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Invariance principles for homogeneous sums of free random variables

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We extend, in the free probability framework, an invariance principle for multilinear homogeneous sums with low influences recently established by Mossel, O'Donnel and Oleszkiewicz in [*Ann. of Math.* (2) **171** (2010) 295–341]. We then deduce several universality phenomenons, in the spirit of the paper [*Ann. Probab.* **38** (2010) 1947–1985] by Nourdin, Peccati and Reinert.

Keywords: central limit theorems; chaos; free Brownian motion; free probability; homogeneous sums; Lindeberg principle; universality; Wigner chaos

1. Introduction and background

Motivation and main goal. Our starting point is the following weak version (which is enough for our purpose) of an invariance principle for multilinear homogeneous sums with low influences, recently established in [7].

Theorem 1.1 (Mossel–O'Donnel–Oleszkiewicz). *Let* $(\Omega, \mathcal{F}, P)$ *be a probability space (in the classical sense). Let* $X_1, X_2, \dots$ *(resp.,* $Y_1, Y_2, \dots$*) be a sequence of independent centered random variables with unit variance satisfying moreover*

$$\sup_{i\geq 1} E\bigl[|X_i|^r\bigr] < \infty \qquad \Bigl(\textit{resp.}, \sup_{i\geq 1} E\bigl[|Y_i|^r\bigr] < \infty\Bigr) \textit{ for all } r \geq 1.$$

Fix $d \geq 1$*, and consider a sequence of functions* $f_N \colon \{1, \dots, N\}^d \to \mathbb{R}$ *satisfying the following two assumptions for each* N *and each* $i_1, \dots, i_d = 1, \dots, N$:

(i) *(full symmetry)* $f_N(i_1, \dots, i_d) = f_N(i_{\sigma(1)}, \dots, i_{\sigma(d)})$ *for all* $\sigma \in \mathfrak{S}_d$;
(ii) *(normalization)* $d! \sum_{j_1,\dots,j_d=1}^{N} f_N(j_1, \dots, j_d)^2 = 1$.

Also, set

$$Q_N(x_1, \dots, x_N) = \sum_{i_1,\dots,i_d=1}^{N} f_N(i_1, \dots, i_d) x_{i_1} \cdots x_{i_d} \tag{1}$$

and

$$\mathrm{Inf}_i(f_N) = \sum_{j_2,\dots,j_d=1}^{N} f_N(i, j_2, \dots, j_d)^2, \qquad i = 1, \dots, N.$$

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Then, for any integer $m \geq 1$,

$$E\big[Q_N(X_1,\dots,X_N)^m\big] - E\big[Q_N(Y_1,\dots,Y_N)^m\big] = \mathrm{O}\big(\tau_N^{1/2}\big), \tag{2}$$

where $\tau_N = \max_{1\leq i\leq N} \mathrm{Inf}_i(f_N)$.

In [7], the authors were motivated by solving two conjectures, namely the *Majority Is Stablest* conjecture from theoretical computer science and the *It Ain't Over Till It's Over* conjecture from social choice theory. It is worthwhile noting that there is another striking consequence of Theorem 1.1, more in the spirit of the classical central limit theorem. Indeed, in article [11] Nourdin, Peccati and Reinert combined Theorem 1.1 with the celebrated *Fourth Moment theorem* of Nualart and Peccati [12], and deduced that multilinear homogenous sums of general centered independent random variables with unit variance enjoy the following universality phenomenon.

Theorem 1.2 (Nourdin–Peccati–Reinert). *Let* $(\Omega, \mathcal{F}, P)$ *be a probability space* (*in the classical sense*). *Let* $G_1, G_2, \dots$ *be a sequence of i.i.d.* $\mathcal{N}(0,1)$ *random variables. Fix* $d \geq 2$ *and consider a sequence of functions* $f_N\colon \{1,\dots,N\}^d \to \mathbb{R}$ *satisfying the following three assumptions for each* N *and each* $i_1,\dots,i_d = 1,\dots,N$:

(i) (*full symmetry*) $f_N(i_1,\dots,i_d) = f_N(i_{\sigma(1)},\dots,i_{\sigma(d)})$ *for all* $\sigma \in \mathfrak{S}_d$;
(ii) (*vanishing on diagonals*) $f_N(i_1,\dots,i_d) = 0$ *if* $i_k = i_l$ *for some* $k \neq l$;
(iii) (*normalization*) $d!\sum_{j_1,\dots,j_d=1}^{N} f_N(j_1,\dots,j_d)^2 = 1$.

Also, let $Q_N(x_1,\dots,x_N)$ *be given by* (1). *Then, the following two conclusions are equivalent as* $N \to \infty$:

(A) $Q_N(G_1,\dots,G_N) \overset{\text{law}}{\to} \mathcal{N}(0,1)$;
(B) $Q_N(X_1,\dots,X_N) \overset{\text{law}}{\to} \mathcal{N}(0,1)$ *for* any *sequence* $X_1, X_2, \dots$ *of i.i.d. centered random variables with unit variance and all moments.*

In the present paper, our goal is twofold. We shall first extend Theorem 1.1 in the context of free probability and we shall then investigate whether a result such as Theorem 1.2 continues to hold true in this framework. We are motivated by the fact that there is often a close correspondence between classical probability and free probability, in which the Gaussian law (resp., the classical notion of independence) has the semicircular law (resp., the notion of free independence) as an analogue.

Free probability in a nutshell. Before going into the details and for the sake of clarity, let us first introduce some of the central concepts in the theory of free probability. (See [9] for a systematic presentation.)

A *non-commutative probability space* is a von Neumann algebra $\mathcal{A}$ (i.e., an algebra of operators on a real separable Hilbert space, closed under adjoint and convergence in the weak operator topology) equipped with a *trace* φ, that is, a unital linear functional (meaning preserving the identity) which is weakly continuous, positive (meaning $\varphi(X) \geq 0$ whenever X is a non-negative element of $\mathcal{A}$; i.e., whenever $X = YY^*$ for some $Y \in \mathcal{A}$), faithful (meaning that if $\varphi(YY^*) = 0$ then $Y = 0$), and tracial (meaning that $\varphi(XY) = \varphi(YX)$ for all $X, Y \in \mathcal{A}$, even though in general $XY \neq YX$).

In a non-commutative probability space, we refer to the self-adjoint elements of the algebra as *random variables*. Any random variable X has a *law*: this is the unique probability measure μ on $\mathbb{R}$ with the same moments as X; in other words, μ is such that

$$\int_{\mathbb{R}} Q(x)\,\mathrm{d}\mu(x) = \varphi\big(Q(X)\big) \tag{3}$$

for any real polynomial Q.

In a non-commutative probability setting, the central notion of *free independence* (introduced by Voiculescu in [14]) goes as follows. Let $\mathcal{A}_1, \dots, \mathcal{A}_p$ be unital subalgebras of $\mathcal{A}$. Let $X_1, \dots, X_m$ be elements chosen among the $\mathcal{A}_i$'s such that, for $1 \le j < m$, two consecutive elements X_j and X_{j+1} do not come from the same $\mathcal{A}_i$, and such that $\varphi(X_j) = 0$ for each j. The subalgebras $\mathcal{A}_1, \dots, \mathcal{A}_p$ are said to be *free* or *freely independent* if, in this circumstance,

$$\varphi(X_1 X_2 \cdots X_m) = 0. \tag{4}$$

Random variables are called freely independent if the unital algebras they generate are freely independent. If X, Y are freely independent, then their joint moments are determined by the moments of X and Y separately as in the classical case.

The *semicircular distribution* $\mathcal{S}(m, \sigma^2)$ with mean $m \in \mathbb{R}$ and variance $\sigma^2 > 0$ is the probability distribution

$$\mathcal{S}\big(m, \sigma^2\big)(\mathrm{d}x) = \frac{1}{2\pi\sigma^2}\sqrt{4\sigma^2 - (x-m)^2}\,\mathbf{1}_{\{|x-m| \le 2\sigma\}}\,\mathrm{d}x.$$

If $m = 0$, this distribution is symmetric around 0, and therefore its odd moments are all 0. A simple calculation shows that the even centered moments are given by (scaled) Catalan numbers: for non-negative integers k,

$$\int_{m-2\sigma}^{m+2\sigma} (x-m)^{2k} \mathcal{S}\big(m, \sigma^2\big)(\mathrm{d}x) = C_k \sigma^{2k},$$

where $C_k = \frac{1}{k+1}\binom{2k}{k}$ (see, e.g., [9], Lecture 2).

Our main results. We are now in a position to state our first main result, which is nothing but a suitable generalization of Theorem 1.1 in the free probability setting.

Theorem 1.3. *Let* $(\mathcal{A}, \varphi)$ *be a non-commutative probability space. Let* $X_1, X_2, \dots$ (*resp.*, $Y_1, Y_2, \dots$) *be a sequence of centered free random variables with unit variance* (*i.e.*, *such that* $\varphi(X_i^2) = \varphi(Y_i^2) = 1$ *for all* i), *satisfying moreover*

$$\sup_{i \ge 1} \varphi\big(|X_i|^r\big) < \infty \qquad \Big(\textit{resp.}, \sup_{i \ge 1} \varphi\big(|Y_i|^r\big) < \infty\Big) \textit{ for all } r \ge 1,$$

where $|X| = \sqrt{X^* X}$. *Fix* $d \ge 1$, *and consider a sequence of functions* $f_N \colon \{1, \dots, N\}^d \to \mathbb{R}$ *satisfying the following three assumptions for each* N *and each* $i_1, \dots, i_d = 1, \dots, N$:

(i) (*mirror-symmetry*) $f_N(i_1, \dots, i_d) = f_N(i_d, \dots, i_1)$;

(ii) (*vanishing on diagonals*) $f_N(i_1,\dots,i_d)=0$ *if* $i_k=i_l$ *for some* $k\neq l$;
(iii) (*normalization*) $\sum_{j_1,\dots,j_d=1}^N f_N(j_1,\dots,j_d)^2=1$.

Also, set

$$Q_N(x_1,\dots,x_N)=\sum_{i_1,\dots,i_d=1}^N f_N(i_1,\dots,i_d)x_{i_1}\cdots x_{i_d} \tag{5}$$

and

$$\mathrm{Inf}_i(f_N)=\sum_{l=1}^d\sum_{j_1,\dots,j_{d-1}=1}^N f_N(j_1,\dots,j_{l-1},i,j_l,\dots,j_{d-1})^2,\qquad i=1,\dots,N.$$

Then, for any integer $m\geq 1$,

$$\varphi\big(Q_N(X_1,\dots,X_N)^m\big)-\varphi\big(Q_N(Y_1,\dots,Y_N)^m\big)=\mathrm{O}\big(\tau_N^{1/2}\big), \tag{6}$$

where $\tau_N=\max_{1\leq i\leq N}\mathrm{Inf}_i(f_N)$.

Due to the lack of commutativity of the variables involved, the proof of Theorem 1.3 raises new difficulties with respect to its commutative counterpart. Moreover, it is worthwhile noting that it contains the free central limit theorem as an immediate corollary. Indeed, let us choose $d=1$ (in this case, assumptions (i) and (ii) are of course immaterial), $Y_1,Y_2,\dots\sim\mathcal{S}(0,1)$ and $f_N(i)=\frac{1}{\sqrt{N}}$, $i=1,\dots,N$. We then have $Q_N(Y_1,\dots,Y_N)\sim\mathcal{S}(0,1)\overset{\text{law}}{=}Y_1$ (thanks to (iii) as well as the fact that a sum of freely independent semicircular random variables remains semicircular) and $\tau_N\to 0$ as $N\to\infty$, so that, thanks to (6),

$$\varphi\bigg[\bigg(\frac{X_1+\cdots+X_N}{\sqrt{N}}\bigg)^m\bigg]\to\varphi\big(Y_1^m\big)$$

for each $m\geq 1$ as $N\to\infty$, which is exactly what the free central limit theorem asserts.

When $d\geq 2$, by combining Theorem 1.3 with the main finding of [4], we will prove the following free counterpart of Theorem 1.2.

Theorem 1.4. *Let* $(\mathcal{A},\varphi)$ *be a non-commutative probability space. Let* $S_1,S_2,\dots$ *be a sequence of free* $\mathcal{S}(0,1)$ *random variables. Fix* $d\geq 2$ *and consider a sequence of functions* $f_N\colon\{1,\dots,N\}^d\to\mathbb{R}$ *satisfying the following three assumptions for each* N *and each* $i_1,\dots,i_d=1,\dots,N$:

(i) (*full symmetry*) $f_N(i_1,\dots,i_d)=f_N(i_{\sigma(1)},\dots,i_{\sigma(d)})$ *for all* $\sigma\in\mathfrak{S}_d$;
(ii) (*vanishing on diagonals*) $f_N(i_1,\dots,i_d)=0$ *if* $i_k=i_l$ *for some* $k\neq l$;
(iii) (*normalization*) $\sum_{j_1,\dots,j_d=1}^N f_N(j_1,\dots,j_d)^2=1$.

Also, let $Q_N(x_1,\dots,x_N)$ *be the polynomial in non-commuting variables given by* (5). *Then, the following two conclusions are equivalent as* $N\to\infty$:

(A) $Q_N(S_1, \dots, S_N) \stackrel{\text{law}}{\to} \mathcal{S}(0,1)$;

(B) $Q_N(X_1, \dots, X_N) \stackrel{\text{law}}{\to} \mathcal{S}(0,1)$ *for any sequence* $X_1, X_2, \dots$ *of free identically distributed and centered random variables with unit variance.*

Although a weak 'mirror-symmetry' assumption would have been undoubtedly more natural, we impose in Theorem 1.4 the same 'full symmetry' assumption (i) than in Theorem 1.2. This is unfortunately not insignificant in our non-commutative framework. But we cannot expect better by using our strategy of proof, as is illustrated by a concrete counterexample in Section 2.

Theorem 1.4 may be seen as a free universality phenomenon, in the sense that the semicircular behavior of $Q_N(X_1, \dots, X_N)$ is asymptotically insensitive to the distribution of its summands. In reality, this is more subtle, as the following explicit situation well illustrates in the case $d = 2$ (quadratic case). Indeed, let us consider

$$Q_N(x_1, \dots, x_N) = \frac{1}{\sqrt{2N-2}} \sum_{i=2}^{N} (x_1 x_i + x_i x_1), \qquad N \geq 2,$$

let $S_1, S_2, \dots$ be a sequence of free $\mathcal{S}(0,1)$ random variables and let $X_1, X_2, \dots$ be a sequence of free Rademacher random variables (i.e., the law of X_1 is given by $\frac{1}{2}\delta_1 + \frac{1}{2}\delta_{-1}$). Then $Q_N(X_1, \dots, X_N) \stackrel{\text{law}}{\to} \mathcal{S}(0,1)$ as $N \to \infty$, but

$$Q_N(S_1, \dots, S_N) \stackrel{\text{law}}{\to} \frac{1}{\sqrt{2}}(S_1 S_2 + S_2 S_1) \not\sim \mathcal{S}(0,1).$$

(See Section 2 for the details.) This means that it is possible to have $Q_N(X_1, \dots, X_N)$ converging in law to $\mathcal{S}(0,1)$ for a *particular* centered distribution of X_1, without having the same phenomenon for *every* centered distribution with variance one. The question of which are the distributions that enjoy such a universality phenomenon is still an open problem. (In the commutative case, it is known that the Gaussian and the Poisson distributions both lead to universality, see [11,13]. Yet there are no other examples.)

Organization of the paper. The rest of our paper is organized as follows. In Section 2, we deduce from Theorem 1.3 several results connected with the universality phenomenon and we study the limitations of Theorem 1.4. Section 3 is devoted to the proof of Theorem 1.3.

2. Free universality

In this section, we show how Theorem 1.3 leads to several results connected with the universality phenomenon. We also study the limitations of Theorem 1.4: Can we replace the role played by the semicircular distribution by any other law? Can we replace the full symmetry assumption (i) by a more natural one?

To do so, we first need to recall some facts proven in references [1,4].

Convergence of Wigner integrals. For $1 \leq p \leq \infty$, we write $L^p(\mathcal{A}, \varphi)$ to indicate the L^p space obtained as the completion of $\mathcal{A}$ with respect to the norm $\|A\|_p = \varphi(|A|^p)^{1/p}$, where $|A| = \sqrt{A^*A}$, and $\|\cdot\|_\infty$ stands for the operator norm. For every integer $q \geq 2$, the space $L^2(\mathbb{R}_+^q)$

is the collection of all real-valued functions on $\mathbb{R}_+^q$ that are square-integrable with respect to the Lebesgue measure. Given $f \in L^2(\mathbb{R}_+^q)$, we write $f^*(t_1, t_2, \dots, t_q) = f(t_q, \dots, t_2, t_1)$, and we call f^* the *adjoint* of f. We say that an element of $L^2(\mathbb{R}_+^q)$ is *mirror symmetric* whenever $f = f^*$ as a function. Given $f \in L^2(\mathbb{R}_+^q)$ and $g \in L^2(\mathbb{R}_+^p)$, for every $r = 1, \dots, p \wedge q$ we define the rth *contraction* of f and g as the element of $L^2(\mathbb{R}_+^{p+q-2r})$ given by

$$\begin{aligned} &f \overset{r}{\frown} g(t_1, \dots, t_{p+q-2r}) \\ &\quad = \int_{\mathbb{R}_+^{p+q-2r}} f(t_1, \dots, t_{p-r}, x_1, \dots, x_r) g(x_r, \dots, x_1, t_{p-r+1}, \dots, t_{p+q-2r}) \, \mathrm{d}x_1 \cdots \mathrm{d}x_r. \end{aligned} \tag{7}$$

One also writes $f \frown^0 g(t_1, \dots, t_{p+q}) = f \otimes g(t_1, \dots, t_{p+q}) = f(t_1, \dots, t_q) g(t_{q+1}, \dots, t_{p+q})$. In the following, we shall use the notation $f \frown^0 g$ and $f \otimes g$ interchangeably. Observe that, if $p = q$, then $f \frown^p g = \langle f, g^* \rangle_{L^2(\mathbb{R}_+^q)}$.

A *free Brownian motion* S on $(\mathcal{A}, \varphi)$ consists of: (i) a filtration $\{\mathcal{A}_t \colon t \geq 0\}$ of von Neumann sub-algebras of $\mathcal{A}$ (in particular, $\mathcal{A}_u \subset \mathcal{A}_t$ for $0 \leq u < t$), (ii) a collection $S = (S_t)_{t \geq 0}$ of self-adjoint operators such that:

- $S_t \in \mathcal{A}_t$ for every t;
- for every t, S_t has a semicircular distribution $\mathcal{S}(0, t)$;
- for every $0 \leq u < t$, the increment $S_t - S_u$ is freely independent of $\mathcal{A}_u$, and has a semicircular distribution $\mathcal{S}(0, t - u)$.

For every integer $q \geq 1$, the collection of all random variables of the type $I_q(f)$, $f \in L^2(\mathbb{R}_+^q)$, is called the qth *Wigner chaos* associated with S, and is defined according to [1], Section 5.3, namely:

- first define $I_q(f) = (S_{b_1} - S_{a_1}) \cdots (S_{b_q} - S_{a_q})$ for every function f having the form
$$f(t_1, \dots, t_q) = \mathbf{1}_{(a_1, b_1)}(t_1) \times \cdots \times \mathbf{1}_{(a_q, b_q)}(t_q), \tag{8}$$
where the intervals (a_i, b_i), $i = 1, \dots, q$, are pairwise disjoint;
- extend linearly the definition of $I_q(f)$ to simple functions vanishing on diagonals, that is, to functions f that are finite linear combinations of indicators of the type (8);
- exploit the isometric relation
$$\langle I_q(f_1), I_q(f_2) \rangle_{L^2(\mathcal{A}, \varphi)} = \varphi\big(I_q(f_1)^* I_q(f_2)\big) = \varphi\big(I_q\big(f_1^*\big) I_q(f_2)\big) = \langle f_1, f_2 \rangle_{L^2(\mathbb{R}_+^q)}, \tag{9}$$
where f_1, f_2 are simple functions vanishing on diagonals, and use a density argument to define $I_q(f)$ for a general $f \in L^2(\mathbb{R}_+^q)$.

Observe that relation (9) continues to hold for every pair $f_1, f_2 \in L^2(\mathbb{R}_+^q)$. Moreover, the above sketched construction implies that $I_q(f)$ is self-adjoint if and only if f is mirror symmetric. We recall the following fundamental multiplication formula, proven in [1]. For every $f \in L^2(\mathbb{R}_+^p)$ and $g \in L^2(\mathbb{R}_+^q)$, where $p, q \geq 1$, we have

$$I_p(f) I_q(g) = \sum_{r=0}^{p \wedge q} I_{p+q-2r}(f \overset{r}{\frown} g). \tag{10}$$

Let $S_1, S_2, \dots \sim \mathcal{S}(0,1)$ be freely independent, fix $d \geq 2$, and consider a sequence of functions $f_N\colon \{1,\dots,N\}^d \to \mathbb{R}$ satisfying assumptions (ii) and (iii) of Theorem 1.4 as well as

$$f_N(i_1,\dots,i_d) = f_N(i_d,\dots,i_1) \qquad \text{for all } N \geq 1 \text{ and } i_1,\dots,i_d \in \{1,\dots,N\}. \tag{11}$$

Let also $Q_N(x_1,\dots,x_N)$ be the polynomial in non-commuting variables given by (5). Set $e_i = \mathbf{1}_{[i-1,i]} \in L^2(\mathbb{R}_+)$, $i \geq 1$. For each N, one has

$$Q_N(S_1,\dots,S_N) \stackrel{\text{law}}{=} Q_N\big(I_1(e_1),\dots,I_1(e_N)\big). \tag{12}$$

By applying the multiplication formula (10) and by taking into account assumption (ii), it is straightforward to check that

$$Q_N\big(I_1(e_1),\dots,I_1(e_N)\big) = I_d(g_N), \tag{13}$$

where

$$g_N = \sum_{i_1,\dots,i_d=1}^{N} f_N(i_1,\dots,i_d) e_{i_1} \otimes \cdots \otimes e_{i_d}. \tag{14}$$

The function g_N is mirror-symmetric (due to (11)) and has an $L^2(\mathbb{R}_+^d)$-norm equal to 1 (due to (iii)). Using both Theorems 1.3 and 1.6 of [4] (see also [10]), we deduce that the following equivalence holds true as $N \to \infty$:

$$Q_N(S_1,\dots,S_N) \stackrel{\text{law}}{\to} \mathcal{S}(0,1) \Longleftrightarrow \|g_N \stackrel{r}{\frown} g_N\|_{L^2(\mathbb{R}_+^{2d-2r})} \to 0 \quad \text{for all } r \in \{1,\dots,d-1\}. \tag{15}$$

For $r = d-1$, observe that

$$\begin{aligned}
&\|g_N \stackrel{d-1}{\frown} g_N\|_{L^2(\mathbb{R}_+^2)} \\
&= \left\| \sum_{i,j=1}^{N} \left(\sum_{k_2,\dots,k_d=1}^{N} f_N(i,k_2,\dots,k_d) f_N(k_d,\dots,k_2,j) \right) e_i \otimes e_j \right\|_{L^2(\mathbb{R}_+^2)} \\
&= \sqrt{\sum_{i,j=1}^{N} \left(\sum_{k_2,\dots,k_d=1}^{N} f_N(i,k_2,\dots,k_d) f_N(k_d,\dots,k_2,j) \right)^2} \\
&\geq \sqrt{\sum_{i=1}^{N} \left(\sum_{k_2,\dots,k_d=1}^{N} f_N(i,k_2,\dots,k_d)^2 \right)^2} \qquad \big(\text{by setting } j = i \text{ and using (11)}\big) \\
&\geq \max_{i=1,\dots,N} \sum_{k_2,\dots,k_d=1}^{N} f_N(i,k_2,\dots,k_d)^2.
\end{aligned} \tag{16}$$

Proof of Theorem 1.4. Of course, only the implication (A) → (B) has to be shown. Assume that (A) holds. Then, using (15) (condition (i) implies in particular (11)), we get that $\|g_N \frown^{d-1} g_N\|_{L^2(\mathbb{R}_+^2)} \to 0$ as $N \to \infty$. Using (16) and since f_N is fully-symmetric, we deduce that the quantity τ_N of Theorem 1.3 tends to zero as N goes to infinity. This, combined with assumption (A) and (6), leads to (B). □

A counterexample. In Theorem 1.4, can we replace the role played by the semicircular distribution by any other law? The answer is no in general. Indeed, let us take a look at the following situation. Fix $d = 2$ and consider

$$Q_N(x_1, \dots, x_N) = \frac{1}{\sqrt{2N-2}} \sum_{i=2}^{N} (x_1 x_i + x_i x_1), \qquad N \geq 2.$$

Let $S_1, S_2, \dots$ be a sequence of free $\mathcal{S}(0,1)$ random variables and let $X_1, X_2, \dots$ be a sequence of free Rademacher random variables (i.e., the law of X_1 is given by $\frac{1}{2}\delta_1 + \frac{1}{2}\delta_{-1}$). Then, using the free central limit theorem, it is clear on one hand that

$$\begin{aligned} Q_N(X_1, \dots, X_N) &= \frac{1}{\sqrt{2}} X_1 \left(\frac{1}{\sqrt{N-1}} \sum_{i=2}^{N} X_i \right) + \frac{1}{\sqrt{2}} \left(\frac{1}{\sqrt{N-1}} \sum_{i=2}^{N} X_i \right) X_1 \\ &\overset{\text{law}}{\to} \frac{1}{\sqrt{2}} (X_1 S_1 + S_1 X_1) \qquad \text{as } N \to \infty, \end{aligned}$$

with X_1 and S_1 freely independent. By Proposition 1.10 and identity (1.10) of Nica and Speicher [8], it turns out that $\frac{1}{\sqrt{2}}(X_1 S_1 + S_1 X_1) \sim \mathcal{S}(0,1)$. But, on the other hand,

$$\begin{aligned} Q_N(S_1, \dots, S_N) &= \frac{1}{\sqrt{2}} S_1 \left(\frac{1}{\sqrt{N-1}} \sum_{i=2}^{N} S_i \right) + \frac{1}{\sqrt{2}} \left(\frac{1}{\sqrt{N-1}} \sum_{i=2}^{N} S_i \right) S_1 \\ &\overset{\text{law}}{=} \frac{1}{\sqrt{2}} (S_1 S_2 + S_2 S_1). \end{aligned}$$

The random variable $\frac{1}{\sqrt{2}}(S_1 S_2 + S_2 S_1)$ being *not* $\mathcal{S}(0,1)$ distributed (its law is indeed the so-called *tetilla law*, see [2]), we deduce that one cannot replace the role played by the semicircular distribution in Theorem 1.4 by the Rademacher distribution.

Another counterexample. In Theorem 1.4, can we replace the full symmetry assumption (i) by the mirror-symmetry assumption? Unfortunately, we have not been able to answer this question. But if the answer is yes, what is sure is that we cannot use the same arguments as in the fully-symmetric case to show such a result. Indeed, when f_N is fully-symmetric we have

$$\tau_N = d \times \max_{i=1,\dots,N} \sum_{k_2,\dots,k_d=1}^{N} f_N(i, k_2, \dots, k_d)^2,$$

allowing us to prove Theorem 1.4 by using the following set of implications: as $N \to \infty$,

$$\begin{aligned} Q_N(S_1, \dots, S_N) \xrightarrow{\text{law}} \mathcal{S}(0,1) \quad &\overset{(15)}{\Longrightarrow} \quad \|g_N \overset{d-1}{\frown} g_N\|_{L^2(\mathbb{R}_+^2)} \to 0 \overset{(16)}{\Longrightarrow} \tau_N \to 0 \\ &\overset{\text{Theorem 1.3}}{\Longrightarrow} Q_N(X_1, \dots, X_N) \xrightarrow{\text{law}} \mathcal{S}(0,1). \end{aligned} \tag{17}$$

Unfortunately, when f_N is only mirror-symmetric the implication

$$\|g_N \overset{d-1}{\frown} g_N\|_{L^2(\mathbb{R}_+^2)} \to 0 \Longrightarrow \tau_N \to 0, \tag{18}$$

that plays a crucial role in (17), is no longer true in general. To see why, let us consider the following counterexample (for which we fix $d = 3$). Define first a sequence of functions $f'_N \colon \{1, \dots, N\}^2 \to \mathbb{R}$ according to the formula

$$f'_N(i, i+1) = f'_N(i+1, i) = \frac{1}{\sqrt{2N-2}},$$

and $f'_N(i, j) = 0$ whenever $i = j$ or $|j - i| \geq 2$. Next, for $i, j, k \in \{1, \dots, N\}$, set

$$\begin{aligned} &f_N(i, j, k) \\ &\quad = \begin{cases} 0, & \text{if } j \geq 2 \text{ or } (j = 1 \text{ and } i = 1) \text{ or } (j = 1 \text{ and } k = 1), \\ f'_{N-1}(i-1, k-1), & \text{otherwise.} \end{cases} \end{aligned} \tag{19}$$

Easy-to-check properties of f_N include mirror-symmetry, vanishing on diagonals property,

$$\sum_{i,j,k=1}^{N} f_N(i, j, k)^2 = \sum_{i,k=1}^{N-1} f'_{N-1}(i, k)^2 = 1$$

and

$$\begin{aligned} &\sum_{i,j=1}^{N} \left(\sum_{k,l=1}^{N} f_N(i, k, l) f_N(l, k, j) \right)^2 \\ &\quad = \sum_{i,j=1}^{N} \left(\sum_{l=1}^{N-1} f'_{N-1}(i, l) f'_{N-1}(l, j) \right)^2 \to 0. \end{aligned} \tag{20}$$

Let g_N be given by (14), that is,

$$g_N = \frac{1}{\sqrt{2N-4}} \sum_{i=1}^{N-2} (e_{i+1} \otimes e_1 \otimes e_{i+2} + e_{i+2} \otimes e_1 \otimes e_{i+1}).$$

The limit (20) can be readily translated into $\|g_N \frown^2 g_N\|^2_{L^2(\mathbb{R}_+^2)} \to 0$ as $N \to \infty$. On the other hand, we have

$$\tau_N = \max_{1\le j\le N} \mathrm{Inf}_j(f_N) = \max_{1\le j\le N} \sum_{i,k=1}^N \left\{f_N(i,j,k)^2 + f_N(j,i,k)^2 + f_N(i,k,j)^2\right\}$$

$$\ge \max_{1\le j\le N} \sum_{i,k=1}^N f_N(i,j,k)^2 = \sum_{i,k=1}^N f_N(i,1,k)^2 = 1,$$

which contradicts (18), as announced.

It is also worth noting that the sequence of functions f_N defined by (19) provides an explicit counterexample to the so-called *Wiener-Wigner transfer principle* (see [4], Theorem 1.8) in a non-fully-symmetric situation. Indeed, on one hand, we have

$$\|g_N \overset{1}{\frown} g_N\|^2_{L^2(\mathbb{R}_+^2)} = \|g_N \overset{2}{\frown} g_N\|^2_{L^2(\mathbb{R}_+^2)} \to 0 \qquad \text{as } N \to \infty,$$

which, due to (15), entails that $Q_N(S_1,\dots,S_N) \overset{\text{law}}{\to} \mathcal{S}(0,1)$. On the other hand, let $G_1,\dots,G_N \sim \mathcal{N}(0,1)$ be independent random variables defined on a (classical) probability space $(\Omega,\mathcal{F},P)$. One has

$$Q_N(G_1,\dots,G_N) = G_1 \times \left(\frac{2}{\sqrt{2N-4}} \sum_{i=2}^{N-1} G_i G_{i+1}\right),$$

and it is easily checked that $\frac{2}{\sqrt{2N-4}} \sum_{i=2}^{N-1} G_i G_{i+1} \overset{\text{law}}{\to} \mathcal{N}(0,2)$ (apply, e.g., the Fourth Moment theorem of [12]). As a result, the sequence $Q_N(G_1,\dots,G_N)$ converges in law to $\sqrt{2}G_1G_2$, which is not Gaussian. This leads to our desired contradiction.

Free CLT for homogeneous sums. As an application of Theorem 1.3, let us also highlight the following practical convergence criterion for multilinear polynomials, which can be readily derived from (15).

Theorem 2.1. *Let $(\mathcal{A},\varphi)$ be a non-commutative probability space. Let $X_1, X_2, \dots$ be a sequence of centered free random variables with unit variance satisfying $\sup_{i\ge 1}\varphi(|X_i|^r) < \infty$ for all $r \ge 1$. Fix $d \ge 1$, and consider a sequence of functions $f_N\colon \{1,\dots,N\}^d \to \mathbb{R}$ satisfying the three basic assumptions* (i)–(ii)–(iii) *of Theorem* 1.3. *Assume moreover that, as N tends to infinity, $\max_{1\le j\le N} \mathrm{Inf}_j(f_N) \to 0$ and $\|g_N \frown^r g_N\|_{L^2(\mathbb{R}_+^{2d-2r})} \to 0$ for all $r \in \{1,\dots,d-1\}$, where g_N is defined through* (14). *Then one has*

$$\sum_{i_1,\dots,i_d=1}^N f_N(i_1,\dots,i_d) X_{i_1}\cdots X_{i_d} \overset{\text{law}}{\to} \mathcal{S}(0,1). \tag{21}$$

For instance, thanks to this result one can easily check that, given a positive integer k, one has

$$\frac{1}{\sqrt{N}}\sum_{i=1}^{N-k}\{X_i X_{i+1}\cdots X_{i+k} + X_{i+k}X_{i+k-1}\cdots X_i\} \stackrel{\text{law}}{\to} \mathcal{S}(0,1) \qquad \text{as } N\to\infty$$

for any sequence (X_i) of centered free random variables with unit variance satisfying $\sup_{i\geq 1}\varphi(|X_i|^r)<\infty$ for all $r\geq 1$.

3. Proof of Theorem 1.3

As in [7], our strategy is essentially based on a generalization of the classical Lindeberg method, which was originally designed for linear sums of (classical) random variables (see [6]). Before we turn to the details of the proof, let us briefly report the two main differences with the arguments displayed in [7] for commuting random variables.

First, in this non-commutative context, we can no longer rely on some classical Taylor expansion as a starting point of our study. This issue can be easily overcome though, by resorting to abstract expansion formulae (see (24)) together with appropriate Hölder-type estimates (see (28)). As far as this particular point is concerned, the situation is quite similar to what can be found in [3], even if the latter reference is only concerned with the linear case, that is, $d=1$.

Another additional difficulty raised by this free background lies in the transposition of the hypercontractivity property, which is at the core of the procedure. In [7], the proof of hypercontractivity for multilinear polynomials heavily depends on the fact that the variables do commute (see, e.g., the proof of [7], Proposition 3.11). Hence, new arguments are needed here and we postpone this point to Section 3.2.

3.1. General strategy

For the rest of the section, we fix two sequences $(X_i),(Y_i)$ of random variables in a non-commutative probability space $(\mathcal{A},\varphi)$, two integers $N,m\geq 1$, as well as a function $f_N\colon \{1,\ldots,N\}^d\to\mathbb{R}$ giving rise to a polynomial Q_N through (1), and we assume that all of these objects meet the requirements of Theorem 1.3. In accordance with the Lindeberg method, we are first prompted to introduce some additional notation.

Notation. For every $i\in\{1,\ldots,N+1\}$, let us consider the vector

$$Z^{N,(i)} := (Y_1,\ldots,Y_{i-1},X_i,\ldots,X_N).$$

In particular, $Z^{N,(1)}=(X_1,\ldots,X_N)$ and $Z^{N+1,(N)}=(Y_1,\ldots,Y_N)$, so that

$$Q_N(X_1,\ldots,X_N)^m - Q_N(Y_1,\ldots,Y_N)^m = \sum_{i=1}^{N}\bigl[Q_N\bigl(Z^{N,(i)}\bigr)^m - Q_N\bigl(Z^{N,(i+1)}\bigr)^m\bigr]. \tag{22}$$

Since the only difference between the vectors $Z^{N,(i)}$ and $Z^{N,(i+1)}$ is their ith-component, it is readily checked that

$$Q_N\big(Z^{N,(i)}\big) = U_N^{(i)} + V_N^{(i)}(X_i) \quad \text{and} \quad Q_N\big(Z^{N,(i+1)}\big) = U_N^{(i)} + V_N^{(i)}(Y_i),$$

where $U_N^{(i)}$ stands for the multilinear polynomial

$$U_N^{(i)} := \sum_{j_1,\dots,j_d \in \{1,\dots,N\}\setminus\{i\}} f_N(j_1,\dots,j_d) Z_{j_1}^{N,(i)} \cdots Z_{j_d}^{N,(i)},$$

and $V_N^{(i)}\colon \mathcal{A} \to \mathcal{A}$ is the linear operator defined, for every $x \in \mathcal{A}$, by

$$V_N^{(i)}(x) := \sum_{l=1}^{d} \sum_{j_1,\dots,j_{d-1} \in \{1,\dots,N\}\setminus\{i\}} f_N(j_1,\dots,j_{l-1},i,j_l\dots,j_{d-1}) Z_{j_1}^{N,(i)} \cdots Z_{j_{l-1}}^{N,(i)} x Z_{j_l}^{N,(i)} \cdots Z_{j_{d-1}}^{N,(i)}.$$

Expansion. Once endowed with the above notation, the problem reduces to examining the differences

$$\varphi\big(\big(U_N^{(i)} + V_N^{(i)}(X_i)\big)^m\big) - \varphi\big(\big(U_N^{(i)} + V_N^{(i)}(Y_i)\big)^m\big) \tag{23}$$

for $i \in \{1,\dots,N-1\}$. In a commutative context, this could be handled with the classical binomial formula. Although such a mere formula is not available here, one can still assert that for every $A, B \in \mathcal{A}$,

$$(A+B)^m = A^m + \sum_{n=1}^{m} \sum_{(r,\mathbf{i}_{r+1},\mathbf{j}_r) \in \mathcal{D}_{m,n}} c_{m,n,r,\mathbf{i}_{r+1},\mathbf{j}_r} A^{i_1} B^{j_1} A^{i_2} B^{j_2} \cdots A^{i_r} B^{j_r} A^{i_{r+1}}, \tag{24}$$

where

$$\mathcal{D}_{m,n} := \left\{ (r,\mathbf{i}_{r+1},\mathbf{j}_r) \in \{1,\dots,m\} \times \mathbb{N}^{r+1} \times \mathbb{N}^r \colon \sum_{l=1}^{r+1} i_l = n, \sum_{l=1}^{r} j_l = m-n \right\}$$

and the $c_{m,n,r,\mathbf{i}_{r+1},\mathbf{j}_r}$'s stand for appropriate combinatorial coefficients (independent of A and B). The sets $\mathcal{D}_{m,n}$ must of course be understood as follows: given $(r,\mathbf{i}_{r+1},\mathbf{j}_r) \in \mathcal{D}_{m,n}$, the product $A^{i_1} B^{j_1} A^{i_2} B^{j_2} \cdots A^{i_r} B^{j_r} A^{i_{r+1}}$ contains A exactly n times and B exactly $(m-n)$ times, both counted with multiplicity.

Let us go back to (23) and let us apply formula (24) in order to expand $(U_N^{(i)} + V_N^{(i)}(X_i))^m$ (resp., $(U_N^{(i)} + V_N^{(i)}(Y_i))^m$). The first and second order terms (i.e., for $n = 1, 2$ in (24)) of the resulting sum happen to vanish, as a straightforward use of the following lemma shows.

Lemma 3.1. *Let Y and Z be two centered random variables with unit variance. Then, for every integer $k \geq 1$ and every sequence (X_i) of centered freely independent random variables indepen-*

dent of Y and Z, one has

$$\varphi(X_{i_1}\cdots X_{i_r}YX_{i_{r+1}}\cdots X_{i_k}) = \varphi(X_{i_1}\cdots X_{i_r}ZX_{i_{r+1}}\cdots X_{i_k}) = 0 \tag{25}$$

and

$$\varphi(X_{i_1}\cdots X_{i_r}YX_{i_{r+1}}\cdots X_{i_s}YX_{i_{s+1}}\cdots X_{i_k}) = \varphi(X_{i_1}\cdots X_{i_r}ZX_{i_{r+1}}\cdots X_{i_s}ZX_{i_{s+1}}\cdots X_{i_k}) \tag{26}$$

for all $0\le r\le s\le k$ *and* $(i_1,\dots,i_k)\in\mathbb{N}^k$.

Proof. Let us first focus on (25). For $k=1$, this is obvious. Assume that the result holds true up to $k-1$ and write

$$\varphi(X_{i_1}\cdots X_{i_r}YX_{i_{r+1}}\cdots X_{i_k}) = \varphi\big(X_{i'_1}^{m_1}\cdots X_{i'_{r'}}^{m_{r'}}YX_{i'_{r'+1}}^{m_{r'+1}}\cdots X_{i'_{s'}}^{m_{s'}}\big)$$

with $i'_{p+1}\neq i'_p$ for $p\in\{1,\dots,s'-1\}\setminus\{r'\}$, $i'_{s'}\neq i'_1$ and $m_p\ge 1$ for every $p\in\{1,\dots,s'\}$. Center successively every random variable $X_{i'_{p_1}}^{m_{p_1}},\dots,X_{i'_{p_t}}^{m_{p_t}}$ for which $m_{p_i}\ge 2$: together with an induction argument, this yields

$$\begin{aligned}
&\varphi\big(X_{i'_1}^{m_1}\cdots X_{i'_{r'}}^{m_{r'}}YX_{i'_{r'+1}}^{m_{r'+1}}\cdots X_{i'_{s'}}^{m_{s'}}\big)\\
&\quad=\varphi\big(X_{i'_1}\cdots X_{i'_{p_1-1}}\big(X_{i'_{p_1}}^{m_{p_1}}-\varphi\big(X_{i'_{p_1}}^{m_{p_1}}\big)\big)X_{i'_{p_1+1}}^{m_{p_1+1}}\cdots X_{i'_{r'}}^{m_{r'}}YX_{i'_{r'+1}}^{m_{r'+1}}\cdots X_{i'_{s'}}^{m_{s'}}\big)\\
&\quad=\varphi\big(X_{i'_1}\cdots X_{i'_{p_1-1}}\big(X_{i'_{p_1}}^{m_{p_1}}-\varphi\big(X_{i'_{p_1}}^{m_{p_1}}\big)\big)X_{i'_{p_1+1}}\cdots X_{i'_{p_2-1}}\big(X_{i'_{p_2}}^{m_{p_2}}-\varphi\big(X_{i'_{p_2}}^{m_{p_2}}\big)\big)\\
&\qquad X_{i'_{p_2+1}}^{m_{p_2+1}}\cdots X_{i'_{r'}}^{m_{r'}}YX_{i'_{r'+1}}^{m_{r'+1}}\cdots X_{i'_{s'}}^{m_{s'}}\big)=\cdots=0
\end{aligned}$$

owing to free independence. Identity (26) can be easily derived from a similar induction procedure. □

Let us go back to the proof of Theorem 1.3. As a consequence of the previous lemma, it now suffices to establish that, either for $W=X_i$ or for $W=Y_i$, one has, as soon as $\sum_l j_l\ge 3$,

$$\big|\varphi\big(\big(U_N^{(i)}\big)^{i_1}\big(V_N^{(i)}(W)\big)^{j_1}\big(U_N^{(i)}\big)^{i_2}\big(V_N^{(i)}(W)\big)^{j_2}\cdots\big(U_N^{(i)}\big)^{i_r}\big(V_N^{(i)}(W)\big)^{j_r}\big)\big|\le c_{m,d}\,\mathrm{Inf}_i(f_N)^{3/2} \tag{27}$$

for some constant $c_{m,d}$. Indeed, in this case, by combining (22), (24) and (27) with the identities in the statement of Lemma 3.1, we get

$$\begin{aligned}
\big|\varphi\big(Q_N(X_1,\dots,X_N)^m\big)-\varphi\big(Q_N(Y_1,\dots,Y_N)^m\big)\big| &\le C_{m,d}\sum_{i=1}^N \mathrm{Inf}_i(f_N)^{3/2}\\
&\le C_{m,d}\tau_N^{1/2}\sum_{i=1}^N\mathrm{Inf}_i(f_N)=C_{m,d}\tau_N^{1/2},
\end{aligned}$$

which is precisely the expected bound of Theorem 1.3.

In order to prove (27), let us first resort to the following Hölder-type inequality, borrowed from [3], Lemma 12:

$$\begin{aligned}&\left|\varphi\left(\left(U_N^{(i)}\right)^{i_1}\left(V_N^{(i)}(W)\right)^{j_1}\cdots\left(U_N^{(i)}\right)^{i_r}\left(V_N^{(i)}(W)\right)^{j_r}\right)\right|\\&\quad\le\varphi\left(\left(U_N^{(i)}\right)^{2^r i_1}\right)^{2^{-r}}\varphi\left(\left(V_N^{(i)}(W)\right)^{2^r j_1}\right)^{2^{-r}}\cdots\varphi\left(\left(U_N^{(i)}\right)^{2^r i_r}\right)^{2^{-r}}\varphi\left(\left(V_N^{(i)}(W)\right)^{2^r j_r}\right)^{2^{-r}}.\end{aligned}\tag{28}$$

Now, let the key (forthcoming) Proposition 3.5 come into the picture. Thanks to it, we can simultaneously assert that, for every $p \ge 1$,

$$\varphi\left(\left(U_N^{(i)}\right)^{2p}\right) \le C_{p,d} \quad \text{and} \quad \varphi\left(V_N^{(i)}(X_i)^{2p}\right) \le C_{p,d} \cdot \mathrm{Inf}_i(f_N)^p$$

for some constant $C_{p,d}$. Going back to (28), we deduce that for every (j_l) such that $\sum_l j_l \ge 3$,

$$\begin{aligned}\left|\varphi\left(\left(U_N^{(i)}\right)^{i_1}\left(V_N^{(i)}(X_i)\right)^{j_1}\cdots\left(U_N^{(i)}\right)^{i_r}\left(V_N^{(i)}(X_i)\right)^{j_r}\right)\right| &\le C'_{r,d}\cdot\mathrm{Inf}_i(f_N)^{2^{-1}(j_1+\cdots+j_r)}\\&\le C'_{r,d}\cdot\mathrm{Inf}_i(f_N)^{3/2}\end{aligned}$$

since $\mathrm{Inf}_i(f_N) \le 1$, and so the proof of Theorem 1.3 is done.

3.2. Hypercontractivity

In order to prove the forthcoming Proposition 3.5 (which played an important role in the proof of Theorem 1.3), we first need a technical lemma. To state it, a few additional notation must be introduced.

Definition 3.2. *Fix integers $n_1, \ldots, n_r \ge 1$. Any set of disjoint blocks of points in $\{1, \ldots, n_1 + \cdots + n_r\}$ is called a* graph *of $\{1, \ldots, n_1 + \cdots + n_r\}$. A graph is* complete *if the union of its blocks covers the whole set $\{1, \ldots, n_1 + \cdots + n_r\}$. Besides, a graph is said to* respect $n_1 \otimes \cdots \otimes n_r$ *if each of its blocks contains at most one point in each set $\{1, \ldots, n_1\}, \{n_1 + 1, \ldots, n_2\}, \ldots, \{n_1 + \cdots + n_{r-1} + 1, \ldots, n_1 + \cdots + n_r\}$.*

Finally, we denote by $\mathcal{G}_*(n_1 \otimes \cdots \otimes n_r)$ the set of graphs respecting $n_1 \otimes \cdots \otimes n_r$ and containing no singleton (i.e., no block with exactly one element), and by $\mathcal{G}_*^c(n_1 \otimes \cdots \otimes n_r)$ the subset of complete graphs in $\mathcal{G}_*(n_1 \otimes \cdots \otimes n_r)$.

Now, given a graph γ of $\{1, \ldots, n\}$ with p vertices ($p \le n$) and a function $f\colon \{1, \ldots, N\}^n \to \mathbb{R}$, we call *contraction* of f with respect to γ the function $C_\gamma(f)\colon \{1, \ldots, N\}^{n-p} \to \mathbb{R}$ defined for every $(j_1, \ldots, j_{n-p})$ by the formula

$$\begin{aligned}&C_\gamma(f)(j_1, \ldots, j_{n-p})\\&\quad:= \sum_{i_1,\ldots,i_p=1}^{N} f(j_1, \ldots, i_1, \ldots, i_p, \ldots, j_{n-p}) \cdot \delta(\gamma, j_1, \ldots, i_1, \ldots, i_p, \ldots, j_{n-p}),\end{aligned}$$

where:

- the (fixed) positions of the i_k's in $(j_1, \dots, i_1, \dots, i_p, \dots, j_{n-p})$ correspond to the positions of the vertices of γ;
- $\delta(\gamma, j_1, \dots, i_1, \dots, i_p, \dots, j_{n-p}) = 1$ if all i_k, i_l in a same block of γ are equal, and 0 otherwise.

With these notation in hand, we can prove the following lemma.

Lemma 3.3. *For every $\gamma \in \mathcal{G}_*(n_1 \otimes \cdots \otimes n_r)$ and all $f_i \in \ell^2(\{1, \dots, N\}^{n_i})$ $(i = 1, \dots, r)$, one has*

$$\left\| C_\gamma(f_1 \otimes \cdots \otimes f_r) \right\|_{\ell^2} \le \prod_{i=1}^{r} \| f_i \|_{\ell^2}.$$

Proof. We use an induction procedure on r. When $r = 1$, $C_\gamma(f_1) = f_1$. Fix now $r \ge 2$ and $\gamma \in \mathcal{G}_*(n_1 \otimes \cdots \otimes n_r)$. Denote by $\tilde{\gamma} \in \mathcal{G}_*(n_2 \otimes \cdots \otimes n_r)$ the restriction of γ to $n_2 \otimes \cdots \otimes n_r$ (i.e., the graph that one obtains from γ by getting rid of the blocks with vertices in $\{1, \dots, n_1\}$). If γ has no vertex in $\{1, \dots, n_1\}$, then

$$C_\gamma(f_1 \otimes \cdots \otimes f_r) = f_1 \otimes C_{\tilde{\gamma}}(f_2 \otimes \cdots \otimes f_r)$$

and we can conclude by induction. Otherwise, it is easily seen that $\|C_\gamma(f_1 \otimes \cdots \otimes f_r)\|_{\ell^2}^2$ can be decomposed as

$$\begin{aligned} &\left\| C_\gamma(f_1 \otimes \cdots \otimes f_r) \right\|_{\ell^2}^2 \\ &\quad = \sum_{i_1, \dots, i_l, j_1, \dots, j_m} \Bigg(\sum_{k_1, \dots, k_q} f_1(i_1, \dots, k_1, \dots, k_q, \dots, i_l) \\ &\qquad\qquad \times C_{\tilde{\gamma}}(f_2 \otimes \cdots \otimes f_r)(j_1, \dots, k_{\sigma(1)}, \dots, k_{\sigma(p)}, \dots, j_m) \Bigg)^2, \end{aligned}$$

where:

- l (resp., m) is the number of points in $\{1, \dots, n_1\}$ (resp., $\{n_1 + 1, \dots, n_1 + \cdots + n_r\}$) which are not assigned by γ;
- in $f_1(i_1, \dots, k_1, \dots, k_q, \dots, i_l)$, the (fixed) positions of the k_i's correspond to the positions of the q vertices of γ in $\{1, \dots, n_1\}$;
- σ: $\{1, \dots, p\} \to \{1, \dots, q\}$ $(p \ge q)$ is a surjective mapping, meaning that each k_i appears at least once in $(k_{\sigma(1)}, \dots, k_{\sigma(p)})$. Here, we use the fact that γ respects $n_1 \otimes \cdots \otimes n_r$ and contains no singleton.

Then, by applying Cauchy–Schwarz inequality over the set of indices $(k_1, \dots, k_q)$, we get

$$\left\| C_\gamma(f_1 \otimes \cdots \otimes f_r) \right\|_{\ell^2}^2 \le \| f_1 \|_{\ell^2}^2 \left\| C_{\tilde{\gamma}}(f_2 \otimes \cdots \otimes f_r) \right\|_{\ell^2}^2,$$

where we have used (possibly several times) the trivial property: for any g: $\{1, \dots, N\}^2 \to \mathbb{R}$, $\sum_{k=1}^N g(k,k)^2 \le \sum_{k_1,k_2=1}^N g(k_1,k_2)^2$. We can now conclude by induction. □

Let us finally turn to the proof of Proposition 3.5, which is the hypercontractivity property for homogeneous sums of free random variables. We shall use Lemma 3.3 as a main ingredient. The following elementary lemma will also be needed at some point.

Lemma 3.4. *For every integer $r \geq 1$ and every sequence $X = (X_i)$ of random variables, one has $|\varphi(X_{i_1} \cdots X_{i_{2r}})| \leq \mu^X_{2^r-1}$, where $\mu^X_k := \sup_{1 \leq l \leq k, i \geq 1} \varphi(X_i^{2l})$.*

Proof. For $r = 1$, this corresponds to Cauchy–Schwarz inequality (see [9]). Assume that the result holds true up to $r - 1$ ($r \geq 2$) for any sequence of random variables. By using Cauchy–Schwarz inequality, we first get

$$\begin{aligned} &\left|\varphi(X_{i_1} \cdots X_{i_{2r}})\right| \\ &\quad = \left|\varphi\big((X_{i_1} \cdots X_{i_r})(X_{i_{r+1}} \cdots X_{i_{2r}})\big)\right| \\ &\quad \leq \varphi\big(X_{i_1}^2 \cdots X_{i_{r-1}} X_{i_r}^2 X_{i_{r-1}} \cdots X_{i_2}\big)^{1/2} \varphi\big(X_{i_{r+1}}^2 \cdots X_{i_{2r-1}} X_{i_{2r}}^2 X_{i_{2r-1}} \cdots X_{i_{r+2}}\big)^{1/2}. \end{aligned} \tag{29}$$

Denote by X^2 the sequence $X_1, X_1^2, X_2, X_2^2, \ldots$. Then by induction, we deduce from (29) that $|\varphi(X_{i_1} \cdots X_{i_{2r}})| \leq \mu^{X^2}_{2^{r-2}} \leq \mu^X_{2^r-1}$, which concludes the proof. □

Proposition 3.5. *Let $X_1, \ldots, X_N$ be centered freely independent random variables and denote by (μ^N_k) the sequence of larger even moments, that is, $\mu^N_k := \sup_{1 \leq i \leq N, 1 \leq l \leq k} \varphi(X_i^{2l})$. Fix $d \geq 1$, and consider a sequence of functions $f_N\colon \{1, \ldots, N\}^d \to \mathbb{R}$ satisfying the three basic assumptions* (i)–(ii)–(iii) *of Theorem 1.3. Define Q_N through (1). Then for every $r \geq 1$, there exists a constant $C_{r,d}$ such that*

$$\varphi\big(Q_N(X_1, \ldots, X_N)^{2r}\big) \leq C_{r,d}\, \mu^N_{2rd-1} \left(\sum_{j_1, \ldots, j_d = 1}^{N} f_N(j_1, \ldots, j_d)^2 \right)^r. \tag{30}$$

Proof. The argument is in spirit quite close to ideas of [5]. Owing to Lemma 3.1, it holds that

$$\begin{aligned} &\varphi\big(Q_N(X_1, \ldots, X_N)^{2r}\big) \\ &\quad = \sum_{\substack{1 \leq j_1^1, \ldots, j_d^1 \leq N \\ \vdots \\ 1 \leq j_1^{2r}, \ldots, j_d^{2r} \leq N}} f_N\big(j_1^1, \ldots, j_d^1\big) \cdots f_N\big(j_1^{2r}, \ldots, j_d^{2r}\big) \varphi\big((X_{j_1^1} \cdots X_{j_d^1}) \cdots (X_{j_1^{2r}} \cdots X_{j_d^{2r}})\big) \\ &\quad = \sum_{(j_1^1, \ldots, j_d^{2r}) \in \mathcal{A}^N_{2rd}} f_N\big(j_1^1, \ldots, j_d^1\big) \cdots f_N\big(j_1^{2r}, \ldots, j_d^{2r}\big) \varphi\big((X_{j_1^1} \cdots X_{j_d^1}) \cdots (X_{j_1^{2r}} \cdots X_{j_d^{2r}})\big), \end{aligned}$$

where we have set, for every $R \geq 1$,

$$\mathcal{A}^N_R := \big\{(j_1, \ldots, j_R) \in \{1, \ldots, N\}^R\colon \text{ for each } i_1, \text{ there exists } i_2 \neq i_1 \text{ such that } j_{i_1} = j_{i_2}\big\}.$$

Bounding each term of the form $\varphi((X_{j_1^1}\cdots X_{j_d^1})\cdots(X_{j_1^{2r}}\cdots X_{j_d^{2r}}))$ of this sum by means of Lemma 3.4 leads to

$$\varphi\big(Q_N(X_1,\dots,X_N)^{2r}\big) \le \mu_{2rd-1}^N \sum_{(j_1^1,\dots,j_d^{2r})\in\mathcal{A}_{2rd}^N} \big|f_N\big(j_1^1,\dots,j_d^1\big)\big|\cdots\big|f_N\big(j_1^{2r},\dots,j_d^{2r}\big)\big|.$$

Recall the notation $\mathcal{G}_*^c(d^{\otimes 2r})$ and C_γ from the beginning of Section 3.2. By taking into account that f_N is assumed to vanish on diagonals, it is easily seen that the above sum is equal to

$$\sum_{(j_1^1,\dots,j_d^{2r})\in\mathcal{A}_{2rd}^N} \big|f_N\big(j_1^1,\dots,j_d^1\big)\big|\cdots\big|f_N\big(j_1^{2r},\dots,j_d^{2r}\big)\big| = \sum_{\gamma\in\mathcal{G}_*^c(d^{\otimes 2r})} C_\gamma\big(|f_N|^{\otimes 2r}\big).$$

Therefore, we may apply Lemma 3.3 so as to deduce that

$$\varphi\big(Q_N(X_1,\dots,X_N)^{2r}\big) \le \mu_{2rd-1}^N \cdot \big|\mathcal{G}_*^c\big(d^{\otimes 2r}\big)\big| \cdot \|f_N\|_{\ell^2(\{1,\dots,N\}^d)}^{2r},$$

which is precisely (30) with $C_{r,d} = |\mathcal{G}_*^c(d^{\otimes 2r})|$. □

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