Bernoulli **20**(2), 2014, 990–1005
DOI: 10.3150/13-BEJ513

Conditions for convergence of random coefficient AR(1) processes and perpetuities in higher dimensions

TORKEL ERHARDSSON

Department of Mathematics, Linköping University, S-581 83 Linköping, Sweden.
E-mail: torkel.erhardsson@liu.se

A d-dimensional RCA(1) process is a generalization of the d-dimensional AR(1) process, such that the coefficients $\{M_t; t = 1, 2, \ldots\}$ are i.i.d. random matrices. In the case $d = 1$, under a nondegeneracy condition, Goldie and Maller gave necessary and sufficient conditions for the convergence in distribution of an RCA(1) process, and for the almost sure convergence of a closely related sum of random variables called a *perpetuity*. We here prove that under the condition $\|\prod_{t=1}^n M_t\| \xrightarrow{\text{a.s.}} 0$ as $n \to \infty$, most of the results of Goldie and Maller can be extended to the case $d > 1$. If this condition does not hold, some of their results cannot be extended.

Keywords: AR(1) process; convergence; higher dimensions; matrix norm; matrix product; perpetuity; random coefficient; random difference equation; random matrix; RCA(1) process

1. Introduction

In this paper, we consider a discrete time stochastic process called the *d-dimensional* RCA(1) *process*, or *random coefficient autoregressive process of order 1*, which is a generalization of the d-dimensional AR(1) process. We also consider a closely related infinite sum of d-dimensional random variables, called a *perpetuity*. Since the appearance of [15], different aspects of the RCA(1) process and the perpetuity have been studied by many authors; see, for example, [1,3,4,7,8,12–14,21] and the references therein. In the present work, we will focus on conditions for convergence in distribution of the RCA(1) process, and for almost sure convergence of the perpetuity.

For each positive integer p, the d-dimensional RCA(p) process is defined as follows. Let $\{(M_{t,1}, \ldots, M_{t,p}); t = 1, 2, \ldots\}$ be an i.i.d. sequence of p-tuples of random matrices of dimension $d \times d$ (the *coefficients*); let $\{Z_t; t = 1, 2, \ldots\}$ be i.i.d. d-dimensional random variables independent of the random matrices (the *error* variables); and let Z_0 be a d-dimensional random variable independent of everything else (the *initial state*). Define the d-dimensional RCA(p) process $\{X_t; t = 1, 2, \ldots\}$ by

$$X_0 = Z_0; \qquad X_t = \sum_{i=1}^{p\wedge t} M_{t,i} X_{t-i} + Z_t \qquad \forall t = 1, 2, \ldots.$$

If the distribution of $(M_{1,1}, \dots, M_{1,p})$ is degenerate at a constant matrix p-tuple, the usual d-dimensional AR(p) process is obtained. However, for the AR(p) process it is often assumed that the error variables have finite second moments. Here, we make no such assumption.

The AR(p) process was originally proposed as a statistical model for time series, and it is today one of the most widely used such models. The RCA(p) process was first considered as a statistical model in [2]. A much studied problem is under what conditions on the coefficients there exists an RCA(p) or AR(p) process which is (wide sense) stationary. For some answers to this problem, and more information on these processes, see [2,3,5,6,20], and the references therein.

The case $p = 1$ has received special attention, since the RCA(1) process is easily seen to be a *Markov chain* on the state space $(\mathbb{R}^d, \mathscr{R}^d)$. For such a process, it is natural to ask under what conditions on the error variables and the random coefficient the process is (Harris) recurrent, positive, or convergent in distribution. For some partial answers to these questions, see [19] and the references therein. See also [10] for a connection between RCA(1) processes and Dirichlet processes; this connection was exploited in [9] to construct a new method to carry out Bayesian inference for an unknown finite measure, when a number of integrals with respect to this measure has been observed.

The *perpetuity* associated with a d-dimensional RCA(1) process is defined as the almost sure limit (if the limit exists) of the d-dimensional random sequence $\{V_t; t = 1, 2, \dots\}$, defined by:

$$V_t = \sum_{i=1}^{t} \prod_{j=1}^{i-1} M_j Z_i \qquad \forall t = 1, 2, \dots.$$

The existence of the perpetuity is closely related to the convergence in distribution of the d-dimensional RCA(1) process. In particular, it is shown in Section 2 that if $\|\prod_{t=1}^{n} M_t\| \xrightarrow{\text{a.s.}} 0$ as $n \to \infty$ (a condition to be called C0 below), then the two convergence statements are equivalent. Moreover, in the case $d = 1$, if $\mathbb{P}(Z_1 = 0) < 1$, it was shown in [12] that the existence of the perpetuity implies C0.

The main result in [12], their Theorem 2.1, is a *complete solution* in the case $d = 1$ to the problem: under what conditions on the error variables and the random coefficients does the perpetuity exist? Five different conditions on the random variables are given, which, if $\mathbb{P}(Z_1 = 0) < 1$, are shown to be equivalent, and to imply both the existence of the perpetuity, and C0. Furthermore, it is shown that under a certain "nondegeneracy" condition, the five conditions are necessary for the convergence in distribution of the associated RCA(1) process.

The main result of the present paper, Theorem 2.1, is a generalization of most of Theorem 2.1 in [12] to the case $d > 1$. All except one of the conditions in the latter theorem are considered. (It is unclear how the remaining condition, which involves the finiteness of a particular integral, should be generalized to the case $d > 1$, if indeed this is possible at all.) It is shown that if C0 is assumed, the remaining conditions of Theorem 2.1 are equivalent, and imply the existence of the perpetuity. However, contrary to the case $d = 1$, the conditions do not imply C0, and if C0 is not assumed, they are not all equivalent. Similarly, under C0, the existence of the perpetuity is equivalent to the convergence in distribution of the associated d-dimensional RCA(1) process; not so without C0.

The remaining part of the paper is structured as follows: in Section 2, the main result is stated and proven; in Section 3, some counterexamples and special cases are collected; and Section 4 contains some suggestions for future research.

2. Main result and proof

Let d be a positive integer. Denote by $|\cdot|$ the Euclidean norm on the space $\mathbb{R}^d$. Let $\mathbb{R}^{d\times d}$ be the space of $d \times d$-matrices with elements in $\mathbb{R}$, and denote by $\|\cdot\|$ the matrix norm induced by $|\cdot|$, that is, $\|A\| = \max_{|x|=1} |Ax|$. (This is known as the *spectral norm*, and is equal to the largest singular value of A.) Denote by I_d the identity $d \times d$-matrix. The following notation will be used for matrix products:

$$\prod_{j=m}^{n} M_j = \begin{cases} M_m M_{m+1} \cdots M_n, & \text{if } m \le n; \\ I_d, & \text{if } m > n. \end{cases}$$

In particular, $\prod_{j=m}^{n-1} M_{n-j} = M_{n-m} M_{n-m-1} \cdots M_1$ for each $m < n$, and $\prod_{j=m}^{n-1} M_{n-j} = I_d$ for each $m \ge n$. Lastly, by convention a minimum over an empty set is defined as ∞.

Theorem 2.1. *Let* $\{(M_t, Z_t); t = 1, 2, \ldots\}$ *be i.i.d. random elements in* $(\mathbb{R}^{d\times d} \times \mathbb{R}^d, \mathscr{R}^{d\times d} \times \mathscr{R}^d)$, *and let* Z_0 *be a random element in* $(\mathbb{R}^d, \mathscr{R}^d)$ *independent of* $\{(M_t, Z_t); t = 1, 2, \ldots\}$. *Define the random sequence* $\{X_t; t = 1, 2, \ldots\}$ *by*

$$X_0 = Z_0; \qquad X_t = M_t X_{t-1} + Z_t \qquad \forall t = 1, 2, \ldots.$$

Under the condition C0: $\|\prod_{t=1}^{n} M_t\| \xrightarrow{\text{a.s.}} 0$ *as* $n \to \infty$, *the following are equivalent*:

(i) X_t *converges in distribution* *as* $t \to \infty$;

(ii) $$\sum_{t=1}^{\infty} \left| \prod_{j=1}^{t-1} M_j Z_t \right| < \infty \qquad a.s.;$$

(iii) $$\sum_{i=1}^{t} \prod_{j=1}^{i-1} M_j Z_i \text{ converges a.s.} \qquad \text{as } t \to \infty;$$

(iv) $$\prod_{j=1}^{t-1} M_j Z_t \xrightarrow{a.s.} 0 \qquad \text{as } t \to \infty;$$

(v) $$\sup_{t=1,2,\ldots} \left| \prod_{j=1}^{t-1} M_j Z_t \right| < \infty \qquad a.s.;$$

(vi) $$\sum_{t=1}^{\infty} \mathbb{P}\left(\min_{k=1,\ldots,t-1} \left| \prod_{j=k}^{t-1} M_j Z_t \right| > x \right) < \infty \qquad \forall x > 0.$$

Remark 2.1. Clearly, the implications (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v) remain valid even if C0 does not hold, and, as will be seen from the proof, so does the implication (iv) $\Rightarrow$ (vi). It will be shown in Example 3.4 that the implication (v) $\Rightarrow$ (vi) need not hold if C0 does not hold. On the other hand, in the case $d = 1$, it was shown in [12] that if $\mathbb{P}(Z_1 = 0) < 1$, then (vi) implies C0, and if also $\mathbb{P}(|M_1| = 1) < 1$, then (v) implies C0; see Example 3.1 below. – The almost sure limit of the sum in (iii) is called a *perpetuity*. Hence, (iii) is the statement that the perpetuity exists.

Proof of Theorem 2.1. (iii) $\Rightarrow$ (i). As is easily shown by induction, we can write

$$X_t = \sum_{i=0}^{t-1} \prod_{j=0}^{i-1} M_{t-j} Z_{t-i} + \prod_{j=0}^{t-1} M_{t-j} Z_0 \qquad \forall t = 1, 2, \ldots.$$

Replacing (M_{t-i}, Z_{t-i}) by (M_{i+1}, Z_{i+1}) for $i = 0, 1, \ldots, t-1$, we get, since the random sequence $\{(M_t, Z_t); t = 1, 2, \ldots\}$ is i.i.d.,

$$X_t \stackrel{d}{=} \sum_{i=1}^{t} \prod_{j=1}^{i-1} M_j Z_i + \prod_{j=1}^{t} M_j Z_0 \qquad \forall t = 1, 2, \ldots. \tag{2.1}$$

C0 implies that $\prod_{t=1}^{n} M_t Z_0 \xrightarrow{\text{a.s.}} 0$ as $n \to \infty$. Hence, the desired conclusion follows from (2.1) and the Cramér–Slutsky theorem.

(i) $\Rightarrow$ (iii). C0 implies that $\prod_{t=1}^{n} M_t Z_0 \xrightarrow{\text{a.s.}} 0$ as $n \to \infty$, so by (2.1) and the Cramér–Slutsky theorem, $\sum_{i=1}^{t} \prod_{j=1}^{i-1} M_j Z_i$ converges in distribution as $t \to \infty$. We need to prove that it also converges a.s. We define, for brevity of notation,

$$S_{m,n} = \sum_{i=m+1}^{n} \prod_{j=1}^{i-1} M_j Z_i \qquad \forall 0 \le m \le n,$$

where $S_{n,n} = 0$ for each $n \ge 0$. The following facts will be important:

$$S_{m,n} = \sum_{i=m+1}^{n} \prod_{j=1}^{i-1} M_j Z_i = \prod_{j=1}^{m} M_j \sum_{i=m+1}^{n} \prod_{j=m+1}^{i-1} M_j Z_i \qquad \forall 0 \le m < n \tag{2.2}$$

and

$$\sum_{i=m+1}^{n} \prod_{j=m+1}^{i-1} M_j Z_i \stackrel{d}{=} \sum_{i=1}^{n-m} \prod_{j=1}^{i-1} M_j Z_i \qquad \forall 0 \le m < n. \tag{2.3}$$

Also, since $\sum_{i=1}^{t} \prod_{j=1}^{i-1} M_j Z_i$ converges in distribution as $t \to \infty$, the associated sequence of distributions is tight. Therefore, for each $\delta > 0$, there exists $K < \infty$ such that

$$\mathbb{P}\left(\left|\sum_{i=1}^{t} \prod_{j=1}^{i-1} M_j Z_i\right| > K\right) < \frac{\delta}{2} \qquad \forall t = 1, 2, \ldots. \tag{2.4}$$

For each $\varepsilon > 0$, each $\delta > 0$, and each $n > m$, we get, if K is chosen as in (2.4) and m is chosen large enough,

$$\begin{aligned}\mathbb{P}\big(|S_{m,n}| > \varepsilon\big) &\leq \mathbb{P}\left(\left\|\prod_{j=1}^{m} M_j\right\| > \frac{\varepsilon}{K}\right) + \mathbb{P}\left(\left|\sum_{i=m+1}^{n}\prod_{j=m+1}^{i-1} M_j Z_i\right| > K\right)\\ &= \mathbb{P}\left(\left\|\prod_{j=1}^{m} M_j\right\| > \frac{\varepsilon}{K}\right) + \mathbb{P}\left(\left|\sum_{i=1}^{n-m}\prod_{j=1}^{i-1} M_j Z_i\right| > K\right) \leq \frac{\delta}{2} + \frac{\delta}{2} = \delta.\end{aligned}$$

Here, we used (2.2) in the first inequality, (2.3) in the equality, and C0 in the second inequality. We conclude that

$$\sup_{n>m} \mathbb{P}\big(|S_{m,n}| > \varepsilon\big) \to 0 \text{ as } m \to \infty \qquad \forall \varepsilon > 0. \tag{2.5}$$

Our next goal is to show that, for each $\varepsilon > 0$ and $m \geq 0$, if K is chosen so that (2.4) is satisfied with $\delta = 2(1-c)$, where $0 < c < 1$, then:

$$c\mathbb{P}\Big(\sup_{n>m} |S_{m,n}| > 2\varepsilon\Big) \leq \sup_{n>m} \mathbb{P}\big(|S_{m,n}| > \varepsilon\big) + \mathbb{P}\left(\bigcup_{k=m+1}^{\infty}\left\{\left\|\prod_{j=1}^{k} M_j\right\| > \frac{\varepsilon}{K}\right\}\right). \tag{2.6}$$

To this end, we fix $\varepsilon > 0$ and $m \geq 0$, and note that with this particular choice of K, (2.3) implies:

$$\mathbb{P}\left(\left|\sum_{i=k+1}^{n}\prod_{j=k+1}^{i-1} M_j Z_i\right| \leq K\right) \geq c \qquad \forall 0 \leq k \leq n,$$

which in turn gives

$$\begin{aligned}&\sum_{k=m+1}^{n} \mathbb{P}\left(\bigcap_{j=m+1}^{k-1}\big\{|S_{m,j}| \leq 2\varepsilon\big\} \cap \big\{|S_{m,k}| > 2\varepsilon\big\}\right)\mathbb{P}\left(\left|\sum_{i=k+1}^{n}\prod_{j=k+1}^{i-1} M_j Z_i\right| \leq K\right)\\ &\geq c\mathbb{P}\Big(\max_{m<k\leq n} |S_{m,k}| > 2\varepsilon\Big) \qquad \forall n \geq m.\end{aligned} \tag{2.7}$$

In order to obtain an upper bound for the left-hand side of (2.7), we note that, by the triangle inequality, $|S_{m,k}| - |S_{k,n}| \leq |S_{m,n}|$ for each $m \leq k \leq n$. This implies:

$$\begin{aligned}&\sum_{k=m+1}^{n} \mathbb{P}\left(\bigcap_{j=m+1}^{k-1}\big\{|S_{m,j}| \leq 2\varepsilon\big\} \cap \big\{|S_{m,k}| > 2\varepsilon\big\} \cap \big\{|S_{k,n}| \leq \varepsilon\big\}\right)\\ &= \mathbb{P}\left(\bigcup_{k=m+1}^{n}\left(\bigcap_{j=m+1}^{k-1}\big\{|S_{m,j}| \leq 2\varepsilon\big\} \cap \big\{|S_{m,k}| > 2\varepsilon\big\} \cap \big\{|S_{k,n}| \leq \varepsilon\big\}\right)\right)\end{aligned}$$

$$\le \mathbb{P}\Bigg(\bigcup_{k=m+1}^{n} \{|S_{m,k}| > 2\varepsilon\} \cap \{|S_{k,n}| \le \varepsilon\}\Bigg)$$

$$\le \mathbb{P}\big(|S_{m,n}| > \varepsilon\big) \qquad \forall n \ge m.$$

Moreover, by (2.2),

$$\left\{\left\|\prod_{j=1}^{k} M_j\right\| \le \frac{\varepsilon}{K}\right\} \cap \left\{\left|\sum_{i=k+1}^{n} \prod_{j=k+1}^{i-1} M_j Z_i\right| \le K\right\} \subset \{|S_{k,n}| \le \varepsilon\} \qquad \forall m \le k \le n.$$

Combining the last two results with the fact that the random sequence $\{(M_t, Z_t); t = 1, 2, \ldots\}$ is i.i.d., we get the desired upper bound:

$$\begin{aligned}
&\sum_{k=m+1}^{n} \mathbb{P}\Bigg(\bigcap_{j=m+1}^{k-1} \{|S_{m,j}| \le 2\varepsilon\} \cap \{|S_{m,k}| > 2\varepsilon\}\Bigg) \mathbb{P}\Bigg(\left|\sum_{i=k+1}^{n} \prod_{j=k+1}^{i-1} M_j Z_i\right| \le K\Bigg) \\
&\quad = \sum_{k=m+1}^{n} \mathbb{P}\Bigg(\bigcap_{j=m+1}^{k-1} \{|S_{m,j}| \le 2\varepsilon\} \cap \{|S_{m,k}| > 2\varepsilon\} \\
&\qquad\qquad \cap \left\{\left|\sum_{i=k+1}^{n} \prod_{j=k+1}^{i-1} M_j Z_i\right| \le K\right\}\Bigg) \\
&\quad = \sum_{k=m+1}^{n} \mathbb{P}\Bigg(\bigcap_{j=m+1}^{k-1} \{|S_{m,j}| \le 2\varepsilon\} \cap \{|S_{m,k}| > 2\varepsilon\} \cap \left\{\left\|\prod_{j=1}^{k} M_j\right\| \le \frac{\varepsilon}{K}\right\} \\
&\qquad\qquad \cap \left\{\left|\sum_{i=k+1}^{n} \prod_{j=k+1}^{i-1} M_j Z_i\right| \le K\right\}\Bigg) \\
&\qquad + \sum_{k=m+1}^{n} \mathbb{P}\Bigg(\bigcap_{j=m+1}^{k-1} \{|S_{m,j}| \le 2\varepsilon\} \cap \{|S_{m,k}| > 2\varepsilon\} \cap \left\{\left\|\prod_{j=1}^{k} M_j\right\| > \frac{\varepsilon}{K}\right\} \\
&\qquad\qquad \cap \left\{\left|\sum_{i=k+1}^{n} \prod_{j=k+1}^{i-1} M_j Z_i\right| \le K\right\}\Bigg) \\
&\quad \le \mathbb{P}\big(|S_{m,n}| > \varepsilon\big) + \mathbb{P}\Bigg(\bigcup_{k=m+1}^{n} \left\{\left\|\prod_{j=1}^{k} M_j\right\| > \frac{\varepsilon}{K}\right\}\Bigg) \qquad \forall n \ge m.
\end{aligned}$$

Letting $n \to \infty$ (and remembering that $m \ge 0$ is fixed), the last result and (2.7) together imply (2.6).

Finally, by (2.6) and the triangle inequality,

$$\begin{aligned}\mathbb{P}\Big(\sup_{\substack{m<k,\ell\\k<\ell}}|S_{k,\ell}|>4\varepsilon\Big)&\le\mathbb{P}\Big(\sup_{n>m}|S_{m,n}|>2\varepsilon\Big)\\&\le\frac{1}{c}\sup_{n>m}\mathbb{P}\big(|S_{m,n}|>\varepsilon\big)\\&\quad+\frac{1}{c}\mathbb{P}\Bigg(\bigcup_{k=m+1}^{\infty}\Bigg\{\Bigg\|\prod_{j=1}^{k}M_j\Bigg\|>\frac{\varepsilon}{K}\Bigg\}\Bigg)\qquad\forall\varepsilon>0,m\ge 0.\end{aligned}$$

By (2.5), the first term on the right-hand side converges to 0 as $m\to\infty$, while the second term converges to 0 as $m\to\infty$ by C0. Hence, $\sup_{\substack{m<k,\ell\\k<\ell}}|S_{k,\ell}|$ converges in probability to 0 as $m\to\infty$. However, by definition, $\sup_{\substack{m<k,\ell\\k<\ell}}|S_{k,\ell}|$ decreases monotonically a.s. to a nonnegative random variable as $m\to\infty$. To avoid a contradiction, this random variable must be 0 with probability 1. It follows that, with probability 1, $\{\sum_{i=1}^{t}\prod_{j=1}^{i-1}M_jZ_i;t=1,2,\dots\}$ is a Cauchy sequence, so $\lim_{t\to\infty}\sum_{i=1}^{t}\prod_{j=1}^{i-1}M_jZ_i$ exists a.s.

(ii) ⇒ (iii) ⇒ (iv) ⇒ (v). Immediate.

(iv) ⇒ (vi). As stated in Remark 2.1, C0 is not needed to prove this implication. Instead, we use the theorem in [17], also known as the *Kochen–Stone lemma*. By this theorem (or lemma), for any sequence of events $\{A_t;t=1,2,\dots\}$ such that $\sum_{t=1}^{\infty}\mathbb{P}(A_t)=\infty$ and

$$\limsup_{n\to\infty}\frac{(\sum_{t=1}^{n}\mathbb{P}(A_t))^2}{\sum_{r=1}^{n}\sum_{t=1}^{n}\mathbb{P}(A_r\cap A_t)}=c>0,\tag{2.8}$$

it holds that $\mathbb{P}(A_t\text{ i.o.})\ge c$. Define the random sequence $\{Y_t;t=1,2,\dots\}$ by:

$$Y_t=\min_{k=1,\dots,t-1}\Bigg|\prod_{j=k}^{t-1}M_jZ_t\Bigg|\qquad\forall t=1,2,\dots.$$

Recall that by definition $Y_1=\infty$ (since it is the minimum over an empty set). Let $x>0$, and define the events $\{A_t;t=1,2,\dots\}$ by: $A_t=\{Y_t>x\}$ $\forall t=1,2,\dots$. We note that if (vi) does not hold, then $\sum_{t=1}^{\infty}\mathbb{P}(A_t)=\infty$ for some $x>0$. We will show that in this case (2.8) holds with $c\ge\frac{1}{2}$, implying that $\mathbb{P}(|\prod_{j=1}^{t-1}M_jZ_t|>x\text{ i.o.})\ge\mathbb{P}(Y_t>x\text{ i.o.})\ge\frac{1}{2}>0$. Hence, (iv) does not hold.

For the probabilities in the denominator of (2.8), we get, if $1\le r<t$,

$$\begin{aligned}\mathbb{P}\big(\{Y_r>x\}\cap\{Y_t>x\}\big)&=\mathbb{P}\Bigg(\{Y_r>x\}\cap\Bigg\{\min_{k=1,\dots,t-1}\Bigg|\prod_{j=k}^{t-1}M_jZ_t\Bigg|>x\Bigg\}\Bigg)\\&\le\mathbb{P}(Y_r>x)\mathbb{P}\Bigg(\min_{k=r+1,\dots,t-1}\Bigg|\prod_{j=k}^{t-1}M_jZ_t\Bigg|>x\Bigg)\\&=\mathbb{P}(Y_r>x)\mathbb{P}(Y_{t-r}>x).\end{aligned}$$

This implies that

$$\sum_{r=1}^{n}\sum_{t=1}^{n}\mathbb{P}\big(\{Y_r > x\}\cap\{Y_t > x\}\big)$$
$$\leq \sum_{r=1}^{n}\mathbb{P}(Y_r > x) + 2\sum_{r=1}^{n-1}\mathbb{P}(Y_r > x)\sum_{t=r+1}^{n}\mathbb{P}(Y_{t-r} > x)$$
$$\leq \sum_{r=1}^{n}\mathbb{P}(Y_r > x) + 2\sum_{r=1}^{n}\mathbb{P}(Y_r > x)\sum_{s=1}^{n}\mathbb{P}(Y_s > x).$$

Hence, we obtain:

$$\limsup_{n\to\infty}\frac{(\sum_{t=1}^{n}\mathbb{P}(Y_t > x))^2}{\sum_{r=1}^{n}\sum_{t=1}^{n}\mathbb{P}(\{Y_r > x\}\cap\{Y_t > x\})}$$
$$\geq \lim_{n\to\infty}\frac{(\sum_{t=1}^{n}\mathbb{P}(Y_t > x))^2}{\sum_{r=1}^{n}\mathbb{P}(Y_r > x) + 2(\sum_{t=1}^{n}\mathbb{P}(Y_t > x))^2} = \frac{1}{2}.$$

(vi) $\Rightarrow$ (ii). This part of the proof is divided into several steps. First, we prove that if $\|\prod_{t=1}^{n} M_t\| \xrightarrow{\text{a.s.}} 0$ as $n\to\infty$, then

$$\sum_{t=1}^{\infty}\mathbb{P}\left(\min_{k=1,\dots,t-1}\left\|\prod_{j=k}^{t-1} M_j\right\| > x\right) < \infty \qquad \forall x > 0. \tag{2.9}$$

We use the Kochen–Stone lemma, as in the preceding part of the proof. Let

$$U_t = \min_{k=1,\dots,t-1}\left\|\prod_{j=k}^{t-1} M_j\right\| \qquad \forall t = 1, 2, \dots.$$

Let $x > 0$, and define the events $\{A_t; t = 1, 2, \dots\}$ by: $A_t = \{U_t > x\}$ $\forall t = 1, 2, \dots$. Assume that $\sum_{t=1}^{\infty}\mathbb{P}(A_t) = \infty$. As before, for the probabilities in the denominator of (2.8), we get:

$$\mathbb{P}\big(\{U_r > x\}\cap\{U_t > x\}\big) \leq \mathbb{P}(U_r > x)\mathbb{P}(U_{t-r} > x) \qquad \forall 1 \leq r < t,$$

implying that

$$\limsup_{n\to\infty}\frac{(\sum_{t=1}^{n}\mathbb{P}(U_t > x))^2}{\sum_{r=1}^{n}\sum_{t=1}^{n}\mathbb{P}(\{U_r > x\}\cap\{U_s > x\})} \geq \frac{1}{2},$$

so $\mathbb{P}(U_t > x \text{ i.o.}) \geq \frac{1}{2}$. Hence, it cannot hold that $\|\prod_{t=1}^{n} M_t\| \xrightarrow{\text{a.s.}} 0$ as $n\to\infty$.

Next, let as before $Y_t = \min_{k=1,\dots,t-1} |\prod_{j=k}^{t-1} M_j Z_t| \ \forall t = 1, 2, \dots$. Since

$$\left|\prod_{j=1}^{t-1} M_j Z_t\right| \le \left\|\prod_{j=1}^{k-1} M_j\right\| \left|\prod_{j=k}^{t-1} M_j Z_t\right| \qquad \forall t = 1, 2, \dots; k = 1, \dots, t-1,$$

it holds that

$$\left|\prod_{j=1}^{t-1} M_j Z_t\right| \le \sup_{n\ge 0} \left\|\prod_{i=1}^{n} M_i\right\| \min_{k=1,\dots,t-1} \left|\prod_{j=k}^{t-1} M_j Z_t\right| \qquad \forall t = 1, 2, \dots,$$

where, since $\|\prod_{t=1}^{n} M_t\| \xrightarrow{\text{a.s.}} 0$ as $n \to \infty$, $\sup_{n\ge 0} \|\prod_{i=1}^{n} M_i\| < \infty$ a.s. This implies that in order to prove (ii), it is sufficient to prove that $\sum_{t=1}^{\infty} Y_t < \infty$ a.s.

Furthermore, by Fubini's theorem,

$$\begin{aligned}
\mathbb{E}\big(Y_t I\{Y_t \le 1\}\big) &= \int_{(0,1]} y \,\mathrm{d}F_{Y_t}(y) \\
&= \int_0^1 \mathbb{P}(x < Y_t \le 1)\,\mathrm{d}x \qquad (2.10) \\
&\le \int_0^1 \mathbb{P}(Y_t > x)\,\mathrm{d}x \qquad \forall t = 1, 2, \dots,
\end{aligned}$$

implying that

$$\sum_{t=1}^{\infty} \mathbb{E}\big(Y_t I\{Y_t \le 1\}\big) \le \int_0^1 \sum_{t=1}^{\infty} \mathbb{P}(Y_t > x)\,\mathrm{d}x. \qquad (2.11)$$

We note that, by (vi), $\sum_{t=1}^{\infty} \mathbb{P}(Y_t > x) < \infty$ for each $x > 0$. We will prove that the right-hand side of (2.11) is finite. By monotone convergence, this will imply that

$$\mathbb{E}\left(\sum_{t=1}^{\infty} Y_t I\{Y_t \le 1\}\right) = \sum_{t=1}^{\infty} \mathbb{E}\big(Y_t I\{Y_t \le 1\}\big) < \infty,$$

from which it will follow that $\sum_{t=1}^{\infty} Y_t I\{Y_t \le 1\} < \infty$ a.s. Since, by (vi) and the Borel–Cantelli lemma, $Y_t \xrightarrow{\text{a.s.}} 0$ as $t \to \infty$, we will be able to conclude that $\sum_{t=1}^{\infty} Y_t < \infty$ a.s.

Define $\{\widetilde{Y}_t; t = 1, 2, \dots\}$ by $\widetilde{Y}_t = \min_{k=1,\dots,t-1} |\prod_{j=k}^{t-1} M_{t-j} \widetilde{Z}_1| \ \forall t = 1, 2, \dots$, where $\widetilde{Z}_1$ is a random variable independent of $\{(M_t, Z_t); t = 1, 2, \dots\}$ such that $\widetilde{Z}_1 \stackrel{d}{=} Z_1$. By definition, $\{\widetilde{Y}_t; t = 1, 2, \dots\}$ is a nonincreasing random sequence, while clearly also $\widetilde{Y}_t \stackrel{d}{=} Y_t \ \forall t = 1, 2, \dots$ (in particular, $\widetilde{Y}_1 = Y_1 = \infty$, since they are both minima over empty sets). Define, for each $x > 0$, the random variable

$$T_x = \inf\{t = 1, 2, \dots; \widetilde{Y}_t \le x\} = \inf\left\{t = 1, 2, \dots; \left|\prod_{j=1}^{t-1} M_{t-j} \widetilde{Z}_1\right| \le x\right\}.$$

Clearly, T_x is a stopping time with respect to the filtration $\{\mathcal{G}_t; t = 1, 2, \ldots\}$, defined by: $\mathcal{G}_t = \sigma(\widetilde{Z}_1; M_1, \ldots, M_{t-1})$ $\forall t = 1, 2, \ldots$. Moreover,

$$\sum_{t=1}^{\infty} \mathbb{P}(Y_t > x) = \sum_{t=1}^{\infty} \mathbb{P}(\widetilde{Y}_t > x) = \sum_{t=1}^{\infty} \mathbb{P}(T_x > t) = \mathbb{E}(T_x) - 1, \tag{2.12}$$

so (vi) implies that $\mathbb{E}(T_x) < \infty$ for each $x > 0$. Define, for each $x > 0$, the random variables $T_x^{(1)} = T_1$ and

$$T_x^{(2)} = \inf\left\{ t = 1, 2, \ldots; \left\| \prod_{j=1}^{t} M_{T_1+t-j} \right\| \le x \right\}.$$

Since $\{M_t; t = 1, 2, \ldots\}$ are i.i.d. and independent of $\widetilde{Z}_1$, it holds that $\{M_s; s = t, t+1, \ldots\}$ are independent of $\mathcal{G}_t$ for each $t = 1, 2, \ldots$. Since T_1 is an a.s. finite stopping time with respect to $\{\mathcal{G}_t; t = 1, 2, \ldots\}$, we get:

$$\begin{aligned}
&\mathbb{P}\big(\big\{T_x^{(2)} > t\big\} \cap \{T_1 = r\}\big) \\
&\quad = \mathbb{P}\left(\left\{ \min_{k=1,\ldots,t} \left\| \prod_{j=k}^{t} M_{T_1+t-j} \right\| > x \right\} \cap \{T_1 = r\}\right) \\
&\quad = \mathbb{P}\left(\left\{ \min_{k=1,\ldots,t} \left\| \prod_{j=k}^{t} M_{r+t-j} \right\| > x \right\} \cap \{T_1 = r\}\right) \\
&\quad = \mathbb{P}\left(\min_{k=1,\ldots,t} \left\| \prod_{j=k}^{t} M_j \right\| > x \right) \mathbb{P}(T_1 = r) \qquad \forall t = 1, 2, \ldots; r = 1, 2, \ldots.
\end{aligned}$$

In particular,

$$\mathbb{P}\big(T_x^{(2)} > t\big) = \mathbb{P}\left(\min_{k=1,\ldots,t} \left\| \prod_{j=k}^{t} M_j \right\| > x \right) \qquad \forall t = 1, 2, \ldots$$

and

$$\mathbb{E}\big(T_x^{(2)}\big) - 1 = \sum_{t=1}^{\infty} \mathbb{P}\big(T_x^{(2)} > t\big) = \sum_{t=1}^{\infty} \mathbb{P}\left(\min_{k=1,\ldots,t} \left\| \prod_{j=k}^{t} M_j \right\| > x \right) < \infty \qquad \forall x > 0,$$

where finiteness follows from (2.9).

Repeating this process, we define recursively, for each $x > 0$, the random variables $\{T_x^{(k)}; k = 2, 3, \ldots\}$ by:

$$T_x^{(k)} = \inf\left\{ t = 1, 2, \ldots; \left\| \prod_{j=1}^{t} M_{S_x^{(k-1)}+t-j} \right\| \le x \right\} \qquad \forall k = 2, 3, \ldots,$$

where $S_x^{(k)} = \sum_{i=1}^k T_x^{(i)}$ $\forall k = 1, 2, \ldots$. Since $\{M_s; s = t, t+1, \ldots\}$ are independent of $\mathscr{G}_t$ for each $t = 1, 2, \ldots$, and since $\{S_x^{(k)}; k = 1, 2, \ldots\}$ are stopping times with respect to $\{\mathscr{G}_t; t = 1, 2, \ldots\}$, we see that $\{T_x^{(k)}; k = 2, 3, \ldots\}$ are i.i.d. with finite mean.

We now observe that by the submultiplicative property,

$$\left|\prod_{j=1}^{S_x^{(k+1)}-1} M_{S_x^{(k+1)}-j}\widetilde{Z}_1\right| \le \left|\prod_{j=1}^{T_1-1} M_{T_1-j}\widetilde{Z}_1\right| \prod_{i=2}^{k+1}\left\|\prod_{j=1}^{T_x^{(i)}} M_{S_x^{(i)}-j}\right\| \le x^k \qquad \forall k = 1, 2, \ldots; x > 0,$$

which implies that

$$T_x \le S_{x^{1/k}}^{(k+1)} = T_1 + T_{x^{1/k}}^{(2)} + \cdots + T_{x^{1/k}}^{(k+1)} \qquad \forall k = 1, 2, \ldots; x > 0.$$

Taking expectations on both sides in this inequality gives:

$$\mathbb{E}(T_x) \le \mathbb{E}(T_1) + k\mathbb{E}\big(T_{x^{1/k}}^{(2)}\big) \qquad \forall k = 1, 2, \ldots; x > 0.$$

Choosing $a \in (0, 1)$ and letting $k_x = \lceil \frac{\log x}{\log a} \rceil$ $\forall x \in (0, 1)$, we get:

$$x^{1/k_x} = \exp\left(\frac{\log x}{\lceil \log x / \log a \rceil}\right) \ge a \qquad \forall x \in (0, 1),$$

implying that

$$\mathbb{E}(T_x) \le \mathbb{E}(T_1) + k_x\mathbb{E}\big(T_a^{(2)}\big) \le \mathbb{E}(T_1) + \mathbb{E}\big(T_a^{(2)}\big)\left(\frac{\log x}{\log a} + 1\right) \qquad \forall x \in (0, 1).$$

This combined with (2.12) implies that the right-hand side of (2.11) is finite, since

$$\int_0^1 \log x \, \mathrm{d}x = \lim_{\epsilon \to 0} \int_\epsilon^1 \log x \, \mathrm{d}x = \lim_{\epsilon \to 0} [x \log x - x]_\epsilon^1 = -1.$$

(v) $\Rightarrow$ (iv). Since

$$\left|\prod_{j=1}^{t-1} M_j Z_t\right| \le \left\|\prod_{j=1}^{m-1} M_j\right\| \left|\prod_{j=m}^{t-1} M_j Z_t\right| \qquad \forall 1 \le m \le t,$$

it holds for each $\varepsilon > 0$ and $K > 0$ that

$$\mathbb{P}\left(\bigcup_{t=m}^{n}\left\{\left|\prod_{j=1}^{t-1} M_j Z_t\right| > \varepsilon\right\}\right) \le \mathbb{P}\left(\left\|\prod_{j=1}^{m-1} M_j\right\| > \frac{\varepsilon}{K}\right) + \mathbb{P}\left(\bigcup_{t=m}^{n}\left\{\left|\prod_{j=m}^{t-1} M_j Z_t\right| > K\right\}\right) \qquad \forall 1 \le m \le n.$$

For the second term on the right-hand side, since the random sequence $\{(M_t, Z_t); t = 1, 2, \ldots\}$ is i.i.d.,

$$\begin{aligned}\mathbb{P}\left(\bigcup_{t=m}^{n}\left\{\left|\prod_{j=m}^{t-1} M_j Z_t\right| > K\right\}\right) &= \mathbb{P}\left(\bigcup_{t=m}^{n}\left\{\left|\prod_{j=m}^{t-1} M_{j-m+1} Z_{t-m+1}\right| > K\right\}\right)\\ &= \mathbb{P}\left(\bigcup_{t=m}^{n}\left\{\left|\prod_{j=1}^{t-m} M_j Z_{t-m+1}\right| > K\right\}\right)\\ &= \mathbb{P}\left(\bigcup_{t=1}^{n-m+1}\left\{\left|\prod_{j=1}^{t-1} M_j Z_t\right| > K\right\}\right)\\ &\le \mathbb{P}\left(\sup_{t=1,2,\ldots}\left|\prod_{j=1}^{t-1} M_j Z_t\right| > K\right) \qquad \forall 1 \le m \le n.\end{aligned}$$

Fixing $m \ge 1$ and letting $n \to \infty$, we get:

$$\begin{aligned}\mathbb{P}\left(\bigcup_{t=m}^{\infty}\left\{\left|\prod_{j=1}^{t-1} M_j Z_t\right| > \varepsilon\right\}\right) &\le \mathbb{P}\left(\left\|\prod_{j=1}^{m-1} M_j\right\| > \frac{\varepsilon}{K}\right)\\ &\quad + \mathbb{P}\left(\sup_{t=1,2,\ldots}\left|\prod_{j=1}^{t-1} M_j Z_t\right| > K\right) \qquad \forall m \ge 1.\end{aligned}$$

For each $\delta > 0$, by (v), the second term on the right-hand side can be made less than $\frac{\delta}{2}$ by choosing K large enough. Similarly, using C0, the first term on the right-hand side can be made less than $\frac{\delta}{2}$ by choosing m large enough. This gives:

$$\mathbb{P}\left(\bigcup_{t=m}^{\infty}\left\{\left|\prod_{j=1}^{t-1} M_j Z_t\right| > \varepsilon\right\}\right) \le \frac{\delta}{2} + \frac{\delta}{2} = \delta,$$

which implies (iv). □

3. Counterexamples and special cases

In this section, we consider some counterexamples, some special cases, and a condition on the matrices $\{M_t; t = 1, 2, \ldots\}$ which is only sufficient for C0, but somewhat easier to validate. In Example 3.1, it is shown that in the case $d > 1$, (ii) in Theorem 2.1 does not imply C0. In Examples 3.2–3.4, it is shown that in the case $d > 1$, if C0 does not hold, not all of the conclusions of Theorem 2.1 hold. The special cases considered are the case $d = 1$ (completely solved in [12]), and the case when $M_t = M$ $\forall t = 1, 2, \ldots$, where M is a (deterministic) constant matrix.

Example 3.1. Consider first the case $d = 1$. This case was completely solved in [12], where it was shown that if $\mathbb{P}(Z_1 = 0) < 1$, then (vi) implies C0, and if also $\mathbb{P}(|M_1| = 1) < 1$, then (v) implies C0. Moreover, if $\mathbb{P}(Z_1 = 0) < 1$, then clearly (iv) implies that $\mathbb{P}(|M_1| = 1) < 1$. As a consequence, if $d = 1$ and $\mathbb{P}(Z_1 = 0) < 1$, then (ii), (iii), (iv), (v) *combined with* $\mathbb{P}(|M_1| = 1) < 1$, and (vi) are equivalent, and they all imply C0.

However, if $d > 1$, the following counterexample shows that even if $\mathbb{P}(Z_1 = 0) < 1$, (ii) does not imply C0. Let $d = 2$, and let v_1 and v_2 be orthonormal column vectors in $\mathbb{R}^2$. Let $0 < \alpha < 1$. Define $M_t = \alpha v_1 v_1^T + v_2 v_2^T$ $\forall t = 1, 2, \ldots$, and $Z_t = v_1$ $\forall t = 1, 2, \ldots$. Then, $\prod_{j=1}^{t-1} M_j Z_t = \alpha^{t-1} v_1$ $\forall t = 1, 2, \ldots$, so (ii) holds. On the other hand, $\|\prod_{j=1}^{t} M_j\| = 1$ $\forall t = 1, 2, \ldots$, which does not converge to 0 a.s. as $t \to \infty$.

Example 3.2. If $d > 1$ and C0 does not hold, then the implication (ii) $\Rightarrow$ (i) does not hold. To see this, let $d = 2$, and let v_1 and v_2 be orthonormal column vectors in $\mathbb{R}^2$. Let $0 < \alpha < 1 < \beta < \infty$. Define $M_t = \alpha v_1 v_1^T + \beta v_2 v_2^T$ $\forall t = 1, 2, \ldots$, and $Z_t = v_1$ $\forall t = 1, 2, \ldots$. Let $Z_0 = v_2$. Then, $\prod_{j=1}^{t-1} M_j Z_t = \alpha^{t-1} v_1$ $\forall t = 1, 2, \ldots$, so (ii) holds, and $\sum_{i=1}^{t} \prod_{j=1}^{i-1} M_j Z_t$ converges a.s. to $\frac{1}{1-\alpha} v_1$ (a deterministic vector) as $t \to \infty$. On the other hand, $\|\prod_{j=1}^{t} M_j\| = \beta^t$ $\forall t = 1, 2, \ldots$, which does not converge to 0 a.s. as $t \to \infty$. If (i) holds, then by (2.1), (ii) and the Cramér–Slutsky theorem, $\prod_{j=1}^{t} M_j Z_0$ must converge in distribution as $t \to \infty$. However, $\prod_{j=1}^{t} M_j Z_0 = \beta^t v_2$ $\forall t = 1, 2, \ldots$, which does not converge in distribution as $t \to \infty$ (the corresponding sequence of distributions is not tight). Hence, (i) does not hold.

Example 3.3. If C0 does not hold, then the implications (i) $\Rightarrow$ (v) and (i) $\Rightarrow$ (vi) do not hold. To see this, let $d = 1$, $|\beta| > 1$ and $c > 0$. Define $M_t = \beta$ $\forall t = 1, 2, \ldots$, $Z_t = (1 - \beta)c$ $\forall t = 1, 2, \ldots$, and $Z_0 = c$. (This is an example where the "nondegeneracy" condition (2.7) in [12] does not hold.) Then

$$\sum_{i=1}^{t} \prod_{j=1}^{i-1} M_j Z_t + \prod_{j=1}^{t} M_j Z_0 = (1 - \beta)c \frac{1 - \beta^t}{1 - \beta} + c\beta^t = c \qquad \forall t = 1, 2, \ldots,$$

so by (2.1) (i) holds. On the other hand, $\|\prod_{j=1}^{t} M_j\| = |\beta|^t$ $\forall t = 1, 2, \ldots$, which does not converge to 0 a.s. as $t \to \infty$. Also, $|\prod_{j=1}^{t-1} M_j Z_t| = |(1 - \beta)|c|\beta|^{t-1}$ $\forall t = 1, 2, \ldots$, so neither (v) nor (vi) holds.

Example 3.4. If $d > 1$ and C0 does not hold, then the implication (v) $\Rightarrow$ (vi) does not hold. To see this, we use the same setup as in Example 3.1, except that we now define $Z_t = v_2$ $\forall t = 1, 2, \ldots$. Then, $\prod_{j=1}^{t-1} M_j Z_t = v_2$ $\forall t = 1, 2, \ldots$, so (v) holds, but not (vi). Moreover, $\|\prod_{j=1}^{t} M_j\| = 1$ $\forall t = 1, 2, \ldots$.

Remark 3.1 (An open problem). Despite some effort, we have not been able to find a counterexample showing that if $d > 1$ and C0 does not hold, the implication (vi) $\Rightarrow$ (v) does not hold. It is therefore possible that, if $d > 1$, even when C0 does not hold, (vi) implies one or several of (ii), (iii), (iv) or (v). We leave it as an open problem to prove these assertions, or to disprove them by means of counterexamples.

Remark 3.2. Consider again the case $d > 1$. As pointed out in Remark 2.13 in [12], a *sufficient* condition for (ii) to hold is that $\sum_{t=1}^{\infty}\prod_{j=1}^{t-1}\|M_j\||Z_t| < \infty$ a.s. By Theorem 2.1 in [12] (see also Example 3.1 above), the latter condition is equivalent to

$$\sum_{t=1}^{\infty}\mathbb{P}\left(\min_{k=1,\dots,t-1}\prod_{j=k}^{t-1}\|M_j\||Z_t| > x\right) < \infty \qquad \forall x > 0$$

and to $\prod_{j=1}^{t-1}\|M_j\||Z_t| \xrightarrow{\text{a.s.}} 0$ as $t \to \infty$. If $\mathbb{P}(Z_1 = 0) < 1$, these equivalent conditions all imply that $\prod_{j=1}^{t}\|M_j\| \xrightarrow{\text{a.s.}} 0$ as $n \to \infty$, which clearly implies C0.

However, C0 does not imply that $\prod_{j=1}^{t}\|M_j\| \xrightarrow{\text{a.s.}} 0$ as $t \to \infty$, as the following counterexample shows. Let $d = 2$, and let v_1 and v_2 be orthonormal column vectors in $\mathbb{R}^2$. Let $\{\alpha_t; t = 1, 2, \dots\}$ be an i.i.d. random sequence such that $\mathbb{P}(\alpha_t = 1) = \mathbb{P}(\alpha_t = \frac{1}{2}) = \frac{1}{2}$ $\forall t = 1, 2, \dots$, and let $K_t = \sum_{j=1}^{t} I\{\alpha_j = 1\}$ $\forall t = 1, 2, \dots$. Define $M_t = \alpha_t v_1 v_1^T + (\frac{3}{2} - \alpha_t)v_2 v_2^T$ $\forall t = 1, 2, \dots$. Then

$$\left\|\prod_{j=1}^{t} M_j\right\| = \max\left(\frac{1}{2^{t-K_t}}, \frac{1}{2^{K_t}}\right) \qquad \forall t = 1, 2, \dots.$$

By the second Borel–Cantelli lemma, $K_t \xrightarrow{\text{a.s.}} \infty$ and $t - K_t \xrightarrow{\text{a.s.}} \infty$ as $t \to \infty$, implying that $\|\prod_{j=1}^{t} M_j\| \xrightarrow{\text{a.s.}} 0$ as $t \to \infty$. On the other hand, $\prod_{j=1}^{t}\|M_j\| = 1$ $\forall t = 1, 2, \dots$.

Remark 3.3. As noted in Remark 3.2, the condition $\prod_{j=1}^{t}\|M_j\| \xrightarrow{\text{a.s.}} 0$ as $t \to \infty$ implies C0. By Proposition 2.6 in [12] (see also Section 4 in [12]), the former condition holds *if and only if* one of the following two conditions hold:

$$\text{(i)}\quad \mathbb{E}\big(\big|\log\|M_1\|\big|\big) < \infty \quad \text{and} \quad \mathbb{E}\big(\log\|M_1\|\big) < 0;$$

$$\text{(ii)}\quad \mathbb{E}\big(\log^-\|M_1\|\big) = \infty \quad \text{and} \quad \mathbb{E}\left(\frac{\log^+\|M_1\|}{A_M(\log^+\|M_1\|)}\right) < \infty,$$

where $A_M(y) = \int_0^y \mathbb{P}(-\log\|M_1\| > x)\,\mathrm{d}x$ $\forall y > 0$, $\log^+ x = \log(x \vee 1)$ $\forall x > 0$, and $\log^- x = -\log(x \wedge 1)$ $\forall x > 0$.

Remark 3.4. Under the condition $\mathbb{E}(\log^+\|M_1\|) < \infty$, Kingman's subadditive ergodic theorem can be used to show that

$$\frac{1}{t}\log\left\|\prod_{j=1}^{t} M_j\right\| \xrightarrow{\text{a.s.}} \lambda = \lim_{n\to\infty}\frac{1}{n}\mathbb{E}\left(\log\left\|\prod_{j=1}^{n} M_j\right\|\right) \qquad \text{as } t \to \infty,$$

where $\lambda \in [-\infty, \infty)$ is a deterministic constant; see Theorem 6 in [16] and Theorem 2 in [11]. (Recall that the matrix norm used in these papers is equivalent to the spectral norm.) The constant λ is sometimes called the *maximal Lyapunov exponent*. In particular, if $\mathbb{E}(\log^+\|M_1\|) < \infty$, then C0 holds if $\lambda < 0$, and does not hold if $\lambda > 0$. For more information, see [11,16] and the references therein.

Remark 3.5. Finally, consider the case when $\mathscr{L}(M_1)$ is degenerate at a constant $d \times d$-matrix M, that is, the case when the RCA(1) process $\{X_t; t = 1, 2, \ldots\}$ is an AR(1) process. In this case, $\prod_{j=1}^{t} M_j = M^t \; \forall t = 1, 2, \ldots$, and the following *spectral representation* holds:

$$M^t = \sum_{k=1}^{s} \sum_{j=0}^{m_k - 1} \left[\frac{d^j}{dx^j} x^t \right]_{x=\lambda_k} Z_{k,j} \qquad \forall t = 1, 2, \ldots, \tag{3.1}$$

where $\{\lambda_k; k = 1, \ldots, s\}$ are the distinct eigenvalues of M, and $\{m_k; k = 1, \ldots, s\}$ are the multiplicities (all positive integers) of the eigenvalues as zeros of the *minimal annihilating polynomial* of M. Moreover, $\{Z_{k,j}; k = 1, \ldots, s; j = 0, \ldots, m_k - 1\}$ are linearly independent $d \times d$-matrices called the *components* of M; for more information, see Section 9.5 in [18]. Assuming that λ_1 is an eigenvalue of maximum modulus, there are two possible cases. If $|\lambda_1| < 1$, then, applying the triangle inequality to the right-hand side of (3.1), we see that $\|M^t\| \to 0$ as $t \to \infty$. On the other hand, if $|\lambda_1| \geq 1$, then $\|M^t\| \geq |M^t v_1| = |\lambda_1|^t \geq 1 \; \forall t = 1, 2, \ldots$, where v_1 is a normalized eigenvector corresponding to λ_1. Hence, C0 holds if and only if $|\lambda_1| < 1$.

4. Suggestions for future research

We mention two possible research directions. First, the open problem stated in Remark 3.1: to determine whether, in the case $d > 1$, (vi) in Theorem 2.1 implies one or several of (ii), (iii), (iv) or (v), without condition C0 (or replacing C0 with an even less restrictive condition). Second, to find a natural generalization (if it exists) of the integral condition (2.1) in Theorem 2.1 in [12] to higher dimensions.

References

[1] Alsmeyer, G. and Iksanov, A. (2009). A log-type moment result for perpetuities and its application to martingales in supercritical branching random walks. *Electron. J. Probab.* **14** 289–312. MR2471666

[2] Anděl, J. (1976). Autoregressive series with random parameters. *Math. Operationsforsch. Statist.* **7** 735–741. MR0428649

[3] Bougerol, P. and Picard, N. (1992). Strict stationarity of generalized autoregressive processes. *Ann. Probab.* **20** 1714–1730. MR1188039

[4] Brandt, A. (1986). The stochastic equation $Y_{n+1} = A_n Y_n + B_n$ with stationary coefficients. *Adv. in Appl. Probab.* **18** 211–220. MR0827336

[5] Brockwell, P.J. and Davis, R.A. (1991). *Time Series: Theory and Methods*, 2nd ed. *Springer Series in Statistics*. New York: Springer. MR1093459

[6] Brockwell, P.J. and Lindner, A. (2010). Strictly stationary solutions of autoregressive moving average equations. *Biometrika* **97** 765–772. MR2672497

[7] de Saporta, B., Guivarc'h, Y. and Le Page, E. (2004). On the multidimensional stochastic equation $Y_{n+1} = A_n Y_n + B_n$. *C. R. Math. Acad. Sci. Paris* **339** 499–502.

[8] Embrechts, P. and Goldie, C.M. (1994). Perpetuities and random equations. In *Asymptotic Statistics* (*Prague*, 1993) (P. Mandl and M. Huškovà, eds.). *Contrib. Statist*. 75–86. Heidelberg: Physica. MR1311930

[9] Erhardsson, T. (2008). Non-parametric Bayesian inference for integrals with respect to an unknown finite measure. *Scand*. *J*. *Stat*. **35** 369–384. MR2418747

[10] Feigin, P.D. and Tweedie, R.L. (1989). Linear functionals and Markov chains associated with Dirichlet processes. *Math*. *Proc*. *Cambridge Philos*. *Soc*. **105** 579–585. MR0985694

[11] Furstenberg, H. and Kesten, H. (1960). Products of random matrices. *Ann*. *Math*. *Statist*. **31** 457–469. MR0121828

[12] Goldie, C.M. and Maller, R.A. (2000). Stability of perpetuities. *Ann*. *Probab*. **28** 1195–1218. MR1797309

[13] Grincevičius, A.K. (1980). Products of random affine transformations. *Lith*. *Math*. *J*. **20** 279–282.

[14] Grincevičius, A.K. (1981). A random difference equation. *Lith*. *Math*. *J*. **21** 302–306.

[15] Kesten, H. (1973). Random difference equations and renewal theory for products of random matrices. *Acta Math*. **131** 207–248. MR0440724

[16] Kingman, J.F.C. (1973). Subadditive ergodic theory. *Ann*. *Probab*. **1** 883–909. MR0356192

[17] Kochen, S. and Stone, C. (1964). A note on the Borel–Cantelli lemma. *Illinois J*. *Math*. **8** 248–251. MR0161355

[18] Lancaster, P. and Tismenetsky, M. (1985). *The Theory of Matrices*, 2nd ed. *Computer Science and Applied Mathematics*. Orlando, FL: Academic Press. MR0792300

[19] Meyn, S. and Tweedie, R.L. (2009). *Markov Chains and Stochastic Stability*, 2nd ed. Cambridge: Cambridge Univ. Press. MR2509253

[20] Nicholls, D.F. and Quinn, B.G. (1982). *Random Coefficient Autoregressive Models*: *An Introduction*. *Lecture Notes in Statistics* **11**. New York: Springer. MR0671255

[21] Vervaat, W. (1979). On a stochastic difference equation and a representation of nonnegative infinitely divisible random variables. *Adv. in Appl*. *Probab*. **11** 750–783. MR0544194

Received July 2012 and revised January 2013