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On the Mellin transforms of the perpetuity and the remainder variables associated to a subordinator

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Results about the laws of the perpetuity and remainder variables associated to a subordinator are presented, with particular emphasis on their Mellin transforms, and multiplicative infinite divisibility property. Previous results by Bertoin–Yor (*Electron. Commun. Probab.* **6** (2001) 95–106) are incorporated in our discussion; important examples when the subordinator is the inverse local time of a diffusion are exhibited. Results of Urbanik (*Probab. Math. Statist.* **15** (1995) 493–513) are also discussed in detail; they appear to be too little known, despite the fact that quite a few of them have priority upon other works in this area.

Keywords: inverse local time; Mellin transform; multiplicative infinite divisibility; perpetuity

1. Introduction

Let $(\xi_l, l \ge 0)$ denote a subordinator, with Laplace–Bernstein exponent $\Phi \not\equiv 0$:

$$\mathbb{E}\big[\exp(-s\xi_l)\big] = \exp\big(-l\Phi(s)\big).$$

We take the slightly unusual notation l for the time parameter, instead of t, because we have in mind, among other examples, those $(\xi_l, l \ge 0)$ which are inverse local times, i.e.:

$$\xi_l = \inf\{t; L_t > l\},$$

where $(L_t, t \ge 0)$ is (a choice of) the local time at 0 for some 1-dimensional diffusion (see, in particular, Subsection 4.6, for the case where the diffusion is a *radial Ornstein–Uhlenbeck process*).

The present paper is a general study in the spirit of Bertoin–Yor [10]. So, we first recall the main results of Bertoin–Yor [10] (see also, for property i), Carmona et al. ([12], Proposition 3.3)):

i) The *perpetuity*:

$$\mathcal{I} \equiv \mathcal{I}_\xi \equiv \int_0^\infty \exp(-\xi_l)\,\mathrm{d}l$$

has integral moments of all orders, which determine its law:

$$\mathbb{E}\big[\mathcal{I}^n\big] = \frac{n!}{\Phi(1)\cdots\Phi(n)}, \ n \geq 1. \tag{1.1}$$

ii) There exists a random variable $\mathcal{R} \equiv \mathcal{R}_\xi$ (the *remainder*) whose law is also determined by its integral moments:

$$\mathbb{E}\big[\mathcal{R}^n\big] = \Phi(1)\cdots\Phi(n), \ n \geq 1. \tag{1.2}$$

In Bertoin–Yor [10], the variable $\mathcal{R}$ is defined via the relation:

$$\mathbb{E}\big[\exp(-t\mathcal{R})\big] = \mathbb{E}\left[\frac{1}{X_t}\right], \ t \geq 0$$

where X denotes the Lamperti process, starting from 1, associated with the subordinator ξ.

iii) There is the factorization:

$$\mathbf{e} \overset{\text{(law)}}{=} \mathcal{I} \cdot \mathcal{R}, \tag{1.3}$$

where $\mathbf{e}$ denotes a standard exponential variable, and, on the RHS of (1.3), $\mathcal{I}$ and $\mathcal{R}$ are independent.

We recall that Berg–Durán ([6], Theorem 1.3) deduced that the right-hand sides of (1.1) and (1.2) are Stieltjes moment sequences, from their general result:
if $(f(s), s > 0)$ is a completely monotone function, then:

$$\frac{1}{f(1)\cdots f(n)}, \ n = 1, 2, \ldots, \quad \text{is a moment-sequence}$$

(for (1.1), take $f_1(s) = \Phi(s)/s$, and, for (1.2), take $f_2(s) = 1/\Phi(s)$).
In the sequel, we shall often refer to the following *trivial example*:

$$\Phi(s) = s; \ \xi_l = l; \ \mathcal{I} = 1; \ \mathcal{R} \overset{\text{(law)}}{=} \mathbf{e}; \ \mathbb{E}\big[\mathcal{R}^n\big] = n!, \ n \in \mathbb{N}.$$

In the present paper, we discuss a number of precisions and improvements about the previous results i) and ii):
In Section 2, we introduce, for $r > 0$, the Mellin transforms:

$$I(r) = \mathbb{E}\big[\mathcal{I}^{r-1}\big], \ R(r) = \mathbb{E}\big[\mathcal{R}^{r-1}\big].$$

Note that, in the trivial example, $R(r) = \Gamma(r), r > 0$. We prove that, in the general case, the following functional equations hold:

$$I(r+1) = \frac{r}{\Phi(r)} I(r), \ R(r+1) = \Phi(r)R(r).$$

This leads us to a study of the Mellin transforms I and R, following the same method as Artin [2] in his study of the Gamma function.

In Section 3, we show the infinite divisibility of the random variable $\log(\mathcal{R})$. This leads to some integral representations of $\log(R(r))$ and $\log(I(r))$. (Note that the integral representation of $\log(R(r))$ was also obtained by Berg ([5], Theorem 2.2).) As a consequence, we obtain a characterization of the infinite divisibility and of the self-decomposability of the random variable $\log(\mathcal{I})$, in terms of the measure whose Laplace transform is Φ'/Φ.

In Section 4, we discuss many examples, in particular when $(\xi_l, l \geq 0)$ is the inverse local time of a radial Ornstein–Uhlenbeck process.

In Section 5, we present the main results of Urbanik [27], a number of which predate some of the results found in the previous sections.

To conclude this introduction, we note that the topic of this paper has attracted a lot of attention since at least the mid-nineties, which stems among other origins from applications to models in insurance, telecommunications, and many other applied domains.

Although the papers [4], [5], [10], [11], [12], [27] deal with topics studied in the present paper, we believe the subject still warrants some new exposition. In particular, the wealth of results found in Urbanik [27] has not been properly appreciated, perhaps because of some unusual (for probabilists) notation adopted by Urbanik in his work.

We also note that the perpetuities involved in our discussion are a very particular class among the family of exponential functionals of the form:

$$\int_0^\infty \exp(-\xi_{l-})\,\mathrm{d}\eta_l,$$

where the pair (ξ, η) is a 2-dimensional Lévy process, for which the integral makes sense (see, e.g., Carmona et al. [12]).

2. Functional equations

2.1. Notation

First, we fix the notation. In the sequel, we consider a subordinator $(\xi_l, l \geq 0)$ with *Laplace–Bernstein exponent* $\Phi \not\equiv 0$, i.e.:

$$\forall l \geq 0,\ \forall s \geq 0, \quad \mathbb{E}\bigl[\exp(-s\xi_l)\bigr] = \exp\bigl(-l\Phi(s)\bigr).$$

One has:

$$\forall s \geq 0, \quad \Phi(s) = as + \int_{(0,+\infty)} \bigl(1 - \mathrm{e}^{-sx}\bigr)\lambda(\mathrm{d}x) \tag{2.1}$$

for some $a \geq 0$ and a measure λ which satisfies:

$$\int_{(0,+\infty)} (x \wedge 1)\lambda(\mathrm{d}x) < \infty.$$

Definition 2.1. *In the sequel, we call* Bernstein function *any such function* Φ *satisfying* (2.1), *with* $a \geq 0$ *and* λ *a measure on* $(0, +\infty)$ *such that* $\int_{(0,+\infty)} (x \wedge 1)\lambda(\mathrm{d}x) < \infty$. *In particular, in* *this paper*, a Bernstein function is always assumed to be null at 0. *The measure* λ *is called the* Lévy measure *of* ξ *or* Φ.

We denote by μ_l the law of ξ_l, and by ρ the potential measure defined by:

$$\rho = \int_0^\infty \mu_l \, \mathrm{d}l.$$

We also set:

$$\widehat{\lambda}(\mathrm{d}x) = x\lambda(\mathrm{d}x) \quad \text{and} \quad \overline{\lambda}(\mathrm{d}x) = \lambda\big((x, +\infty)\big)\,\mathrm{d}x.$$

Finally, we set: $\kappa = (a\varepsilon + \widehat{\lambda}) * \rho$, where ε denotes the Dirac measure at 0 and $*$ the convolution of measures.

The following proposition is easily proven.

Proposition 2.1. *The functions*: $\Phi'(s)$, $\frac{\Phi(s)}{s}$, $\frac{1}{\Phi(s)}$, $\frac{\Phi'(s)}{\Phi(s)}$, *are completely monotone. They are respectively the Laplace transforms of*: $a\varepsilon + \widehat{\lambda}$, $a\varepsilon + \overline{\lambda}$, ρ, κ.

In particular, the measure κ, which plays an important role in the sequel, may also be defined by:

$$\forall s > 0, \quad \frac{\Phi'(s)}{\Phi(s)} = \int_{\mathbb{R}_+} \mathrm{e}^{-sx} \kappa(\mathrm{d}x).$$

Since $\lim_{s\to\infty} \frac{\Phi'(s)}{\Phi(s)} = 0$, one has: $\kappa(\{0\}) = 0$.

The perpetuity and remainder variables are defined in Section 1 and denoted by $\mathcal{I}$ and $\mathcal{R}$, respectively. We now introduce the Mellin transforms:

$$I(r) := \mathbb{E}\big[\mathcal{I}^{r-1}\big], \quad R(r) := \mathbb{E}\big[\mathcal{R}^{r-1}\big], \quad \text{for } r > 0.$$

2.2. Functional equations

Proposition 2.2. *The following functional equations hold*:

$$\forall r > 0, \quad I(r+1) = \frac{r}{\Phi(r)} I(r), \tag{2.2}$$

$$\forall r > 0, \quad R(r+1) = \Phi(r) R(r). \tag{2.3}$$

We note that equation (2.2) is a particular case of Proposition 3.1 in Carmona et al. [12].

Proof.

1) We set, for $t \geq 0$,

$$\mathcal{I}_t = \int_t^\infty \exp(-\xi_l)\,\mathrm{d}l = \left(\int_0^\infty \exp\big(-(\xi_{l+t} - \xi_t)\big)\,\mathrm{d}l\right) \exp(-\xi_t).$$

Hence, for $r > 0$,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\big[\mathcal{I}_t^r\big] &= -r\mathbb{E}\big[\mathcal{I}_t^{r-1}\exp(-\xi_t)\big] \\ &= -r\mathbb{E}\bigg[\bigg(\int_0^\infty \exp\big(-(\xi_{l+t}-\xi_t)\big)\,\mathrm{d}l\bigg)^{r-1}\exp(-r\xi_t)\bigg] \\ &= -r\mathbb{E}\big[\mathcal{I}^{r-1}\big]\exp\big(-t\Phi(r)\big).\end{aligned}$$

Integrating between 0 and ∞, we obtain (2.2).

2) By (1.3),

$$\forall r > 0, \quad I(r)R(r) = \Gamma(r). \tag{2.4}$$

Then, (2.3) follows easily from (2.2) and (2.4). □

In the trivial example, $\Phi(r) = r$, $R(r) = \Gamma(r)$, and (2.3) is the classical equation satisfied by the Gamma function. In the next subsection, we use Artin's method based on logarithmic convexity, to characterize the Mellin transforms I and R.

2.3. Logarithmic convexity

The following theorem extends the classical Bohr–Mollerup theorem (see, for instance, Andrews et al. ([1], Theorem 1.9.3)).

Theorem 2.1. *The Mellin transform*: $r > 0 \longrightarrow R(r)$, *is the unique function* f *from* $(0, +\infty)$ *into* $(0, +\infty)$ *such that*:

i) $f(1) = 1$,
ii) $\forall r > 0$, $f(r+1) = \Phi(r)f(r)$,
iii) f *is log-convex* (*i.e.*, $\log f$ *is convex*) *on* $(0, +\infty)$.

Proof.

1) Clearly, $f = R$ satisfies properties i) and iii) by the definition, and satisfies property ii) by Proposition 2.2.
2) Let f satisfying properties i), ii) and iii), $r \in (0, 1]$ and $n \geq 2$. By iteration of ii), we get:

$$f(n+r) = \Phi(r)\Phi(r+1)\cdots\Phi(r+n-1)f(r),$$
$$f(n) = \Phi(1)\cdots\Phi(n-1).$$

Moreover, by property iii),

$$f(n)\bigg(\frac{f(n)}{f(n-1)}\bigg)^r \leq f(n+r) \leq f(n)\bigg(\frac{f(n+1)}{f(n)}\bigg)^r.$$

Hence,

$$\frac{1}{\Phi(r)}\prod_{j=1}^{n-1}\frac{\Phi(j)}{\Phi(j+r)}\big(\Phi(n-1)\big)^r \le f(r) \le \frac{1}{\Phi(r)}\prod_{j=1}^{n-1}\frac{\Phi(j)}{\Phi(j+r)}\big(\Phi(n)\big)^r.$$

Since R also satisfies properties i), ii) and iii), we obtain:

$$\left(\frac{\Phi(n-1)}{\Phi(n)}\right)^r \le \frac{f(r)}{R(r)} \le \left(\frac{\Phi(n)}{\Phi(n-1)}\right)^r.$$

Letting n tend to ∞, we get:

$$\forall 0 < r \le 1, \quad f(r) = R(r).$$

Using again property ii), we obtain by induction that $f = R$. □

The same proof, replacing $\Phi(s)$ by $s/\Phi(s)$, yields the following theorem.

Theorem 2.2. *The Mellin transform*: $r > 0 \longrightarrow I(r)$, *is the unique function* g *from* $(0, +\infty)$ *into* $(0, +\infty)$ *such that*:

i) $g(1) = 1$,
ii) $\forall r > 0$, $g(r+1) = \frac{r}{\Phi(r)} g(r)$,
iii) g *is log-convex on* $(0, +\infty)$.

2.4. Representation as limits

The next theorem generalizes the following classical formula:

$$\forall r > 0, \quad \Gamma(r) = \lim_{n\to\infty} \frac{n!}{r(r+1)\cdots(r+n-1)} n^{r-1}. \tag{2.5}$$

Theorem 2.3. *There is the asymptotic formula*:

$$\forall r > 0, \quad R(r) = \lim_{n\to\infty} \prod_{j=0}^{n-1} \frac{\Phi(j+1)}{\Phi(j+r)} \big(\Phi(n)\big)^{r-1}.$$

Proof.

1) We set, for $n \ge 1$ and $r > 0$,

$$h(n, r) = \prod_{j=0}^{n-1} \frac{\Phi(j+1)}{\Phi(j+r)} \big(\Phi(n)\big)^{r-1}.$$

2) Suppose $r \in (0,1]$. Then

$$\frac{h(n+1,r)}{h(n,r)} = \frac{\Phi(n+1)^r \Phi(n)^{1-r}}{\Phi(n+r)} \leq 1$$

since, as $1/\Phi$ is completely monotone, $1/\Phi$ is log-convex and therefore, $\log \Phi$ is concave. Thus, $n \longrightarrow h(n,r)$ is decreasing and, as it was seen in the proof of Theorem 2.1, $h(n,r) \geq R(r)$. Therefore, $h(r) := \lim_{n\uparrow\infty} \downarrow h(n,r)$ exists for $r \in (0,1]$ and is > 0.

3) One has, for $n \geq 1$ and $r > 0$,

$$h(n, r+1) = \Phi(r)\left(\frac{\Phi(n)}{\Phi(n+1)}\right)^r h(n+1, r).$$

Hence, taking into account the above step 2), we obtain by induction:

$$\forall r > 0, \quad h(r) := \lim_{n\to\infty} h(n,r) \text{ exists, and } h(r+1) = \Phi(r)h(r).$$

4) As $1/\Phi$ is log-convex, h is log-convex too. Thus, h satisfies the conditions i), ii) and iii) in Theorem 2.1, and consequently, $h = R$. □

Theorem 2.4. *There is the asymptotic formula*:

$$\forall r > 0, \quad I(r) = \Gamma(r) \lim_{n\to\infty} \prod_{j=0}^{n-1} \frac{\Phi(j+r)}{\Phi(j+1)} \big(\Phi(n)\big)^{1-r}.$$

Proof. This is a direct consequence of Theorem 2.3 and (2.4). Another proof consists in using the same proof as that of Theorem 2.3 (replacing $1/\Phi(s)$ by $\Phi(s)/s$) and (2.5). □

2.5. Convolution equations satisfied by the laws of $\mathcal{I}$ and $\mathcal{R}$

In this subsection, we assume, for simplicity, $a = 0$. We also denote by $\overline{\lambda}$ the density of the measure $\overline{\lambda}$: $\overline{\lambda}(x) = \lambda((x, +\infty))$ for $x \geq 0$.

Property 1 in the following theorem is a particular case (Example B) of Carmona et al. ([12], Proposition 2.1). However, the proofs are different.

Theorem 2.5. 1. *The law of $\mathcal{I}$ admits a density which we denote by* $(\theta(v), v > 0)$ *and which satisfies*:

$$\forall v > 0, \quad \theta(v) = \int_v^\infty \theta(y)\overline{\lambda}\left(\log\left(\frac{y}{v}\right)\right) \mathrm{d}y. \tag{2.6}$$

2. *The law of $\mathcal{R}$ satisfies*:

$$\forall v > 0, \quad \mathbb{P}(\mathcal{R} > v) = \mathbb{E}\left[\frac{1}{\mathcal{R}} 1_{(\mathcal{R}>v)} \overline{\lambda}\left(\log\left(\frac{\mathcal{R}}{v}\right)\right)\right]. \tag{2.7}$$

3. *If the potential measure* ρ *admits a density* (*still denoted by* ρ), *then the law of* $\mathcal{R}$ *admits a density which we denote by* $(\zeta(v), v > 0)$ *and which satisfies*:

$$\forall v > 0, \quad \zeta(v) = \int_v^\infty \zeta(y)\rho\left(\log\left(\frac{y}{v}\right)\right) \mathrm{d}y. \tag{2.8}$$

Proof.

1) Let θ be the law of $\mathcal{I}$. By Proposition 2.1, we have, for $r > 0$,

$$\frac{\Phi(r)}{r} I(r+1) = \iint \mathrm{e}^{-xr} y^r \overline{\lambda}(x)\,\mathrm{d}x\theta(\mathrm{d}y).$$

The change of variable (from x to v): $x = \log(\frac{y}{v})$ yields:

$$\frac{\Phi(r)}{r} I(r+1) = \int_0^\infty v^{r-1}\left[\int_v^\infty \overline{\lambda}\left(\log\left(\frac{y}{v}\right)\right)\theta(\mathrm{d}y)\right]\mathrm{d}v.$$

Now,

$$I(r) = \int v^{r-1}\theta(\mathrm{d}v).$$

Then, from the functional equation (2.2), the measures:

$$\theta(\mathrm{d}v) \quad \text{and} \quad \left[\int_v^\infty \overline{\lambda}\left(\log\left(\frac{y}{v}\right)\right)\theta(\mathrm{d}y)\right]\mathrm{d}v$$

admit the same Mellin transform, hence they are equal.

2) We set $\eta(y) = \mathbb{P}(\mathcal{R} > y)$. We have, for $r > 0$,

$$\frac{\Phi(r)}{r} R(r) = -\int\int \mathrm{e}^{-xr} y^{r-1}\overline{\lambda}(x)\,\mathrm{d}x\,\mathrm{d}\eta(y).$$

The change of variable (from x to v): $x = \log(\frac{y}{v})$ yields:

$$\frac{\Phi(r)}{r} R(r) = -\int_0^\infty v^{r-1}\left[\int_v^\infty \frac{1}{y}\overline{\lambda}\left(\log\left(\frac{y}{v}\right)\right)\mathrm{d}\eta(y)\right]\mathrm{d}v.$$

Now,

$$\frac{1}{r} R(r+1) = -\frac{1}{r}\int v^r\,\mathrm{d}\eta(v) = \int_0^\infty v^{r-1}\eta(v)\,\mathrm{d}v.$$

Then (2.7) follows from the functional equation (2.3) by injectivity of the Mellin transform.

3) The proof of point 3 is quite similar to that of point 1. □

2.6. The symmetry case

Definition 2.2. *For any Bernstein function $\Phi \not\equiv 0$, we define Φ^* by*:

$$\forall s > 0,\ \Phi^*(s) = \frac{s}{\Phi(s)} \quad \text{and} \quad \Phi^*(0) = \lim_{s\to 0} \Phi^*(s).$$

We denote by Σ the set of Bernstein functions $\Phi \not\equiv 0$ such that Φ^* is a Bernstein function. (*In particular* (*see Definition* 2.1), *if $\Phi \in \Sigma$, then $\Phi^*(0) = 0$.*)

In the terminology of Schilling et al. [26], *Σ is the set of* special Bernstein functions Φ *such that*

$$\Phi(0) = 0 \quad \text{and} \quad \lim_{s\to 0} \frac{\Phi(s)}{s} = +\infty.$$

We call symmetry case *the situation where $\Phi \in \Sigma$. In this case*:

$$\forall s \geq 0, \quad \Phi(s)\Phi^*(s) = s.$$

Obviously, if $\Phi \in \Sigma$, then $\Phi^* \in \Sigma$ and $\Phi^{**} = \Phi$. We then say that (Φ, Φ^*) is a pair of conjugate Bernstein functions.

Our interest in the symmetry case stems from the next proposition.

Proposition 2.3. *Suppose that (Φ, Φ^*) is a pair of conjugate Bernstein functions. We denote by $\mathcal{I}_\Phi$ and $\mathcal{R}_\Phi$, respectively the perpetuity and the remainder variables related to the Lévy process whose Laplace–Bernstein exponent is Φ, and likewise for $\mathcal{I}_{\Phi^*}$ and $\mathcal{R}_{\Phi^*}$. Then*:

$$\mathcal{I}_\Phi \overset{\text{(law)}}{=} \mathcal{R}_{\Phi^*} \quad \text{and} \quad \mathcal{R}_\Phi \overset{\text{(law)}}{=} \mathcal{I}_{\Phi^*}.$$

Proof. By Theorem 2.1 and Theorem 2.2, $\mathcal{I}_\Phi$ and $\mathcal{R}_{\Phi^*}$ have the same Mellin transform. Likewise, $\mathcal{R}_\Phi$ and $\mathcal{I}_{\Phi^*}$ have the same Mellin transform. The desired result follows from the injectivity of the Mellin transform. □

We now state a useful characterization of the symmetry case (see also Schilling et al. ([26], Theorem 10.3)).

Theorem 2.6. *Let $\Phi \not\equiv 0$ be a Bernstein function. Following Subsection* 2.1, *we still denote by ρ the related potential measure. Then, $\Phi \in \Sigma$ if and only if there exist $b \geq 0$ and $h : (0, +\infty) \longrightarrow \mathbb{R}_+$ such that*:

$$\begin{gathered} h \text{ is decreasing},\ \lim_{x\to\infty} h(x) = 0 \quad \text{and} \\ \rho(\mathrm{d}x) = b\varepsilon(\mathrm{d}x) + h(x)\,\mathrm{d}x. \end{gathered} \tag{2.9}$$

Moreover, if these properties are satisfied, then:

$$\Phi^*(s) = bs - \int_0^\infty \left(1 - \mathrm{e}^{-sx}\right) \mathrm{d}h(x),\ s \geq 0. \tag{2.10}$$

Proof.

1) Suppose $\Phi \in \Sigma$. Then

$$\Phi^*(s) = a^* s + \int \big(1 - e^{-sx}\big)\lambda^*(dx). \tag{2.11}$$

Since $\frac{1}{\Phi(s)} = \frac{\Phi^*(s)}{s}$, Proposition 2.1 yields:

$$\rho(dx) = a^* \varepsilon(dx) + \lambda^*\big((x, +\infty)\big)\, dx.$$

This shows the "only if" part, setting:

$$b = a^* \quad \text{and} \quad h(x) = \lambda^*\big((x, +\infty)\big). \tag{2.12}$$

Moreover, (2.11) and (2.12) entail (2.10).

2) Conversely, suppose that (2.9) holds. Since ρ is a Radon measure on $\mathbb{R}_+$, then

$$\int_0^1 h(x)\, dx = \int_0^\infty (y \wedge 1)\big(-dh(y)\big) < \infty.$$

Therefore, the Stieltjes measure $-dh$ is a Lévy measure and the function:

$$\Psi(s) = bs - \int_0^\infty \big(1 - e^{-sx}\big)\, dh(x), \;\; s \geq 0$$

is a Bernstein function. Using again Proposition 2.1 and (2.9), we obtain:

$$\forall s > 0, \qquad \frac{\Psi(s)}{s} = \frac{1}{\Phi(s)}.$$

Hence, $\Phi \in \Sigma$ and $\Phi^* = \Psi$. $\square$

Now, we exhibit a large convex cone Λ contained in Σ (see also Schilling et al. ([26], Theorem 10.11)).

Theorem 2.7. *We denote by Λ the convex cone consisting of functions Ψ of the form*:

$$\Psi(s) = bs - \int_0^\infty \big(1 - e^{-sx}\big)\, dh(x), \;\; s \geq 0,$$

for some $b \geq 0$ and some decreasing log-convex function h on $(0, +\infty)$ satisfying:

$$\int_0^1 h(x)\, dx < \infty, \; \int_0^\infty h(x)\, dx = +\infty \quad \text{and} \quad \lim_{x\to\infty} h(x) = 0.$$

Then, $\Lambda \subset \Sigma$.

Proof. Let $b \geq 0$ and let h be a decreasing log-convex function on $(0, +\infty)$ satisfying $\int_0^1 h(x)\,\mathrm{d}x < \infty$ and $\int_0^\infty h(x)\,\mathrm{d}x = +\infty$. We set:

$$\rho(\mathrm{d}x) = b\varepsilon(\mathrm{d}x) + h(x)\,\mathrm{d}x.$$

By Hirsch ([17], Théorème 2) (see also Itô [19]), ρ is the potential measure associated with a Bernstein function Φ. By Theorem 2.6, if moreover $\lim_{x\to\infty} h(x) = 0$, then $\Phi \in \Sigma$ and

$$\Phi^*(s) = bs - \int_0^\infty \big(1 - \mathrm{e}^{-sx}\big)\,\mathrm{d}h(x), \ s \geq 0.$$

But, as we have noted before, $\Psi := \Phi^* \in \Sigma$. □

Definition 2.3. *Following Schilling et al.* [26], *we call* complete Bernstein function *any function* Φ *of the form*:

$$\Phi(s) = b + as + \int_0^\infty \big(1 - \mathrm{e}^{-sx}\big) m(x)\,\mathrm{d}x, \ s \geq 0, \tag{2.13}$$

for some $a \geq 0$, $b \geq 0$, *and some completely monotone function* m *on* $(0, +\infty)$ *satisfying*:

$$\int_0^\infty (x \wedge 1) m(x)\,\mathrm{d}x < \infty.$$

We denote by $\widehat{\mathcal{S}}$ *the convex cone of complete Bernstein functions.*

We also denote by $\mathcal{S}$ *the convex cone consisting of functions* $\Phi \in \widehat{\mathcal{S}}$ *such that*:

$$\Phi(0) = 0 \quad \textit{and} \quad \Phi^*(0) = \lim_{s\to 0} \frac{s}{\Phi(s)} = 0,$$

or, with the notation of (2.13),

$$b = 0 \quad \textit{and} \quad \int_0^\infty x m(x)\,\mathrm{d}x = \infty. \tag{2.14}$$

Using Bernstein's theorem, we obtain that $\Phi \in \mathcal{S}$ *if and only if there exist* $a \geq 0$ *and a measure* μ *on* $(0, +\infty)$ *so that*:

$$\int_{(0,+\infty)} \frac{1}{1+t}\mu(\mathrm{d}t) < \infty, \ \int_{(0,+\infty)} \frac{1}{t}\mu(\mathrm{d}t) = +\infty \quad \textit{and}$$

$$\Phi(s) = as + \int_{(0,+\infty)} \frac{s}{s+t}\mu(\mathrm{d}t), \ s \geq 0.$$

In other words, $\Phi \in \mathcal{S}$ *if and only if* $\frac{\Phi(s)}{s}$ *is a* Stieltjes transform S *such that*

$$\lim_{s\to 0} S(s) = +\infty \quad \textit{and} \quad \lim_{s\to 0} sS(s) = 0.$$

Proposition 2.4. *There is the double inclusion*: $\mathcal{S} \subset \Lambda \subset \Sigma$. *Moreover,* Φ *belongs to* $\mathcal{S}$ *if and only if* Φ^* *belongs to* $\mathcal{S}$.

Proof.

1) Suppose $\Phi \in \mathcal{S}$ and Φ given by (2.13) and (2.14). Set:

$$\forall x > 0, \quad h(x) = \int_x^\infty m(y)\,\mathrm{d}y.$$

Then,

$$\int_0^1 h(x)\,\mathrm{d}x = \int_0^\infty (x \wedge 1) m(x)\,\mathrm{d}x < \infty \quad \text{and}$$
$$\int_0^\infty h(x)\,\mathrm{d}x = \int_0^\infty x m(x)\,\mathrm{d}x = +\infty.$$

Moreover, $\lim_{x\to\infty} h(x) = 0$, h is decreasing, and, since h is completely monotone, then h is log-convex. Therefore, $\Phi \in \Lambda$.

2) By Schilling et al. ([26], Proposition 7.1), if $\Phi \not\equiv 0$ is a complete Bernstein function, then so is Φ^*. This gives another proof of $\mathcal{S} \subset \Sigma$ and shows that, if $\Phi \in \mathcal{S}$, then $\Phi^* \in \mathcal{S}$ too.

□

The cone $\widehat{\mathcal{S}}$ of complete Bernstein functions has a deep probabilistic interpretation. Indeed, using Krein's theory of strings, an open problem in Itô–Mc Kean concerning the precise class of subordinators which are inverse local times of a regular diffusion on $[0, +\infty)$, instantaneously reflecting at 0, has been solved simultaneously and independently in 1981 by Kotani–Watanabe [21] and Knight [20]. See also Bertoin [7], Küchler [22], Küchler–Salminen [23]. In fact, this class is precisely the class of subordinators whose Laplace–Bernstein exponent belongs to $\widehat{\mathcal{S}}$. Interestingly, in Bertoin's Saint-Flour course [8], Chapter 9, there is the description of Φ^*, where Φ is the Bernstein function associated to:

$$A_{\tau_l} := \int_0^{\tau_l} f(B_t)\,\mathrm{d}t, \quad l \ge 0,$$

with $f \ge 0$ with support in $(0, +\infty)$, and $(\tau_l, l \ge 0)$ the inverse local time at 0 of the Brownian motion $(B_t, t \ge 0)$. Precisely, Corollary 9.7, page 78, asserts that in that case: $\Phi^* \in \widehat{\mathcal{S}}$. More generally, the discussion of Krein's theory of strings and its relationship with *inverse local times of generalized diffusions* is expounded in an exhaustive manner in Chapter 14 of Schilling et al. [26]. The Krein table (that is: identifying the generalized diffusion whose Laplace–Bernstein exponent of the inverse local time is a given element of $\widehat{\mathcal{S}}$) established by Donati-Martin and Yor ([14] and [15]) is reproduced in Schilling et al. [26] on pp. 201–202. Note that it has relatively few entries, and would certainly deserve to be completed.

3. Multiplicative infinite divisibility

3.1. Multiplicative infinite divisibility of $\mathcal{R}$

Definition 3.1. *A positive random variable X is said to be* multiplicatively infinitely divisible (*m.i.d.*) *if, for any $n \in \mathbb{N}$ with $n \geq 2$, there exist independent identically distributed positive random variables*: $X_1, \ldots, X_n$, *such that*:

$$X \stackrel{\text{(law)}}{=} X_1 \cdots X_n.$$

Obviously, a strictly positive random variable X is m.i.d. if and only if $\log X$ is infinitely divisible.

In the sequel, the notation of the previous section is still in force. The following proposition was also proven by C. Berg ([4], Theorem 1.8).

Proposition 3.1. *The remainder $\mathcal{R}$ is m.i.d.*

Proof. For $n \geq 2$, let $\Phi_n := (\Phi)^{1/n}$. As it is well known, Φ_n also is a Bernstein function (associated with the subordinate Lévy process: $\xi^{(1/n)} := (\xi_{\tau_l^{(1/n)}}, l \geq 0)$ where $\tau^{(1/n)}$ denotes a $(1/n)$-stable subordinator independent of ξ). Let $\mathcal{R}_n$ be the remainder related to $\xi^{(1/n)}$. Then, we deduce easily from Theorem 2.1 and from the injectivity of the Mellin transform:

$$\mathcal{R} \stackrel{\text{(law)}}{=} \mathcal{R}_n^{(1)} \cdots \mathcal{R}_n^{(n)}$$

where $\mathcal{R}_n^{(1)}, \ldots, \mathcal{R}_n^{(n)}$ are n independent copies of $\mathcal{R}_n$. Thus, by Definition 3.1, $\mathcal{R}$ is m.i.d. $\square$

3.2. Integral representations

We shall deduce from Proposition 3.1 a representation of the Mellin transform R of $\mathcal{R}$. We recall (cf. Subsection 2.1) that κ denotes the measure whose Laplace transform is Φ'/Φ.

Theorem 3.1. *We have, for $r > 0$,*

$$R(r) = \Phi(1)^{r-1} \exp\left[\int_{(0,+\infty)} \frac{\mathrm{e}^{-(r-1)x} - 1 - (r-1)(\mathrm{e}^{-x} - 1)}{x(\mathrm{e}^x - 1)} \kappa(\mathrm{d}x)\right].$$

Proof. We shall give two proofs. We mention that C. Berg stated a more general result (see Berg ([5], Theorem 2.2)), with a different proof.

First proof. By Proposition 3.1 and the Lévy–Khintchine formula, there exist $a, \sigma \in \mathbb{R}$ and a measure Θ on $\mathbb{R} \setminus \{0\}$ satisfying

$$\int \frac{x^2}{1+x^2} \Theta(\mathrm{d}x) < \infty$$

such that

$$\forall u \in \mathbb{R}, \quad \mathbb{E}\big[\mathcal{R}^{\mathrm{i}u}\big] = \exp\big(-\psi(u)\big)$$

with

$$\psi(u) = \mathrm{i}au + \frac{1}{2}\sigma^2 u^2 + \int\big(1 - \mathrm{e}^{\mathrm{i}ux} + \mathrm{i}ux 1_{\{|x|<1\}}\big)\Theta(\mathrm{d}x).$$

Since the Mellin transform $R(r)$ is defined for $r > 0$, then ψ continuously extends to $\mathrm{i}(-\infty, 1)$. Hence,

$$\forall s > 0, \int_{(1,+\infty)} \mathrm{e}^{sx}\Theta(\mathrm{d}x) < \infty \quad \text{and} \quad \forall 0 < s < 1, \int_{(-\infty,-1)} \mathrm{e}^{-sx}\Theta(\mathrm{d}x) < \infty$$

and, for $r > 0$,

$$\log\big(R(r)\big) = -a(r-1) + \frac{1}{2}\sigma^2(r-1)^2 + \int\big(\mathrm{e}^{(r-1)x} - 1 - (r-1)x 1_{\{|x|<1\}}\big)\Theta(\mathrm{d}x). \tag{3.1}$$

We deduce then from (3.1) and (2.3):

$$\log\Phi(r) = -a + \frac{1}{2}\sigma^2(2r-1) + \int\big(\mathrm{e}^{rx} - \mathrm{e}^{(r-1)x} - x 1_{\{|x|<1\}}\big)\Theta(\mathrm{d}x),$$

and, by differentiation,

$$\frac{\Phi'(r)}{\Phi(r)} = \sigma^2 + \int x\big(\mathrm{e}^{rx} - \mathrm{e}^{(r-1)x}\big)\Theta(\mathrm{d}x).$$

Therefore, $\sigma = 0$ and the measure $x(1 - \mathrm{e}^{-x})\Theta(\mathrm{d}x)$ is the image of κ by $x \longrightarrow -x$. (In particular, Θ is carried by $\mathbb{R}_-$.)

Then, (3.1) becomes:

$$\log\big(R(r)\big) = -a(r-1) + \int \frac{(\mathrm{e}^{-(r-1)x} - 1 + (r-1)x 1_{\{0<x<1\}})}{x(\mathrm{e}^x - 1)}\kappa(\mathrm{d}x) \tag{3.2}$$

and, in particular, for $r = 2$,

$$\log\big(\Phi(1)\big) = -a + \int \frac{(\mathrm{e}^{-x} - 1 + x 1_{\{0<x<1\}})}{x(\mathrm{e}^x - 1)}\kappa(\mathrm{d}x). \tag{3.3}$$

The desired result follows directly from (3.2) and (3.3).

Second proof. We set, for $r > 0$,

$$f(r) = \Phi(1)^{r-1}\exp\bigg[\int_{(0,+\infty)} \frac{\mathrm{e}^{-(r-1)x} - 1 - (r-1)(\mathrm{e}^{-x} - 1)}{x(\mathrm{e}^x - 1)}\kappa(\mathrm{d}x)\bigg].$$

Obviously, f is a log-convex function such that $f(1)=1$. Therefore, by Theorem 2.1, we only have to prove:

$$\forall r>0, \quad f(r+1)=\Phi(r)f(r).$$

Now, a simple computation yields:

$$\begin{aligned}\log f(r+1)-\log f(r) &= \log\Phi(1)+\int\frac{e^{-x}-e^{-rx}}{x}\kappa(dx)\\ &= \log\Phi(1)+\int_1^r\left(\int e^{-sx}\kappa(dx)\right)ds\\ &= \log\Phi(1)+\int_1^r\frac{\Phi'(s)}{\Phi(s)}\,ds=\log\Phi(r).\end{aligned}$$ □

In the trivial example, Theorem 3.1 yields a classical representation of the Gamma function (see, for instance, Andrews et al. ([1], Theorem 1.6.2)):

$$\Gamma(r)=\exp\left[\int_0^{+\infty}\frac{e^{-(r-1)x}-1-(r-1)(e^{-x}-1)}{x(e^x-1)}\,dx\right]. \tag{3.4}$$

We now state a result which was seen in the first proof of Theorem 3.1.

Proposition 3.2. *The Lévy measure of the infinite divisible variable* $\log\mathcal{R}$ *is the image by the map* $x\longrightarrow -x$ *of the measure on* $(0,+\infty)$:

$$x^{-1}\left(e^x-1\right)^{-1}\kappa(dx).$$

As a direct consequence of Theorem 3.1 and (2.4), we obtain the following theorem.

Theorem 3.2. *For* $r>0$,

$$I(r)=\Gamma(r)\Phi(1)^{-r+1}\exp\left[-\int_{(0,+\infty)}\frac{e^{-(r-1)x}-1-(r-1)(e^{-x}-1)}{x(e^x-1)}\kappa(dx)\right].$$

Note that, by (3.4), we also have, for $r>0$,

$$I(r)=\Phi(1)^{-r+1}\exp\left[\int_{(0,+\infty)}\frac{e^{-(r-1)x}-1-(r-1)(e^{-x}-1)}{x(e^x-1)}\big(dx-\kappa(dx)\big)\right]. \tag{3.5}$$

3.3. Multiplicative infinite divisibility of $\mathcal{I}$

In the next theorem, we state a characterization of the multiplicative infinite divisibility of $\mathcal{I}$ (see also Berg ([4], Theorem 1.9)).

Theorem 3.3. *The variable $\mathcal{I}$ is m.i.d. if and only if $\kappa(\mathrm{d}x) \le \mathrm{d}x$. Besides, if $\mathcal{I}$ is m.i.d., then the Lévy measure of the infinitely divisible variable $\log \mathcal{I}$ is*:

$$1_{(-\infty,0)}(x)x^{-1}\big(1-\mathrm{e}^{-x}\big)^{-1}\big(1-k(-x)\big)\,\mathrm{d}x$$

where k denotes the density: $\frac{\kappa(\mathrm{d}x)}{\mathrm{d}x}$.

Proof. We deduce from formula (3.5) that:

$$\forall u \in \mathbb{R}, \quad \mathbb{E}\big[\mathcal{I}^{\mathrm{i}u}\big] = \exp\big(-\eta(u)\big)$$

with

$$\eta(u) = \mathrm{i}u \log \Phi(1) + \int_{(0,+\infty)} \frac{1-\mathrm{e}^{-\mathrm{i}ux} + \mathrm{i}u(\mathrm{e}^{-x}-1)}{x(\mathrm{e}^{x}-1)}\big(\mathrm{d}x - \kappa(\mathrm{d}x)\big).$$

Then the result follows from the Lévy–Khintchine formula. □

We now give a sufficient condition.

Proposition 3.3. *If $\Phi \in \Sigma$ (see Definition 2.2), then the variable $\mathcal{I}$ is m.i.d.*

Proof. If $\Phi \in \Sigma$, then, with obvious notation,

$$\kappa(\mathrm{d}x) + \kappa^*(\mathrm{d}x) = 1_{\mathbb{R}_+}(x)\,\mathrm{d}x.$$

In particular, $\kappa(\mathrm{d}x) \le \mathrm{d}x$, and Theorem 3.3 applies.

Another proof consists in using jointly Proposition 2.3 and Proposition 3.1. □

Another sufficient condition is the following proposition.

Proposition 3.4. *Suppose that, for every $\alpha \in (0,1)$, the function $s^{1-\alpha}\Phi^{\alpha}(s)$ is a Bernstein function, then the variable $\mathcal{I}$ is m.i.d.*

Proof. For $n \in \mathbb{N}$, $n \ge 2$, we denote by Φ_n the Bernstein function $s^{1-\alpha}\Phi^{\alpha}(s)$ with $\alpha = 1/n$. Then

$$\frac{s}{\Phi(s)} = \left(\frac{s}{\Phi_n(s)}\right)^n.$$

We denote by $\mathcal{I}_n$ the perpetuity related to the Bernstein function Φ_n. Then, we deduce easily from Theorem 2.2 and from the injectivity of the Mellin transform:

$$\mathcal{I} \overset{(\mathrm{law})}{=} \mathcal{I}_n^{(1)} \cdots \mathcal{I}_n^{(n)}$$

where $\mathcal{I}_n^{(1)} \cdots \mathcal{I}_n^{(n)}$ are n independent copies of $\mathcal{I}_n$. Thus, by Definition 3.1, $\mathcal{I}$ is m.i.d. □

Corollary 3.1. *Suppose that* Φ *is a complete Bernstein function* (*see Definition* 2.3), *null at* 0. *Then*, *the variable* $\mathcal{I}$ *is m.i.d.*

Proof. By Schilling et al. ([26], Proposition 7.10.) (see also Berg [3]), a complete Bernstein function Φ satisfies the condition of Proposition 3.4, which entails the desired result.

Note that if moreover $\Phi^*(0) = 0$, then $\Phi \in \mathcal{S}$, and the multiplicative infinite divisibility of $\mathcal{I}$ also follows from Proposition 2.4 and Proposition 3.3. $\square$

We end this section by a straightforward consequence of Proposition 3.2 and Theorem 3.3.

Proposition 3.5. *The variable* $\log \mathcal{R}$ *is self-decomposable if and only if there exists a positive decreasing function* $\jmath$ *on* $(0, +\infty)$ *such that*:

$$\kappa(\mathrm{d}x) = 1_{(0,+\infty)}(x)\big(\mathrm{e}^x - 1\big)\jmath(x)\,\mathrm{d}x.$$

The variable $\log \mathcal{I}$ *is self-decomposable if and only if there exists a positive decreasing function* ℓ *on* $(0, +\infty)$ *such that*:

$$\kappa(\mathrm{d}x) = 1_{(0,+\infty)}(x)\big[1 - \big(\mathrm{e}^x - 1\big)\ell(x)\big]\,\mathrm{d}x.$$

4. Examples

4.1. α-stable subordinator

Let $\alpha \in (0, 1)$ and let ξ be an α-stable subordinator. Then, $\Phi(s) = s^\alpha$. Consequently, $\Phi \in \mathcal{S}$ and $\Phi^*(s) = s^{1-\alpha}$. We have: $\kappa(\mathrm{d}x) = \alpha 1_{\mathbb{R}_+}(x)\,\mathrm{d}x$. Therefore, $\log \mathcal{I}$ and $\log \mathcal{R}$ are self-decomposable. Moreover, we obtain easily:

$$I(r) = \big[\Gamma(r)\big]^{1-\alpha} \quad \text{and} \quad R(r) = \big[\Gamma(r)\big]^{\alpha}.$$

The diffusion whose inverse local time is an α-stable subordinator is a Bessel process of dimension $d = 2(1 - \alpha)$ (see, for instance, Molchanov–Ostrovski [24]).

4.2. Exponential compound Poisson process

Let $c > 0$ and let ξ be a compound Poisson process whose Lévy measure is $c1_{\mathbb{R}_+}(x)\mathrm{e}^{-cx}\,\mathrm{d}x$. Then, $\Phi(s) = s(s + c)^{-1}$. Consequently, Φ is a complete Bernstein function ($m(x) = c\mathrm{e}^{-cx}$), but $\Phi \notin \mathcal{S}$ (since $\Phi^*(s) = s + c$ and $\Phi^*(0) = c \neq 0$). We have: $\kappa(\mathrm{d}x) = 1_{\mathbb{R}_+}(x)(1 - \mathrm{e}^{-cx})\,\mathrm{d}x$. Thus $\mathcal{I}$ is m.i.d. More precisely, we deduce from Proposition 3.5 that the variables $\log \mathcal{I}$ and $\log \mathcal{R}$ are self-decomposable. Moreover, we obtain easily (for example by Theorem 2.5):

$$I(r) = \frac{\Gamma(c + r)}{\Gamma(c + 1)} \quad \text{and} \quad R(r) = \frac{\Gamma(c + 1)\Gamma(r)}{\Gamma(c + r)} = cB(r, c).$$

Therefore, $\mathcal{I} \overset{\text{(law)}}{=} \gamma_{c+1}$ and $\mathcal{R} \overset{\text{(law)}}{=} \beta_{1,c}$ where γ_u denotes a gamma variable of parameter u and $\beta_{u,v}$ denotes a beta variable of parameters u, v.

4.3. Geometric compound Poisson process

Let $0 \le c < q < 1$. Following Bertoin et al. [9], we consider a compound Poisson process $(\xi_l, l \ge 0)$ whose Lévy measure is the geometric probability:

$$\lambda = \sum_{n=1}^{\infty} (c/q)^{n-1}(1 - c/q)\varepsilon_{-n \log q}$$

where ε_x denotes the Dirac measure at point x. (In particular, if $c = 0$, then $\lambda = \varepsilon_{-\log q}$ and $\xi_l = -(\log q)N_l$ where $(N_l, l \ge 0)$ denotes the standard Poisson process.)

We obtain easily:

$$\Phi(s) = \frac{1 - q^s}{1 - cq^{s-1}}.$$

Then, by Theorems 2.3 and 2.4, we have:

$$R(r) = \prod_{j=0}^{\infty} \frac{(1 - q^{j+1})(1 - cq^{j+r-1})}{(1 - q^{j+r})(1 - cq^j)}$$

$$\text{and} \quad I(r) = \Gamma(r) \prod_{j=0}^{\infty} \frac{(1 - q^{j+r})(1 - cq^j)}{(1 - q^{j+1})(1 - cq^{j+r-1})}.$$

These formulae are proven in Bertoin et al. [9], where they are related to the so-called q-calculus (see, for instance, Gasper–Rahman [16]).

It is easy to see that the measure κ is given by:

$$\kappa = -\log q \sum_{n=1}^{\infty} \bigl(1 - (c/q)^n\bigr)\varepsilon_{-n \log q}.$$

Hence, by Theorem 3.3, $\mathcal{I}$ is not m.i.d. However, it is proven in Bertoin et al. [9] that the variable $\mathcal{I}_0$, corresponding to $c = 0$, i.e.:

$$\mathcal{I}_0 = \int_0^{\infty} q^{N_l}\, \mathrm{d}l,$$

is self-decomposable. Moreover, one has:

$$\mathcal{I}_0 \overset{\text{(law)}}{=} c\mathcal{I}_0 + \mathcal{I}$$

where, in the RHS, the variables are assumed to be independent (see Bertoin et al. ([9], Proposition 3.1)). This entails that the perpetuity $\mathcal{I}$ is infinitely divisible.

4.4. Gamma process

We assume here that $(\xi_l, l \geq 0)$ is the Gamma process. Thus,

$$\forall l \geq 0, \quad \xi_l \overset{\text{(law)}}{=} \gamma_l \quad \text{and} \quad \Phi(s) = \log(1+s).$$

In particular, Φ is a complete Bernstein function ($m(x) = \mathrm{e}^{-x}/x$), but, since $\Phi^*(0) = 1$, $\Phi \notin \mathcal{S}$. Nevertheless, by Corollary 3.1, $\mathcal{I}$ is m.i.d.

We now determine the measure κ. An easy Laplace transform computation (see also Remark 5.1 and formula (5.7) below) shows that the density k of κ is given by:

$$k(x) = \mathrm{e}^{-x} \int_0^\infty \frac{x^l}{\Gamma(l+1)} \,\mathrm{d}l, \tag{4.1}$$

which entails:

$$k'(x) = \mathrm{e}^{-x} \int_0^1 \frac{x^{l-1}}{\Gamma(l)} \,\mathrm{d}l, \tag{4.2}$$

and therefore:

$$k(x) = \int_0^x \int_0^1 \frac{\mathrm{e}^{-y} y^{l-1}}{\Gamma(l)} \,\mathrm{d}l \,\mathrm{d}y = \mathbb{P}(\gamma_U \leq x)$$

where U is uniform on (0, 1) and independent from $(\gamma_l, l \geq 0)$. Consequently, k is an increasing function, and $\lim_{x\to\infty} k(x) = 1$. In particular, the function: $\ell(x) = (\mathrm{e}^x - 1)^{-1}(1 - k(x))$ is decreasing and hence, by Proposition 3.5, $\log \mathcal{I}$ is self-decomposable. Let: $\jmath(x) = (\mathrm{e}^x - 1)^{-1} k(x)$. We deduce easily from (4.1) and (4.2) that:

$$\jmath'(x) \leq \left(\mathrm{e}^x - 1\right)^{-2}\left(1 - \mathrm{e}^{-x} - x\right) \int_0^1 \frac{x^{l-1}}{\Gamma(l)} \,\mathrm{d}l \leq 0$$

and hence, by Proposition 3.5, $\log \mathcal{R}$ is self-decomposable.

The diffusion whose inverse local time is a Gamma process was determined by Donati-Martin and Yor [14].

4.5. Some examples from Bertoin–Yor

The next examples appear in Bertoin–Yor [10].

4.5.1.

Let $\alpha \in (0, 1)$, $c > 1$ and

$$\Phi(s) = \frac{\alpha s \Gamma(\alpha(s - 1 + c))}{\Gamma(\alpha(s + c))}.$$

By Bertoin–Yor [10], Φ is a Bernstein function such that

$$a = 0, \quad \lambda(\mathrm{d}x) = m(x)\,\mathrm{d}x \quad \text{and} \quad m(x) = -h'(x)$$

with

$$h(x) = \frac{1}{\Gamma(\alpha)} \frac{\mathrm{e}^{-(c-1)x}}{(1-\mathrm{e}^{-x/\alpha})^{1-\alpha}}.$$

Then, h is a completely monotone function, hence m is a completely monotone function and Φ is a complete Bernstein function. Consequently, by Corollary 3.1, $\mathcal{I}$ is m.i.d.

We now determine the measure κ. We first remark that formula (3.4) yields, by differentiation, the following classical formula:

$$\frac{\Gamma'}{\Gamma}(r) = \int_0^\infty \left(\frac{\mathrm{e}^{-x}}{x} - \frac{\mathrm{e}^{-rx}}{1-\mathrm{e}^{-x}} \right) \mathrm{d}x. \tag{4.3}$$

We deduce therefrom, by a simple computation,

$$\kappa(\mathrm{d}x) = \frac{1 - \mathrm{e}^{-\frac{1}{\alpha}x} + \mathrm{e}^{-cx} - \mathrm{e}^{-(c-1)x}}{1 - \mathrm{e}^{-\frac{1}{\alpha}x}} \,\mathrm{d}x.$$

Then, with the notation of Proposition 3.5, one has:

$$\ell(x) = \frac{\mathrm{e}^{-cx}}{1 - \mathrm{e}^{-\frac{1}{\alpha}x}}.$$

Clearly, ℓ is a decreasing function, therefore, by Proposition 3.5, $\log \mathcal{I}$ is self-decomposable. Moreover, we obtain easily by Theorem 2.2 and (2.4):

$$I(r) = \alpha^{1-r} \frac{\Gamma(\alpha(r-1+c))}{\Gamma(\alpha c)} \quad \text{and} \quad R(r) = \alpha^{r-1} \frac{\Gamma(r)\Gamma(\alpha c)}{\Gamma(\alpha(r-1+c))}.$$

Therefore, $\mathcal{I} \overset{\text{(law)}}{=} \alpha^{-1}\gamma_{\alpha c}^{\alpha}$ and the law of $\mathcal{R}$ may also be made explicit (see Bertoin–Yor [10]).

It may be noted that, in the case $\alpha = 1/2$, the subordinator whose Laplace–Bernstein exponent is the above complete Bernstein function Φ, appears in Comtet et al. ([13], Example 5.2).

4.5.2.

Let $\alpha \in (0, 1)$, $1 < b \le c$ and

$$\Phi(s) = \frac{s\Gamma(\alpha(s+c))}{(b+s-1)\Gamma(\alpha(s-1+c))}, \quad \Phi^*(s) = \frac{(b+s-1)\Gamma(\alpha(s-1+c))}{\Gamma(\alpha(s+c))}.$$

We have: $\Phi(0) = 0$ and $\Phi^*(0) = (b-1)\Gamma(\alpha(c-1))/\Gamma(\alpha c) \neq 0$. One can prove that:

$$\Phi^*(s) = \Phi^*(0) - \int_0^\infty \left(1 - \mathrm{e}^{-sx}\right) \mathrm{e}^{-(b-1)x} h'(x)\,\mathrm{d}x$$

with

$$h(x) = \frac{1}{\Gamma(1+\alpha)} \frac{e^{-(c-b)x}}{(1-e^{-x/\alpha})^{1-\alpha}}.$$

Then, h is a completely monotone function, hence Φ^* is a complete Bernstein function. By Schilling et al. ([26], Proposition 7.1), Φ also is a complete Bernstein function. Consequently, by Corollary 3.1, $\mathcal{I}$ is m.i.d. It also may be seen that:

$$\kappa(dx) = \left[1 - \left(e^{-(b-1)x} - e^{-(c-1)x} \frac{1-e^{-x}}{1-e^{-\frac{1}{\alpha}x}}\right)\right] dx.$$

Moreover, we obtain by Theorem 2.1 and (2.4):

$$R(r) = \frac{b-1}{\Gamma(\alpha c)} B(r, b-1) \Gamma\big(\alpha(r-1+c)\big) \quad \text{and} \quad I(r) = \frac{\Gamma(\alpha c)\Gamma(r-1+b)}{\Gamma(b)\Gamma(\alpha(r-1+c))}.$$

Therefore, $\mathcal{R} \overset{\text{(law)}}{=} \beta_{1,b-1}\gamma_{\alpha c}^{\alpha}$ with the variables $\beta_{1,b-1}$ and $\gamma_{\alpha c}^{\alpha}$ independent, and the law of $\mathcal{I}$ may also be determined (see Bertoin–Yor [10]).

4.6. Inverse local time of a radial Ornstein–Uhlenbeck process

In this subsection, we assume that $(\xi_l, l \geq 0)$ is an inverse local time process, i.e.:

$$\xi_l = \inf\{t; L_t > l\},$$

where $(L_t, t \geq 0)$ is (a choice of) the local time at 0 for a *radial Ornstein–Uhlenbeck process*, with dimension $\delta = 2(1-\alpha) \in (0,2)$ and parameter $\mu > 0$. This process is defined as the square-root of the $\mathbb{R}_+$-valued diffusion $(Z_t, t \geq 0)$ which solves:

$$Z_t = 2\int_0^t \sqrt{Z_s}\, d\beta_s - 2\mu \int_0^t Z_s\, ds + \delta t$$

where $(\beta_s, s \geq 0)$ denotes a standard real-valued Brownian motion starting from 0. This family of subordinators ξ was studied in Pitman–Yor [25]. We have also devoted some study to this process in [18].

By Pitman–Yor [25] (see also Hirsch–Yor [18]), the Laplace–Bernstein exponent of ξ is, for a suitable choice of the local time,

$$\Phi(s) = \frac{\Gamma(\frac{s}{2\mu} + \alpha)}{\Gamma(\frac{s}{2\mu})}$$

and

$$a = 0 \quad \text{and} \quad \lambda(dx) = \frac{2\mu\alpha}{\Gamma(1-\alpha)} \frac{e^{-2\mu\alpha x}}{(1-e^{-2\mu x})^{1+\alpha}}\, dx.$$

Clearly, Φ is a complete Bernstein function, but $\Phi \notin \mathcal{S}$. By Corollary 3.1, $\mathcal{I}$ is m.i.d. A simple computation, from formula (4.3), yields:

$$\kappa(\mathrm{d}x) = \frac{1-\mathrm{e}^{-2\mu\alpha x}}{1-\mathrm{e}^{-2\mu x}}\,\mathrm{d}x.$$

Then, with the notation of Proposition 3.5, one has:

$$\ell(x) = \frac{\mathrm{e}^{-2\mu\alpha x} - \mathrm{e}^{-2\mu x}}{(1-\mathrm{e}^{-2\mu x})(\mathrm{e}^{x}-1)}.$$

It is not difficult to see that ℓ is a decreasing function. Then, by Proposition 3.5, $\log\mathcal{I}$ is self-decomposable.

- If $2\mu\alpha = 1$, then by Theorem 2.1 and formula (2.4),

$$R(r) = \frac{\Gamma(\alpha r)}{\Gamma(\alpha)} \quad \text{and} \quad I(r) = \frac{\Gamma(\alpha)\Gamma(r)}{\Gamma(\alpha r)}.$$

  Consequently, $\mathcal{R} \overset{\text{(law)}}{=} \gamma_\alpha^\alpha$ and the law of $\mathcal{I}$ is: $(\mathbb{E}[\tau_\alpha^{-\alpha}])^{-1} x \mathbb{P}_{\tau_\alpha^{-\alpha}}(\mathrm{d}x)$, where τ_α denotes a standard α-stable positive variable, i.e.,

$$\mathbb{E}\bigl[\exp(-s\tau_\alpha)\bigr] = \exp\bigl(-s^\alpha\bigr),$$

  and $\mathbb{P}_{\tau_\alpha^{-\alpha}}(\mathrm{d}x)$ denotes the law of $\tau_\alpha^{-\alpha}$.
  Besides, with the notation of Proposition 3.5, one has:

$$\jmath(x) = \frac{\mathrm{e}^{-x}}{1-\mathrm{e}^{-2\mu x}}.$$

  Clearly, $\jmath$ is a decreasing function. Then, by Proposition 3.5, $\log\mathcal{R}$ also is a self-decomposable variable.
- If $2\mu(1-\alpha) = 1$, then by Theorem 2.2 and formula (2.4),

$$I(r) = (1-\alpha)^{1-r}\Gamma\bigl((1-\alpha)(r-1)+1\bigr)$$
$$\text{and} \quad R(r) = \frac{(1-\alpha)^{r-1}\Gamma(r)}{\Gamma((1-\alpha)(r-1)+1)}.$$

  Consequently, $\mathcal{I} \overset{\text{(law)}}{=} (1-\alpha)^{-1}\mathbf{e}^{1-\alpha}$ and $\mathcal{R} \overset{\text{(law)}}{=} (1-\alpha)\tau_{1-\alpha}^{\alpha-1}$.
  Besides, with the notation of Proposition 3.5, one has:

$$\jmath(x) = \frac{1}{\mathrm{e}^{x}-1} - \frac{1}{\mathrm{e}^{\frac{1}{1-\alpha}x}-1}.$$

  It is not difficult to show that $\jmath$ is a decreasing function. Then, by Proposition 3.5, $\log\mathcal{R}$ also is a self-decomposable variable.

Note that, in this case, the Bernstein function Φ writes:

$$\Phi(s) = \frac{\Gamma((1-\alpha)(s-1)+1)}{\Gamma((1-\alpha)s)},$$

and therefore, Φ is the Bernstein function of example 4.5.1 with $c = \alpha^{-1}$, after changing α into $1-\alpha$.

5. Relating our results to Urbanik's

As we wrote in the Introduction, some of our main results, notably those of Section 3, may also be found in Urbanik's paper [27] (for convenience, we shall simply write $\underline{\mathrm{U}}$ when referring to this paper). However, in order that our reader may have some cosiness in comparing our results with those in $\underline{\mathrm{U}}$, we first need to recall and explain the main notation in $\underline{\mathrm{U}}$, which we undertake in the next Subsection 5.1, while Subsection 5.2 is devoted to the statement and explanation (both in $\underline{\mathrm{U}}$'s manner, and with our notation) of the main relevant results in $\underline{\mathrm{U}}$. Finally, in Subsection 5.3, we compare some of $\underline{\mathrm{U}}$'s results with ours.

5.1. Basic notation in $\underline{\mathrm{U}}$

a) In $\underline{\mathrm{U}}$, a subordinator $(\xi_l, l \geq 0)$[1], and any quantity related to it, is always referred to, or tagged, by its representing measure $M(\mathrm{d}x)$ on $\mathbb{R}_+$, as follows:

$$\mathbb{E}\big[\exp(-z\xi_l)\big] = \exp\big(-l\Phi(z)\big), \quad l, z \geq 0,$$

where

$$\Phi(z) = \int_{\mathbb{R}_+} \frac{1-\mathrm{e}^{-zx}}{1-\mathrm{e}^{-x}} M(\mathrm{d}x), \tag{5.1}$$

with the function: $x \longrightarrow \frac{1-\mathrm{e}^{-zx}}{1-\mathrm{e}^{-x}}$ being taken equal to z for $x=0$. (In fact, in $\underline{\mathrm{U}}$, the Bernstein function $\Phi(z)$ is denoted: $\langle M\rangle(z)$.) Note that (5.1) writes:

$$\Phi(z) = zM\big(\{0\}\big) + \int_{(0,+\infty)} \frac{1-\mathrm{e}^{-zx}}{1-\mathrm{e}^{-x}} M(\mathrm{d}x). \tag{5.2}$$

Clearly, from (5.2), the Lévy measure – which we shall denote as $\lambda(\mathrm{d}x)$ – on $(0,+\infty)$, associated to $(\xi_l, l \geq 0)$ is

$$\lambda(\mathrm{d}x) = 1_{(0,+\infty)}(x)\big(1-\mathrm{e}^{-x}\big)^{-1} M(\mathrm{d}x). \tag{5.3}$$

However, in $\underline{\mathrm{U}}$, the Lévy measure is never mentioned, all reference being made to $M(\mathrm{d}x)$, which is a *bounded* measure on $\mathbb{R}_+$.

[1] We keep using, whenever convenient, our notation and we confront it / compare it / with that in $\underline{\mathrm{U}}$.

b) As an important example of the notation in U which we presented so far, we note (U, p. 494) that the representing measure of the standard gamma process $(\gamma_l, l \geq 0)$ is denoted as:

$$\Pi(\mathrm{d}x) = \frac{\mathrm{e}^{-x}(1-\mathrm{e}^{-x})}{x}\,\mathrm{d}x. \tag{5.4}$$

Indeed, the Lévy measure $\lambda_\gamma(\mathrm{d}x)$ of the gamma process $(\gamma_l, l \geq 0)$ is:

$$\lambda_\gamma(\mathrm{d}x) = \frac{\mathrm{e}^{-x}}{x}\,\mathrm{d}x.$$

Also, formulae (1.3) and (1.4) in U may read:

$$\Phi_\gamma(z) \equiv \langle \Pi \rangle(z) = \log(1+z). \tag{5.5}$$

c) In order to understand the main results in U, and their connection with ours, we still need to recall concepts related to the S-transform on positive bounded measures on $\mathbb{R}_+$: $M \longrightarrow SM$ as introduced in U. Rather than describing the S-transform directly, we prefer first to indicate how it acts on Bernstein functions. In U, formula (1.21), we find:

$$\Phi_{SM}(z) = \log\bigl(\Phi_{\overline{M}}(1+z)\bigr) \tag{5.6}$$

where: $\overline{M} = (M(\mathbb{R}_+))^{-1}M$. It is easily seen, and this is confirmed by U, formula (1.20), that the subordinator $(\xi_{SM}(l), l \geq 0)$ may be constructed as $(\widetilde{\xi}_{\gamma_l}, l \geq 0)$, that is by γ-subordination of the subordinator $(\widetilde{\xi}_l, l \geq 0)$, which itself is the Esscher transform of the subordinator $\xi_{\overline{M}}$; precisely,

$$\mathbb{P}^{\widetilde{\xi}}_{|\mathcal{F}_l} = \exp(-X_l + l) \cdot \mathbb{P}^{\xi_{\overline{M}}}_{|\mathcal{F}_l}$$

where $\mathbb{P}^{\widetilde{\xi}}$ and $\mathbb{P}^{\xi_{\overline{M}}}$ are the laws of the subordinators on the canonical Skorokhod space $D([0,+\infty))$, where $X_l(\omega) = \omega(l)$, and $\mathcal{F}_l = \sigma\{X_m; m \leq l\}$. Finally, for concreteness, let us give the explicit form of the measure SM; this is (U, formula (1.25)):

$$SM(\mathrm{d}x) = \bigl(1-\mathrm{e}^{-x}\bigr)\mathrm{e}^{-x}\int_0^\infty l^{-1}e_+(lM)(\mathrm{d}x)\,\mathrm{d}l, \tag{5.7}$$

where $e_+(lM)$ is the notation in U for the law of $\xi_M(l)$, for given l (which we denoted as μ_l in Subsection 2.1).

Remark 5.1. A simple computation from (5.7) shows that, for $s > 0$,

$$\int_{\mathbb{R}_+} \mathrm{e}^{-sx}\bigl(1-\mathrm{e}^{-x}\bigr)^{-1}\mathrm{e}^{x}x\,SM(\mathrm{d}x) = \frac{\Phi'_M(s)}{\Phi_M(s)}.$$

Hence, the measure: $(1-\mathrm{e}^{-x})^{-1}\mathrm{e}^{x}x\,SM(\mathrm{d}x)$ is our measure κ associated to Φ_M (see Proposition 2.1).

5.2. Main results in U

We are now in a right position to state clearly and precisely the main results in U, before comparing them to ours.

a) We first write:

$$\frac{1}{1+z} = \frac{1}{\Phi_{\overline{M}}(1+z)}\left(\frac{\Phi_{\overline{M}}(1+z)}{1+z}\right),$$

and note that the left-hand side, which is the Laplace transform of a standard exponential variable **e**, appears on the right-hand side as a product of two Laplace transforms. Precisely, there is the following theorem.

Theorem 5.1 (U, Proposition 2.1, Theorem 2.3). i) *To every positive, bounded measure M, we may associate a positive r.v. Y_M such that*:

$$\mathbf{e} \overset{\text{(law)}}{=} \xi_{SM}(1) + Y_M, \tag{5.8}$$

where, in the right-hand side Y_M is independent of $\xi_{SM}(1)$.

ii) *Y_M is infinitely divisible if and only if $SM \le \Pi$, in which case, there is the identity in law*:

$$Y_M \overset{\text{(law)}}{=} \xi_{\Pi - SM}(1)$$

and, of course the identity in law (5.8) *extends at the level of subordinators*:

$$(\gamma_l, l \ge 0) \overset{\text{(law)}}{=} (\xi_{SM}(l) + \xi_{\Pi - SM}(l), l \ge 0) \tag{5.9}$$

where, in the right-hand side, the two subordinators are independent.

Besides, in U, Theorem 2.7, a characterization of the measures $\Pi - SM$, assumed to be ≥ 0, is given.

b) We are now – almost – in the right position to state and prove another important result found in U, Theorem 3.3, pertaining to the m.i.d. property of $\mathcal{I}_M$.

Theorem 5.2 (U, Theorem 3.1 and Theorem 3.3). i) *Set*:

$$\mathcal{I}_M = \int_0^\infty \exp\bigl(-\xi_M(l)\bigr)\,\mathrm{d}l.$$

There exists a random variable $\mathcal{R}_M$, independent of $\mathcal{I}_M$, such that:

$$\mathbf{e} \overset{\text{(law)}}{=} \mathcal{I}_M \cdot \mathcal{R}_M.$$

ii) $\mathcal{R}_M$ *is m.i.d. and*

$$-\log \mathcal{R}_M \overset{\text{(law)}}{=} e(r_M, SM) \quad \textit{with} \quad r_M = -\log M(\mathbb{R}_+) + l(SM).$$

iii) $\mathcal{I}_M$ *is m.i.d. if and only if*: $SM \le \Pi$.

iv) *When the condition*: $SM \le \Pi$ *is satisfied, then*:

$$-\log \mathcal{I}_M \overset{\text{(law)}}{=} e(i_M, \Pi - SM) \quad \textit{with} \quad i_M = \log M(\mathbb{R}_+) + l(\Pi - SM).$$

In order that the statement of Theorem 5.2 be totally understandable for our reader, it only remains to explain about the notation $e(a, M)$ and $l(M)$, which is featured twice (in ii) and iv)) in the above Theorem 5.2.

The notation $e(a, M)$ indicates a real-valued r.v., or rather its law, which is infinitely divisible, and whose characteristic function $\widetilde{e}(a, M)$, in the notation of U, is given by:

$$\widetilde{e}(a, M)(s) = \exp\left[\mathrm{i}as + \int_{\mathbb{R}_+} \left(\mathrm{e}^{\mathrm{i}sx} - 1 - \frac{\mathrm{i}sx}{1+x^2}\right) \frac{M(\mathrm{d}x)}{(1-\mathrm{e}^{-x})^2}\right],$$

whereas $l(M)$ is a real number defined as:

$$l(M) = \int_{\mathbb{R}_+} \left(\mathrm{e}^{-x} - 1 + \frac{x}{1+x^2}\right) \frac{M(\mathrm{d}x)}{(1-\mathrm{e}^{-x})^2}.$$

5.3. Comparison of U's results with ours

We are now able to see that our main results of Section 3 appear in the above Theorem 5.2. Indeed, by Remark 5.1,

$$SM(\mathrm{d}x) = \frac{1-\mathrm{e}^{-x}}{x\mathrm{e}^{x}} \kappa(\mathrm{d}x) \tag{5.10}$$

where κ still denotes the measure whose Laplace transform is Φ_M'/Φ_M. In particular, by formula (5.4), U's condition: $SM \le \Pi$ is equivalent to our condition: $\kappa(\mathrm{d}x) \le \mathrm{d}x$. On the other hand, we also obtain from (5.10):

$$\widetilde{e}\big(l(SM), SM\big)(s) = \exp\left[\int_{(0,+\infty)} \frac{\mathrm{e}^{\mathrm{i}sx} - 1 + \mathrm{i}s(\mathrm{e}^{-x}-1)}{x(\mathrm{e}^{-x}-1)} \kappa(\mathrm{d}x)\right]. \tag{5.11}$$

Moreover,

$$\widetilde{e}\big(l(\Pi), \Pi\big)(s) = \exp\left[\int_0^{+\infty} \frac{\mathrm{e}^{\mathrm{i}sx} - 1 + \mathrm{i}s(\mathrm{e}^{-x}-1)}{x(\mathrm{e}^{-x}-1)}\, \mathrm{d}x\right] \tag{5.12}$$

and, by (5.2),

$$M(\mathbb{R}_+) = \Phi_M(1). \tag{5.13}$$

Thus, in view of the above formulae (5.11), (5.12) and (5.13), property ii) in Theorem 5.2 corresponds to our Theorem 3.1, while properties iii) and iv) in Theorem 5.2 correspond to our formula (3.5), Theorem 3.2 and Theorem 3.3.

Moreover, in U, Theorem 2.5, it is stated that, if Φ_M is a complete Bernstein function, then $\mathcal{I}_M$ is m.i.d., which is our Corollary 3.1.

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