsilon(\bar{\zeta})\,\mathrm{d}\bar{\zeta}, \\ \mathcal{J}_2(x; y) &= \int\int_{y-\gamma(x,\bar{\zeta})}^\infty g\big(x + \gamma(x, \bar{\zeta}); \zeta\big)\,\mathrm{d}\zeta\, h_\varepsilon(\bar{\zeta})\,\mathrm{d}\bar{\zeta} - 2\int_y^\infty g(x; \zeta)\,\mathrm{d}\zeta \int h_\varepsilon(\zeta)\,\mathrm{d}\zeta. \end{aligned} \tag{4.5}$$

Remark 4.1. Note that if $\mathrm{supp}(\nu) \cap \{\zeta\colon \gamma(x, \zeta) \geq y\} = \varnothing$ (so that it is not possible to reach the level y from x with only one jump), then $A_1(x; y) = 0$ and $\mathbb{P}(X_t(x) \geq x + y) = \mathrm{O}(t^2)$ as $t \to 0$. If, in addition, it is possible to reach the level y from x with two jumps, then $\mathcal{J}_2(x; y) \neq 0$, implying that $\mathbb{P}(X_t(x) \geq x + y)$ decreases at the order of t^2. These observations are consistent with the results in Ishikawa [16] and Picard [27].

Remark 4.2. In the case that the coefficient $\gamma(x, \zeta)$ does not depend on x, we get the following expansion for $\mathbb{P}(X_t(x) \geq x + y)$:

$$\begin{aligned} \mathbb{P}\big(X_t(x) \geq x + y\big) &= t\int_y^\infty g(\zeta)\,\mathrm{d}\zeta + \frac{b_\varepsilon(x) + b_\varepsilon(x+y)}{2} g(y) t^2 \\ &\quad - \bigg(\frac{\sigma^2(x) + \sigma^2(x+y)}{2} g'(y) + 2\sigma(x+y)\sigma'(x+y) g(y)\bigg)\frac{t^2}{2} \end{aligned}$$

$$+\int\left(\int_{y-\gamma(\bar{\zeta})}^{\infty} g(\zeta)\,\mathrm{d}\zeta-\int_{y}^{\infty} g(\zeta)\,\mathrm{d}\zeta-2g(y)\gamma(\bar{\zeta})\right)\bar{h}_{\varepsilon}(\bar{\zeta})\,\mathrm{d}\bar{\zeta}\,t^{2}$$
$$+\left(\int\int_{y-\gamma(\bar{\zeta})}^{\infty} g(\zeta)\,\mathrm{d}\zeta\, h_{\varepsilon}(\bar{\zeta})\,\mathrm{d}\bar{\zeta}-2\int_{y}^{\infty} g(\zeta)\,\mathrm{d}\zeta\int h_{\varepsilon}(\zeta)\,\mathrm{d}\zeta\right)\frac{t^{2}}{2}+\mathrm{O}\big(t^{3}\big).$$

The leading term in the above expression is determined by the jump component of the process and it has a natural interpretation: if within a very short time interval there is a "large" positive move (say, a move by more than y), this move must be due to a "large" jump. It is until the second term, when the diffusion and drift terms of the process $X(x)$ appear. If, for instance, b and σ are constants, the effect of a positive "drift" $b_{\varepsilon}>0$ is to increase the probability of a "large" positive move of more than y by $b_{\varepsilon}g(y)t^{2}(1+\mathrm{o}(1))$. Similarly, since typically $g'(y)<0$ when $y>0$, the effect of a non-zero spot volatility σ is to increase the probability of a "large" positive move by $\frac{\sigma^{2}}{2}|g'(y)|t^{2}(1+\mathrm{o}(1))$.

5. Expansion for the transition densities

Our goal here is to obtain a second-order small-time approximation for the transition densities $\{p_t(\cdot;x)\}_{t\geq 0}$ of $\{X_t(x)\}_{t\geq 0}$. As it was done in the previous section, the idea is to work with the expansion (4.2) by first showing that each term there is differentiable with respect to y, and then determining their rates of convergence to 0 as $t\to 0$. One of the main difficulties of this approach comes from controlling the term corresponding to no "large" jumps. As in the case of purely diffusion processes, Malliavin calculus is proved to be the key tool for this task. This analysis is presented in the following subsection before our main result is presented in Section 5.2.

5.1. Density estimates for SDE with bounded jumps

In this part, we analyze the term corresponding to $N_t^{\varepsilon}=0$:

$$\mathbb{P}\big(X_t(x)\geq x+y|N_t^{\varepsilon}=0\big)=\mathbb{P}\big(X_t(\varepsilon,\varnothing,x)\geq x+y\big).$$

We will prove that, for any fixed positive integer N and $\eta>0$, there exist an $\varepsilon_0>0$ and a constant $C<\infty$ (both only depending on N and η) such that the density $p_t(\cdot;\varepsilon,\varnothing,x)$ of $X_t(\varepsilon,\varnothing,x)$ satisfies

$$\sup_{|y-x|>\eta,\varepsilon<\varepsilon_0} p_t(y;\varepsilon,\varnothing,x)<Ct^{N}\tag{5.1}$$

for all $0<t\leq 1$.

To simplify notation, in this subsection, we write X_t^x for $X_t(\varepsilon,\varnothing,x)$. Recall that X_t^x satisfies an equation of the following general form

$$X_t^x=x+\int_0^t b_{\varepsilon}\big(X_{s-}^x\big)\,\mathrm{d}s+\int_0^t\sigma\big(X_{s-}^x\big)\,\mathrm{d}W_s+\int_0^t\int\gamma\big(X_{s-}^x,\zeta\big)\bar{M}'(\mathrm{d}s,\mathrm{d}\zeta),\tag{5.2}$$

where, $M'(\mathrm{d}s, \mathrm{d}\zeta)$ is a Poisson random measure on $\mathbb{R}_+ \times \mathbb{R} \setminus \{0\}$ with mean measure $\mu'(\mathrm{d}s, \mathrm{d}\zeta) = \nu'(\mathrm{d}\zeta)\,\mathrm{d}s = \bar{h}_\varepsilon(\zeta)\,\mathrm{d}\zeta\,\mathrm{d}s$ and $\bar{M}' = M' - \mu'$ is its compensated measure. Since there are no "big jumps" for X_t^x, $\bar{h}_\varepsilon$ is supported in a ball $B(0, \varepsilon)$.

Malliavin calculus is the main tool to analyze the existence and smoothness of density for X_t^x. Throughout this subsection, we follow closely the presentation of Bichteler, Gravereaux and Jacod [6], Chapter IV (see also Appendix A in Figueroa-López, Luo and Ouyang [12] for an introduction to this theory). As described therein, there are different ways to define a Malliavin operator for Wiener–Poisson spaces. For our purposes, it suffices to consider the Malliavin operator corresponding to $\rho = 0$ (see Bichteler, Gravereaux and Jacod [6], Section 9a–9c, for the details). The intuitive explanation of $\rho = 0$ is that when making perturbation of the sample path on the Wiener–Poisson space, we only perturb the Brownian path.

Let us start by noting that our assumption on the coefficients of (5.2) ensures that $x \to X_t^x$ is a C^2-diffeomorphism with a continuous density (see Bichteler, Gravereaux and Jacod [6] for more details). Define

$$U_t := \Gamma\big(X_t^x, X_t^x\big) = \left\{\int_0^t \sigma^2\big(X_s^x\big)\mathbf{J}_s(x)^{-2}\,\mathrm{d}s\right\}\mathbf{J}_t(x)^2. \tag{5.3}$$

In the above, we use the standard notation:

$$\mathbf{J}_t(x) = \frac{\mathrm{d}X_t^x}{\mathrm{d}x}. \tag{5.4}$$

Remark 5.1. Under the condition (C4) of Section 2, $\mathbf{J}_t(x)$ admits an inverse $Y_t := \mathbf{J}_t(x)^{-1}$, almost surely. Indeed, one can show that (cf. Bichteler, Gravereaux and Jacod [6])

$$\begin{aligned} d\mathbf{J}_t(x) = 1 &+ \partial_x b_\varepsilon\big(X_{t-}^x\big)\mathbf{J}_{t-}(x)\,\mathrm{d}t + \partial_x \sigma\big(X_{t-}^x\big)\mathbf{J}_{t-}(x)\,\mathrm{d}W_t \\ &+ \partial_x \gamma\big(X_{t-}^x, \zeta\big)\mathbf{J}_{t-}(x)\bar{M}'(\mathrm{d}t, \mathrm{d}\zeta), \end{aligned}$$

while $Y_t = \mathbf{J}_t(x)^{-1}$ satisfies an equation of the form:

$$dY_t = 1 + Y_{t-}D_t\,\mathrm{d}t + Y_{t-}E_t\,dW_t + Y_{t-}F_t\bar{M}'(\mathrm{d}t, \mathrm{d}\zeta).$$

Here D_t, E_t and F_t are determined by $b_\varepsilon(x), \sigma(x), \gamma(x, \zeta)$ and X_t^x. As a consequence, together with our assumption on b, σ and γ, one has

$$\mathbb{E}\sup_{0\le t\le 1}\mathbf{J}_t(x)^p \quad \text{and} \quad \mathbb{E}\sup_{0\le t\le 1}\mathbf{J}_t(x)^{-p} < \infty$$

for all $p > 1$.

The main result of this section is Theorem 5.1 below. For this purpose, we state some preliminary known results. Let us start with the following integration by parts formula (the main ingredient for existence and smoothness of the density of X_t^x), which is a special case of Lemma 4–14 in Bichteler, Gravereaux and Jacod [6] together with the discussion of Chapter IV therein.

Proposition 5.1 (Integration by parts). *For any $f \in C_c^\infty(\mathbb{R})$, there exists a random variable $G_t \in L^p$ for all $p \in \mathbb{N}$, such that*

$$\mathbb{E}\partial_x f\big(X_t^x\big) = \mathbb{E}G_t U_t^{-2} f\big(X_t^x\big).$$

The following existence and regularity result for the density of a finite measure is well known (see, e.g., Theorem 5.3 in Shigekawa [32]).

Proposition 5.2. *Let m be a finite measure supported in an open set $\mathrm{O} \subset \mathbb{R}$. Take any $p > 1$. Suppose that there exists $g \in \mathbb{L}^p(m)$ such that*

$$\int_{\mathbb{R}} \partial_x f \,\mathrm{d}m = \int_{\mathbb{R}} f g \,\mathrm{d}m, \qquad f \in C_c^\infty(\mathrm{O}).$$

Then m has a bounded density function $q \in C_b(\mathrm{O})$ satisfying

$$\|q\|_\infty \le C \|g\|_{\mathbb{L}^p(m)} m(\mathrm{O})^{1-1/p}.$$

Here the constant C depends on p.

The following lemma is the main ingredient in proving Theorem 5.1.

Lemma 5.1. *Recall $U_t = \Gamma(X_t, X_t)$. Under the condition* (C4) *of Section* 2, *we have*

$$\mathbb{E}U_t^{-p} \le C t^{-p},$$

for all $p > 1$.

Proof. The proof is a direct consequence of assumption (C4) and Remark 5.1. More precisely,

$$\begin{aligned}
\mathbb{E}U_t^{-p} &= \mathbb{E}\frac{\mathbf{J}_t(x)^{-2p}}{(\int_0^t \mathbf{J}_s(x)^{-2}\sigma(X_s^x)^2\,\mathrm{d}s)^p} \le \frac{1}{t^p}\mathbb{E}\frac{\mathbf{J}_t(x)^{-2p}}{\delta^{2p}\inf_{0\le s\le t}\mathbf{J}_s(x)^{-2p}} \\
&= \frac{1}{t^p}\delta^{-2p}\mathbb{E}\Big(\mathbf{J}_t(x)^{-2p}\sup_{0\le s\le 1}\mathbf{J}_s(x)^{2p}\Big).
\end{aligned}$$

The proof is completed. □

Remark 5.2. The above lemma is where condition (C4)(ii) is used. It could be relaxed to include degenerate diffusion coefficients. But in the degenerate case, we need to take a non-trivial ρ (as opposed to $\rho = 0$ in the present setting) in the construction of Malliavin operator on the Wiener–Poisson space. In this case, the process U_t becomes

$$\begin{aligned}
U_t &:= \mathbf{J}_t(x)^2\int_0^t \sigma^2\big(X_s^x\big)\mathbf{J}_s(x)^{-2}\,\mathrm{d}s \\
&\quad + \mathbf{J}_t(x)^2\int_{\mathbb{R}}\int_0^t \mathbf{J}_{s-}(x)^{-2}\big(1+\partial_x\gamma\big(X_{s-}^x,\zeta\big)\big)^2\big(\partial_\zeta\gamma\big(X_{s-}^x,\zeta\big)\big)^2\rho(\zeta)M'(\mathrm{d}s,\mathrm{d}\zeta).
\end{aligned}$$

Under suitable conditions on ρ, the above is well-defined and it is also possible to obtain an estimate of the form:

$$\mathbb{E}U_t^{-p} \le Ct^{-N(p)}.$$

Finally, we can state and prove our main result of this section.

Theorem 5.1. *Assume the condition* (C3) *of Section* 2 *is satisfied. Let* $\{X_t^x\}_{t\ge 0}$ *be the solution to equation* (5.2) *and denote the density of* X_t^x *by* $p_t(y;x)$. *Fix* $\eta > 0$ *and* $N > 0$. *Then, there exists* $r(\eta, N) > 0$ *such that, if* ν' *is supported in* $B(0,r)$ *with* $r \le r(\eta, N)$, *we have, for all* $0 \le t \le 1$,

$$\sup_{|x-y|\ge\eta} p_t(y;x) \le C(\eta, N)t^N.$$

Proof. For a fix $t \ge 0$, define a finite measure m_t^η on $\mathbb{R}$ by

$$m_t^\eta(A) = \mathbb{P}\big(\big\{X_t^x \in A \cap \bar{B}^c(x,\eta)\big\}\big), \qquad A \subset \mathbb{R},$$

where $\bar{B}^c(x,r)$ denotes the complement of the closure of $B(x,r)$. Thus, to prove our result it suffices to prove that m_t^η admits a density that has the desired bound. To this end, for any smooth function f compactly supported in $\bar{B}^c(x,\eta)$, we have:

$$\begin{aligned}\int_{\mathbb{R}} (\partial_x f)(y) m_t^\eta(\mathrm{d}y) &= \mathbb{E}\partial_x f\big(X_t^x\big) = \mathbb{E}G_t U_t^{-2} f\big(X_t^x\big) \\ &= \int_{\mathbb{R}} \mathbb{E}\big[G_t U_t^{-2} | X_t^x = y\big] f(y) m_t^\eta(\mathrm{d}y),\end{aligned}$$

where the second equality follows from integration by parts. Now by an application of Proposition 3.1 to X_t^x, one has, for any $p > 0$,

$$m_t^\eta(\mathbb{R}) \le \mathbb{P}\Big(\sup_{0\le s\le t} \big|X_s^x - x\big| \ge \eta\Big) \le C(\eta, p)t^p.$$

The rest of the proof follows from Proposition 5.2 and Lemma 5.1. □

5.2. Expansion for the transition density

We are ready to state the main result of this section, namely, the second-order expansion for the transition densities $\{p_t(\cdot;x)\}_{t\ge 0}$ of the process $\{X_t(x)\}_{t\ge 0}$ in terms of the Lévy density $g(x;y)$ defined in (4.3). The proof is presented in Appendix B.

Theorem 5.2. *Let* $x \in \mathbb{R}$ *and* $y > 0$. *Then, under the hypothesis of Theorem* 4.1, *we have*

$$p_t(x+y;x) := -\frac{\partial \mathbb{P}(X_t(x) \ge x+y)}{\partial y} = t a_1(x;y) + \frac{t^2}{2} a_2(x;y) + \mathrm{O}\big(t^3\big) \tag{5.5}$$

as $t \to 0$, *where* $a_1(x; y)$ *and* $a_2(x; y)$ *admit the following representations* (*for* $\varepsilon > 0$ *small enough*):

$$a_1(x; y) := g(x; y), \qquad a_2(x; y) := \eth(x; y) + \Im_1(x; y) + \Im_2(x; y),$$

with

$$\begin{aligned}
\eth(x; y) &= -\frac{\partial}{\partial y}\mathcal{D}(x; y), \\
\Im_1(x; y) &= \int \Big(g\big(x + \gamma(x, \bar{\zeta}); y - \gamma(x, \bar{\zeta})\big) + g\big(x; \bar{\gamma}(x + y, \bar{\zeta}) - x\big)\partial_u \bar{\gamma}(x + y, \bar{\zeta}) \\
&\quad - 2g(x; y) - \gamma(x, \bar{\zeta})\partial_x g(x; y) + \gamma(x, \bar{\zeta})\partial_y g(x; y) \\
&\quad + \partial_y\big(\gamma(x + y, \bar{\zeta})g(x; y)\big)\Big)\bar{h}_\varepsilon(\bar{\zeta})\,\mathrm{d}\bar{\zeta}, \\
\Im_2(x; y) &= \int g\big(x + \gamma(x, \zeta); y - \gamma(x, \zeta)\big)h_\varepsilon(\zeta)\,\mathrm{d}\zeta - 2g(x; y)\int h_\varepsilon(\zeta)\,\mathrm{d}\zeta,
\end{aligned} \tag{5.6}$$

and $\mathcal{D}(x, y)$ *be given as in* (4.5).

6. The first-order term of the option price expansion

In this section, we use our previous results to derive the leading term of the small-time expansion for option prices of out-of-the-money (OTM) European call options. This can be achieved by either the asymptotics of the tail distributions or the transition density. Given that the former requires less stringent conditions on the coefficients of the SDE, we choose the former approach.

It is well known by practitioners that the market implied volatility skewness is more pronounced as the expiration time approaches. Such a phenomenon indicates that a jump risk should be included into classical purely-continuous financial models (e.g., local volatility models and stochastic volatility models) to reproduce more accurately the implied volatility skews observed in short-term option prices. Moreover, further studies have shown that accurate modeling of the option market and asset prices requires a mixture of a continuous diffusive component and a jump component (see Aït-Sahalia and Jacod [1], Aït-Sahalia and Jacod [2], Barndorff-Nielsen and Shephard [4], Podolskij [28], Carr and Wu [7], and Medvedev and Scaillet [22]). The study of small-time asymptotics of option prices and implied volatilities has grown significantly during the last decade, as it provides a convenient tool for testing various pricing models and calibrating parameters in each model (see, e.g., Gatheral et al. [15], Feng, Forde and Fouque [8], Forde and Jacquier [13], Berestyki, Busca and Florent [5], Figueroa-López and Forde [9], Roper [30], Tankov [33], Gao and Lee [14], Muhle-Karbe and Nutz [23], Figueroa-López, Gong and Houdré [10]). In spite of the ample literature on the asymptotic behavior of the transition densities and option prices for either purely-continuous or purely-jump models, results on local jump-diffusion models are scarce. Our result in this section is thus a first attempt in this direction.

Throughout this section, let $\{S_t\}_{t\geq 0}$ be the stock price process and let $X_t = \log S_t$ for each $t \geq 0$. We assume that $\mathbb{P}$ is the option pricing measure and that under this measure the process

$\{X_t\}_{t\geq 0}$ is of the form in (2.1). As usual, without loss of generality we assume that the risk-free interest rate r is 0. In particular, in order for $S_t = \exp X_t$ to be a $\mathbb{Q}$-(local) martingale, we fix

$$b(x) := -\frac{1}{2}\sigma^2(x) - \int \left(\mathrm{e}^{\gamma(x,z)} - 1 - \gamma(x,\zeta)\mathbf{1}_{\{|\zeta|\leq 1\}}\right)h(z)\,\mathrm{d}z.$$

We assume that σ and γ are such that the conditions (C1)–(C4) of Section 2 are satisfied. We also impose an extra condition for $h(z)$ and $\gamma(x,z)$ in order to derive option price expansion, as we are working with the exponential of a jump-diffusion now:

(C5) $h(z)$ and $\gamma(x,z)$ are such that $\sup_{x\in\mathbb{R}}\int_{|z|\geq 1}\mathrm{e}^{3|\gamma(x,z)|}h(z)\,\mathrm{d}z < \infty$.

Note that this condition ensures immediately that $b(x)$ above is well defined.

By the Markov property of the system, it will suffice to compute a small-time expansion for

$$v_t = \mathbb{E}(S_t - K)_+ = \mathbb{E}\left(\mathrm{e}^{X_t} - K\right)_+.$$

In particular, using the well-known formula

$$\mathbb{E}U\mathbf{1}_{\{U>K\}} = K\mathbb{P}\{U > K\} + \int_K^\infty \mathbb{P}\{U > s\}\,\mathrm{d}s,$$

we can write

$$\mathbb{E}\left(\mathrm{e}^{X_t} - K\right)_+ = \int_K^\infty \mathbb{P}\{S_t > s\}\,\mathrm{d}s = S_0\int_{K/S_0}^\infty \mathbb{P}\{X_t - x > \log s\}\,\mathrm{d}s,$$

where $x = X_0 = \log S_0$. Recall that

$$\mathbb{P}(X_t - x \geq y) = \mathrm{e}^{-\lambda_\varepsilon t}\sum_{n=0}^\infty \mathbb{P}\left(X_t - x \geq y | N_t^\varepsilon = n\right)\frac{(\lambda_\varepsilon t)^n}{n!}, \tag{6.1}$$

where $\lambda_\varepsilon := \int \phi_\varepsilon(\zeta)h(\zeta)\,\mathrm{d}\zeta$ is the jump intensity of $\{N_t^\varepsilon\}_{t\geq 0}$. We proceed as in Section 4. First, note that

$$v_t = S_0\int_{K/S_0}^\infty \mathbb{P}\{X_t - x > \log s\}\,\mathrm{d}s = S_0\mathrm{e}^{-\lambda_\varepsilon t}(I_1 + I_2 + I_3), \tag{6.2}$$

where

$$\begin{aligned}
I_1 &= \int_{K/S_0}^\infty \mathbb{P}\left\{X_t - x \geq \log s | N_t^\varepsilon = 0\right\}\mathrm{d}s = \int_{K/S_0}^\infty \mathbb{P}\left\{X_t(\varepsilon, \varnothing, x) - x \geq \log s\right\}\mathrm{d}s,\\
I_2 &= \lambda_\varepsilon t\int_{K/S_0}^\infty \mathbb{P}\left\{X_t - x \geq \log s | N_t^\varepsilon = 1\right\}\mathrm{d}s,\\
I_3 &= \lambda_\varepsilon^2 t^2\sum_{n=2}^\infty \frac{(\lambda_\varepsilon t)^{n-2}}{n!}\int_{K/S_0}^\infty \mathbb{P}\left\{X_t - x \geq \log s | N_t^\varepsilon = n\right\}\mathrm{d}s.
\end{aligned}$$

It is clear that $I_1/t \to 0$ as $t \to 0$ by Lemma 3.1. We show that the same is true for I_3, which is the content of the following lemma. Its proof is given in Appendix C.

Lemma 6.1. *With the above notation, we have*

$$\sup_{n\in\mathbb{N},t\in[0,1]} \frac{1}{n!} \int_0^\infty \mathbb{P}\big(|X_t - x| \geq \log y | N_t^\varepsilon = n\big)\, \mathrm{d}y < \infty.$$

As a consequence, $I_3/t \to 0$ *as* $t \to 0$.

Note that the above lemma actually implies that $\mathbb{E}\mathrm{e}^{|X_t - x|} < \infty$ for all $t \in [0, 1)$. We are ready to state the main result of this section.

Theorem 6.1. *Let* $v_t = \mathbb{E}(S_t - K)_+$ *be the price of a European call option with strike* $K > S_0$. *Under the conditions* (C1)–(C5), *we have*

$$\lim_{t\to 0} \frac{1}{t} v_t = \int_{-\infty}^{\infty} \big(S_0 \mathrm{e}^{\gamma(x,\zeta)} - K\big)_+ h(\zeta)\, \mathrm{d}\zeta. \tag{6.3}$$

Proof. We use the notation introduced in (6.2). Following a similar argument as in the proof of Lemma 6.1, one can show that

$$\int_{K/S_0}^{\infty} \sup_{t\in[0,1]} \mathbb{P}\big\{X_t - x \geq \log s | N_t^\varepsilon = 1\big\}\, \mathrm{d}s < \infty. \tag{6.4}$$

Also, it is clear that I_1/t converges to 0 when t approaches to 0 by Lemma 3.1. Using the latter fact, equation (6.4), Lemma 6.1, equation (6.2), and dominated convergence theorem, we have

$$\lim_{t\to 0} \frac{v_t}{t} = \lim_{t\to 0} \frac{S_0 I_2}{t} = \lambda_\varepsilon S_0 \int_{K/S_0}^{\infty} \lim_{t\to 0} \mathbb{P}\big\{X_t - x \geq \log s | N_t^\varepsilon = 1\big\}\, \mathrm{d}s.$$

Next, using Theorem 4.1, it follows that

$$\lim_{t\to 0} \frac{v_t}{t} = S_0 \int_{K/S_0}^{\infty} A_1(x, \log s)\, \mathrm{d}s = S_0 \int_{K/S_0}^{\infty} \int_{\{\gamma(x,\zeta)\geq \log s\}} h(\zeta)\, \mathrm{d}\zeta\, \mathrm{d}s.$$

Finally, (6.3) follows from applying Fubini's theorem to the right-hand side of the above equality. □

Remark 6.1. As a special case of our result, let $\gamma(x, \zeta) = \zeta$. The model reduces to an exponential Lévy model. The above first-order asymptotics becomes to

$$\lim_{t\to 0} \frac{1}{t} v_t = \int_{-\infty}^{\infty} \big(S_0 \mathrm{e}^{\zeta} - K\big)_+ h(\zeta)\, \mathrm{d}\zeta.$$

This recovers the well-known first-order asymptotic behavior for exponential Lévy model (see, e.g., Roper [30] and Tankov [33]).

Appendix A: Proof of the tail distribution expansion

The proof of Theorem 4.1 is decomposed into three steps described in the following three subsections. For future use in obtaining the expansion for the transition densities, we will write explicitly the remainder terms when applying Dynkin's formula (3.7) or in any other type of approximation.

A.1. Key lemma to control the tail of the process with one large jump

The following result will allow us to obtain the second-order expansion for the process with one large jump. Below, we recall that $J := J^{\varepsilon}$ represents the jump size of the big-jump component $Z(\varepsilon)$; that is, a random variable with density $\breve{h}_{\varepsilon}(\zeta) := h_{\varepsilon}(\zeta)/\lambda_{\varepsilon} := \phi_{\varepsilon}(\zeta)h(\zeta)/\lambda_{\varepsilon}$.

Lemma A.1. *Under the setting and conditions* (C1)–(C4) *of Section* 2,

$$\mathbb{P}\big(X_t\big(\varepsilon, \varnothing, z+\gamma(z, J)\big) \geq \vartheta\big) = H_0(z;\vartheta) + tH_1(z;\vartheta) + t^2\breve{\mathcal{R}}_t(z;\vartheta) \tag{A.1}$$

for any $z, \vartheta \in \mathbb{R}$, *where*

$$\begin{aligned} H_0(z;\vartheta) &:= \mathbb{P}\big(\gamma(z, J) + z \geq \vartheta\big), \qquad H_1(z;\vartheta) := D(z;\vartheta) + I(z;\vartheta), \\ D(z;\vartheta) &:= \widetilde{\Gamma}(\vartheta; z)b_{\varepsilon}(\vartheta) - \partial_{\vartheta}\widetilde{\Gamma}(\vartheta; z)v(\vartheta) - \widetilde{\Gamma}(\vartheta; z)v'(\vartheta), \\ I(z;\vartheta) &:= \int\big[\mathbb{P}\big(z+\gamma(z, J) \geq \bar{\gamma}(\vartheta, \zeta)\big) - \mathbb{P}\big(z+\gamma(z, J) \geq \vartheta\big) - \widetilde{\Gamma}(\vartheta; z)\gamma(\vartheta, \zeta)\big]\bar{h}_{\varepsilon}(\zeta)\,\mathrm{d}\zeta, \end{aligned} \tag{A.2}$$

and, for $\varepsilon > 0$ *small enough*,

$$\limsup_{t\to 0}\sup_{z\in\mathbb{R}}\big|\breve{\mathcal{R}}_t^1(z;\vartheta)\big| < \infty, \qquad \sup_{z,\vartheta}\big|H_1(z;\vartheta)\big| < \infty.$$

The idea to obtain (A.1) consists of approximating the function $\mathbf{1}_{\{X_t(\varepsilon,\varnothing,z+\gamma(z,J))\geq\vartheta\}}$ by a smooth sequence of functions $f_{\delta}(X_t(\varepsilon, \varnothing, z+\gamma(z, J)))$, $\delta > 0$. Concretely, we let

$$f_{\delta}(w) := k_{\vartheta} * \varphi_{\delta}(w) = \int_{-\infty}^{w-\vartheta}\varphi_{\delta}(u)\,\mathrm{d}u,$$

where $*$ denotes the convolution operation, $k_{\vartheta}(w) := \mathbf{1}_{w\geq\vartheta}$, and $\varphi_{\delta}(w) := \delta^{-1}\varphi(\delta^{-1}w)$ for a density function $\varphi \in C^{\infty}$ with $\operatorname{supp}(\varphi) = [-1, 1]$. In particular, as $\delta \to 0$,

$$f_{\delta}(w) \to k_{\vartheta}(w) = \mathbf{1}_{\{w\geq\vartheta\}} \quad \text{and} \quad \int g(w)f_{\delta}'(w)\,\mathrm{d}w = \int g(w)\varphi_{\delta}(w-\vartheta)\,\mathrm{d}w \to g(\vartheta), \tag{A.3}$$

whenever $w \neq \vartheta$ and g is bounded and continuous at ϑ. It is then natural to apply Dynkin's formula to $f_{\delta}(X_t(\varepsilon, \varnothing, z+\gamma(z, J)))$ and show that each of the resulting terms is convergent when $\delta \to 0$. The following result, whose proof is presented in Appendix C, is needed to formalize the last step.

Lemma A.2. *Let $\widetilde{\Gamma}(\cdot; z)$ be the density of the random variable $z + \gamma(z, J)$ and let $p_t(\cdot; \varepsilon, \varnothing, \zeta)$ be the density of $X_t(\varepsilon, \varnothing, \zeta)$. Then, under the conditions* (C1)–(C4) *of Section 2, there exists an $\varepsilon > 0$ small enough such that for any compact set $K \subset \mathbb{R}$,*

$$\limsup_{t\to 0} \sup_{z\in\mathbb{R}} \sup_{\eta\in K} \left| \frac{\partial^k}{\partial \eta^k} \int \widetilde{\Gamma}(\zeta; z) p_t(\eta; \varepsilon, \varnothing, \zeta)\, \mathrm{d}\zeta \right| < \infty, \qquad k \geq 0. \tag{A.4}$$

Furthermore, (A.4) *holds also true with $\partial_\eta p_t(\eta; \varepsilon, \varnothing, \zeta)$ in place of $p_t(\eta; \varepsilon, \varnothing, \zeta)$ inside the integral.*

We are now in position to show (A.1).

Proof of Lemma A.1. Throughout, $\partial_y \gamma$ and $\partial_\zeta \gamma$ will denote the partial derivatives of $\gamma(y, \zeta)$ with respect to its first and second arguments, respectively. By dominated convergence theorem, we have

$$\mathbb{P}\big(X_t\big(\varepsilon, \varnothing, z + \gamma(z, J)\big) \geq \vartheta\big) = \lim_{\delta\downarrow 0} \mathbb{E} f_\delta\big(X_t\big(\varepsilon, \varnothing, z + \gamma(z, J)\big)\big). \tag{A.5}$$

Note that

$$\mathbb{E} f_\delta\big(X_t\big(\varepsilon, \varnothing, z + \gamma(z, J)\big)\big) = \int \widetilde{\Gamma}(\zeta; z) \mathbb{E} f_\delta\big(X_t(\varepsilon, \varnothing, \zeta)\big)\, \mathrm{d}\zeta, \tag{A.6}$$

and, thus, an application of the Dynkin's formula (3.7) with $n = 2$ to the expectation in the above integral yields

$$\begin{aligned} &\mathbb{E} f_\delta\big(X_t\big(\varepsilon, \varnothing, z + \gamma(z, J)\big)\big) \\ &\quad = \int \widetilde{\Gamma}(\zeta; z) f_\delta(\zeta)\, \mathrm{d}\zeta + t \int \widetilde{\Gamma}(\zeta; z) L_\varepsilon f_\delta(\zeta)\, \mathrm{d}\zeta \qquad (\text{A.7}) \\ &\qquad + t^2 \int \widetilde{\Gamma}(\zeta; z) \int_0^1 (1 - \alpha) \mathbb{E}(L_\varepsilon)^2 f_\delta\big(X_{\alpha t}(\varepsilon, \varnothing, \zeta)\big)\, \mathrm{d}\alpha\, \mathrm{d}\zeta. \qquad (\text{A.8}) \end{aligned}$$

We analyze the limit of each of the three terms on the right-hand side of the previous equation. By dominated convergence theorem, the leading term of (A.5) is given by

$$H_0(z; \vartheta) := \lim_{\delta\downarrow 0} \int \widetilde{\Gamma}(\zeta; z) f_\delta(\zeta)\, \mathrm{d}\zeta = \int \widetilde{\Gamma}(\zeta; z) I_{[\vartheta,\infty)}(\zeta)\, \mathrm{d}\zeta = \mathbb{P}\big(\gamma(z, J) + z \geq \vartheta\big).$$

To compute the limit of the second term, recall that $L_\varepsilon f_\delta = \mathcal{D}_\varepsilon f_\delta + \mathcal{I}_\varepsilon f_\delta$ with $\mathcal{D}_\varepsilon$ and $\mathcal{I}_\varepsilon$ defined as in (3.4). Then, the term of order t has the following two contributions:

$$A_\delta := \int \widetilde{\Gamma}(\zeta; z) \mathcal{D}_\varepsilon f_\delta(\zeta)\, \mathrm{d}\zeta, \qquad B_\delta := \int \widetilde{\Gamma}(\zeta; z) \mathcal{I}_\varepsilon f_\delta(\zeta)\, \mathrm{d}\zeta.$$

Using that $f'_\delta(\zeta) = \varphi_\delta(\zeta - \vartheta)$ and by integration by parts, it follows that

$$A_\delta = \int \big(\widetilde{\Gamma}(\zeta; z) b_\varepsilon(\zeta) - \partial_\zeta \widetilde{\Gamma}(\zeta; z) v(\zeta) - \widetilde{\Gamma}(\zeta; z) v'(\zeta)\big) \varphi_\delta(\zeta - \vartheta)\, \mathrm{d}\zeta,$$

where we recall that $v(x) := \sigma^2(x)/2$ and $b_\varepsilon(x) := b(x) - \int_{|\zeta|\le 1} \gamma(x,\zeta) h_\varepsilon(\zeta)\,\mathrm{d}\zeta$. Applying (A.3) and Lemma 2.1(2),

$$\lim_{\delta\downarrow 0} A_\delta = \widetilde{\Gamma}(\vartheta; z) b_\varepsilon(\vartheta) - \partial_\vartheta \widetilde{\Gamma}(\vartheta; z) v(\vartheta) - \widetilde{\Gamma}(\vartheta; z) v'(\vartheta).$$

We now analyze the limit of the second term B_δ. Since $f_\delta'(\cdot) = \varphi_\delta(\cdot - \vartheta)$ has compact support, we can apply (C.9) below to write B_δ as

$$\begin{aligned} B_\delta &= \int \varphi_\delta(w-\vartheta) \widetilde{\mathcal{H}}_\varepsilon \widetilde{\Gamma}(w; z)\,\mathrm{d}w \\ &= \int \varphi_\delta(w-\vartheta) \int \biggl(\int_{\bar\gamma(w,\zeta)}^{w} \widetilde{\Gamma}(\eta; z)\,\mathrm{d}\eta - \widetilde{\Gamma}(w; z)\gamma(w,\zeta) \biggr) \bar h_\varepsilon(\zeta)\,\mathrm{d}\zeta\,\mathrm{d}w. \end{aligned} \tag{A.9}$$

Since

$$\begin{aligned} \partial_\zeta^2 \biggl(\int_{\bar\gamma(w,\zeta)}^{w} \widetilde{\Gamma}(\eta; z)\,\mathrm{d}\eta - \widetilde{\Gamma}(w; z)\gamma(w,\zeta) \biggr) &= -\partial_\zeta \widetilde{\Gamma}\bigl(\bar\gamma(w,\zeta); z\bigr)\bigl(\partial_\zeta \bar\gamma(w,\zeta)\bigr)^2 \\ &\quad - \widetilde{\Gamma}\bigl(\bar\gamma(w,\zeta); z\bigr)\partial_\zeta^2 \bar\gamma(w,\zeta) - \widetilde{\Gamma}(w; z)\partial_\zeta^2 \gamma(w,\zeta), \end{aligned}$$

the factor multiplying $\varphi_\delta(w-\vartheta)$ in (A.9) can be written as

$$\begin{aligned} \widetilde{\mathcal{H}}_\varepsilon \widetilde{\Gamma}(w; z) = -\int\int_0^1 \bigl[&\partial_\zeta \widetilde{\Gamma}\bigl(\bar\gamma(w,\zeta\beta); z\bigr)\bigl(\partial_\zeta \bar\gamma(w,\zeta\beta)\bigr)^2 + \widetilde{\Gamma}\bigl(\bar\gamma(w,\zeta\beta); z\bigr)\partial_\zeta^2 \bar\gamma(w,\zeta\beta) \\ &+ \widetilde{\Gamma}(w; z)\partial_\zeta^2 \gamma(w,\zeta\beta)\bigr](1-\beta)\,\mathrm{d}\beta\,\zeta^2 \bar h_\varepsilon(\zeta)\,\mathrm{d}\zeta, \end{aligned}$$

which shows that $\widetilde{\mathcal{H}}_\varepsilon \widetilde{\Gamma}(w; z)$ is bounded and continuous in w in light of conditions (C2) and (C4). Thus, using (A.3),

$$\lim_{\delta\downarrow 0} B_\delta = \int \biggl(\int_{\bar\gamma(\vartheta,\zeta)}^{\vartheta} \widetilde{\Gamma}(\eta; z)\,\mathrm{d}\eta - \widetilde{\Gamma}(\vartheta; z)\gamma(\vartheta,\zeta) \biggr) \bar h_\varepsilon(\zeta)\,\mathrm{d}\zeta =: B_0(z; \vartheta).$$

Recalling that $\widetilde{\Gamma}(\zeta; z)$ is the density of $\tilde J := z + \gamma(z, J)$, $B_0(z; \vartheta)$ can also be written as

$$B_0(z; \vartheta) = \int \Bigl(\mathbb{P}\bigl(z + \gamma(z,J) \ge \bar\gamma(\vartheta,\zeta)\bigr) - \mathbb{P}\bigl(z + \gamma(z,J) \ge \vartheta\bigr) - \widetilde{\Gamma}(\vartheta; z)\gamma(\vartheta,\zeta) \Bigr) \bar h_\varepsilon(\zeta)\,\mathrm{d}\zeta.$$

Putting together the previous two limits, we obtain the term of order t:

$$H_1(z; \vartheta) := \lim_{\delta\downarrow 0} \int \widetilde{\Gamma}(\zeta; z) L_\varepsilon f_\delta(\zeta)\,\mathrm{d}\zeta = D(z; \vartheta) + I(z; \vartheta),$$

with $D(z; \vartheta)$ and $I(z; \vartheta)$ given as in the statement of the lemma.

Finally, we estimate the remainder term

$$\check{\mathcal{R}}_t(z; \vartheta) := \lim_{\delta\downarrow 0} \int \widetilde{\Gamma}(\zeta; z) \int_0^1 (1-\alpha)\mathbb{E}(L_\varepsilon)^2 f_\delta\bigl(X_{\alpha t}(\varepsilon, \varnothing, \zeta)\bigr)\,\mathrm{d}\alpha\,\mathrm{d}\zeta \tag{A.10}$$

and show that this is uniformly bounded for t small enough. Let $\breve{\mathcal{R}}_t(z;\vartheta;\delta,\varepsilon)$ be the expression following $\lim_{\delta\downarrow 0}$ and note that

$$\begin{aligned}\breve{\mathcal{R}}_t(z;\vartheta,\delta,\varepsilon) &= \int \widetilde{\Gamma}(\zeta;z)\int_0^1 (1-\alpha)\mathbb{E}(\mathcal{D}_\varepsilon)^2 f_\delta\big(X_{\alpha t}(\varepsilon,\varnothing,\zeta)\big)\,\mathrm{d}\alpha\,\mathrm{d}\zeta \\ &\quad+\int \widetilde{\Gamma}(\zeta;z)\int_0^1 (1-\alpha)\mathbb{E}(\mathcal{I}_\varepsilon)^2 f_\delta\big(X_{\alpha t}(\varepsilon,\varnothing,\zeta)\big)\,\mathrm{d}\alpha\,\mathrm{d}\zeta \\ &\quad+\int \widetilde{\Gamma}(\zeta;z)\int_0^1 (1-\alpha)\mathbb{E}\mathcal{I}_\varepsilon\mathcal{D}_\varepsilon f_\delta\big(X_{\alpha t}(\varepsilon,\varnothing,\zeta)\big)\,\mathrm{d}\alpha\,\mathrm{d}\zeta \\ &\quad+\int \widetilde{\Gamma}(\zeta;z)\,\mathrm{d}\zeta\int_0^1 (1-\alpha)\mathbb{E}\mathcal{D}_\varepsilon\mathcal{I}_\varepsilon f_\delta\big(X_{\alpha t}(\varepsilon,\varnothing,\zeta)\big)\,\mathrm{d}\alpha\,\mathrm{d}\zeta.\end{aligned}\tag{A.11}$$

The idea is to use Lemmas A.2 and C.2 to deal with the four terms on the right-hand side of the previous equation. For simplicity, we only give the details for second term, that we denote hereafter $\bar{I}_t^{(2)}(\vartheta;\delta,\varepsilon,z)$. The other terms can similarly be handled. First, let us show that $\mathcal{I}_\varepsilon f_\delta(\cdot)$ has compact support in light of our condition (2.4) and the fact that f_δ' has compact support. Indeed, writing $\mathcal{I}_\varepsilon f_\delta$ as

$$\begin{aligned}\mathcal{I}_\varepsilon f_\delta(y) = \int\int_0^1 &\Big(f_\delta''\big(y+\gamma(y,\zeta\beta)\big)\big(\partial_\zeta\gamma(y,\zeta\beta)\big)^2 + f_\delta'\big(y+\gamma(y,\zeta\beta)\big)\partial_\zeta^2\gamma(y,\zeta\beta) \\ &- f_\delta'(y)\partial_\zeta^2\gamma(y,\zeta\beta)\Big)(1-\beta)\,\mathrm{d}\beta\,\zeta^2\bar{h}_\varepsilon(\zeta)\,\mathrm{d}\zeta,\end{aligned}$$

it is clear that $\mathcal{I}_\varepsilon f_\delta(y)=0$ if $y\notin \operatorname{supp} f_\delta'$ and $y+\gamma(y,\zeta\beta)\notin\mathcal{S}:=(\operatorname{supp} f_\delta')\cap(\operatorname{supp} f_\delta'')$ for any ζ,β. Since $|1+\partial_y\gamma(y,\zeta)|\geq\delta$, it follows that, for y large enough, $y+\gamma(y,\zeta\beta)\notin\mathcal{S}$ regardless of ζ and β. Next, since $\mathcal{I}_\varepsilon f_\delta(\cdot)$ has compact support, we can apply (C.8) to get

$$\bar{I}_t^{(2)}(z;\vartheta,\delta,\varepsilon) = \int \widetilde{\Gamma}(\zeta;z)\int_0^1 (1-\alpha)\int \mathcal{I}_\varepsilon f_\delta(w)\widetilde{\mathcal{I}}_\varepsilon p_{\alpha t}(w;\varepsilon,\varnothing,\zeta)\,\mathrm{d}w\,\mathrm{d}\alpha\,\mathrm{d}\zeta.$$

Next, let $\tilde{p}_t(\eta;\zeta):=\widetilde{\mathcal{I}}_\varepsilon p_t(\eta;\varepsilon,\varnothing,\zeta)$. An application of the identity (C.9) followed by Fubini leads to

$$\begin{aligned}&\bar{I}_t^{(2)}(z;\vartheta,\delta,\varepsilon) \\ &\quad= \int f_\delta'(w)\int_0^1(1-\alpha)\int\bigg(\int_{\bar{\gamma}(w,\tilde{\zeta})}^{w}\int\widetilde{\Gamma}(\zeta;z)\tilde{p}_{\alpha t}(\eta;\zeta)\,\mathrm{d}\zeta\,\mathrm{d}\eta \\ &\qquad\qquad - \int\widetilde{\Gamma}(\zeta;z)\tilde{p}_{\alpha t}(w;\zeta)\,\mathrm{d}\zeta\,\gamma(w,\tilde{\zeta})\bigg)\bar{h}_\varepsilon(\tilde{\zeta})\,\mathrm{d}\tilde{\zeta}\,\mathrm{d}\alpha\,\mathrm{d}w.\end{aligned}$$

Now, fix $\breve{p}_t(\eta;z,\varepsilon):=\int\widetilde{\Gamma}(\zeta;z)p_t(\eta;\varepsilon,\varnothing,\zeta)\,\mathrm{d}\zeta$ and note that

$$\breve{p}_t'(\eta;z,\varepsilon) = \partial_\eta\int\widetilde{\Gamma}(\zeta;z)p_t(\eta;\varepsilon,\varnothing,\zeta)\,\mathrm{d}\zeta = \int\widetilde{\Gamma}(\zeta;z)p_t'(\eta;\varepsilon,\varnothing,\zeta)\,\mathrm{d}\zeta,\tag{A.12}$$

in light of the last statement of Lemma A.2, which will allow us to pass the derivative into the integration sign. Using (A.12) and Fubini's theorem, it follows that

$$\int \widetilde{\Gamma}(\zeta; z)\tilde{p}_{\alpha t}(\eta; \zeta)\,\mathrm{d}\zeta = \int \widetilde{\Gamma}(\zeta; z)\widetilde{\mathcal{I}}_{\varepsilon} p_{\alpha t}(\eta; \varepsilon, \varnothing, \zeta)\,\mathrm{d}\zeta = \widetilde{\mathcal{I}}_{\varepsilon}\breve{p}_{\alpha t}(\eta; z, \varepsilon). \tag{A.13}$$

Therefore,

$$\bar{I}_t^{(2)}(z; \vartheta, \delta, \varepsilon) = \sum_{j=1}^{2} \int f_\delta'(w)\bar{I}_t^{(2,j)}(w; z, \varepsilon)\,\mathrm{d}w, \tag{A.14}$$

where

$$\begin{aligned}
&\bar{I}_t^{(2,1)}(w; z, \varepsilon)\\
&\quad = -\int_0^1 (1-\alpha)\int\int_0^1 (\widetilde{\mathcal{I}}_{\varepsilon}\breve{p}_{\alpha t})'\big(\bar{\gamma}(w, \tilde{\zeta}\tilde{\beta}); z, \varepsilon\big)(\partial_\zeta \bar{\gamma})(w, \tilde{\zeta}\tilde{\beta})(1-\tilde{\beta})\,\mathrm{d}\tilde{\beta}\tilde{\zeta}^2\bar{h}_{\varepsilon}(\tilde{\zeta})\,\mathrm{d}\tilde{\zeta}\,\mathrm{d}\alpha,\\
&\bar{I}_t^{(2,2)}(w; z, \varepsilon) = -\int_0^1 (1-\alpha)\widetilde{\mathcal{I}}_{\varepsilon}\breve{p}_{\alpha t}(w; z, \varepsilon)\int\int_0^1 \big(\partial_\zeta^2 \gamma\big)(w, \tilde{\zeta}\tilde{\beta})(1-\tilde{\beta})\,\mathrm{d}\tilde{\beta}\tilde{\zeta}^2\bar{h}_{\varepsilon}(\tilde{\zeta})\,\mathrm{d}\tilde{\zeta}\,\mathrm{d}\alpha.
\end{aligned}$$

Now, let us define the operator

$$\hat{\mathcal{I}}g(y; \zeta) := g\big(\bar{\gamma}(y, \zeta)\big)\partial_y\bar{\gamma}(y, \zeta) - \big(1 + \partial_y\gamma(y, \zeta)\big)g(y) - g'(y)\gamma(y, \zeta).$$

By writing $\widetilde{\mathcal{I}}_{\varepsilon}g(y)$ as

$$\widetilde{\mathcal{I}}_{\varepsilon}g(y) = \int\int_0^1 \big(\partial_\zeta^2\hat{\mathcal{I}}g\big)(y; \bar{\zeta}\bar{\beta})(1-\bar{\beta})\,\mathrm{d}\bar{\beta}\bar{\zeta}^2\bar{h}_{\varepsilon}(\bar{\zeta})\,\mathrm{d}\bar{\zeta},$$

it is not hard to see that $\widetilde{\mathcal{I}}_{\varepsilon}\breve{p}_{\alpha t}(w; z, \varepsilon)$ can be expressed as follows

$$\begin{aligned}
\widetilde{\mathcal{I}}_{\varepsilon}\breve{p}_{\alpha t}(w; z, \varepsilon) &= \sum_{k=0}^{2}\int\int_0^1 \breve{p}_{\alpha t}^{(k)}\big(\bar{\gamma}(w, \bar{\zeta}\bar{\beta}); z, \varepsilon\big)\mathcal{D}_k^{(1)}(w; \bar{\zeta}\bar{\beta})(1-\bar{\beta})\,\mathrm{d}\bar{\beta}\bar{\zeta}^2\bar{h}_{\varepsilon}(\bar{\zeta})\,\mathrm{d}\bar{\zeta}\\
&\quad + \sum_{k=0}^{1}\breve{p}_{\alpha t}^{(k)}(w; z, \varepsilon)\int\int_0^1 \mathcal{D}_k^{(2)}(w; \bar{\zeta}\bar{\beta})(1-\bar{\beta})\,\mathrm{d}\bar{\beta}\bar{\zeta}^2\bar{h}_{\varepsilon}(\bar{\zeta})\,\mathrm{d}\bar{\zeta},
\end{aligned} \tag{A.15}$$

where $\mathcal{D}_j^1(w; \zeta)$ is a finite sum of terms, which consists of the product of partial derivatives of $\bar{\gamma}(w; \zeta)$. Similarly, $\mathcal{D}_j^2(w; \zeta)$ is a finite sum of terms, which consists of the product of partial derivatives of $\gamma(w; \zeta)$. In particular, both $\mathcal{D}_j^1(w; \zeta)$ and $\mathcal{D}_j^2(w; \zeta)$ are uniformly bounded and continuous and, also, in light of Lemma A.2, $(\widetilde{\mathcal{I}}_{\varepsilon}\breve{p}_{\alpha t})'(w; z, \varepsilon)$ will also be of the same form as (A.15).

Upon the substitutions of (A.15) (and the analog representation for $(\widetilde{\mathcal{I}}_\varepsilon \breve{p}_{\alpha t})'(w; z, \varepsilon)$) into (A.14), we can represent $\bar{I}_t^{(2)}(z; \vartheta, \delta, \varepsilon)$ as the sum of terms of the form

$$\int f_\delta'(w) \int_0^1 (1-\alpha) \tilde{I}_{\alpha t}(w; z, \varepsilon) \,\mathrm{d}\alpha \,\mathrm{d}w,$$

where $\tilde{I}_{\alpha t}(w; z, \varepsilon)$ will take one of the following four generic forms with some function $\widetilde{D}(w, \zeta)$ in $C_b^{\geq 1}(\mathbb{R} \times \mathbb{R})$:

$$\begin{aligned}
\tilde{I}_{\alpha t}^{(1)}(w; z, \varepsilon) = {} & \int \int_0^1 \int \int_0^1 \breve{p}_{\alpha t}^{(k)}\big(\bar{\gamma}\big(\bar{\gamma}(w, \tilde{\zeta}\tilde{\beta}), \bar{\zeta}\bar{\beta}\big); z, \varepsilon\big) \widetilde{\mathcal{D}}\big(\bar{\gamma}(w, \tilde{\zeta}\tilde{\beta}); \bar{\zeta}\bar{\beta}\big) \\
& \times (1-\bar{\beta}) \,\mathrm{d}\bar{\beta} \bar{\zeta}^2 \bar{h}_\varepsilon(\bar{\zeta}) \,\mathrm{d}\bar{\zeta} (\partial_\zeta \bar{\gamma})(w, \tilde{\zeta}\tilde{\beta})(1-\tilde{\beta}) \,\mathrm{d}\tilde{\beta} \tilde{\zeta}^2 \bar{h}_\varepsilon(\tilde{\zeta}) \,\mathrm{d}\tilde{\zeta}, \\
\tilde{I}_{\alpha t}^{(2)}(w; z, \varepsilon) = {} & \int \int_0^1 \int \int_0^1 \widetilde{\mathcal{D}}\big(\bar{\gamma}(w, \tilde{\zeta}\tilde{\beta}); \bar{\zeta}\bar{\beta}\big)(1-\bar{\beta}) \,\mathrm{d}\bar{\beta} \bar{\zeta}^2 \bar{h}_\varepsilon(\bar{\zeta}) \,\mathrm{d}\bar{\zeta} \\
& \times \breve{p}_{\alpha t}^{(k)}\big(\bar{\gamma}(w, \tilde{\zeta}\tilde{\beta}); z, \varepsilon\big)(\partial_\zeta \bar{\gamma})(w, \tilde{\zeta}\tilde{\beta})(1-\tilde{\beta}) \,\mathrm{d}\tilde{\beta} \tilde{\zeta}^2 \bar{h}_\varepsilon(\tilde{\zeta}) \,\mathrm{d}\tilde{\zeta}, \\
\tilde{I}_{\alpha t}^{(3)}(w; z, \varepsilon) = {} & \int \int_0^1 \breve{p}_{\alpha t}^{(k)}\big(\bar{\gamma}(w, \bar{\zeta}\bar{\beta}); z, \varepsilon\big) \widetilde{\mathcal{D}}(w; \bar{\zeta}\bar{\beta})(1-\bar{\beta}) \,\mathrm{d}\bar{\beta} \bar{\zeta}^2 \bar{h}_\varepsilon(\bar{\zeta}) \,\mathrm{d}\bar{\zeta} \\
& \times \int \int_0^1 \big(\partial_\zeta^2 \gamma\big)(w, \tilde{\zeta}\tilde{\beta})(1-\tilde{\beta}) \,\mathrm{d}\tilde{\beta} \tilde{\zeta}^2 \bar{h}_\varepsilon(\tilde{\zeta}) \,\mathrm{d}\tilde{\zeta}, \\
\tilde{I}_{\alpha t}^{(4)}(w; z, \varepsilon) = {} & \breve{p}_{\alpha t}^{(k)}(w; z, \varepsilon) \int \int_0^1 \widetilde{\mathcal{D}}(w; \bar{\zeta}\bar{\beta})(1-\bar{\beta}) \,\mathrm{d}\bar{\beta} \bar{\zeta}^2 \bar{h}_\varepsilon(\bar{\zeta}) \,\mathrm{d}\bar{\zeta} \\
& \times \int \int_0^1 \big(\partial_\zeta^2 \gamma\big)(w, \tilde{\zeta}\tilde{\beta})(1-\tilde{\beta}) \,\mathrm{d}\tilde{\beta} \tilde{\zeta}^2 \bar{h}_\varepsilon(\tilde{\zeta}) \,\mathrm{d}\tilde{\zeta}.
\end{aligned} \tag{A.16}$$

Using Lemma A.2, it is now clear that each $\tilde{I}_{\alpha t}^{(i)}(w; z, \varepsilon)$ is uniformly bounded in w and z for t small enough. Concretely, using (A.4), it follows that, for $\varepsilon, t > 0$ small enough,

$$\sup_{z \in \mathbb{R}, w \in \operatorname{supp} f_1} \big|\tilde{I}_t^{(i)}(w; z, \varepsilon)\big| < \infty. \tag{A.17}$$

Due to the continuity $\tilde{I}_t^{(i)}(w; z, \varepsilon)$ and uniformly boundedness condition (A.17), it turns out that

$$\lim_{\delta \to 0} \int f_\delta'(w) \int_0^1 (1-\alpha) \tilde{I}_{\alpha t}(w; z, \varepsilon) \,\mathrm{d}\alpha \,\mathrm{d}w = \int_0^1 (1-\alpha) \tilde{I}_{\alpha t}(\vartheta; z, \varepsilon) \,\mathrm{d}\alpha, \tag{A.18}$$

which is uniformly bounded in z for any fixed ϑ and $0 < t < t_0$ with $t_0 > 0$ small enough. □

A.2. The leading term

In order to determine the leading term of (4.1), we analyze the second term in (4.2) corresponding to $n = 1$ (only one "large" jump). Again, we emphasize that in order to obtain the expansion for the transition densities below, we will need to write explicitly the remainder terms when applying Dynkin's formula (3.7).

By conditioning on the time of the jump (necessarily uniformly distributed on $[0, t]$),

$$\mathbb{P}\big(X_t(x) \geq x + y | N_t^{\varepsilon} = 1\big) = \frac{1}{t} \int_0^t \mathbb{P}\big(X_t\big(\varepsilon, \{s\}, x\big) \geq x + y\big)\, \mathrm{d}s. \tag{A.19}$$

Conditioning on $\mathcal{F}_{s^-}$,

$$\mathbb{P}\big(X_t\big(\varepsilon, \{s\}, x\big) \geq x + y\big) = \mathbb{E}\big(G_{t-s}\big(X_{s^-}(\varepsilon, \varnothing, x)\big)\big) = \mathbb{E}\big(G_{t-s}\big(X_s(\varepsilon, \varnothing, x)\big)\big), \tag{A.20}$$

where

$$G_t(z) := G_t(z; x, y) := \mathbb{P}\big[X_t\big(\varepsilon, \varnothing, z + \gamma(z, J)\big) \geq x + y\big]. \tag{A.21}$$

Using Lemma A.1,

$$\begin{aligned} &\mathbb{P}\big(X_t\big(\varepsilon, \{s\}, x\big) \geq x + y\big) \\ &\quad = \mathbb{E} H_0\big(X_s(\varepsilon, \varnothing, x); x + y\big) + (t - s)\mathbb{E} H_1\big(X_s(\varepsilon, \varnothing, x); x + y\big) \\ &\qquad + (t - s)^2 \mathbb{E} \mathcal{R}^1_{t-s}\big(X_s(\varepsilon, \varnothing, x); x, y\big), \end{aligned} \tag{A.22}$$

where $\mathcal{R}^1_t(w; x, y) := \breve{\mathcal{R}}_t(w; x + y)$. Next, we apply the Dynkin's formula (3.7) with $n = 2$ to $\mathbb{E} H_0(X_s(\varepsilon, \varnothing, x); x + y)$, which is valid since $H_0(z; x + y) = \mathbb{P}(\gamma(z, J) + z \geq x + y)$ is C_b^4 in light of Lemma 2.1(3). By (3.7),

$$\mathbb{E} H_0\big(X_s(\varepsilon, \varnothing, x); x + y\big) = H_{0,0}(x; y) + s H_{0,1}(x; y) + s^2 \mathcal{R}^2_s(x; y), \tag{A.23}$$

where

$$\begin{aligned} H_{0,0}(x; y) &:= H_0(x; x + y) = \mathbb{P}\big[\gamma(x, J) \geq y\big], \\ H_{0,1}(x; y) &:= (L_\varepsilon H_0)(x; x + y) = b_\varepsilon(x) \frac{\partial H_0(z; x + y)}{\partial z}\bigg|_{z=x} \\ &\qquad + \frac{\sigma^2(x)}{2} \frac{\partial^2 H_0(z; x + y)}{\partial z^2}\bigg|_{z=x} \\ &\qquad + \int \bigg(H_0\big(x + \gamma(x, \zeta); x + y\big) - H_0(x; x + y) \\ &\qquad\qquad - \gamma(x, \zeta) \frac{\partial_z H_0(z; x + y)}{\partial z}\bigg|_{z=x}\bigg) \bar{h}_\varepsilon(\zeta)\, \mathrm{d}\zeta, \end{aligned} \tag{A.24}$$

$$\mathcal{R}^2_s(x; y) := \int_0^1 (1 - \alpha) \mathbb{E}\big(L_\varepsilon^2 H_0\big)\big(X_{\alpha s}(\varepsilon, \varnothing, x); x + y\big)\, \mathrm{d}\alpha.$$

Note that $\sup_{s<1,x,y} |\mathcal{R}_s^2(x; y)| < \infty$ in light of Lemma 3.2 and, also, by writing $\mathbb{P}[\tilde{\gamma}(z, J) \geq x + y] = \mathbb{P}[\gamma(z, J) \geq x + y - z]$ as $G(z, x + y - z)$ with $G(x, y) = \mathbb{P}(\gamma(x, J) \geq y)$, we have

$$
\begin{aligned}
\frac{\partial H_0(z; x+y)}{\partial z}\bigg|_{z=x} &= \frac{\partial \mathbb{P}[\tilde{\gamma}(z, J) \geq x + y]}{\partial z}\bigg|_{z=x} = \frac{\partial \mathbb{P}[\gamma(x, J) \geq y]}{\partial x} + \Gamma(y; x), \\
\frac{\partial^2 H_0(z; x+y)}{\partial z^2}\bigg|_{z=x} &= \frac{\partial^2 \mathbb{P}[\tilde{\gamma}(z, J) \geq x + y]}{\partial z^2}\bigg|_{z=x} \\
&= \frac{\partial^2 \mathbb{P}[\gamma(x, J) \geq y]}{\partial x^2} + 2\frac{\partial \Gamma(y; x)}{\partial x} - \frac{\partial \Gamma(y; x)}{\partial y}.
\end{aligned}
$$

Substituting the previous identities in (A.24), we can write $H_{0,1}(x; y)$ as

$$
\begin{aligned}
H_{0,1}(x; y) = b_\varepsilon(x)&\left(\frac{\partial \mathbb{P}[\gamma(x, J) \geq y]}{\partial x} + \Gamma(y; x)\right) \\
&+ \frac{\sigma^2(x)}{2}\left(\frac{\partial^2 \mathbb{P}[\gamma(x, J) \geq y]}{\partial x^2} + 2\frac{\partial \Gamma(y; x)}{\partial x} - \frac{\partial \Gamma(y; x)}{\partial y}\right) + \hat{H}_{0,1}(x; y),
\end{aligned}
\tag{A.25}
$$

with $\hat{H}_{0,1}(x; y)$ given by

$$
\begin{aligned}
\hat{H}_{0,1}(x; y) = \int \bigg(&\mathbb{P}\big[\gamma\big(x + \gamma(x, \zeta), J\big) \geq y - \gamma(x, \zeta)\big] - \mathbb{P}\big[\gamma(x, J) \geq y\big] \\
&- \gamma(x, \zeta)\left(\frac{\partial \mathbb{P}[\gamma(x, J) \geq y]}{\partial x} + \Gamma(y; x)\right)\bigg)\bar{h}_\varepsilon(\zeta)\, d\zeta.
\end{aligned}
\tag{A.26}
$$

Plugging (A.23) in (A.22) and recalling from Lemma A.1 that the second and third terms on the right-hand side of (A.22) are bounded for t small enough, we get that

$$
\mathbb{P}\big(X_t\big(\varepsilon, \{s\}, x\big) \geq x + y\big) = \mathbb{P}\big[\gamma(x, J) \geq y\big] + \mathrm{O}(t).
$$

The latter can then be plugged in (A.19) to get

$$
\mathbb{P}\big(X_t(x) \geq x + y | N_t^\varepsilon = 1\big) = \mathbb{P}\big[\gamma(x, J) \geq y\big] + \mathrm{O}(t).
$$

Finally, (4.2) can be written as

$$
\begin{aligned}
\mathbb{P}\big(X_t(x) \geq x + y\big) &= \mathrm{e}^{-\lambda_\varepsilon t} t \lambda_\varepsilon \mathbb{P}\big[\gamma(x, J) \geq y\big] + \mathrm{O}\big(t^2\big) \\
&= t \int \mathbf{1}_{\{\gamma(x,\zeta) \geq y\}} h(\zeta)\, d\zeta + \mathrm{O}\big(t^2\big),
\end{aligned}
\tag{A.27}
$$

where, in the first equality, we used (3.3) to justify that $\mathbb{P}(X_t(x) \geq x + y | N_t^\varepsilon = 0) = \mathbb{P}(X_t(\varepsilon, \varnothing, x) \geq x + y) = \mathrm{O}(t^2)$ while, in the second equality above, we take $\varepsilon > 0$ small enough. Equation (A.27) gives first-order asymptotic expansion of the tail probability $\mathbb{P}(X_t(x) \geq x + y)$. We now proceed to obtain the second-order term.

A.3. Second-order term

In addition to (A.23), we also consider the leading terms in the term $\mathbb{E}H_1(X_s(\varepsilon,\varnothing,x);x+y)$ of (A.22) and the term $\mathbb{P}(X_t(x)\geq x+y|N_t^\varepsilon=2)$ of (4.2). Let us first show that $z\to H_1(z;x+y)$ is C_b^2. To this end, let

$$\begin{aligned}\mathcal{K}(\zeta;x,y,z) &:= \mathbb{P}\big[z+\gamma(z,J)\geq\bar{\gamma}(x+y,\zeta)\big]-\mathbb{P}\big[z+\gamma(z,J)\geq x+y\big]\\ &\quad-\widetilde{\Gamma}(x+y;z)\gamma(x+y,\zeta),\end{aligned}$$

and recall that

$$\begin{aligned}H_1(z;x+y) &= \widetilde{\Gamma}(x+y;z)b_\varepsilon(x+y)-(\partial_\zeta\widetilde{\Gamma})(x+y;z)v(x+y)\\ &\quad-\widetilde{\Gamma}(x+y;z)v'(x+y)+\int\mathcal{K}(\zeta;x,y,z)\bar{h}_\varepsilon(\zeta)\,\mathrm{d}\zeta,\end{aligned}$$

where $\partial_\zeta\widetilde{\Gamma}$ and $\partial_z\widetilde{\Gamma}$ denote the partial derivatives of the density $\widetilde{\Gamma}(\zeta;z)$. Obviously, the first three terms on the right-hand side of the previous expression are C_b^2 in light of Lemma 2.1(2). Hence, for the derivative $\partial_z H_1(z;x+y)$ to exist, it suffices to show that $\partial_z\mathcal{K}(\zeta;x,y,z)$ exists and that

$$\sup_{z,x,y}\left|\frac{\partial\mathcal{K}(\zeta;x,y,z)}{\partial z}\right|<C|\zeta|^2 \tag{A.28}$$

for any $|\zeta|<\varepsilon$ and some constant $C<\infty$. Recalling that

$$\begin{aligned}\mathcal{K}(\zeta;x,y,z) &= \int_{\bar{\gamma}(x+y,\zeta)}^{x+y}\widetilde{\Gamma}(\eta;z)\,\mathrm{d}\eta-\widetilde{\Gamma}(x+y;z)\gamma(x+y,\zeta)\\ &= \int_0^1\big[(\partial_\zeta\widetilde{\Gamma})\big(\bar{\gamma}(x+y,\zeta\beta);z\big)(\partial_\zeta\bar{\gamma})(x+y,\zeta\beta)\\ &\qquad-\widetilde{\Gamma}(x+y;z)\big(\partial_\zeta^2\gamma\big)(x+y,\zeta\beta)\big](1-\beta)\,\mathrm{d}\beta\,\zeta^2\end{aligned}$$

and using that $\widetilde{\Gamma}(\eta;z)\in C_b^\infty$, we can write $\partial_z\mathcal{K}(\zeta;x,y,z)$ as

$$\begin{aligned}&\int_0^1\big(\big(\partial_{z,\zeta}^2\widetilde{\Gamma}\big)\big(\bar{\gamma}(x+y,\zeta\beta);z\big)(\partial_\zeta\bar{\gamma})(x+y,\zeta\beta)\\ &\qquad-(\partial_z\widetilde{\Gamma})(x+y;z)\big(\partial_\zeta^2\gamma\big)(x+y,\zeta\beta)\big)(1-\beta)\,\mathrm{d}\beta\,\zeta^2.\end{aligned}$$

Therefore, in light of Lemma 2.1 and the fact that $\gamma\in C_b^{\geq 1}$, there exists a constant C such that (A.28) holds. We can similarly prove that $\partial_z^2 H_1(z;x,y)$ exists and is bounded.

Using Dynkin's formula (3.7) with $n=1$ and that $\widetilde{\Gamma}(\zeta;z)=\Gamma(\zeta-z;z)$, we get

$$\mathbb{E}H_1\big(X_s(\varepsilon,\varnothing,x);x,y\big)=H_{1,0}(x,y)+s\mathcal{R}_s^3(x;y), \tag{A.29}$$

where

$$
\begin{aligned}
H_{1,0}(x;y) &:= H_1(x;x+y) = \mathcal{D}_{1,0}(x;y) + \hat{H}_{1,0}(x;y) \qquad \text{with} \\
\mathcal{D}_{1,0}(x;y) &:= \Gamma(y;x)b_\varepsilon(x+y) - (\partial_\zeta\Gamma)(y;x)v(x+y) - \Gamma(y;x)v'(x+y), \\
\hat{H}_{1,0}(x;y) &:= \int \big(\mathbb{P}\big[x+\gamma(x,J) \geq \bar{\gamma}(x+y,\zeta)\big] - \mathbb{P}\big[\gamma(x,J) \geq y\big] \\
&\qquad - \Gamma(y;x)\gamma(x+y,\zeta)\big)\bar{h}_\varepsilon(\zeta)\,\mathrm{d}\zeta, \\
\mathcal{R}_s^3(x;y) &:= \int_0^1 \mathbb{E}L_\varepsilon H_1\big(X_{\alpha s}(\varepsilon,\varnothing,x);x+y\big)\,\mathrm{d}\alpha = \mathrm{O}(1) \qquad \text{as } s \to 0.
\end{aligned}
\tag{A.30}
$$

In order to handle $\mathbb{P}(X_t(x) \geq x+y|N_t^\varepsilon = 2)$, we again condition on the times of the jumps, which are necessarily distributed as the order statistics of two independent uniform $[0,t]$ random variables. Concretely,

$$
\mathbb{P}\big(X_t(x) \geq x+y|N_t^\varepsilon = 2\big) = \frac{2}{t^2}\int_0^t \int_{s_1}^t \mathbb{P}\big(X_t\big(\varepsilon,\{s_1,s_2\},x\big) \geq x+y\big)\,\mathrm{d}s_2\,\mathrm{d}s_1. \tag{A.31}
$$

Next, we determine the leading term of $\mathbb{P}(X_t(\varepsilon,\{s_1,s_2\},x) \geq x+y)$. By conditioning on $\mathcal{F}_{s_2^-}$,

$$
\mathbb{P}\big(X_t\big(\varepsilon,\{s_1,s_2\},x\big) \geq x+y\big) = \mathbb{E}\big(G_{t-s_2}\big(X_{s_2}\big(\varepsilon,\{s_1\},x\big)\big)\big),
$$

where, by Lemma A.1,

$$
\begin{aligned}
G_t(z) &= \mathbb{P}\big[X_t\big(\varepsilon,\varnothing,z+\gamma(z,J)\big) \geq x+y\big] \\
&= H_0(z;x+y) + tH_1(z;x+y) + t^2\check{\mathcal{R}}_t(z;x+y).
\end{aligned}
\tag{A.32}
$$

Then, for $\varepsilon > 0$ and t small enough,

$$
\begin{aligned}
&\mathbb{P}\big(X_t\big(\varepsilon,\{s_1,s_2\},x\big) \geq x+y\big) \\
&\quad = \mathbb{E}\big(H_0\big(X_{s_2}\big(\varepsilon,\{s_1\},x\big);x+y\big)\big) + (t-s_2)\mathbb{E}\mathcal{R}_{t-s_2}^4\big(X_{s_2}\big(\varepsilon,\{s_1\},x\big);x,y\big),
\end{aligned}
\tag{A.33}
$$

with

$$
\mathcal{R}_t^4(z;x,y) := H_1(z;x+y) + t\check{\mathcal{R}}_t(z;x+y).
$$

Again, conditioning on $\mathcal{F}_{s_1^-}$,

$$
\mathbb{E}\big(H_0\big(X_{s_2}\big(\varepsilon,\{s_1\},x\big);x+y\big)\big) = \mathbb{E}\big(\widehat{G}_{s_2-s_1}\big(X_{s_1}(\varepsilon,\varnothing,x);x+y\big)\big),
$$

where

$$
\widehat{G}_t(z;x+y) := \mathbb{E}H_0\big(X_t\big(\varepsilon,\varnothing,z+\gamma(z,J)\big);x+y\big).
$$

Since $z \to H_0(z; x+y) = \mathbb{P}(z + \gamma(z, J) \geq x + y)$ is C_b^∞ by Lemma 2.1(3), we can apply Dynkin's formula (3.7) with $n = 1$ to deduce

$$\begin{aligned}
\widehat{G}_t(z; x+y) &= \int \widetilde{\Gamma}(\zeta; z) \mathbb{E} H_0\big(X_t(\varepsilon, \varnothing, \zeta); x+y\big) \, \mathrm{d}\zeta \\
&= \int \widetilde{\Gamma}(\zeta; z) H_0(\zeta; x+y) \, \mathrm{d}\zeta + t \mathcal{R}_t^6(z; x, y) \\
&=: H_2(z; x+y) + t\mathcal{R}_t^6(z; x, y),
\end{aligned}$$

where, denoting two independent copies of J by J_1, J_2,

$$\begin{aligned}
H_2(z; x+y) &:= \mathbb{P}\big(z + \gamma(z, J_1) + \gamma\big(z + \gamma(z, J_1), J_2\big) \geq x + y\big), \\
\mathcal{R}_t^6(z; x, y) &:= \int \widetilde{\Gamma}(\zeta; z) \int_0^1 \mathbb{E} L_\varepsilon H_0\big(X_{\alpha t}(\varepsilon, \varnothing, \zeta); x+y\big) \, \mathrm{d}\alpha \, \mathrm{d}\zeta.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\mathbb{P}\big(X_t\big(\varepsilon, \{s_1, s_2\}, x\big) \geq x + y\big) \\
&\quad = \mathbb{E}\big(H_2\big(X_{s_1}(\varepsilon, \varnothing, x); x+y\big)\big) + (s_2 - s_1)\mathbb{E}\mathcal{R}^6_{s_2 - s_1}\big(X_{s_1}(\varepsilon, \varnothing, x); x, y\big) \\
&\qquad + (t - s_2)\mathbb{E}\mathcal{R}^4_{t - s_2}\big(X_{s_2}\big(\varepsilon, \{s_1\}, x\big); x, y\big).
\end{aligned}$$

Applying again Dynkin's formula (3.7) with $n = 1$ to the first term on the right-hand side of the previous equation, we can write

$$\begin{aligned}
&\mathbb{P}\big(X_t\big(\varepsilon, \{s_1, s_2\}, x\big) \geq x + y\big) \\
&\quad = H_{2,0}(x; y) + s_1 \mathcal{R}^5_{s_1}(x; y) \\
&\qquad + (s_2 - s_1)\mathbb{E}\mathcal{R}^6_{s_2 - s_1}\big(X_{s_1}(\varepsilon, \varnothing, x); x, y\big) \\
&\qquad + (t - s_2)\mathbb{E}\mathcal{R}^4_{t - s_2}\big(X_{s_2}\big(\varepsilon, \{s_1\}, x\big); x, y\big),
\end{aligned} \tag{A.34}$$

where

$$\begin{aligned}
H_{2,0}(x; y) &:= H_2(x; x+y) = \mathbb{P}\big(\gamma(x, J_1) + \gamma\big(x + \gamma(x, J_1), J_2\big) \geq y\big), \\
\mathcal{R}^5_{s_1}(x; y) &:= \int_0^1 \mathbb{E} L_\varepsilon H_2\big(X_{\alpha s_1}(\varepsilon, \varnothing, x); x+y\big) \, \mathrm{d}\alpha.
\end{aligned}$$

Therefore, we conclude that

$$\mathbb{P}\big(X_t(x) \geq x + y | N_t^\varepsilon = 2\big) = H_{2,0}(x; y) + \mathrm{O}(t). \tag{A.35}$$

In light of (A.19), (A.22)–(A.25), (A.29), and (A.35), we have the following second-order decomposition of the tail distribution $\mathbb{P}(X_t(x) \geq x + y)$:

$$\begin{aligned}
&\mathbb{P}\bigl(X_t(x) \geq x + y\bigr) \\
&\quad = \mathrm{e}^{-\lambda_\varepsilon t}\lambda_\varepsilon t H_{0,0}(x; y) + \mathrm{e}^{-\lambda_\varepsilon t}\frac{\lambda_\varepsilon t^2}{2}\bigl(H_{0,1}(x; y) + H_{1,0}(x; y)\bigr) \\
&\qquad + \mathrm{e}^{-\lambda_\varepsilon t}\frac{(\lambda_\varepsilon t)^2}{2} H_{2,0}(x; y) + \mathrm{O}\bigl(t^3\bigr) \\
&\quad = \lambda_\varepsilon t H_{0,0}(x; y) + \frac{t^2}{2}\bigl\{\lambda_\varepsilon\bigl[H_{0,1}(x; y) + H_{1,0}(x; y)\bigr] + \lambda_\varepsilon^2\bigl[H_{2,0}(x; y) - 2H_{0,0}(x; y)\bigr]\bigr\} \\
&\qquad + \mathrm{O}\bigl(t^3\bigr),
\end{aligned}$$

where, in the first equality above, we had again used (3.3) to justify that

$$\mathbb{P}\bigl(X_t(x) \geq x + y | N_t^\varepsilon = 0\bigr) = \mathbb{P}\bigl(X_t(\varepsilon, \varnothing, x) \geq x + y\bigr) = \mathrm{O}\bigl(t^3\bigr)$$

for ε small enough. The expressions in (4.5) follows from the fact that,

$$\begin{aligned}
\lambda_\varepsilon \mathbb{P}\bigl[\gamma(x, J) \geq y\bigr] &= \int_y^\infty \lambda_\varepsilon \Gamma_\varepsilon(\zeta; x)\,\mathrm{d}\zeta = \int_{\{\zeta:\gamma(x,\zeta)\geq y\}} h(\zeta)\phi_\varepsilon(\zeta)\,\mathrm{d}\zeta \\
&=: \int_y^\infty g_\varepsilon(x; \zeta)\,\mathrm{d}\zeta
\end{aligned} \tag{A.36}$$

for some function $g_\varepsilon(x; \zeta)$. Thus, for fixed $x \in \mathbb{R}$ and $y > 0$,

$$\lambda_\varepsilon \Gamma_\varepsilon(y; x) = g_\varepsilon(x; y). \tag{A.37}$$

Furthermore, by differentiation of the last equality in (A.36) and using that $\gamma(x, 0) = 0$, it follows that, for $\varepsilon > 0$ small enough, $g_\varepsilon(x; y)$ admits the representation on the right-hand side of (4.3). Using (A.36)–(A.37), it then follows that

$$\begin{aligned}
\lambda_\varepsilon H_{0,0}(x; y) &= \int_y^\infty g(x; \zeta)\,\mathrm{d}\zeta, \\
\lambda_\varepsilon\bigl[\hat{H}_{0,1}(x; y) + \hat{H}_{1,0}(x; y)\bigr] &= \mathcal{J}_1(x; y), \\
\lambda_\varepsilon\bigl[H_{0,1}(x; y) + H_{1,0}(x; y)\bigr] &= \mathcal{D}(x; y) + \mathcal{J}_1(x; y), \\
\lambda_\varepsilon^2\bigl[H_{2,0}(x; y) - 2H_{0,0}(x; y)\bigr] &= \mathcal{J}_2(x; y),
\end{aligned}$$

with $\mathcal{D}(x; y)$, $\mathcal{J}_1(x; y)$, and $\mathcal{J}_2(x; y)$ given as in the statement of the theorem. This concludes the result of Theorem 4.1.

Appendix B: Proof of the expansion for the transition densities

The following result will allow us to control the higher-order terms of the expansion (4.2) (see Appendix C for its proof):

Lemma B.1. *Let*

$$\bar{\mathcal{R}}_t(x,y) := \mathrm{e}^{-\lambda_\varepsilon t} \sum_{n=3}^{\infty} \mathbb{P}\big(X_t(x) \geq x + y | N_t^\varepsilon = n\big) \frac{(\lambda_\varepsilon t)^n}{n!}. \tag{B.1}$$

Then, under the conditions of Theorem 5.2, there exists $\varepsilon > 0$ small enough as well as $t_0 := t_0(\varepsilon) > 0$ and $B = B(\varepsilon) < \infty$ such that, for any $0 < t < t_0$,

$$\big|\partial_y \bar{\mathcal{R}}_t(x,y)\big| \leq Bt^3.$$

Proof of Theorem 5.2. Let us consider the terms corresponding to one and two "large" jumps in (4.2). From (A.19), (A.22), (A.23), and (A.29), it follows that

$$\begin{aligned} &\mathbb{P}\big(X_t(x) \geq x + y | N_t^\varepsilon = 1\big) \\ &\quad = H_{0,0}(x;y) + \frac{t}{2}\big[H_{0,1}(x;y) + H_{1,0}(x;y)\big] \qquad \text{(B.2)} \\ &\qquad + \frac{1}{t}\int_0^t \big\{s^2 \mathcal{R}_s^2(x;y) + (t-s)s\mathcal{R}_s^3(x;y) + (t-s)^2 \mathbb{E}\mathcal{R}_{t-s}^1\big(X_s(\varepsilon,\varnothing,x);x,y\big)\big\}\,\mathrm{d}s. \end{aligned}$$

Similarly, from (A.31), (A.33), and (A.34), we have

$$\begin{aligned} &\mathbb{P}\big(X_t(x) \geq x + y | N_t^\varepsilon = 2\big) \\ &\quad = H_{2,0}(x;y) + \frac{2}{t^2}\int_0^t \int_{s_1}^t \big\{s_1 \mathcal{R}_{s_1}^5(x;y) + (s_2 - s_1)\mathbb{E}\mathcal{R}_{s_2-s_1}^6\big(X_{s_1}(\varepsilon,\varnothing,x);x,y\big) \qquad \text{(B.3)} \\ &\qquad\qquad + (t-s_2)\mathbb{E}\mathcal{R}_{t-s_2}^4\big(X_{s_2}\big(\varepsilon,\{s_1\},x\big);x,y\big)\big\}\,\mathrm{d}s_2\,\mathrm{d}s_1. \end{aligned}$$

Equations (B.2)–(B.3) show that in order for the derivatives

$$\hat{a}_1(x;y) := \frac{\partial}{\partial y}\mathbb{P}\big(X_t(x) \geq x + y | N_t^\varepsilon = 1\big), \qquad \hat{a}_2(x;y) := \frac{\partial}{\partial y}\mathbb{P}\big(X_t(x) \geq x + y | N_t^\varepsilon = 2\big)$$

to exist, it suffices that the partial derivatives with respect to y of the functions $H_{i,j}(x;y)$ exist and also that the partial derivatives with respect to y of the two types of functions, $\mathcal{R}_t^i(x;y)$ with $i = 2,3,5$ and $\mathcal{R}_t^j(w;x,y)$ with $j = 1,4,6$, exist and are uniformly bounded on $w \in \mathbb{R}$ and on a neighborhood of y. Furthermore, under the later boundedness property, we will then be able to conclude that

$$\hat{a}_1(x;y) = \frac{\partial H_{0,0}(x;y)}{\partial y} + \frac{t}{2}\left[\frac{\partial H_{0,1}(x;y)}{\partial y} + \frac{\partial H_{1,0}(x;y)}{\partial y}\right] + \mathrm{O}\big(t^2\big) \qquad (t \to 0), \tag{B.4}$$

$$\hat{a}_2(x;y) = \frac{\partial H_{2,0}(x;y)}{\partial y} + \mathrm{O}(t) \qquad (t \to 0). \tag{B.5}$$

Note that (B.4)–(B.5) suffices to obtain the conclusion of the theorem, namely equation (5.5), in light of (4.2), Theorem 5.1, and Lemma B.1. We now proceed to verify the differentiability of the functions $H_{i,j}(x,y)$ and the remainder terms.

(1) *Differentiability of* $H_{i,j}(x;y)$: The desired differentiability essentially follows from Lemma 2.1. Indeed, Lemma 2.1(2) implies that $\partial_y H_{0,0}(x;y) = \partial_y \mathbb{P}[\gamma(x,J) \geq y] = -\Gamma(y;x)$ and also, recalling the formula of $H_{0,1}(x,y)$ given in equations (A.25)–(A.26),

$$\begin{aligned}
\partial_y H_{0,1}(x;y) := {} & \frac{\sigma^2(x)}{2}\left(-\frac{\partial^2\Gamma(y;x)}{\partial x^2} + 2\frac{\partial^2\Gamma(y;x)}{\partial y\,\partial x} - \frac{\partial^2\Gamma(y;x)}{\partial y^2}\right) \\
& + b_\varepsilon(x)\left(-\frac{\partial\Gamma(y;x)}{\partial x} + \frac{\partial\Gamma(y;x)}{\partial y}\right) \\
& + \int \Big(\Gamma(y;x) - \Gamma\Big(y-\gamma(x,\zeta); x+\gamma(x,\zeta) \\
& \qquad - \gamma(x,\zeta)\left(\frac{\partial\Gamma(y;x)}{\partial y} - \frac{\partial\Gamma(y;x)}{\partial x}\right)\Big)\Big)\bar{h}_\varepsilon(\zeta)\,\mathrm{d}\zeta.
\end{aligned}$$

Similarly, recalling the definition of $H_{1,0}(x;y)$ given in (A.30),

$$\begin{aligned}
\partial_y H_{1,0}(x;y) := {} & \partial_y\big(\Gamma(y;x)b_\varepsilon(x+y) - (\partial_\zeta\Gamma)(y;x)v(x+y) - \Gamma(y;x)v'(x+y)\big) \\
& + \int \big(\Gamma(y;x) - \Gamma\big(\bar{\gamma}(x+y,\zeta)-x;x\big)\partial_y\bar{\gamma}(x+y,\zeta) \\
& \qquad - \partial_y\big(\Gamma(y;x)\gamma(x+y,\zeta)\big)\big)\bar{h}_\varepsilon(\zeta)\,\mathrm{d}\zeta.
\end{aligned}$$

To compute $\partial_y H_{2,0}(x;y)$, note that

$$\begin{aligned}
\frac{\partial}{\partial y}H_{2,0}(x;y) &= \frac{\partial}{\partial y}\int \mathbb{P}\big(\gamma\big(x+\gamma(x,\zeta_1),J_2\big) \geq y-\gamma(x,\zeta_1)\big)h_\varepsilon(\zeta_1)\,\mathrm{d}\zeta_1 \\
&= \int \frac{\partial}{\partial y}\int_{y-\gamma(x,\zeta_1)}^{\infty} \Gamma\big(\zeta_2; x+\gamma(x,\zeta_1)\big)\,\mathrm{d}\zeta_2 h_\varepsilon(\zeta_1)\,\mathrm{d}\zeta_1 \\
&= -\int \Gamma\big(y-\gamma(x,\zeta_1); x+\gamma(x,\zeta_1)\big)h_\varepsilon(\zeta_1)\,\mathrm{d}\zeta_1,
\end{aligned}$$

where the second equality above again follows from Lemma 2.1(2). Finally, the representations in (5.6) can be deduced for ε small enough from the relationships (A.36)–(A.37).

(2) *Boundedness of* $\partial_y\mathcal{R}^i(w;x,y)$: Analyzing the remainder terms $\mathcal{R}^2(x;y)$, $\mathcal{R}_t^3(x;y)$, $\mathcal{R}_t^5(x;y)$, and $\mathcal{R}_t^6(w;x,y)$, it transpires that it suffices to show that $\partial_y L_\varepsilon^2 H_0(w;x+y)$, $\partial_y L_\varepsilon H_0(w;x+y)$, $\partial_y L_\varepsilon H_1(w;x+y)$, and $\partial_y L_\varepsilon H_2(w;x+y)$ exist and are uniformly bounded in w and y. From the definition of L_ε in (3.4), one can see that, for any function $H(w;y):\mathbb{R}^2 \to \mathbb{R}$ in $C_b^\infty(\mathbb{R}^2)$, $\partial_y(L_\varepsilon H(w;y))$ exists and

$$\partial_y\big(L_\varepsilon H(w;y)\big) = L_\varepsilon(\partial_y H)(w;y), \qquad \sup_{w,y}\big|\partial_y L_\varepsilon H(w;y)\big| < \infty.$$

From Lemma 2.1(4) and the relationship (A.28), one can verify that $H_0(w; x+y)$, $H_1(w; x+y)$, $H_2(w; x+y)$ are C_b^∞ functions.

In order to show that $\partial_y \mathcal{R}_t^1(w; x, y)$ and $\partial_y \mathcal{R}_t^4(w; x, y)$ exist and are bounded, it suffices that the remainder term $\breve{\mathcal{R}}_t(z; \vartheta)$ of (A.1) is differentiable with respect to ϑ and $\partial_\vartheta \breve{\mathcal{R}}_t(z; \vartheta)$ is bounded. The remainder term is defined as in (A.10), which in turn is defined as the limit as $\delta \to 0$ of each of the four terms in (A.11). We will show that the limit as $\delta \to 0$ of the second term, which was therein denoted by $\bar{I}_t^{(2)}(z; \vartheta, \delta, \varepsilon)$, is indeed differentiable with respect to ϑ and its derivative is bounded. The other three terms can be dealt with similarly. As shown in the proof of Lemma A.1 (see (A.18) and arguments before), the limit of the second term in (A.11) can be expressed as the sum of terms of the form $\int_0^1 (1-\alpha)\tilde{I}_{\alpha t}(\vartheta; z, \varepsilon)\,\mathrm{d}\alpha$, where $\tilde{I}_{\alpha t}(\vartheta; z, \varepsilon)$ takes one of the four generic terms listed in (A.17). So, we only need to show that each of these terms is differentiable with respect to w and that their respective derivatives are bounded. The latter facts will follow from Lemma A.2 together with the same arguments leading to (A.17). □

Appendix C: Proofs of other lemmas and additional needed results

The following result is needed in order to prove Lemma A.2.

Lemma C.1. *Assume that the conditions* (C1)–(C4) *of Section* 2 *are enforced. Let* $\Phi_t : x \to X_t(\varepsilon, \varnothing, x)$ *be the diffeomorphism associated with the solution of the SDE* (2.11). *Then, for any* $k \geq 1$, $T < \infty$, *and compact* $K \subset \mathbb{R}$,

$$\sup_{t\in(0,T]} \sup_{\eta\in K} \mathbb{E}\left(\left|\frac{\mathrm{d}^i \Phi_t^{-1}}{\mathrm{d}\eta^i}(\eta)\right|^k\right) < \infty, \qquad i = 1, 2. \tag{C.1}$$

Proof. To simplify the notation, we write $\breve{X}(x) = \{\breve{X}_t(x)\}_{t\in(0,T]}$ for $\{X_t(\varepsilon, \varnothing, x)\}_{t\geq 0}$ and fix $Y_t(x) := \breve{X}_{(T-t)-}(x)$ for $0 \leq t < T$ and $Y_T(x) := \breve{X}_0(x) = x$. We follow a similar approach to that in the proof of Lemma 3.1 in Ishikawa [17] based on time-reversibility (see Section VI.4 in Protter [29] for further information). Recall that the time-reversal process of a cádág process $V = \{V_t\}_{0\leq t\leq T}$ is given by the cádlág process

$$\overline{V}_t^T = (V_{(T-t)-} - V_{T-})\mathbf{1}_{0<t<T} + (V_0 - V_{T-})\mathbf{1}_{t=T}. \tag{C.2}$$

Our main tool is Theorem VI.4.22 in Protter [29]. The following notation and definitions are useful for verifying the assumptions in the theorem.

Throughout, $\Phi_{t,T}(\cdot; \omega) : \mathbb{R} \to \mathbb{R}$ denotes the diffeomorphisms defined by $\Phi_{t,T}(x; \omega) := X_{t,T}^\varepsilon(x; \omega)$ where $X_{t,T}^\varepsilon(x; \omega)$ is the unique solution of the SDE

$$\begin{aligned} X_{t,T}^\varepsilon(x) = x &+ \int_t^T \sigma\big(X_{t,u}^\varepsilon(x)\big)\,\mathrm{d}W_u + \int_t^T b_\varepsilon\big(X_{t,u}^\varepsilon(x)\big)\,\mathrm{d}u \\ &+ \sum_{t<u\leq T}^{c} \gamma\big(X_{t,u^-}^\varepsilon(x), \Delta Z_u'\big), \end{aligned} \tag{C.3}$$

where $\sum^c$ denotes the compensated sum. The a.s. existence of this diffeomorphisms is guaranteed from (2.4) as stated in Remark 2.2. As usual, $\mathcal{F}_t = \mathcal{F}_t^0 \vee \mathcal{N}$ and $\mathbb{F} = (\mathcal{F}_t)_{0\le t\le T}$, where $\mathcal{F}_t^0 = \sigma\{W_u, Z'_u; u \le t\}$ $(0 \le t \le T)$ and $\mathcal{N}$ are the $\mathbb{P}$-null sets of $\mathcal{F}_T^0$. We also define the backward filtration $\tilde{\mathbb{H}} = (\mathcal{H}^t)_{0<t\le T}$ by $\mathcal{H}^t = \bigcap_{t<u\le T} \bar{\mathcal{F}}_u \vee \sigma\{\breve{X}_T\}$, where $(\bar{\mathcal{F}}_t)_{0\le t\le T}$ is defined analogously to $(\mathcal{F}_t)_{0\le t\le T}$ by W and Z' replaced with their reversal processes $\bar{W}^T$ and $\bar{Z'}^T$.

We are ready to show the assertions of the lemma. First, note that, by the uniqueness of the solution of (C.3), $\breve{X}_T(x) = \Phi_{t,T}(\breve{X}_t(x))$. Thus, $\breve{X}_t(x) = \Phi_{t,T}^{-1}(\breve{X}_T(x)) \in \mathcal{H}^{T-t}$ and, of course, $\breve{X}_t(x) \in \mathcal{F}_t$, so that $\sigma(\breve{X}_t(x)) \in \mathcal{F}_t \wedge \mathcal{H}^{T-t}$. Also, by Itô's formula, the quadratic covariation of $W = \{W_t\}_{0\le t\le T}$ with $\sigma(\breve{X}) := \{\sigma(\breve{X}_t(x))\}_{0\le t\le T}$ is given by

$$[\sigma(\breve{X}), W]_t = \int_0^t \sigma'(\breve{X}_u(x))\sigma(\breve{X}_u(x))\,\mathrm{d}u = \int_0^t \sigma'(Y_{T-u}(x))\sigma(Y_{T-u}(x))\,\mathrm{d}u. \tag{C.4}$$

Finally, recalling that $W = \{W_t\}_{0\le t\le T}$ is an $(\mathbb{F}, \tilde{\mathbb{H}})$-reversible semimartingale (cf. Theorem VI.4.20 in Protter [29]), the assumptions of Theorem VI.4.22 in Protter [29] are satisfied with $\sigma(\breve{X})$ and W in place of H and Y, respectively. By the theorem, we have

$$\overline{\int_0^{\cdot} \sigma(\breve{X}_u(x))\,\mathrm{d}W_u}^{T}_{t} + \overline{[\sigma(\breve{X}), W]}^{T}_{t} = \int_0^t \sigma(\breve{X}_{T-u}(x))\,\mathrm{d}\bar{W}_u^T,$$

or equivalently, by (C.4) and the change of variable $v = T-u$,

$$\overline{\int_0^{\cdot} \sigma(\breve{X}_{u-}(x))\,\mathrm{d}W_u}^{T}_{t} - \int_0^t \sigma'(Y_v(x))\sigma(Y_v(x))\,\mathrm{d}v = \int_0^t \sigma(Y_u(x))\,\mathrm{d}\overline{W}_u^T. \tag{C.5}$$

Omitting for simplicity the dependence of the processes on x, the first term on the left- hand side of (C.5) can be written as

$$\begin{aligned}
&\overline{\breve{X}_{\cdot} - x - \int_0^{\cdot} b_\varepsilon(\breve{X}_{u-})\,\mathrm{d}u - \sum_{0<u\le\cdot}^{c} \gamma(\breve{X}_{u-}, \Delta Z'_u)}^{T}_{t} \\
&\quad = \breve{X}_{(T-t)-} - \breve{X}_{T-} + \int_{T-t}^{T} b(\breve{X}_u)\,\mathrm{d}u + \sum_{T-t\le u<T}^{c} \gamma(\breve{X}_{u^-}, \Delta Z'_u) \\
&\quad = Y_t - Y_0 + \int_0^t b_\varepsilon(Y_v)\,\mathrm{d}v + \sum_{0<v\le t}^{c} \gamma(\breve{X}_{(T-v)^-}, \Delta Z'_{T-v}),
\end{aligned}$$

where the last equality above is from the change of variable $v = T-u$. Then, (C.5) implies that

$$\begin{aligned}
Y_t(x) &= Y_0(x) - \int_0^t b_\varepsilon(Y_v(x))\,\mathrm{d}v + \int_0^t \sigma'(Y_v(x))\sigma(Y_v(x))\,\mathrm{d}v + \int_0^t \sigma(Y_v(x))\,\mathrm{d}\overline{W}_v^T \\
&\quad - \sum_{0<v\le t}^{c} \gamma(\breve{X}_{(T-v)^-}(x), \Delta Z'_{T-v}), \qquad Y_0(x) = \breve{X}_{T^-}(x).
\end{aligned}$$

Let us write the jump component of Y in a more convenient way. To this end, note that, since $\check{X}_{(T-v)^-}(x) + \gamma(\check{X}_{(T-v)^-}(x), \Delta Z'_{T-v}) = \check{X}_{T-v}(x)$, one can express $\check{X}_{(T-v)^-}(x)$ in terms of the inverse $\bar{\gamma}(u,\zeta)$ of the mapping $z \to u := z + \gamma(z,\zeta)$ as follows

$$Y_v(x) = \check{X}_{(T-v)^-}(x) = \bar{\gamma}\big(\check{X}_{T-v}(x), \Delta Z'_{T-v}\big) = \bar{\gamma}\big(Y_{v^-}(x), \Delta Z'_{T-v}\big).$$

Then,

$$\Delta Y_v(x) = \bar{\gamma}\big(Y_{v^-}(x), \Delta Z'_{T-v}\big) - Y_{v^-}(x) = \bar{\gamma}\big(Y_{v^-}(x), -\Delta \bar{Z}'_v\big) - Y_{v^-}(x) = \gamma_0\big(Y_{v^-}(x), \Delta \bar{Z}'_v\big),$$

where $\gamma_0(u,\zeta) := \bar{\gamma}(u,-\zeta) - u$ and $\bar{Z}'_v := \overline{Z'}^T_v$ is the time-reversal process of $\{Z'_v\}_{0\le v\le T}$. We conclude that

$$\begin{aligned} Y_t(x) = \check{X}_{T^-}(x) &- \int_0^t b_\varepsilon\big(Y_v(x)\big)\,\mathrm{d}v + \int_0^t \sigma'\big(Y_v(x)\big)\sigma\big(Y_v(x)\big)\,\mathrm{d}v + \int_0^t \sigma\big(Y_v(x)\big)\,\mathrm{d}\overline{W}^T_v \\ &+ \sum_{0<v\le t}^{c} \gamma_0\big(Y_{v^-}(x), \Delta \bar{Z}'_v\big). \end{aligned}$$

Now, define the diffeomorphism $\Psi_t : \mathbb{R} \to \mathbb{R}$ as $\Psi_t(\eta) := \check{Y}_t(\eta)$, where $\{\check{Y}_t(\eta)\}_{0\le t\le T}$ is the solution of the SDE

$$\begin{aligned} \check{Y}_t(\eta) = \eta &- \int_0^t b_\varepsilon\big(\check{Y}_v(\eta)\big)\,\mathrm{d}v + \int_0^t \sigma'\big(\check{Y}_v(\eta)\big)\sigma\big(\check{Y}_v(\eta)\big)\,\mathrm{d}v + \int_0^t \sigma\big(\check{Y}_v(\eta)\big)\,\mathrm{d}\overline{W}^T_v \\ &+ \sum_{0<v\le t}^{c} \gamma_0\big(\check{Y}_{v-}(\eta), \Delta \bar{Z}'_v\big). \end{aligned}$$

Since, $\mathbb{P}$-a.s.,

$$\Psi_T\big(\Phi_T(x)\big) = \Psi_T\big(\check{X}_T(x)\big) = \Psi_T\big(\check{X}_{T^-}(x)\big) = Y_T(x) = x \qquad \text{for all } x \in \mathbb{R}, T < \infty,$$

it follows that, $\mathbb{P}$-a.s., $\Psi_t(\eta) = \Phi_t^{-1}(\eta)$ for all $\eta \in \mathbb{R}$. Furthermore, $\{\check{Y}_t(\eta)\}_{t\ge 0}$ solves an SDE of the form (6-2) in Bichteler, Gravereaux and Jacod [6] with their coefficients satisfying the assumptions of Lemma 10-29 therein. Finally, by Lemma 10-29-c in Bichteler, Gravereaux and Jacod [6], with $n = 2$ and $q = 1$,

$$\sup_{0<t\le T}\sup_{\eta\in K}\mathbb{E}\left[\left|\frac{d^i\Phi_t^{-1}(\eta)}{d\eta^i}\right|^k\right] = \sup_{0<t\le T}\sup_{\eta\in K}\mathbb{E}\left[\left|\frac{\mathrm{d}^i\Psi_t(\eta)}{\mathrm{d}\eta^i}\right|^k\right] = \sup_{0<t\le T}\sup_{\eta\in K}\mathbb{E}\left[\left|\frac{d^i\check{Y}_t(\eta)}{d\eta^i}\right|^k\right] < \infty$$

for $i = 1, 2$. □

Proof of Lemma A.2. For simplicity, we write $\widetilde{\Gamma}(\zeta) = \widetilde{\Gamma}(\zeta; z)$ and only show the case $k = 1$ (the other cases can similarly be proved). Using the same ideas as in the proof of Proposition I.2

in Léandre [18], one can show that

$$\int \widetilde{\Gamma}(\zeta)p_t(\eta;\varepsilon,\varnothing,\zeta)\,\mathrm{d}\zeta = \mathbb{E}\big(H_t(\eta)\big),$$

where

$$H_t(\eta) := \widetilde{\Gamma}\big(\Phi_t^{-1}(\eta)\big)\frac{\mathrm{d}\Phi_t^{-1}}{\mathrm{d}\eta}(\eta).$$

Denoting $\bar{J}_t(\eta) := d\Phi_t^{-1}(\eta)/d\eta$, note that

$$H_t'(\eta) = \widetilde{\Gamma}'\big(\Phi_t^{-1}(\eta)\big)\bar{J}_t(\eta)^2 + \widetilde{\Gamma}\big(\Phi_t^{-1}(\eta)\big)\bar{J}_t'(\eta),$$

and, using (C.1) and that $\widetilde{\Gamma} \in C_b^\infty$, it follows that $\sup_{\eta\in K}\mathbb{E}|H_t'(\eta)|^2 < \infty$. In particular,

$$\lim_{h\to 0}\mathbb{E}\left(\frac{H_t(\eta+h)-H_t(\eta)}{h}\right) = \mathbb{E}\left(\lim_{h\to 0}\frac{H_t(\eta+h)-H_t(\eta)}{h}\right) = \mathbb{E}H_t'(\eta), \tag{C.6}$$

since the set of random variables $\{[H_t(\eta+h)-H_t(\eta)]/h\colon |h|<1\}$ is uniformly integrable. Indeed,

$$\sup_{|h|\le 1}\mathbb{E}\left(\frac{H_t(\eta+h)-H_t(\eta)}{h}\right)^2 = \sup_{|h|\le 1}\mathbb{E}\left(\int_0^1 H_t'(\eta+h\beta)\,\mathrm{d}\beta\right)^2 \le \sup_{\substack{|h|\le 1\\ \beta\in[0,1]}}\mathbb{E}\big(H_t'(\eta+h\beta)\big)^2,$$

which is finite in light of (C.1). Then, (C.6) can be written as

$$\frac{\mathrm{d}}{\mathrm{d}\eta}\int \widetilde{\Gamma}(\zeta)p_t(\eta;\varepsilon,\varnothing,\zeta)\,\mathrm{d}\zeta = \mathbb{E}\Big(\widetilde{\Gamma}'\big(\Phi_t^{-1}(\eta)\big)\big(\bar{J}_t(\eta)\big)^2\Big) + \mathbb{E}\Big(\widetilde{\Gamma}\big(\Phi_t^{-1}(\eta)\big)\bar{J}_t'(\eta)\Big).$$

It is now clear that (A.4) will hold true in light of (C.1).

We now show the last assertion of the lemma. First note that, from the non-negativity of $\tilde{\Gamma}$ and p_t, (A.4) implies that there exist a constant $t_0 > 0$ small enough such that for any $t < t_0$,

$$\sup_{z\in\mathbb{R}}\sup_{\eta\in K}\int \big|\widetilde{\Gamma}(\zeta)p_t(\eta;\varepsilon,\varnothing,\zeta)\big|\,\mathrm{d}\zeta < \infty,$$

and, thus, $\widetilde{\Gamma}(\zeta)p_t(\eta;\varepsilon,\varnothing,\zeta)$ is uniformly integrable with respect to ζ. The latter fact together with (A.4) implies that

$$\left|\frac{\partial^k}{\partial\eta^k}\int \widetilde{\Gamma}(\zeta)\frac{\partial p_t}{\partial\eta}(\eta;\varepsilon,\varnothing,\zeta)\,\mathrm{d}\zeta\right| = \left|\frac{\partial^{k+1}}{\partial\eta^{k+1}}\int \widetilde{\Gamma}(\zeta)p_t(\eta,\varepsilon,\varnothing,\zeta)\,\mathrm{d}\zeta\right| < C$$

for some $C > 0$ and any $t < t_0$, $z \in \mathbb{R}$ and $\eta \in K$. Then, (A.4) is also true with $\partial p_t/\partial\eta$ in place of p_t inside the integral of (A.4). $\square$

Lemma C.2. *Assume the conditions* (C1)–(C4) *of Section* 2 *are satisfied and let* $\mathcal{D}_\varepsilon$ *and* $\mathcal{I}_\varepsilon$ *be the operators defined in* (3.4). *Define the following operators*:

$$\widetilde{\mathcal{D}}_\varepsilon g(y) := v(y)g''(y) + \big(2v'(y) - b(y)\big)g'(y) + \big(v''(y) - b'(y)\big)g(y),$$

$$\widetilde{\mathcal{I}}_\varepsilon g(y) := \int \big(g\big(\bar{\gamma}(y,\zeta)\big)\partial_y\bar{\gamma}(y,\zeta) - \big(1 + \partial_y\gamma(y,\zeta)\big)g(y) - g'(y)\gamma(y,\zeta)\big)\bar{h}_\varepsilon(\zeta)\,\mathrm{d}\zeta,$$

$$\widetilde{\mathcal{H}}_\varepsilon g(y) := \int\bigg(\int_{\bar{\gamma}(y,\zeta)}^{y} g(\eta)\,\mathrm{d}\eta - g(y)\gamma(y,\zeta)\bigg)\bar{h}_\varepsilon(\zeta)\,\mathrm{d}\zeta,$$

where hereafter $\bar{\gamma}(u,\zeta)$ *denotes the inverse of the mapping* $y \to u := y + \gamma(y,\zeta)$ *for a fixed* ζ *and whose existence is guaranteed from condition* (C4). *Then*, *the following assertions hold*:

1. $\widetilde{\mathcal{D}}_\varepsilon g$ *is well defined and uniformly bounded for any* $g \in C_b^2$ *and*, *furthermore*, *for any* $f \in C_b^2$ *with compact support*,

$$\int g(y)\mathcal{D}_\varepsilon f(y)\,\mathrm{d}y = \int f(y)\widetilde{\mathcal{D}}_\varepsilon g(y)\,\mathrm{d}y. \tag{C.7}$$

2. $\widetilde{\mathcal{I}}_\varepsilon g$ *is well defined and uniformly bounded for any* $g \in C_b^1$ *and*, *additionally*, *if* g *is integrable*, *then*, *for any* $f \in C_b^1$ *with compact support*,

$$\int g(y)\mathcal{I}_\varepsilon f(y)\,\mathrm{d}y = \int f(y)\widetilde{\mathcal{I}}_\varepsilon g(y)\,\mathrm{d}y. \tag{C.8}$$

3. *For any* $g \in C_b^1$ *and* $f \in C_b^1$ *such that* f' *and* f'' *are integrable*,

$$\int g(y)\mathcal{I}_\varepsilon f(y)\,\mathrm{d}y = \int f'(y)\widetilde{\mathcal{H}}_\varepsilon g(y)\,\mathrm{d}y. \tag{C.9}$$

Proof. The dual relationships essentially follow from a combination of integration by parts and change of variables. Let us show (C.9). First, we show that $\mathcal{I}_\varepsilon f(y)$ is integrable and, thus, the left-hand side of equation (C.9) is well defined. To this end, we write $\mathcal{I}_\varepsilon f(y)$ as

$$\begin{aligned}\mathcal{I}_\varepsilon f(y) = \int\int_0^1 \big(&f''\big(y+\gamma(y,\zeta\beta)\big)\big(\partial_\zeta\gamma(y,\zeta\beta)\big)^2 + f'\big(y+\gamma(y,\zeta\beta)\big)\partial_\zeta^2\gamma(y,\zeta\beta)\\ &- f'(y)\partial_\zeta^2\gamma(y,\zeta\beta)\big)(1-\beta)\,\mathrm{d}\beta\,\bar{h}_\varepsilon(\zeta)\zeta^2\,\mathrm{d}\zeta.\end{aligned}$$

Since $\gamma \in C_b^{\geq 1}$, it is now evident that $\int|\mathcal{I}_\varepsilon f(y)|\,\mathrm{d}y < \infty$ provided that $\int|f^{(k)}(y + \gamma(y,\zeta\beta))|\,\mathrm{d}y < \infty$ for $k = 1, 2$. To verify the latter fact, note that, by changing variables from y to $w := \tilde{\gamma}(y,\zeta\beta) = y + \gamma(y,\zeta\beta)$,

$$\int\big|f^{(k)}\big(y+\gamma(y,\zeta\beta)\big)\big|\,\mathrm{d}y = \int\big|f^{(k)}(w)\big|\frac{1}{|1 + (\partial_y\gamma)(\bar{\gamma}(w,\beta\zeta),\zeta\beta)|}\,\mathrm{d}w < \infty,$$

due to (2.4).

Once we have show that $\mathcal{I}_\varepsilon f(y)$ is integrable, we now prove the equality in equation (C.9). Let us first note that

$$\begin{aligned}&\int g(y)\mathcal{I}_\varepsilon f(y)\,\mathrm{d}y\\&\quad=\lim_{\delta\to0}\int g(y)\int_{|\zeta|\geq\delta}\bigl(f\bigl(y+\gamma(y,\zeta)\bigr)-f(y)-f'(y)\gamma(y,\zeta)\bigr)\bar{h}_\varepsilon(\zeta)\,\mathrm{d}\zeta\,\mathrm{d}y.\end{aligned}\tag{C.10}$$

For each $\delta>0$, fix

$$A_\delta=\int g(y)\int_{|\zeta|\geq\delta}\bigl(f\bigl(y+\gamma(y,\zeta)\bigr)-f(y)\bigr)\bar{h}_\varepsilon(\zeta)\,\mathrm{d}\zeta\,\mathrm{d}y,$$

and note that

$$A_\delta=\int\int_{|\zeta|\geq\delta}\int_0^1 g(y)f'\bigl(y+\gamma(y,\zeta\beta)\bigr)(\partial_\zeta\gamma)(y,\zeta\beta)\,\mathrm{d}\beta\bar{h}_\varepsilon(\zeta)\zeta\,\mathrm{d}\zeta\,\mathrm{d}y.$$

Changing variable from y to $w:=\tilde{\gamma}(y,\zeta\beta)=y+\gamma(y,\zeta\beta)$ and applying Fubini, we get

$$A_\delta=\int f'(w)\int_{|\zeta|\geq\delta}\int_0^1 g\bigl(\bar{\gamma}(w,\zeta\beta)\bigr)\frac{(\partial_\zeta\gamma)(\bar{\gamma}(w,\beta\zeta),\zeta\beta)}{1+(\partial_y\gamma)(\bar{\gamma}(w,\beta\zeta),\zeta\beta)}\,\mathrm{d}\beta\zeta\bar{h}_\varepsilon(\zeta)\,\mathrm{d}\zeta\,\mathrm{d}w.$$

From the identity

$$\partial_\zeta\int_{\bar{\gamma}(w,\zeta)}^{w}g(\eta)\,\mathrm{d}\eta=-g\bigl(\bar{\gamma}(w,\zeta)\bigr)\partial_\zeta\bar{\gamma}(w,\zeta)=g\bigl(\bar{\gamma}(w,\zeta)\bigr)\frac{(\partial_\zeta\gamma)(\bar{\gamma}(w,\zeta),\zeta)}{1+(\partial_y\gamma)(\bar{\gamma}(w,\zeta),\zeta)},$$

we can then write

$$A_\delta=\int f'(w)\int_{|\zeta|\geq\delta}\int_{\bar{\gamma}(w,\zeta)}^{w}g(\eta)\,\mathrm{d}\eta\bar{h}_\varepsilon(\zeta)\,\mathrm{d}\zeta\,\mathrm{d}w.$$

Plugging the previous formula in (C.10), we get

$$\int g(y)\mathcal{I}_\varepsilon f(y)\,\mathrm{d}y=\lim_{\delta\to0}\int f'(y)\int_{|\zeta|\geq\delta}\biggl(\int_{\bar{\gamma}(y,\zeta)}^{y}g(\eta)\,\mathrm{d}\eta-\gamma(y,\zeta)g(y)\biggr)\bar{h}_\varepsilon(\zeta)\,\mathrm{d}\zeta\,\mathrm{d}y.$$

Let

$$B_\delta(y):=\int_{|\zeta|\geq\delta}C(y,\zeta)\bar{h}_\varepsilon(\zeta)\,\mathrm{d}\zeta\qquad\text{with }C(y,\zeta):=\int_{\bar{\gamma}(y,\zeta)}^{y}g(\eta)\,\mathrm{d}\eta-\gamma(y,\zeta)g(y),$$

and note that, for $g\in C_b^1$,

$$\partial_\zeta^2C(y,\zeta)=-g'\bigl(\bar{\gamma}(y,\zeta)\bigr)\bigl(\partial_\zeta\bar{\gamma}(y,\zeta)\bigr)^2-g\bigl(\bar{\gamma}(y,\zeta)\bigr)\partial_\zeta^2\bar{\gamma}(y,\zeta)-g(y)\partial_\zeta^2\gamma(y,\zeta),\tag{C.11}$$

is bounded in light of Lemma 2.1(4). Then, writing

$$\int f'(y) B_\delta(y)\,\mathrm{d}y = \int f'(y) \int_{|\zeta|\geq\delta} \int_0^1 \partial_\zeta^2 C(y,\zeta\beta)(1-\beta)\,\mathrm{d}\beta\zeta^2\bar{h}_\varepsilon(\zeta)\,\mathrm{d}\zeta\,\mathrm{d}y,$$

it is clear that, when f' is integrable,

$$\begin{aligned}\lim_{\delta\to 0}\int f'(y) B_\delta(y)\,\mathrm{d}y &= \int f'(y) \lim_{\delta\to 0} B_\delta(y)\,\mathrm{d}y \\ &= \int f'(y) \int \left(\int_{\bar{\gamma}(y,\zeta)}^{y} g(\eta)\,\mathrm{d}\eta - \gamma(y,\zeta) g(y) \right) \bar{h}_\varepsilon(\zeta)\,\mathrm{d}\zeta\,\mathrm{d}y,\end{aligned}$$

which implies (C.9). □

Proof of Lemma B.1. By conditioning on the times of the jumps, which are necessarily distributed as the order statistics of n independent uniform $[0,t]$ random variables, we have

$$\mathbb{P}\big(X_t(x) \geq x+y \,|\, N_t^\varepsilon = n\big) = \frac{n!}{t^n} \int_\Delta \mathbb{P}\big(X_t\big(\varepsilon, \{s_1,\ldots,s_n\}, x\big) \geq x+y\big)\,\mathrm{d}s_n \cdots \mathrm{d}s_1,$$

where $\Delta := \{(s_1,\ldots,s_n)\colon\ 0 < s_1 < s_2 < \cdots < s_n < t\}$. Hence, conditioning on $\mathcal{F}_{s_n^-}$,

$$\begin{aligned}\mathbb{P}\big(X_t\big(\varepsilon,\{s_1,\ldots,s_n\},x\big) \geq x+y\big) &= \mathbb{E}\big[\mathbb{P}\big(X_t\big(\varepsilon,\{s_1,\ldots,s_n\},x\big) \geq x+y \,|\, \mathcal{F}_{s_n^-}\big)\big] \\ &= \mathbb{E}\big[G_{t-s_n}\big(X_{s_n}\big(\varepsilon,\{s_1,\ldots,s_{n-1}\},x\big); x, y\big)\big],\end{aligned}$$

where $G_t(z;x,y) = \mathbb{P}(X_t(\varepsilon,\varnothing,z+\gamma(z,J)) \geq x+y)$. In terms of the densities $p_t(\cdot;\varepsilon,\varnothing,\zeta)$ and $\widetilde{\Gamma}(\cdot;z)$ of $X_t(\varepsilon,\varnothing,\zeta)$ and $z+\gamma(z,J)$, respectively, we have that

$$\begin{aligned}G_t(z;x,y) &= \int \int_{x+y}^{\infty} p_t(\eta;\varepsilon,\varnothing,\zeta)\,\mathrm{d}\eta\widetilde{\Gamma}(\zeta;z)\,\mathrm{d}\zeta \\ &= \int_{x+y}^{\infty} \int p_t(\eta;\varepsilon,\varnothing,\zeta)\widetilde{\Gamma}(\zeta;z)\,\mathrm{d}\zeta\,\mathrm{d}\eta.\end{aligned}$$

From Lemma A.2, we know that there exists ε small enough such that, for any $\delta > 0$, there exists $B := B(\varepsilon,\delta) < \infty$ and $t_0 := t_0(\varepsilon,\delta) > 0$ for which

$$\sup_{z\in\mathbb{R}} \sup_{\eta\in[x+y-\delta,x+y+\delta]} \int p_t(\eta;\varepsilon,\varnothing,\zeta)\widetilde{\Gamma}(\zeta;z)\,\mathrm{d}\zeta \leq B \tag{C.12}$$

for all $0 < t < t_0$. The uniform bound (C.12) allows us to interchange the differentiation and the other relevant operations (integration, expectation, etc.) so that

$$\mathcal{G}_t^{(n)}(x,y) := \partial_y \mathbb{P}\big(X_t(x) \geq x+y \,|\, N_t^\varepsilon = n\big)$$

can be written as

$$
\begin{aligned}
\mathcal{G}_t^{(n)}(x,y) &= \frac{n!}{t^n}\int_\Delta \partial_y \mathbb{P}\big(X_t\big(\varepsilon,\{s_1,\dots,s_n\},x\big)\geq x+y\big)\,\mathrm{d}s_n\cdots\mathrm{d}s_1 \\
&= \frac{n!}{t^n}\int_\Delta \mathbb{E}\big[\partial_y G_{t-s_n}\big(X_{s_n}\big(\varepsilon,\{s_1,\dots,s_{n-1}\},x\big);x,y\big)\big]\,\mathrm{d}s_n\cdots\mathrm{d}s_1 \\
&= \frac{n!}{t^n}\int_\Delta \mathbb{E}\bigg[\int p_{t-s_n}(x+y;\varepsilon,\varnothing,\zeta)\widetilde{\Gamma}\big(\zeta;X_{s_n}\big(\varepsilon,\{s_1,\dots,s_{n-1}\},x\big)\big)\,\mathrm{d}\zeta\bigg]\,\mathrm{d}s_n\cdots\mathrm{d}s_1
\end{aligned}
$$

and also, for any $0<t<t_0$,

$$
\big|\partial_y \mathbb{P}\big(X_t(x)\geq x+y|N_t^\varepsilon=n\big)\big|\leq B.
$$

Using this bound,

$$
\begin{aligned}
\big|\partial_y \bar{\mathcal{R}}_t(x,y)\big| &\leq \mathrm{e}^{-\lambda_\varepsilon t}\sum_{n=3}^\infty \big|\partial_y \mathbb{P}\big(X_t(x)\geq x+y|N_t^\varepsilon=n\big)\big|\frac{(\lambda_\varepsilon t)^n}{n!} \\
&\leq B\mathrm{e}^{-\lambda_\varepsilon t}\sum_{n=3}^\infty \frac{(\lambda_\varepsilon t)^n}{n!}\leq B\lambda_\varepsilon^3 t^3.
\end{aligned}
$$

The proof is then complete. □

Proof of Lemma 6.1. By conditioning on the times of the jumps, which are necessarily distributed as the order statistics of n independent uniform $[0,t]$ random variables, we have

$$
\mathbb{P}\big(|X_t-x|\geq \log y|N_t^\varepsilon=n\big)=\frac{n!}{t^n}\int_\Delta \mathbb{P}\big(\big|X_t\big(\varepsilon,\{s_1,\dots,s_n\},x\big)-x\big|\geq \log y\big)\,\mathrm{d}s_n\cdots\mathrm{d}s_1,
$$

where $\Delta:=\{(s_1,\dots,s_n)\colon\ 0<s_1<s_2<\cdots<s_n<t\}$. Hence, we only need to bound

$$
\sup_{n\in\mathbb{N},t\in[0,1]}\frac{1}{n!}\int_0^\infty \mathbb{P}\big(\big|X_t\big(\varepsilon,\{s_1,\dots,s_n\},x\big)-x\big|\geq \log y\big)\,\mathrm{d}y
$$

uniformly. By conditioning again,

$$
\begin{aligned}
&\mathbb{P}\big(\big|X_t\big(\varepsilon,\{s_1,\dots,s_n\},x\big)-x\big|\geq \log y\big) \\
&\quad=\mathbb{E}\big[\mathbb{P}\big(\big|X_t\big(\varepsilon,\{s_1,\dots,s_n\},x\big)-x\big|\geq \log y|\mathcal{F}_{s_n^-}\big)\big] \\
&\quad\leq \mathbb{E}\big[\mathbb{P}\big[\big|X_{t-s_n}(\varepsilon,\varnothing,z)-x\big|+\big|\gamma(z,J)\big|\geq \log y\big]|_{z=X_{s_n}(\varepsilon,\{s_1,\dots,s_{n-1}\},x)}\big].
\end{aligned}
$$

Recall the condition (C5), we have for some constant $M>0$ and all $\lambda\leq 3$

$$
\sup_x \mathbb{E}\mathrm{e}^{\lambda|\gamma(x,J)|}\sup_x\leq C\int \mathrm{e}^{|3\gamma(x,z)|}h(z)\,\mathrm{d}z\leq M<\infty.
$$

Now fix any positive constant A and $t \le 1$, we have

$$\begin{aligned}\mathbb{E}e^{|X_t(\varepsilon,\{s_1,\dots,s_n\},x)-x|} &= \int_0^A \mathbb{P}\{|X_t(\varepsilon,\{s_1,\dots,s_n\},x)-x| > \log y\}\,\mathrm{d}y \\ &\quad + \int_A^\infty \mathbb{P}\{|X_t(\varepsilon,\{s_1,\dots,s_n\},x)-x| > \log y\}\,\mathrm{d}y \\ &\le A + 2Me^{(1/2)\lambda_1^2 k(1+\exp(\lambda_1\varepsilon))}\frac{1}{A^\alpha}\frac{1}{\alpha}\big(\mathbb{E}e^{\lambda_1|X_{s_n}(\varepsilon,\{s_1,\dots,s_{n-1}\},x)-x|}\big).\end{aligned}$$

Above, we used (3.2) for the last inequality with $\lambda = \lambda_1 = 1+\alpha$, where $0<\alpha<2$ is to be chosen later. Now we iterate the above procedure by taking $\lambda_i = (1+\alpha)^i$, $i = 1,2,\dots,n$, at each step, and choose $\lambda_n = (1+\alpha)^n = e$. We conclude that there exists a large enough constant C independent of n and t such that

$$\int_0^\infty \mathbb{P}\{|X_t(\varepsilon,\{s_1,\dots,s_n\},x)-x| > \log y\}\,\mathrm{d}y \le C^n\left(\frac{1}{\alpha}\right)^n.$$

In what follows, we only need to show $C^n(1/\alpha)^n/n! \to 0$ as $n\to\infty$. Recall that $\alpha = \mathrm{e}^{1/n}-1$. We have

$$\log\left[C^n\left(\frac{1}{\alpha}\right)^n\right] \sim n\left(C+\log\frac{1}{n}\right) \qquad \text{as } n\to\infty.$$

On the other hand, we know $\log n! \sim n^2/2$ as $n\to\infty$. The proof is then complete. □

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