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Small noise asymptotics and first passage times of integrated Ornstein–Uhlenbeck processes driven by α-stable Lévy processes

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In this paper, we study the asymptotic behaviour of one-dimensional integrated Ornstein–Uhlenbeck processes driven by α-stable Lévy processes of small amplitude. We prove that the integrated Ornstein–Uhlenbeck process converges weakly to the underlying α-stable Lévy process in the Skorokhod M_1-topology which secures the weak convergence of first passage times. This result follows from a more general result about approximations of an arbitrary Lévy process by continuous integrated Ornstein–Uhlenbeck processes in the M_1-topology.

Keywords: absolutely continuous approximations; α-stable Lévy process; first passage times; integrated Ornstein–Uhlenbeck process; Skorokhod M_1-topology; tightness

1. Introduction

Consider a dimensionless Langevin equation for the motion of a particle with a position x^ε subject to a linear friction force $F = -A\dot{x}^\varepsilon$, $A > 0$ (Stokes' law for friction force) and a random noise $\dot{l}$ of a small amplitude $\varepsilon > 0$

$$\ddot{x}^\varepsilon = -A\dot{x}^\varepsilon + \varepsilon\dot{l}. \tag{1.1}$$

Denoting by $v^\varepsilon := \dot{x}^\varepsilon$ the velocity process, we understand this equation as a two-dimensional equation in a phase space (v, x) which can be written in the integral form as

$$v_t^\varepsilon = v_0 - A\int_0^t v_s^\varepsilon \,\mathrm{d}s + \varepsilon l_t, \tag{1.2}$$

$$x_t^\varepsilon = x_0 + \int_0^t v_s^\varepsilon \,\mathrm{d}s, t \geq 0, \qquad x_0, v_0 \in \mathbb{R}. \tag{1.3}$$

The study of the dynamics of x^ε and v^ε in the Gaussian case, that is, when $l = b$ is a standard Brownian motion, has a long history. In this case, the velocity process v^ε is a Ornstein–Uhlenbeck (OU) process, and the displacement process x^ε, being the integrated Ornstein–Uhlenbeck process, is often referred to as Langevin's Brownian motion. For example, it is well known that for zero initial conditions $x_0 = v_0 = 0$, strong friction and large amplitude, $A = \varepsilon \to +\infty$, the displacement process x^ε can be considered as a good physical approximation of the Brownian motion (see, e.g., Chapter 2 in Horsthemke and Lefever [9]).

The dynamics of the integrated OU processes driven by non-Gaussian Lévy processes attracted attention recently in financial mathematics in the context of stochastic volatility models, see Barndorff-Nielsen [4], and Barndorff-Nielsen and Shephard [5]. Garbaczewski and Olkiewicz [7] studied integrated OU processes driven by a 1-stable (Cauchy) process. Al-Talibi, Hilbert and Kolokoltsov [2] established convergence in probability of marginals of an integrated OU process driven by an α-stable Lévy process in the limit of large friction parameter. Chechkin, Gonchar and Szydłowski [6] studied the equation (1.1) (with $\varepsilon = 1$, in two- and three-dimensional setting) in a model of plasma in an external constant magnetic field and subject to an α-stable Lévy electric forcing.

Our present research is mainly motivated by the this paper and focuses on the first passage times of the displacement process x^ε in the limit of small amplitude $\varepsilon \to 0$ under the assumption that the driving process l is a non-Gaussian α-stable Lévy process. We refer the reader to the works by Lefebvre [11] and Hesse [8] where first passage problems for integrated Ornstein–Uhlenbeck processes driven by Brownian motion were studied.

Let us briefly describe the outline of the paper. First, we shall show that on a certain ε-dependent time scale the integrated OU process Ax^ε weakly converges to the driving process l in the sense of finite-dimensional distributions. Further, we shall establish a stronger convergence of the processes in an appropriate topology. We notice that since the driving α-stable Lévy process has càdlàg paths and the integrated Ornstein–Uhlenbeck process is absolutely continuous, no convergence in the uniform topology or in the Skorokhod J_1-topology is possible. Thus, we prove the convergence in the weaker Skorokhod M_1-topology which is still strong enough to ensure the continuity of the running supremum or the inverse function of a process, and to guarantee the convergence of the first passage times. As a by-product, we obtain an approximation result for an arbitrary Lévy process by absolutely continuous integrated OU processes in the M_1-topology.

2. Object of study and main result

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbf{P})$ be a filtered probability space satisfying the usual conditions. On this probability space, consider a Lévy process l with càdlàg paths and a Lévy–Khintchine representation $\mathbf{E}e^{iul_t} = e^{t\Psi(u)}$, where

$$\Psi(u) = -\frac{\sigma^2}{2}u^2 + i\mu u + \int_{\mathbb{R}\backslash\{0\}} \big(e^{iuy} - 1 - iuy\mathbb{I}_{\{|y|\leq 1\}}\big)\nu(dy)), \qquad u \in \mathbb{R}, \tag{2.1}$$

with $\sigma \geq 0$, $\mu \in \mathbb{R}$, and a jump measure ν satisfying the conditions $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}\backslash\{0\}}(y^2 \wedge 1)\nu(dy) < \infty$. In particular, we shall be interested in non-Gaussian strictly α-stable Lévy processes $l^{(\alpha)} = (l_t^{(\alpha)})_{t\geq 0}$, $\alpha \in (0, 2)$, for which the closed-form representation of the characteristic exponent Ψ is known to be equal to

$$\Psi(u) = \begin{cases} -c|u|^\alpha \left(1 - \mathrm{i}\beta \operatorname{sign}(u) \tan \dfrac{\pi\alpha}{2}\right), & \alpha \in (0,1) \cup (1,2), \\ -c|u| \left(1 + \mathrm{i}\beta \dfrac{2}{\pi} \operatorname{sign}(u) \ln u\right), & \alpha = 1, u \in \mathbb{R}, \end{cases}$$

$c > 0$ and $\beta \in [-1, 1]$ being a scale and skewness parameters (see, e.g., Theorem 14.15 in Sato [13]). The well-known self-similarity property of $l^{(\alpha)}$ will be used in the following: $\operatorname{Law}(\varepsilon l^{(\alpha)}_{t/\varepsilon^\alpha}, t \geq 0) = \operatorname{Law}(l^{(\alpha)}_t, t \geq 0)$ for any $\varepsilon > 0$.

For any $A > 0$, $\varepsilon > 0$, any v_0, $x_0 \in \mathbb{R}$, and any Lévy process l (and in particular an α-stable Lévy process) there exists a path-wise unique strong solution of the linear stochastic differential equation (1.2) given by

$$v^\varepsilon_t = v_0 \mathrm{e}^{-At} + \varepsilon \int_0^t \mathrm{e}^{-A(t-s)} \, \mathrm{d}l_s, \tag{2.2}$$

where the last integral is a Lévy–Wiener integral (see Chapter 4.3.5 in Applebaum [3]). It is helpful to recall another representation of v^ε which is obtained with the help of integration by parts, namely

$$v^\varepsilon_t = v_0 \mathrm{e}^{-At} + \varepsilon l_t - \varepsilon A \int_0^t \mathrm{e}^{-A(t-s)} l_s \, \mathrm{d}s. \tag{2.3}$$

It is clear, that the process v^ε is also càdlàg and its jumps coincide with the jumps of the driving process εl.

The equation (1.3) for the displacement process x^ε can be also solved explicitly. Applying the Fubini theorem we obtain

$$\begin{aligned} x^\varepsilon_t &= x_0 + \int_0^t v^\varepsilon_s \, \mathrm{d}s = x_0 + \int_0^t \left[v_0 \mathrm{e}^{-As} + \varepsilon \int_0^s \mathrm{e}^{-A(s-u)} \, \mathrm{d}l_u \right] \mathrm{d}s \\ &= x_0 + \frac{v_0}{A}\left(1 - \mathrm{e}^{-At}\right) + \varepsilon \int_0^t \left[\int_u^t \mathrm{e}^{-A(s-u)} \, \mathrm{d}s \right] \mathrm{d}l_u \\ &= x_0 + \frac{v_0}{A}\left(1 - \mathrm{e}^{-At}\right) + \frac{\varepsilon}{A} \int_0^t \left(1 - \mathrm{e}^{-A(t-u)}\right) \mathrm{d}l_u. \end{aligned} \tag{2.4}$$

From now on, we set the initial conditions $x_0 = v_0 = 0$. At the end of the Section 3, we discuss the generalization of the results to the case of arbitrary initial conditions.

For a real valued stochastic process $y = (y_t)_{t \geq 0}$ and $a > 0$, let $\tau_a(y)$ denote the first passage time

$$\tau_a(y) = \inf\{t \geq 0\colon\ y_t > a\}.$$

The main goal of this paper is to study the law of the first passage times $\tau_a(x^\varepsilon)$ of the displacement process x^ε in the limit $\varepsilon \to 0$.

The asymptotics of $\tau_a(x^\varepsilon)$ can be determined in an especially simple way in the case of an integrated OU process driven by a standard Brownian motion $l = b$, that is a strictly 2-stable continuous Lévy process with the characteristic exponent $\Psi(u) = -u^2/2$, $u \in \mathbb{R}$.

Consider the Polish space $C([0,\infty),\mathbb{R})$ of real-valued continuous functions endowed with the topology U of local uniform convergence associated with the metric

$$d_U\big(x,x'\big) := \int_0^\infty \mathrm{e}^{-T}\Big(1\wedge \sup_{t\in[0,T]}\big|x_t - x'_t\big|\Big)\,\mathrm{d}T, \qquad x,x'\in C\big([0,\infty)\big).$$

The following result about the weak convergence (denoted in the sequel by '⇒') of integrated OU processes to the Brownian motion is well known and is presented here for the sake of completeness.

Proposition 2.1. *Let $l = b = (b_t)_{t\ge 0}$ be a standard Brownian motion, and let x^ε be the integrated OU process satisfying the equations (1.2) and (1.3) with zero initial conditions. Then, for any $A > 0$,*

$$\big(Ax^\varepsilon_{t/\varepsilon^2}\big)_{t\ge 0} \Rightarrow (b_t)_{t\ge 0}$$

in $C([0,\infty),\mathbb{R};U)$ as $\varepsilon\to 0$.

Proof. According to (2.4), the process Ax^ε is determined explicitly as

$$Ax^\varepsilon_t = \varepsilon\int_0^t \big(1-\mathrm{e}^{-A(t-s)}\big)\,\mathrm{d}b_s.$$

Applying the time change $t\mapsto \frac{t}{\varepsilon^2}$ and using the self-similarity of the Brownian motion b, $\mathrm{Law}(\varepsilon b_{t/\varepsilon^2}, t\ge 0) = \mathrm{Law}(b_t, t\ge 0)$, we obtain that for any $\varepsilon > 0$ the process $(Ax^\varepsilon_{t/\varepsilon^2})_{t\ge 0}$ coincides in law with the process AX^ε given by the convolution integral

$$AX^\varepsilon_t = \int_0^t \big(1-\mathrm{e}^{-A(t-s)/\varepsilon^2}\big)\,\mathrm{d}B_s, \tag{2.5}$$

where B is another standard Brownian motion. We show that the process

$$Y^\varepsilon_t = AX^\varepsilon_t - B_t = \int_0^t \mathrm{e}^{-A(t-s)/\varepsilon^2}\,\mathrm{d}B_s$$

converges to zero in probability as $\varepsilon\to 0$ uniformly over $t\in[0,T]$ for any $T>0$. Indeed, Y^ε is a centred Gaussian process with the variance

$$\mathbf{E}\big|Y^\varepsilon_t\big|^2 = \int_0^t \mathrm{e}^{-2A(t-s)/\varepsilon^2}\,\mathrm{d}s = \frac{\varepsilon^2}{2A}\big(1-\mathrm{e}^{-2At/\varepsilon^2}\big) \le \frac{\varepsilon^2}{2A}, \qquad t\in[0,T].$$

Applying Theorem 5.3 from Adler [1], we conclude that for any $T > 0$ there is an absolute constant $C > 0$ such that for any $\Delta > 0$

$$\mathbf{P}\Big(\sup_{t\in[0,T]}\big|Y^\varepsilon_t\big| > \Delta\Big) \le C\Delta\bigg(1-\Phi\bigg(\frac{2A\Delta}{\varepsilon^2}\bigg)\bigg) \to 0, \qquad \varepsilon\to 0,$$

where Φ is the probability distribution function of a standard Gaussian random variable. The weak convergence of $(Ax^\varepsilon_{t/\varepsilon^2})_{t\ge 0} \Rightarrow b$ in $C([0,\infty),\mathbb{R};U)$ follows immediately from the convergence in probability. □

It is clear that the law of the first passage time $\tau_a(x^\varepsilon)$ is determined with the help of the running supremum of the process x^ε,

$$S^\varepsilon_t := \sup_{s\in[0,t]} x^\varepsilon_s, \qquad t\ge 0,$$

namely $\mathbf{P}(\tau_a(x^\varepsilon)\le t) = \mathbf{P}(S^\varepsilon_t \ge a)$. Since the running supremum of a continuous process is a continuous mapping in $C([0,\infty),\mathbb{R};U)$, we obtain the convergence in law of the first passage times.

Corollary 2.1. *For any $a > 0$*

$$\varepsilon^2\tau_a\big(x^\varepsilon\big) \xrightarrow{d} \tau_{a/A}(b) \qquad \text{as } \varepsilon\to 0.$$

The probability density of the first passage time $\tau_{a/A}(b)$ is well known,

$$\mathbf{P}\big(\tau_{a/A}(b)\le t\big) = \frac{a}{A\sqrt{2\pi}}\int_0^t \frac{1}{s^{3/2}}\mathrm{e}^{-a^2/(2A^2s)}\,\mathrm{d}s.$$

If the driving Lévy process $l = l^{(\alpha)}$ is α-stable and non-Gaussian, the situation becomes more complicated. Consider the space $D([0,\infty),\mathbb{R})$ of real valued càdlàg functions. We shall see in Proposition 3.1 that $Ax^\varepsilon_{\cdot/\varepsilon^\alpha}$ converges to $l^{(\alpha)}$ in the sense of finite-dimensional distributions whereas the integrated OU process Ax^ε is absolutely continuous. Thus no weak convergence in the space $D([0,\infty),\mathbb{R})$ equipped with the topology of the local uniform convergence is possible.

In his seminal paper, Skorohod [15] introduced four weaker topologies on the space $D([0,\infty),\mathbb{R})$ different from the uniform topology. The most frequently used topology J_1 is designed to match the jump times and sizes of the approximating processes and their limit, and does not fit in with our setting. Thus, we shall prove convergence in the weaker M_1-topology which is still strong enough to guarantee the continuity of the supremum, and thus the convergence of the first passage times. Essentially this topology linearises the jumps through the introduction of a fictitious time-scale and is appropriate for establishing the convergence of a sequence of continuous processes to a discontinuous limit. It is also worth mentioning that the idea of a fictitious time-scale has been used in some other contexts, see Williams [17] and the references therein. A very detailed treatment of the M_1-convergence can be found in the monograph by Whitt [16].

Let us recall the necessary facts about the space $D([0,\infty),\mathbb{R})$ endowed with the non-uniform topology M_1. For any function $x\in D([0,\infty),\mathbb{R})$ and for any $T>0$ define a *completed graph* Γ_x^T of the restriction of x on $[0,T]$ as a set

$$\Gamma_x^T := \big\{(x_0,0)\big\}\cup\big\{(z,t)\in\mathbb{R}\times(0,T]\colon\ z=cx_{t-}+(1-c)x_t \text{ for some } c,\, c\in[0,1]\big\}.$$

The completed graph is a subset in $\mathbb{R}^2$ containing the graph of x on $[0,T]$ as well as the line segments connecting the points of discontinuity (x_{t-},t) and (x_t,t). On a completed graph we introduce an order saying that $(z,t)\le(z',t')$ if either $t<t'$ or $t=t'$ and $|x_{t-}-z|\le|x_{t-}-z'|$. A parametric representation of the graph is a continuous mapping $(z_u,t_u)\colon\ [0,1]\to\Gamma_x^T$, which in non-decreasing w.r.t. order on the completed graph. Denote Π_x^T the set of all parametric representations of the graph Γ_x^T. The Skorokhod M_1-topology in $D([0,\infty),\mathbb{R})$ is then induced by the metric

$$d_{M_1}\big(x,x'\big) := \int_0^\infty \mathrm{e}^{-T}\big(1\wedge d_{M_1,T}\big(x,x'\big)\big)\,\mathrm{d}T,$$

$$d_{M_1,T}\big(x,x'\big) := \inf_{\substack{(z,t)\in\Pi_x^T,\\ (z',t')\in\Pi_{x'}^T}}\ \max_{u\in[0,1]}\big\{\big|z_u-z'_u\big|,\big|t_u-t'_u\big|\big\},\qquad x,x'\in D\big([0,\infty),\mathbb{R}\big),\ T>0,$$

see Whitt [16], Sections 3.3, 12.3 and 12.9. One can construct a metric equivalent to d_{M_1}, for which the space $D([0,\infty),\mathbb{R};M_1)$ is Polish, see Whitt [16], Section 12.8.

The main result of this paper is the following convergence result.

Theorem 2.1. *Let $l^{(\alpha)}=(l_t^{(\alpha)})_{t\ge0}$ be an α-stable Lévy process, $\alpha\in(0,2)$ and let x^ε be the integrated OU process satisfying the equations* (1.2) *and* (1.3) *with zero initial conditions. Then*

$$\big(Ax^\varepsilon_{t/\varepsilon^\alpha}\big)_{t\ge0}\Rightarrow\big(l_t^{(\alpha)}\big)_{t\ge0}$$

in $D([0,\infty),\mathbb{R};M_1)$ as $\varepsilon\to0$.

The convergence of the first passage times follows immediately.

Corollary 2.2. *Let $l^{(\alpha)}$ be an α-stable process with $\limsup_{t\to\infty}l_t^{(\alpha)}=+\infty$ a.s. Then, for any $a>0$*

$$\varepsilon^\alpha\tau_a\big(x^\varepsilon\big)\xrightarrow{d}\tau_{a/A}\big(l^{(\alpha)}\big)\qquad \text{as } \varepsilon\to0.$$

Proof. As in Corollary 2.1, we define the first exit time with the help of the running supremum $S_t^\varepsilon:=\sup_{s\in[0,t]}x_t^\varepsilon$, $t\ge0$, and the inverse function $I_t^\varepsilon:=\inf\{s\ge0\colon\ S_s^\varepsilon>t\}$, $t\ge0$. Under the condition $x_0=v_0=0$, the inverse function is continuous in the M_1-topology, see Puhalskii and Whitt [12], Lemma 2.1. The continuous mapping theorem yields the result. □

In contrast to the Brownian case, the laws of the first passage times $\tau_a(l^{(\alpha)})$ of an α-stable Lévy process are often not known explicitly. We refer the reader to the recent works by Kuznetsov [10] and Simon [14], and references therein for various results on this topic.

The rest of the paper is organized as follows. Since the α-stable case can be studied with the help of an appropriate time change, which transforms the small noise amplitude into the big friction parameter, we shall study the M_1-convergence of continuous integrated OU processes with big friction parameter driven by arbitrary Lévy processes. This result can be of its own interest. Finally, we prove Theorem 2.1 and discuss the case of arbitrary initial conditions.

3. Absolutely continuous aproximations of Lévy processes in M_1-topology and the proof of the main result

For $A>0$, $\gamma\geq 0$, and a real valued Lévy process L with a characteristic exponent Ψ given by (2.1) we study the system of stochastic differential equations

$$V_t^\gamma = -\gamma A\int_0^t V_s^\gamma\,\mathrm{d}s + L_t, \tag{3.1}$$

$$X_t^\gamma = \gamma\int_0^t V_s^\gamma\,\mathrm{d}s. \tag{3.2}$$

First, we prove the convergence of finite-dimensional marginals of AX^γ to those of L in probability.

Proposition 3.1. *For any $m\geq 1$ and $0\leq t_1<\cdots<t_m<\infty$,*

$$\big(AX_{t_1}^\gamma,\ldots,AX_{t_m}^\gamma\big)\xrightarrow{\mathbf{P}}(L_{t_1},\ldots,L_{t_m})\qquad \textit{as } \gamma\to\infty.$$

Proof. With the help of the formulae (2.2) and (2.4) one can solve the equations (3.1) and (3.2) explicitly:

$$V_t^\gamma=\int_0^t \mathrm{e}^{-\gamma A(t-s)}\,\mathrm{d}L_s,\qquad AX_t^\gamma=\int_0^t\big(1-\mathrm{e}^{-\gamma A(t-s)}\big)\,\mathrm{d}L_s. \tag{3.3}$$

It is clear that the processes AX^γ and V^γ start at the origin a.s., $AX_0^\gamma=V_0^\gamma=L_0=0$. For $m\geq 1$ fix the time instants $0=t_0<t_1<\cdots<t_m<\infty$ and real numbers $u_0,u_1,\ldots,u_m$ and consider the characteristic function

$$\mathbf{E}\exp\left(\mathrm{i}\sum_{k=0}^m u_k\big(AX_{t_k}^\gamma-L_{t_k}\big)\right)=\mathbf{E}\exp\left(-\mathrm{i}\sum_{k=1}^m u_k\int_0^{t_k}\mathrm{e}^{-\gamma A(t_k-s)}\,\mathrm{d}L_s\right). \tag{3.4}$$

We represent the sum in the last exponent as a sum of independent random variables

$$\begin{aligned}\sum_{k=1}^m u_k\big(AX_{t_k}^\gamma-L_{t_k}\big)&=-\sum_{k=1}^m u_k\int_0^{t_1}\mathrm{e}^{-\gamma A(t_k-s)}\,\mathrm{d}L_s\\&\quad-\sum_{k=2}^m u_k\int_{t_1}^{t_2}\mathrm{e}^{-\gamma A(t_k-s)}\,\mathrm{d}L_s-\cdots-u_m\int_{t_{m-1}}^{t_m}\mathrm{e}^{-\gamma A(t_m-s)}\,\mathrm{d}L_s\end{aligned}$$

and show that the characteristic function of every summand converges to 1 as $\gamma \to \infty$. Fix an index j, $1 \le j \le m$. Then by a well-known formula for characteristic functions of convolution integrals w.r.t. a Lévy process (see, e.g., Lemma 17.1 in Sato [13]) we obtain the equality

$$\begin{aligned} \ln \mathbf{E} \exp\left(-\mathrm{i} \sum_{k=j}^{m} u_k \int_{t_{j-1}}^{t_j} \mathrm{e}^{-\gamma A(t_k - s)}\, \mathrm{d}L_s \right) \\ = \int_{t_{j-1}}^{t_j} \Psi\left(-\sum_{k=j}^{m} u_k \mathrm{e}^{-\gamma A(t_k - s)} \right) \mathrm{d}s. \end{aligned} \tag{3.5}$$

For brevity, we denote the argument

$$u_j^\gamma(s) := -\sum_{k=j}^{m} u_k \mathrm{e}^{-\gamma A(t_k - s)}, \qquad t_{j-1} \le s \le t_j,$$

and $u^* = \sum_{k=1}^{m} |u_k| < \infty$. Clearly, $|u_j^\gamma(s)| \le \sum_{k=j}^{m} |u_k| \le u^*$ for $s \in [t_{j-1}, t_j]$, $1 \le j \le m$. The exponent $\Psi(u)$ is continuous and bounded on $[-u^*, u^*]$. On each of the intervals $[t_{j-1}, t_j)$, $1 \le j \le m$, we determine the pointwise limit of $\Psi(u_j^\gamma(s))$ as $\gamma \to \infty$, namely

$$\lim_{\gamma \to \infty} \Psi\big(u_j^\gamma(s)\big) = \Psi(0) = 0.$$

After applying the Lebesgue dominated convergence theorem to the right-hand side of (3.5), we conclude that the term on the left-hand side of (3.4) tends to 1 as $\gamma \to \infty$, and as a well-known consequence the convergence in probability

$$\big(AX_0^\gamma, AX_{t_1}^\gamma, \ldots, AX_{t_m}^\gamma\big) \xrightarrow{\mathbf{P}} (L_0, L_{t_1}, \ldots, L_{t_m})$$

holds as $\gamma \to \infty$. □

In the proof of the next Theorem 3.1 about the convergence of AX^γ to L in the M_1-topology, we shall make use of the following oscillation function. For $x, y \in \mathbb{R}$ denote the segment $[\![x, y]\!] := \{z \in \mathbb{R}\colon\ z = x + c(y - x), c \in [0, 1]\}$ and introduce the oscillation function $M : \mathbb{R}^3 \to [0, \infty)$,

$$M(x_1, x, x_2) := \begin{cases} \min\big\{|x - x_1|, |x_2 - x|\big\}, & \text{if } x \notin [\![x_1, x_2]\!], \\ 0, & x \in [\![x_1, x_2]\!]. \end{cases}$$

In other words, $M(x_1, x, x_2)$ is the Euclidean distance between the point x and the segment $[\![x_1, x_2]\!]$.

Now we prove the main result of this section.

Theorem 3.1. *Let L be an arbitrary real valued Lévy process, and X^γ be a solution of* (3.2). *Then for any $A > 0$,*

$$AX^\gamma \xrightarrow{\mathbf{P}} L \qquad \text{in } D\big([0,\infty),\mathbb{R};M_1\big) \text{ as } \gamma\to\infty.$$

Proof. 1. First, with the help of the Lévy–Itô decomposition we represent L as a sum of a continuous Brownian motion σB and a Lévy process Z without Gaussian part. Due to the linearity of the Langevin equation, we represent the solution V^γ as a sum

$$V_t^\gamma = \sigma\int_0^t \mathrm{e}^{-\gamma A(t-s)}\,\mathrm{d}B_s + \int_0^t \mathrm{e}^{-\gamma A(t-s)}\,\mathrm{d}Z_s,$$

and consequently the process AX^γ as a sum of two continuous processes

$$\begin{aligned} AX_t^\gamma &= AX_t^{\gamma,B} + AX_t^{\gamma,Z} \\ &:= \sigma\int_0^t \big(1-\mathrm{e}^{-\gamma A(t-s)}\big)\,\mathrm{d}B_s + \int_0^t \big(1-\mathrm{e}^{-\gamma A(t-s)}\big)\,\mathrm{d}Z_s. \end{aligned} \tag{3.6}$$

In Proposition 2.1, we proved that $AX^{\gamma,B}$ converges to σB in probability in the local uniform topology (see (2.5) with $\gamma = \frac{1}{\varepsilon^2}$), and consequently in the M_1-topology. Since σB is continuous, due to Corollary 12.7.1 in Whitt [16] it is sufficient to prove the M_1-convergence of $AX^{\gamma,Z}$ to Z. The convergence in probability of finite-dimensional marginals of $AX^{\gamma,Z}$ follows from Proposition 3.1. The Lévy process Z is stochastically continuous at any $T\ge 0$, so that due to Section 3 in the original paper by Skorohod [15] or Chapter 12 in Whitt [16] for the convergence in $D([0,\infty),\mathbb{R};M_1)$ it is sufficient to establish the boundedness of the family $\{AX^{\gamma,Z}\}$, that is to show that for every $T>0$

$$\lim_{K\to\infty}\sup_{\gamma>0}\mathbf{P}\Big(\sup_{t\in[0,T]}\big|AX_t^{\gamma,Z}\big|>K\Big)=0; \tag{3.7}$$

and to control the oscillation function, that is to show that for every $T>0$ and $\Delta>0$

$$\lim_{\delta\downarrow 0}\limsup_{\gamma\to\infty}\mathbf{P}\Big(\sup_{\substack{0\le t_1<t<t_2\le T,\\ t_2-t_1\le\delta}} M\big(AX_{t_1}^{\gamma,Z},AX_t^{\gamma,Z},AX_{t_2}^{\gamma,Z}\big)>\Delta\Big)=0. \tag{3.8}$$

Without loss of generality, we assume from now on that $A=1$. Let $T>0$ be fixed.

2. For the proof of (3.7), we use the representation (3.3) of $X^{\gamma,Z}$. Integrating by parts (compare with (2.3)) yields

$$X_t^{\gamma,Z} = Z_t - \int_0^t \mathrm{e}^{-\gamma(t-s)}\,\mathrm{d}Z_s = \gamma\int_0^t \mathrm{e}^{-\gamma(t-s)}Z_s\,\mathrm{d}s.$$

Thus for any $\gamma \geq 0$, we obtain the estimate

$$\sup_{t\in[0,T]} \big|X_t^{\gamma,Z}\big| \leq \sup_{t\in[0,T]} |Z_t| \sup_{t\in[0,T]} \gamma \int_0^t \mathrm{e}^{-\gamma(t-s)}\,\mathrm{d}s \leq \sup_{t\in[0,T]} |Z_t|, \tag{3.9}$$

so that the condition (3.7) holds true.

3. We now prove the estimate (3.8). Let $\Delta > 0$ be fixed. We show that for any $\theta > 0$ there is $\delta_0 = \delta_0(\Delta, \theta, T)$ such that for any $\delta \in (0, \delta_0]$ there is $\gamma_0 = \gamma_0(\delta, \Delta, \theta, T)$ such that for all $\gamma > \gamma_0$ the inequality

$$\mathbf{P}\Big(\sup_{\substack{0\leq t_1<t<t_2\leq T,\\ t_2-t_1\leq\delta}} M\big(X_{t_1}^{\gamma,Z}, X_t^{\gamma,Z}, X_{t_2}^{\gamma,Z}\big) > \Delta\Big) \leq \theta$$

holds true. The proof of this inequality will consist of three steps.

Step 1. *Reduction to a compound Poisson process with drift.* First, we decompose Z into a sum of a martingale with bounded jumps and small variance and a compound Poisson process with drift.

Let $a = a(\Delta, \theta, T) \in (0, 1]$ be such that $\nu(\{a\}) = \nu(\{-a\}) = 0$ and

$$\frac{16T}{\Delta^2} \int_{\{|y|<a\}} y^2 \nu(\mathrm{d}y) \leq \frac{\theta}{4}.$$

For this a, denote

$$\mu_a := \mu - \int_{a\leq|y|\leq 1} y\nu(\mathrm{d}y)$$

and consider the processes

$$\eta_t := \sum_{s\leq t} \Delta Z_s \mathbb{I}\big(|\Delta Z_s| \geq a\big) + \mu_a t \quad \text{and} \quad \xi_t := Z_t - \eta_t.$$

The processes ξ and η are independent Lévy processes with the respective Lévy–Khintchine representations

$$\mathbf{E}\mathrm{e}^{\mathrm{i}u\xi_1} = \exp\Big(\int_{\{|y|<a\}} \big(\mathrm{e}^{\mathrm{i}uy} - 1 - \mathrm{i}uy\big)\nu(\mathrm{d}y)\Big),$$

$$\mathbf{E}\mathrm{e}^{\mathrm{i}u\eta_1} = \exp\Big(\int_{\{|y|\geq a\}} \big(\mathrm{e}^{\mathrm{i}uy} - 1\big)\nu(\mathrm{d}y) + \mathrm{i}\mu_a u\Big), \qquad u \in \mathbb{R}.$$

Moreover, $Z = \xi + \eta$, η is a compound Poisson process with the drift μ_a, and ξ is a zero mean martingale with the variance $\mathbf{E}\xi_t^2 = t\int_{\{|y|<a\}} y^2\nu(\mathrm{d}y)$. Due to the linearity of equations (3.1) and (3.2), we obtain the representation

$$X^{\gamma,Z} = X^{\gamma,\xi} + X^{\gamma,\eta}$$

with

$$X_t^{\gamma,\xi} = \int_0^t \left(1 - \mathrm{e}^{-\gamma(t-s)}\right) \mathrm{d}\xi_s, \qquad X_t^{\gamma,\eta} = \int_0^t \left(1 - \mathrm{e}^{-\gamma(t-s)}\right) \mathrm{d}\eta_s.$$

Denote the event

$$E_\xi := \left\{\omega\colon \sup_{t\in[0,T]} \left|X_t^{\gamma,\xi}\right| < \frac{\Delta}{4}\right\}.$$

Using the estimate similar to (3.9) and applying the Doob inequality to the martingale ξ we obtain for all $\gamma \geq 0$ that

$$\begin{aligned}\mathbf{P}\left(E_\xi^c\right) &\leq \mathbf{P}\left(\sup_{t\in[0,T]} |\xi_t| > \frac{\Delta}{4}\right) \leq \frac{16T\mathbf{E}|\xi_T|^2}{\Delta^2} \\ &\leq \frac{16T}{\Delta^2} \int_{\{|y|<a\}} y^2 \nu(\mathrm{d}y) \leq \frac{\theta}{4}.\end{aligned}$$

Thus for all $\omega \in E_\xi$ and for all $\gamma \geq 0$ the inequality

$$\sup_{t\in[0,T]} \left|X_t^{\gamma,Z} - X_t^{\gamma,\eta}\right| \leq \sup_{t\in[0,T]} \left|X_t^{\gamma,\xi}\right| < \frac{\Delta}{4}$$

holds true. This implies that for all $\gamma \geq 0$ and $0 \leq t_1 < t < t_2 \leq T$

$$\left|M\left(X_{t_1}^{\gamma,Z}, X_t^{\gamma,Z}, X_{t_2}^{\gamma,Z}\right) - M\left(X_{t_1}^{\gamma,\eta}, X_t^{\gamma,\eta}, X_{t_2}^{\gamma,\eta}\right)\right| \leq \frac{\Delta}{2}.$$

Step 2. *Local extrema of* $X^{\gamma,\eta}$. There exists a level $z = z(a, \theta, T) > 0$ such that for the event

$$E_\eta := \left\{\sup_{t\in[0,T]} |\eta_t| \leq \frac{z}{2}\right\}$$

the inequality

$$\mathbf{P}\left(E_\eta^c\right) \leq \frac{\theta}{4}$$

holds. In particular, this implies that for $\omega \in E_\eta$ the jump sizes of η do not exceed z in absolute value, that is $\sup_{t\in[0,T]} |\Delta\eta_t(\omega)| \leq z$. The process η has the finite jump intensity

$$\beta_a = \int_{|y|\geq a} \nu(\mathrm{d}y) < \infty.$$

For the Lévy process η, consider its counting jump process $N = (N_t)_{t\geq 0}$ which is a Poisson process with intensity β_a. Denote by $\{\tau_k\}_{k\geq 0}$ the sequence of arrival times of η, $\tau_0 = 0$, and by $\{J_k\}_{k\geq 0}$, $J_0 = 0$, the sequence of its jump sizes, $J_k := \eta_{\tau_k} - \eta_{\tau_k-}$. It is easy to see that the process

$X^{\gamma,\eta}$ has the following path-wise representation:

$$X_t^{\gamma,\eta} = \sum_{k=0}^{N_t} J_k\big(1 - \mathrm{e}^{-\gamma(t-\tau_k)}\big) + \mu_a\bigg(t - \frac{1-\mathrm{e}^{-\gamma t}}{\gamma}\bigg), \qquad t \geq 0. \tag{3.10}$$

We choose $m^* \geq 0$ such that

$$\mathbf{P}\big(N_T \leq m^*\big) > 1 - \frac{\theta}{4}.$$

Further, for $\kappa > 0$ and $m = 0, \ldots, m^*$ consider the events

$$\begin{aligned}
C_\kappa^0 &:= C^0 = \{\omega\colon\ N_T = 0\},\\
C_\kappa^m &:= \{\omega\colon\ N_T = m\}\\
&\qquad \cap \{\omega\colon\ \tau_k - \tau_{k-1} \geq 2\kappa \text{ for } k = 1, \ldots, m, \text{ and } T - \tau_m \geq 2\kappa\}, \qquad 1 \leq m \leq m^*,\\
C_\kappa &:= \bigsqcup_{m=0}^{m^*} C_m \subset \big\{N_T \leq m^*\big\}.
\end{aligned}$$

It is well known (see, e.g., Proposition 3.4 in Sato [13]) that conditioned on $\{N_T = m\}$, the jump times $\tau_1, \ldots, \tau_m$ are distributed on the interval $[0, T]$ with the probability law of the order statistics obtained from m samples of the uniform distribution on $[0, T]$. Thus, we are able to choose $\kappa = \kappa(\theta, \Delta, T, m^*) > 0$ small enough, such that

$$\mathbf{P}(C_\kappa) > 1 - \frac{\theta}{3}.$$

For a fixed $m = 0, \ldots, m^*$ consider $\omega \in E_\eta \cap C_\kappa^m$. It is easy to see from the representation (3.10) that

$$X_t^{\gamma,\eta}(\omega) = \begin{cases}
\mu_a\Big(t - \frac{1-\mathrm{e}^{-\gamma t}}{\gamma}\Big), & t \in \big[0, \tau_1(\omega)\big),\\
\sum_{j=1}^{k} J_j(\omega)\big(1 - \mathrm{e}^{-\gamma(t-\tau_j(\omega))}\big) + \mu_a\Big(t - \frac{1-\mathrm{e}^{-\gamma t}}{\gamma}\Big), & \\
 & t \in \big[\tau_{k-1}(\omega), \tau_k(\omega)\big), k = 2, \ldots, m,\\
\sum_{j=1}^{m} J_j(\omega)\big(1 - \mathrm{e}^{-\gamma(t-\tau_j(\omega))}\big) + \mu_a\Big(t - \frac{1-\mathrm{e}^{-\gamma t}}{\gamma}\Big), & \\
 & t \in \big[\tau_m(\omega), T\big].
\end{cases}$$

The process $X^{\gamma,\eta}$ has smooth paths on the intervals $[\tau_{k-1}, \tau_k)$, $k = 1, \ldots, m$, and $[\tau_m, T]$. We show that for γ large enough the paths of $X^{\gamma,\eta}$ are either monotone on these intervals, or have at most one local extremum on each of the intervals. Indeed, $X^{\gamma,\eta}$ is obviously monotone on $t \in [0, \tau_1]$.

Let now $1 \le m \le m^*$. For $t \in (\tau_k, \tau_{k+1})$, $k = 1, \dots, m-1$, and for $t \in (\tau_m, T)$ consider the derivative of $X^{\gamma,\eta}$ w.r.t. t:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}X_t^{\gamma,\eta} &= \gamma \sum_{j=1}^{k} J_j \mathrm{e}^{-\gamma(t-\tau_j)} + \mu_a\big(1-\mathrm{e}^{-\gamma t}\big)\\ &= \gamma \sum_{j=1}^{k-1} J_j \mathrm{e}^{-\gamma(t-\tau_j)} + \gamma J_k \mathrm{e}^{-\gamma(t-\tau_k)} + \mu_a\big(1-\mathrm{e}^{-\gamma t}\big)\\ &= \gamma J_k \mathrm{e}^{-\gamma(t-\tau_k)}\left(1+\sum_{j=1}^{k-1}\frac{J_j}{J_k}\mathrm{e}^{-\gamma(\tau_k-\tau_j)}\right) + \mu_a\big(1-\mathrm{e}^{-\gamma t}\big).\end{aligned}$$

Taking into account that the jump sizes J_k are bounded, $|J_k(\omega)| \in [a, z]$ and the arrival times are separated by 2κ, $\tau_k - \tau_j \ge 2(k-j)\kappa$, $1 \le j \le k-1$, and $T - \tau_m \ge 2\kappa$, we can choose a non-random $\gamma_m = \gamma_m(a, z, \kappa, m)$ such that for $\gamma \ge \gamma_m$ the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}X_t^{\gamma,\eta} = \gamma J_k \mathrm{e}^{-\gamma(t-\tau_k)}\left(1+\sum_{j=1}^{k-1}\frac{J_j}{J_k}\mathrm{e}^{-\gamma(\tau_k-\tau_j)}\right) + \mu_a\big(1-\mathrm{e}^{-\gamma t}\big) = 0$$

has at most one solution on each of the intervals (τ_k, τ_{k+1}), $k = 1, \dots, m-1$, and on (τ_m, T). This unique solution (the local extremum of $X^{\gamma,\eta}$) exists if and only if $\mu_a \ne 0$ and $\frac{J_k}{\mu_a} < 0$, and is located at

$$\begin{aligned}t_k^* = t_k^*(\gamma) &= \frac{1}{\gamma}\ln\left(1+\gamma \mathrm{e}^{\gamma\tau_k}\left|\frac{J_k}{\mu_a}\right|\left(1+\sum_{j=1}^{k-1} J_j \mathrm{e}^{-\gamma(\tau_k-\tau_j)}\right)\right)\\ &\approx \tau_k + \frac{1}{\gamma}\ln\left(\gamma\left|\frac{J_k}{\mu_a}\right|\right), \qquad 1 \le k \le m.\end{aligned} \tag{3.11}$$

Moreover, we can choose γ_m big enough such that for $\gamma \ge \gamma_m$ we have $\tau_k < t_k^* \le \tau_k + \kappa$ for all $k = 1, \dots, m$. Furthermore, we choose γ_m big enough such that for $\gamma \ge \gamma_m$

$$\begin{aligned}&\max_{t\in[\tau_k+\kappa,\tau_{k+1}]}\left|\frac{\mathrm{d}}{\mathrm{d}t}X_t^{\gamma,\eta} - \mu_a\right| \le \frac{\Delta}{4}, \qquad k = 1, \dots, m-1, \quad \text{and}\\ &\max_{t\in[\tau_m+\kappa,T]}\left|\frac{\mathrm{d}}{\mathrm{d}t}X_t^{\gamma,\eta} - \mu_a\right| \le \frac{\Delta}{4}.\end{aligned} \tag{3.12}$$

Additionally for $\mu_a \ne 0$, we can assume that for $\gamma \ge \gamma_m$

$$\begin{aligned}&\max_{t\in[t_k^*,\tau_{k+1}]}\left|\frac{\mathrm{d}}{\mathrm{d}t}X_t^{\gamma,\eta}\right| \le 2|\mu_a| \qquad \text{if } \frac{J_k}{\mu_a} < 0, k = 1, \dots, m-1, \quad \text{and}\\ &\max_{t\in[t_m^*,T]}\left|\frac{\mathrm{d}}{\mathrm{d}t}X_t^{\gamma,\eta}\right| \le 2|\mu|_a \qquad \text{if } \frac{J_m}{\mu_a} < 0.\end{aligned} \tag{3.13}$$

Overall, for $\gamma \ge \gamma_m$ and for $\omega \in E_\eta \cap C_\kappa^m$ the paths of $X^{\gamma,\eta}$ have the following structure: they are continuous on $[0,T]$, smooth on (τ_k,τ_{k+1}), $k=0,\dots,m$, and (τ_m,T) and may have extrema either at arrival times τ_k, $k=1,\dots,m$, or at time instants t_k^* given by (3.11) provided $\frac{J_k}{\mu_a}<0$. The slope of $X^{\gamma,\eta}$ is close to μ_a on the left-hand neighbourhoods of the arrival times τ_k, $k=1,\dots,m$, and T. The derivative of $X^{\gamma,\eta}$ is bounded by a constant, say $2|\mu_a|$, in the right-hand neighbourhoods of the local extrema t_k^*. Let $\gamma^* := \bigvee_{m=1}^{m^*}\gamma_m$.

Step 3. *Estimate of the oscillation function M*. Let $\delta_0 \in (0, \kappa \wedge \frac{\Delta}{8(|\mu_a|+1)})$, $\gamma \ge \gamma^*$, and let $\omega \in E_\eta \cap C_\kappa^m$ for some $m=0,\dots,m^*$.

We estimate the value of the oscillation function $M = M(X_{t_1}^{\gamma,\eta}, X_t^{\gamma,\eta}, X_{t_2}^{\gamma,\eta})$ for $0\le t_1<t<t_2\le T$ and $t_2-t_1\le\delta\le\delta_0$. Let us consider three cases:

(i) If the path of $t\mapsto X_t^{\gamma,\eta}(\omega)$ is monotone on $[t_1,t_2]$, then $M=0$.

(ii) Let $\tau_k\in[t_1,t_2]$ for some $k=1,\dots,m$, and let τ_k be a local extremum. In this case, the maximal value of M over $t\in[t_1,t_2]$ is attained at τ_k and

$$\begin{aligned}M &\le \min\big\{\big|X_{\tau_k}^{\gamma,\eta}-X_{t_1}^{\gamma,\eta}\big|, \big|X_{\tau_k}^{\gamma,\eta}-X_{t_2}^{\gamma,\eta}\big|\big\} \le \big|X_{\tau_k}^{\gamma,\eta}-X_{t_1}^{\gamma,\eta}\big|\\ &\le \big|X_{\tau_k}^{\gamma,\eta}-X_{\tau_k-\delta_0}^{\gamma,\eta}\big|.\end{aligned}$$

Then due to (3.12)

$$M \le \Big(|\mu_a|+\frac{\Delta}{4}\Big)\delta_0 \le \frac{\Delta}{4}.$$

(iii) Let $t\mapsto X^{\gamma,\eta}$ be non-monotone in $[t_1,t_2]$ and a local extremum t_k^* exist and belong to $[t_1,t_2]$ for some $k=1,\dots,m$. Then we estimate with the help of (3.13) that

$$\begin{aligned}M &\le \min\big\{\big|X_{t_k^*}^{\gamma,\eta}-X_{t_1}^{\gamma,\eta}\big|, \big|X_{t_k^*}^{\gamma,\eta}-X_{t_2}^{\gamma,\eta}\big|\big\} \le \big|X_{t_k^*}^{\gamma,\eta}-X_{t_k^*+\delta_0}^{\gamma,\eta}\big|\\ &\le 2|\mu_a|\delta_0 \le \frac{\Delta}{4}.\end{aligned}$$

Overall, these estimates imply, that for all $0<\delta\le\delta_0$ and $\gamma\ge\gamma^*$

$$\mathbf{P}\Big(\sup_{\substack{0\le t_1<t<t_2\le T,\\ t_2-t_1\le\delta}} M\big(X_{t_1}^{\gamma,\eta}, X_t^{\gamma,\eta}, X_{t_2}^{\gamma,\eta}\big) > \frac{\Delta}{2}\Big| E_\eta\cap C_\kappa\Big) = 0$$

and the inequality (3.8) follows:

$$\begin{aligned}&\mathbf{P}\Big(\sup_{\substack{0\le t_1<t<t_2\le T,\\ t_2-t_1\le\delta}} M\big(X_{t_1}^{\gamma,Z}, X_t^{\gamma,Z}, X_{t_2}^{\gamma,Z}\big) > \Delta\Big)\\ &\quad\le \mathbf{P}\Big(\sup_{\substack{0\le t_1<t<t_2\le T,\\ t_2-t_1\le\delta}} M\big(X_{t_1}^{\gamma,\eta}, X_t^{\gamma,\eta}, X_{t_2}^{\gamma,\eta}\big) > \frac{\Delta}{2}, E_\eta\cap C_\kappa\Big) + \mathbf{P}\big(E_\eta^c\big) + \mathbf{P}\big(C_\kappa^c\big) + \mathbf{P}\big(E_\xi^c\big)\\ &\quad\le \frac{\theta}{4}+\frac{\theta}{4}+\frac{\theta}{3} < \theta.\end{aligned}$$

□

Proof of Theorem 2.1. Consider the equations (1.2) and (1.3) with zero initial conditions driven by an α-stable Lévy process $l^{(\alpha)}$, $\alpha \in (0,2)$. Applying the time change $t \mapsto \frac{t}{\varepsilon^\alpha}$ and using the self-similarity of $l^{(\alpha)}$, namely that $\mathrm{Law}(\varepsilon^{1/\alpha} l^{(\alpha)}_{t/\varepsilon^\alpha}, t \geq 0) = \mathrm{Law}(l^{(\alpha)}_t, t \geq 0)$, we obtain that for any $\varepsilon > 0$ the law of the processes $(v^\varepsilon_{t/\varepsilon^\alpha})_{t\geq 0}$ and $(x^\varepsilon_{t/\varepsilon^\alpha})_{t\geq 0}$ coincides with the law of the processes V^{1/ε^α} and X^{1/ε^α} which solve the stochastic differential equations (3.1) and (3.2) driven by a process L being a copy of $l^{(\alpha)}$, $\mathrm{Law}(L) = \mathrm{Law}(l^{(\alpha)})$. Then the statement of Theorem 2.1 follows from Theorem 3.1. □

Let us discuss the weak convergence of integrated OU processes driven by an α-stable Lévy process of small intensity for arbitrary initial conditions.

We start with the generalization of the Theorem 3.1. Consider the system of stochastic differential equations driven by an arbitrary Lévy process L

$$V^\gamma = v_0 - \gamma \int_0^t A V_s \,\mathrm{d}s + L_t, \qquad X^\gamma = x_0 + \gamma \int_0^t V_s \,\mathrm{d}s,$$

γ being a big parameter, and the initial conditions v_0, x_0 being arbitrary. The explicit solutions are given by the formulae

$$V^\gamma_t = v_0 \mathrm{e}^{-\gamma A t} + \int_0^t \mathrm{e}^{-\gamma A(t-s)} \,\mathrm{d}L_s \tag{3.14}$$

and

$$A X^\gamma_t = A x_0 + v_0\bigl(1 - \mathrm{e}^{-\gamma A t}\bigr) + \int_0^t \bigl(1 - \mathrm{e}^{-\gamma A(t-s)}\bigr) \,\mathrm{d}L_s. \tag{3.15}$$

It follows immediately from Theorem 3.1 that for $x_0 \in \mathbb{R}$ and $v_0 = 0$ the processes $(A(X^\gamma_t - x_0))_{t\geq 0}$ converge in probability to L in $D([0,\infty), \mathbb{R}; M_1)$.

The situation becomes a little more complicated for $v_0 \neq 0$. The continuous second summand on the right-hand side of (3.15) does not converge uniformly on the intervals $[0,T]$, $T > 0$, yet has a discontinuous point-wise limit

$$v_0\bigl(1 - \mathrm{e}^{-\gamma A t}\bigr) \to v_0 \mathbb{I}_{(0,\infty)}(t), \qquad t \geq 0.$$

Thus for $v_0 \neq 0$, the limiting process $(L_t - v_0 \mathbb{I}_{\{0\}}(t))_{t\geq 0}$ is discontinuous in probability at the origin and the convergence in probability of finite-dimensional marginals of the process $A(X^\gamma - x_0) - v_0$ to those of L holds only on the set $(0,\infty)$. The M_1-convergence to L still holds on all intervals $[\delta, T]$, $0 < \delta < T$, and we obtain the following result: for arbitrary $x_0, v_0 \in \mathbb{R}$

$$\bigl(A\bigl(X^\gamma_t - x_0\bigr) - v_0\bigr)_{t\geq 0} \xrightarrow{\mathbf{P}} L \qquad \text{in } D\bigl((0,\infty), \mathbb{R}; M_1\bigr) \text{ as } \gamma \to \infty.$$

Consequently, Theorem 2.1 takes the following form. Let $l^{(\alpha)}$ be an α-stable Lévy process and let x^ε be the integrated OU process satisfying equations (1.2) and (1.3) with arbitrary initial conditions $x_0, v_0 \in \mathbb{R}$. Then

$$\bigl(A\bigl(x^\varepsilon_{t/\varepsilon^\alpha} - x_0\bigr) - v_0\bigr)_{t\geq 0} \Rightarrow l^{(\alpha)} \qquad \text{in } D\bigl((0,\infty), \mathbb{R}; M_1\bigr) \text{ as } \varepsilon \to 0.$$

Finally, we direct the reader's attention to Section 13.6.2 in Whitt [16], and Puhalskii and Whitt [12] for more information on the treatment of discontinuities of stochastic processes at the origin, especially on the so-called M_1'-convergence.

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References

[1] Adler, R.J. (1990). *An Introduction to Continuity, Extrema, and Related Topics for General Gaussian Processes. Institute of Mathematical Statistics Lecture Notes—Monograph Series* **12**. Hayward, CA: IMS. MR1088478

[2] Al-Talibi, H., Hilbert, A. and Kolokoltsov, V. (2010). Nelson-type limit for a particular class of Lévy processes. *AIP Conf. Proc.* **1232** 189.

[3] Applebaum, D. (2009). Extending stochastic resonance for neuron models to general Lèvy noise. *IEEE Transactions on Neural Networks* **20** 1993–1995.

[4] Barndorff-Nielsen, O.E. (1998). Processes of normal inverse Gaussian type. *Finance Stoch.* **2** 41–68. MR1804664

[5] Barndorff-Nielsen, O.E. and Shephard, N. (2003). Integrated OU processes and non-Gaussian OU-based stochastic volatility models. *Scand. J. Stat.* **30** 277–295. MR1983126

[6] Chechkin, A.V., Gonchar, V.Y. and Szydłowski, M. (2002). Fractional kinetics for relaxation and superdiffusion in a magnetic field. *Physics of Plasmas* **9** 78–88.

[7] Garbaczewski, P. and Olkiewicz, R. (2000). Ornstein–Uhlenbeck–Cauchy process. *J. Math. Phys.* **41** 6843–6860. MR1781410

[8] Hesse, C.H. (1991). The one-sided barrier problem for an integrated Ornstein–Uhlenbeck process. *Comm. Statist. Stochastic Models* **7** 447–480. MR1123138

[9] Horsthemke, W. and Lefever, R. (1984). *Noise-Induced Transitions: Theory and Applications in Physics, Chemistry, and Biology. Springer Series in Synergetics* **15**. Berlin: Springer. MR0724433

[10] Kuznetsov, A. (2011). On extrema of stable processes. *Ann. Probab.* **39** 1027–1060. MR2789582

[11] Lefebvre, M. (1989). Moment generating function of a first hitting place for the integrated Ornstein–Uhlenbeck process. *Stochastic Process. Appl.* **32** 281–287. MR1014454

[12] Puhalskii, A.A. and Whitt, W. (1997). Functional large deviation principles for first-passage-time processes. *Ann. Appl. Probab.* **7** 362–381. MR1442318

[13] Sato, K. (1999). *Lévy Processes and Infinitely Divisible Distributions. Cambridge Studies in Advanced Mathematics* **68**. Cambridge: Cambridge Univ. Press. Translated from the 1990 Japanese original, revised by the author. MR1739520

[14] Simon, T. (2011). Hitting densities for spectrally positive stable processes. *Stochastics* **83** 203–214. MR2800088

[15] Skorohod, A.V. (1956). Limit theorems for stochastic processes. *Theory Probab. Appl.* **1** 261–290. MR0084897

[16] Whitt, W. (2002). *Stochastic-Process Limits*: *An Introduction to Stochastic-Process Limits and their Application to Queues*. *Springer Series in Operations Research*. New York: Springer. MR1876437
[17] Williams, D.R.E. (2001). Path-wise solutions of stochastic differential equations driven by Lévy processes. *Rev. Mat. Iberoam*. **17** 295–329. MR1891200

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