On hyperbolic Bessel processes and beyond

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We investigate distributions of hyperbolic Bessel processes. We find links between the hyperbolic cosine of hyperbolic Bessel processes and functionals of geometric Brownian motion. We present an explicit formula for the Laplace transform of the hyperbolic cosine of a hyperbolic Bessel process and some other interesting probabilistic representations of this Laplace transform. We express the one-dimensional distribution of a hyperbolic Bessel process in terms of other, known and independent processes. We present some applications including a new proof of Bougerol's identity and its generalization. We characterize the distribution of the process which is the hyperbolic sine of hyperbolic Bessel process.

Keywords: Bessel process; Brownian motion; hyperbolic Bessel process; Laplace transform

1. Introduction

The important role which functionals of Brownian motion play in many fields of mathematics (including mathematical finance, diffusion processes in random environment, probabilistic studies related to hyperbolic spaces etc.) is a motivation to study the wide class of different diffusion processes connected to those functionals somehow (see, e.g., [6,9,12,17] and [18]). In this work, we consider a special class of such diffusions: Brownian motion with a special stochastic drift. It is well known in the literature that processes like Ornstein–Uhlenbeck processes, Bessel processes or radial Ornstein–Uhlenbeck processes are examples of Brownian motion with stochastic drift connected to special functions: parabolic cylinder, Bessel or Kummer functions (see [5]). In this work, we consider another interesting class of such diffusions – hyperbolic Bessel processes which are connected to Legendre functions. We establish the distribution of a hyperbolic Bessel process. Borodin presented the computation of the transition density function for a hyperbolic Bessel process (see formulas (5.4) and (5.5) in [5]). His method relied on the connection between the Laplace transform of the transition density function and two increasing and decreasing solutions of some ordinary differential equation of the second order (for more details of this method see also [6]). Gruet established the transition density functions of hyperbolic processes by methods of planar geometry (see [13,14]). However, the results obtained by both authors are technically very complicated. The role of the hyperbolic Bessel processes in the world of Brownian motion functionals was mentioned also by Byczkowski, Małecki and Ryznar [8].

Our approach to hyperbolic Bessel processes is completely different and purely probabilistic. We present results for hyperbolic Bessel processes with index $\alpha \geq -1/2$ and for fixed time. We find a link between the hyperbolic cosine of a hyperbolic Bessel process and functionals of geometric Brownian motion. We present different probabilistic representations of the Laplace transform of the hyperbolic cosine of a hyperbolic Bessel process R. We also give probabilistic representations of the density of $\cosh R_t$. The link between hyperbolic Bessel processes and

functionals of geometric Brownian motion enables us to establish a simple explicit form of the Laplace transform of the vector $(e^{B_t+kt}, \int_0^t e^{2(B_u+ku)}\,du)$ for a standard Brownian motion B and a nonnegative integer k. These results are obtained for fixed t (and not for stochastic time) and so they can be effectively used in numerical computations. We express the distribution of a hyperbolic Bessel process R in terms of a squared Bessel process X and a vector $(B^{(\mu)}, A^{(\mu)})$ of Brownian motion with drift and the integral of geometric Brownian motion, which is independent of X. The joint distribution of $(B^{(\mu)}, A^{(\mu)})$ is known in the literature and is expressed via the density function (e.g., see [17]). As an interesting example of applications, we find an alternative simple proof of the well known Bougerol identity (for a thorough study, we refer the reader to [1,2] or [11]). We also characterize the distribution of $\sinh(R)$ for R being a hyperbolic Bessel process. It is interesting that this distribution is deduced from the stochastic process which is in a sense a generalization of a squared Bessel process. At the end, we outline the case of exponential random time independent of the Brownian motion driving the hyperbolic Bessel process. In this case, many explicit and computable results are known in the literature (see, e.g., [6,17] or [18]).

2. Hyperbolic Bessel processes with fixed time

We consider a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,\infty)}$ satisfying the usual conditions and $\mathcal{F} = \mathcal{F}_\infty$, and with a standard Brownian motion B. We define a hyperbolic Bessel process R of index $\alpha \in (-\frac{1}{2}; \infty)$ as a nonnegative diffusion, starting from a nonnegative x, given by

$$R_t = x + B_t + \left(\alpha + \frac{1}{2}\right) \int_0^t \coth R_u \, du. \tag{2.1}$$

For $\alpha = -\frac{1}{2}$, we define the hyperbolic Bessel process as the reflected (at 0) Brownian motion starting from a nonnegative x, that is,

$$R_t = |x + B_t|. \tag{2.2}$$

Now, we discuss properties of solutions to (2.1) which follow from general theory on SDE (see, e.g., Cherny and Engelbert [10]). The behavior of R depends on the index α. For $\alpha \in (-\frac{1}{2}, \infty)$, the point 0 is an isolated singular point of (2.1). For $\alpha \in (-\frac{1}{2}, 0)$ and $x \geq 0$ there exists a positive solution of (2.1) defined up to $T_a = \inf\{t \geq 0: R_t = a\}$ for every $a > x \geq 0$ and such a solution is unique. This solution hits zero with a positive probability (see [10], Theorem 2.12), but the point zero is instantaneously reflecting. Indeed we check that $m(\{0\}) = 0$ for m the speed measure of the hyperbolic Bessel process (for the boundary classification see, e.g., Borodin and Salminen [6], pages 14–16). Therefore, by analogy with a SDE defining δ-dimensional Bessel process, in the case $\alpha \in (-\frac{1}{2}, 0)$ we understand (2.1) as SDE

$$dR_t = B_t + \left(\alpha + \frac{1}{2}\right) \mathbf{1}_{\{R_t \neq 0\}} \coth R_t \, dt, \qquad R_0 = x. \tag{2.3}$$

For the case $\alpha \geq 0$, we have two possibilities. If $x > 0$, then there exist a unique strictly positive solution defined up to T_a for every $a > x \geq 0$. If $x = 0$, then there exists a positive solution

of (2.1) defined up to T_a, and such a solution is unique. Moreover, $R > 0$ on $]0, T_a]$ (see [10], Theorem 2.16). Summing up, general theory implies that for $\alpha > -\frac{1}{2}$ and $x \geq 0$ there exists a unique positive solution defined up to the explosion time.

Recall that Gruet [13] defines a hyperbolic Bessel process of index $\alpha > -1/2$ as a nonnegative diffusion R with generator

$$\mathcal{A} = \frac{1}{2} \frac{d^2}{dx^2} + \left(\alpha + \frac{1}{2}\right) \coth(x) \frac{d}{dx}. \tag{2.4}$$

In the case of $-1/2 < \alpha < 0$, the definition is completed by the requirement that 0 is an instantaneously reflecting point. It is also assumed that the starting point is nonnegative. A similar definition of hyperbolic Bessel process can be found in Borodin [5]. In [13], by the identification of the Green function, it is stated that a reasonable candidate for a hyperbolic Bessel process with $\alpha = -1/2$ is a reflected Brownian motion (see the end of Section 3 in [13]).

To start the discussion of the properties of hyperbolic Bessel processes, we define a process θ to be the solution of the SDE

$$d\theta_t = \sqrt{\left|\theta_t^2 - 1\right|} \, dB_t + (\alpha + 1)|\theta_t| \, dt, \tag{2.5}$$

where $\alpha \geq -1/2$, and such that $\theta_0 = x \geq 1$. Observe that for $x \in \mathbb{R}$ and coefficients $\sigma(x) = \sqrt{|x^2 - 1|}$, $b(x) = (\alpha + 1)|x|$ we have

$$\left|\sigma(x)\right| + \left|b(x)\right| \leq (2 + \alpha)\big(|x| + 1\big),$$

so there exists a weak solution, which does not explode. Using Ex. 3.14 [19], Chapter IX (with $g \equiv 1, c = 3, \delta = 1$), we see that pathwise uniqueness holds for (2.5). Thus, the SDE (2.5) has a unique strong nonexploding solution.

Now, we consider the diffusion $\eta_t := \theta_t - 1$. It is a diffusion with drift and diffusion coefficients equal to $\tilde{b}(y) = (\alpha + 1)|y + 1|$ and $\tilde{\sigma}(y) = \sqrt{|y^2 + 2y|}$, respectively. We observe that 0 is an isolated singular point for η. For $\alpha \geq 0$ and $a > 0$, we have

$$\int_0^a \exp\left(\int_x^a \frac{2\tilde{b}(y)}{\tilde{\sigma}^2(y)} \, dy\right) dx = \int_0^a \left(\frac{a^2 + 2a}{x^2 + 2x}\right)^{\alpha+1} dx \geq a^{\alpha+1} \int_0^a x^{-(\alpha+1)} \, dx = \infty.$$

Therefore, from [10], Theorems 2.16 and 2.17, it follows that $\eta_t > 0$ for all $t > 0$, provided $\eta_0 > 0$. So, if $\theta_0 > 1$, then $\theta_t > 1$ for all $t > 0$. Hence, the process θ satisfies for $\alpha \geq 0$ the SDE

$$d\theta_t = \sqrt{\theta_t^2 - 1} \, dB_t + (\alpha + 1)\theta_t \, dt. \tag{2.6}$$

It turns out that in the case of $\alpha \geq 0$ we can recover a hyperbolic Bessel process from the process θ. As usual, by arcosh we denote the inverse function of cosh.

Theorem 2.1. *If $\alpha \geq 0$ and $x > 1$, then the process $R_t = \operatorname{arcosh}(\theta_t)$ is a hyperbolic Bessel process of index α, where θ is the solution of (2.6), $\theta_0 = x$.*

Proof. Observe that, by definition,

$$R_t = \operatorname{ar cosh}(\theta_t) = \ln\left(\theta_t + \sqrt{\theta_t^2 - 1}\right) > 0. \tag{2.7}$$

As $\theta_t > 1$ for all $t \geq 0$, we can use the Itô lemma to conclude that R satisfies (2.1). □

Theorem 2.2. *A hyperbolic Bessel process R of index $\alpha \geq -1/2$ does not explode.*

Proof. For $\alpha \geq 0$, we use Theorem 2.1, which implies that R does not explode since θ does not explode. For $\alpha = -\frac{1}{2}$ the process R is given by (2.2), so it does not explode. For $\alpha \in (-\frac{1}{2}; 0)$, we observe that $\cosh(R)$ and θ have the same generator $\mathcal{A}_\theta$ on C_c^2, the space of twice continuously differentiable functions with compact support. From the uniqueness of solution of the martingale problem induced by $\mathcal{A}_\theta$, we conclude that $\cosh(R)$ and θ have the same distribution. The diffusion θ does not explode, so R does not explode either. □

Now we give a formula for the Laplace transform of cosh of a hyperbolic Bessel process with $\alpha \geq -\frac{1}{2}$, one of the most important results of this paper.

Theorem 2.3. *Fix $\alpha \geq -\frac{1}{2}$. If R is a hyperbolic Bessel process of index α and $R_0 = x \geq 0$, then for $t > 0$ and $\lambda > 0$,*

$$\mathbb{E}\left[\exp(-\lambda \cosh R_t)\right] = \mathbb{E}\left[\exp\left(-\lambda \cosh(x) V_t - \frac{\lambda^2}{2} \int_0^t V_u^2 \, du\right)\right], \tag{2.8}$$

where $V_t = e^{(\alpha+1/2)t + B_t}$ and B is a standard Brownian motion.

Proof. Set $\theta_t = \cosh R_t \geq 1$, so $\theta_0 = \cosh x$. By the definition of R, the properties of the function cosh and the Itô lemma we see that

$$d\theta_t = \sqrt{\theta_t^2 - 1} \, dB_t + a\theta_t \, dt, \tag{2.9}$$

where $a = \alpha + 1$. Using again the Itô lemma, we obtain

$$de^{-\lambda\theta_t} = -\lambda e^{-\lambda\theta_t} \sqrt{\theta_t^2 - 1} \, dB_t - \lambda a e^{-\lambda\theta_t} \theta_t \, dt + \frac{\lambda^2}{2} e^{-\lambda\theta_t} \left(\theta_t^2 - 1\right) dt. \tag{2.10}$$

The process $\int_0^t e^{-\lambda\theta_u} \sqrt{\theta_u^2 - 1} \, dB_u$ is a martingale since $e^{-x} \sqrt{x^2 - 1} \leq e^{-1}$ for $x \geq 1$.

Thus, we infer from (2.10) that

$$\mathbb{E}e^{-\lambda\theta_t} = e^{-\lambda\theta_0} - a\lambda \int_0^t \mathbb{E}\left[e^{-\lambda\theta_u} \theta_u\right] du + \frac{\lambda^2}{2} \int_0^t \mathbb{E}\left[e^{-\lambda\theta_u} \left(\theta_u^2 - 1\right)\right] du. \tag{2.11}$$

Define $p(t, \lambda) := \mathbb{E}e^{-\lambda\theta_t}$, so the function p is bounded. Using (2.11), we deduce that p satisfies the following partial differential equation:

$$\frac{\partial p}{\partial t} = -\frac{\lambda^2}{2} p + a\lambda \frac{\partial p}{\partial \lambda} + \frac{\lambda^2}{2} \frac{\partial^2 p}{\partial \lambda^2}, \tag{2.12}$$

with $p(0, \lambda) = e^{-\lambda \theta_0}$. Set $V_t = V_0 e^{(\alpha+1/2)t+B_t}$. Then

$$dV_t = V_t \, dB_t + aV_t \, dt$$

with $a = \alpha + 1$. The generator of V is

$$\mathcal{A}_V = \frac{x^2}{2} \frac{d^2}{dx^2} + ax \frac{d}{dx}.$$

Observe, by (2.11), that $p \in C^{1,2}((0, \infty) \times (0, \infty))$ and from the Feynman–Kac theorem (see [16], Chapter 5, Theorem 7.6) we know that the only bounded solution of the partial differential equation

$$\frac{\partial u}{\partial t} = \mathcal{A}_V u - \frac{x^2}{2} u, \qquad u(0, x) = e^{-\theta_0 x},$$

that is, (2.12), admits the stochastic representation

$$u(t, x) = \mathbb{E} \exp\left(-\theta_0 V_t - \frac{1}{2} \int_0^t V_u^2 \, du\right),$$

where $V_0 = x$. This implies the assertion of the theorem. $\qquad \square$

As an easy consequence, we obtain the following proposition.

Proposition 2.4. *The Laplace transform of the vector* $(e^{B_t}, \int_0^t e^{2B_u} \, du)$ *is given by*

$$\mathbb{E} \exp\left(-\gamma e^{B_t} - \frac{\lambda^2}{2} \int_0^t e^{2B_u} \, du\right) = \mathbb{E} \exp\left(-\lambda \cosh(x + B_t)\right) \qquad (2.13)$$

for $\gamma > 0$ and $\lambda > 0$, where $x = \operatorname{ar\,cosh} \frac{\gamma}{\lambda}$.

Proof. The conclusion follows by applying Theorem 2.3 with $\alpha = -\frac{1}{2}$, and the observation that $\cosh(R_t) = \cosh(x + B_t)$, since $\cosh$ is an even function. This implies that for any $\lambda > 0$,

$$\mathbb{E} \exp\left(-\lambda \cosh(x + B_t)\right) = \mathbb{E} \exp\left(-\lambda \cosh(R_t)\right). \qquad \square$$

Remark 2.5. The form of the Laplace transform of the vector $(e^{B_t}, \int_0^t e^{2B_u} \, du)$ in the last proposition, that is, (2.13), is very simple when compared to the form of density of this vector obtained by Matsumoto and Yor [17]. Indeed, the density given in [17] has the oscillating nature in the neighbourhood of 0 and is not convenient for computational use (see [4]). The knowledge of the Laplace transform makes it possible to invert it numerically and obtain the density of $(e^{B_t}, \int_0^t e^{2B_u} \, du)$. It is also important to realize that it enables one to obtain the density for $(e^{B_t^{(\mu)}}, \int_0^t e^{2B_u^{(\mu)}} \, du)$, where $B_t^{(\mu)} := B_t + \mu t$, $\mu \in \mathbb{R}$, as we know from [17] that

$$\mathbb{P}\left(B_t^{(\mu)} \in dx, \int_0^t e^{2B_u^{(\mu)}} \, du \in dy\right) = e^{\mu x - \mu^2 t/2} \mathbb{P}\left(B_t \in dx, \int_0^t e^{2B_u} \, du \in dy\right).$$

Theorem 2.6. *Let $k \in \mathbb{N}$, $\gamma > 0$, $\lambda > 0$ and R be a hyperbolic Bessel process*

$$R_t = \operatorname{ar\,cosh}(\gamma/\lambda) + B_t + k \int_0^t \coth R_u \, du. \tag{2.14}$$

For $\lambda > 0$, we have

$$\mathbb{E} \exp(-\lambda \cosh R_t) = (-1)^k e^{(k^2/2)t} \frac{\partial^k p(\gamma, \lambda)}{\partial \gamma^k},$$

where

$$p(\gamma, \lambda) = \mathbb{E} e^{-\lambda \cosh(\operatorname{ar\,cosh}(\gamma/\lambda) + B_t)}.$$

Proof. By Proposition 2.4, for $x = \operatorname{ar\,cosh}(\gamma/\lambda)$,

$$p(\gamma, \lambda) = \mathbb{E} e^{-\lambda \cosh(x + B_t)} = \mathbb{E} \exp\left(-\gamma e^{B_t} - \frac{\lambda^2}{2} \int_0^t e^{2B_u} \, du\right).$$

Define the new probability measure $\mathbb{Q}$ by

$$\left. \frac{d\mathbb{Q}}{d\mathbb{P}} \right|_{\mathcal{F}_t} = e^{kB_t - (k^2/2)t}.$$

Then $V_t = B_t - kt$ is a standard Brownian motion under $\mathbb{Q}$ and

$$p(\gamma, \lambda) = \mathbb{E}_{\mathbb{Q}} \exp\left(-kB_t + \frac{k^2}{2}t - \gamma e^{B_t} - \frac{\lambda^2}{2} \int_0^t e^{2B_u} \, du\right)$$

$$= e^{(k^2/2)t} \mathbb{E}_{\mathbb{Q}} \exp\left(-k(V_t + kt) - \gamma e^{(V_t + kt)} - \frac{\lambda^2}{2} \int_0^t e^{2(V_u + ku)} \, du\right).$$

The result follows from Theorem 2.3 after taking the kth derivative of p with respect to γ. $\square$

Proposition 2.7. *If $k \in \mathbb{N}$ and $Y_t = e^{B_t + kt}$, then for $\lambda > 0$, $\gamma > 0$,*

$$\mathbb{E} \exp\left(-\gamma Y_t - \frac{\lambda^2}{2} \int_0^t Y_u^2 \, du\right) = (-1)^k e^{(k^2/2)t} \frac{\partial^k p(\gamma, \lambda)}{\partial \gamma^k},$$

where $p(\gamma, \lambda)$ is as in Theorem 2.6.

Proof. This follows from Theorems 2.6 and 2.3. $\square$

Theorem 2.8. *If R is a hyperbolic Bessel process of index $\alpha \geq -\frac{1}{2}$ starting from x, then for $\lambda > 0$,*

$$\mathbb{E} \exp(-\lambda \cosh R_t) = \frac{\sqrt{2\pi}}{\sqrt{t}} e^{-(a^2 t/2)} \mathbb{E}\left(1_{\{V_t \geq |B_t|\}} V_t h_t(B_t, x) J_0\left(\lambda \phi(B_t, V_t)\right)\right),$$

where $a = \alpha + \frac{1}{2}$, J_0 is the Bessel function of the first kind of order 0, $h_t(z, x) = \exp(\frac{z^2}{2t} + az - \lambda \cosh(x)e^z)$, $\phi(v, z) = \sqrt{2}e^{v/2}(\cosh z - \cosh v)^{1/2}$ for $z \geq |v|$, and V, B are two independent standard Brownian motions.

Proof. Let $W_t = B_t + at$ and let $\mathbb{Q}$ be the new probability measure given by

$$\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_t} = e^{-aB_t - (a^2/2)t}.$$

From Theorem 2.3, we have

$$\mathbb{E}\exp(-\lambda \cosh R_t) = \mathbb{E}\exp\left(-\lambda \cosh(x)e^{at+B_t} - \frac{\lambda^2}{2}\int_0^t e^{2(B_u+au)}\,du\right)$$

$$= \mathbb{E}_{\mathbb{Q}}\exp\left(aW_t - \frac{a^2}{2}t - \lambda \cosh(x)e^{W_t} - \frac{\lambda^2}{2}\int_0^t e^{2W_u}\,du\right)$$

$$= e^{-(a^2/2)t}\mathbb{E}\exp\left(aB_t - \lambda \cosh(x)e^{B_t} - \frac{\lambda^2}{2}\int_0^t e^{2B_u}\,du\right),$$

so

$$\mathbb{E}\exp(-\lambda \cosh R_t) = e^{-(a^2/2)t}\mathbb{E}\left[e^{aB_t - \lambda \cosh(x)e^{B_t}}\mathbb{E}\left(e^{-(\lambda^2/2)\int_0^t e^{2B_u}\,du}\big| B_t\right)\right].$$

To finish the proof, we use the conditional Laplace transform (see (5.5) in [17])

$$\mathbb{E}\left(e^{-(\lambda^2/2)\int_0^t e^{2B_u}\,du}\big| B_t = x\right)\frac{1}{\sqrt{2\pi t}}e^{-x^2/(2t)} = \int_{|x|}^{\infty}\frac{z}{\sqrt{2\pi t^3}}e^{-z^2/(2t)}J_0\big(\lambda\phi(x, z)\big)\,dz. \qquad \square$$

Now we define a process $\Gamma^{(\alpha)}$. Let B be a standard Brownian motion, $\alpha \in \mathbb{R}$, $\lambda > 0$ and

$$\Gamma_t^{(\alpha)} = \frac{e^{B_t + \alpha t}}{1 + \lambda\int_0^t e^{B_u + \alpha u}\,du}. \tag{2.15}$$

Theorem 2.9. *If R is a hyperbolic Bessel process of index $\alpha \geq -\frac{1}{2}$ starting from x, then for $\lambda > 0$,*

$$\mathbb{E}\exp(-\lambda \cosh R_t) = e^{-\lambda}\mathbb{E}\left[e^{-\lambda(\cosh(x)-1)\Gamma_t^{(\alpha+1/2)}}\left(1 + \lambda\int_0^t e^{B_u + (\alpha+1/2)u}\,du\right)^{-\alpha-1}\right],$$

where B is a standard Brownian motion and $\Gamma_t^{(\alpha)}$ is given by (2.15).

Proof. Set $Y_t^{(\alpha)} = e^{B_t + \alpha t}$. Define the new measure $\mathbb{Q}$ by

$$\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_t} = e^{-\lambda\int_0^t Y_u^{(\alpha+1/2)}\,dB_u - (\lambda^2/2)\int_0^t (Y_u^{(\alpha+1/2)})^2\,du}.$$

Theorem 2.3 implies

$$
\begin{aligned}
\mathbb{E}e^{-\lambda \cosh R_t} &= \mathbb{E}\big[e^{-\cosh(x)\lambda Y_t^{(\alpha+1/2)} - (\lambda^2/2)\int_0^t (Y_u^{(\alpha+1/2)})^2\, du}\big] \\
&= \mathbb{E}_{\mathbb{Q}}\big[e^{-\cosh(x)\lambda Y_t^{(\alpha+1/2)} + \lambda \int_0^t Y_u^{(\alpha+1/2)}\, dB_u}\big] \\
&= \mathbb{E}_{\mathbb{Q}}\big[e^{-\cosh(x)\lambda Y_t^{(\alpha+1/2)} + \lambda(Y_t^{(\alpha+1/2)}-1) - (\alpha+1)\lambda \int_0^t Y_u^{(\alpha+1/2)}\, du}\big] := I,
\end{aligned}
$$

where we have used the fact that

$$
Y_t^{(\alpha+1/2)} = 1 + \int_0^t Y_u^{(\alpha+1/2)}\, dB_u + (\alpha+1)\int_0^t Y_u^{(\alpha+1/2)}\, du. \tag{2.16}
$$

From the Girsanov theorem, the process $V_t = B_t + \lambda \int_0^t Y_u^{(\alpha+1/2)}\, du$ is a standard Brownian motion under $\mathbb{Q}$. By the result of Alili, Matsumoto, and Shiraishi [3], Lemma 3.1, we have

$$
B_t = V_t - \ln\left(1 + \lambda \int_0^t e^{V_u + (\alpha+1/2)u}\, du\right). \tag{2.17}
$$

Thus,

$$
Y_t^{(\alpha+1/2)} = \frac{e^{V_t + (\alpha+1/2)t}}{1 + \lambda \int_0^t e^{V_u + (\alpha+1/2)u}\, du},
$$

and

$$
\begin{aligned}
I &= e^{-\lambda}\mathbb{E}_{\mathbb{Q}}\left[\exp\left(-\lambda\big(\cosh(x)-1\big)\frac{e^{V_t + (\alpha+1/2)t}}{1 + \lambda \int_0^t e^{V_u + (\alpha+1/2)u}\, du} - (\alpha+1)(V_t - B_t)\right)\right] \\
&= e^{-\lambda}\mathbb{E}\left[e^{-\lambda(\cosh(x)-1)\Gamma_t^{(\alpha+1/2)}}\left(1 + \lambda \int_0^t e^{B_u + (\alpha+1/2)u}\, du\right)^{-\alpha-1}\right]. \qquad \square
\end{aligned}
$$

Let us observe that for a hyperbolic Bessel process of index $\alpha \geq -\frac{1}{2}$ starting from $x = 0$ the expectation $\mathbb{E}[\exp(-\lambda \cosh R_t)]$ involves the random variable $\int_0^t e^{B_u + (\alpha+1/2)u}\, du$ and not $\Gamma_t^{(\alpha)}$.

Taking $\alpha = -1/2$ in the previous result, we obtain the following corollary.

Corollary 2.10. *For $\lambda > 0$, $x \geq 0$,*

$$
\mathbb{E}\big[\exp\big(-\lambda \cosh(B_t + x)\big)\big] = e^{-\lambda}\mathbb{E}\left[e^{-\lambda(\cosh(x)-1)\Gamma_t^{(0)}}\left(1 + \lambda \int_0^t e^{B_u}\, du\right)^{-1/2}\right].
$$

We will use the following notation:

$$
A_t^{(\alpha)} = \int_0^t e^{2(B_u + \alpha u)}\, du, \qquad \alpha \in \mathbb{R}, \qquad A_t = A_t^{(0)}, \tag{2.18}
$$

with B being a standard Brownian motion.

Theorem 2.11. *If R is a hyperbolic Bessel process of index $\alpha = 0$ and $x = 0$, then the density of* $\cosh R_t$ *on* $[1, \infty)$, $t > 0$, *has the form*

$$\mathbb{P}(\cosh R_t \in dz) = \mathbb{E}\left[\frac{1}{4A_{t/4}^{(1)}} \exp\left(-\frac{z-1}{4A_{t/4}^{(1)}}\right)\right] dz. \qquad (2.19)$$

Proof. Fix $t > 0$. Using Theorem 2.9 for $\alpha = x = 0$ and $\lambda > 0$ we obtain

$$\mathbb{E}e^{-\lambda(\cosh R_t - 1)} = \mathbb{E}\left[\frac{1}{1 + \lambda \int_0^t e^{B_u + u/2} du}\right].$$

From the last equality and the scaling property of Brownian motion, we have

$$\mathbb{E}e^{-\lambda(\cosh R_{4t} - 1)} = \mathbb{E}\left[\frac{1}{1 + 4\lambda \int_0^t e^{B_{4u} + 2u} du}\right] = \mathbb{E}\left[\frac{1}{1 + 4\lambda A_t^{(1)}}\right]. \qquad (2.20)$$

From Theorem 2.8 in [15], we know that

$$\mathbb{E}\left[\frac{1}{1 + 4\lambda A_t^{(1)}}\right] = 1 - 4\lambda \int_0^\infty G_t(y)e^{-4\lambda y}\, dy,$$

where $G_t(y) = \mathbb{E}(e^{-y/A_t^{(1)}})$.

Since G_t is differentiable, integration by parts yields

$$\mathbb{E}\left[\frac{1}{1 + 4\lambda A_t^{(1)}}\right] = 1 + \int_0^\infty G_t(y)\left(e^{-4\lambda y}\right)'\, dy = -\int_0^\infty G_t'(y)e^{-4\lambda y}\, dy.$$

From the last equality and the definition of G_t, we have

$$\begin{aligned}
\mathbb{E}\left[\frac{1}{1 + 4\lambda A_t^{(1)}}\right] &= \int_0^\infty e^{-4\lambda y}\mathbb{E}\left[\frac{1}{A_t^{(1)}} \exp\left(-\frac{y}{A_t^{(1)}}\right)\right] dy \\
&= \int_0^\infty e^{-\lambda z}\mathbb{E}\left[\frac{1}{4A_t^{(1)}} \exp\left(-\frac{z}{4A_t^{(1)}}\right)\right] dz.
\end{aligned} \qquad (2.21)$$

From (2.20) and (2.21), we conclude that

$$\mathbb{P}\big((\cosh R_{4t} - 1) \in dz\big) = \mathbb{E}\left[\frac{1}{4A_t^{(1)}} \exp\left(-\frac{z}{4A_t^{(1)}}\right)\right] dz$$

for $z \geq 0$, which ends the proof. $\qquad \square$

Remark 2.12. Using the explicit form of the density of $A_t^{(1)}$ and Theorem 2.11, we can obtain the integral form of the density of $\cosh R_t$, with $\alpha = 0$ and $x = 0$.

Proposition 2.13. *If R is a hyperbolic Bessel process of index $\alpha = 0$ and $x = 0$, then on $[1, \infty)$,*

$$\mathbb{P}(\cosh R_t \in dz) = -\frac{1}{4}G'_{t/4}\left(\frac{z-1}{4}\right)dz \qquad (2.22)$$

for $t > 0$, where

$$G_t(y) = e^{-t/2}\mathbb{E}\exp\left(B_t + \frac{1}{2t}\left(B_t^2 - \varphi_y^2(B_t)\right)\right), \qquad (2.23)$$

B is a standard Brownian motion and

$$\varphi_y(z) = \ln\left(ye^{-z} + \cosh(z) + \sqrt{y^2e^{-2z} + \sinh^2(z) + 2ye^{-z}\cosh(z)}\right). \qquad (2.24)$$

Proof. As before, let $G_t(y) = \mathbb{E}(e^{-y/A_t^{(1)}})$. As an easy consequence of the Matsumoto–Yor result [17], Theorem 5.6, we find that G_t is given by (2.23) (see the proof of Theorem 2.5 in [15]). Now the assertion follows from Theorem 2.11 and the observation that

$$\mathbb{E}\left[\frac{1}{4A_{t/4}^{(1)}}\exp\left(-\frac{z-1}{4A_{t/4}^{(1)}}\right)\right] = -\frac{1}{4}G'_{t/4}\left(\frac{z-1}{4}\right). \qquad \square$$

Let us recall that the formula of Laplace transform of a squared Bessel process X of index α has the form

$$\mathbb{E}e^{-\lambda X_t} = (1 + 2\lambda t)^{-(\alpha+1)}\exp\left(-\lambda x/(1 + 2\lambda t)\right) \qquad (2.25)$$

for $\lambda > 0$, $t \geq 0$ and $X_0 = x \geq 0$ (see [19], Chapter XI, page 441). We will use (2.25) in the proof of the next theorem and subsequently.

The next theorem states that, for a fixed t, that $R_t \stackrel{\text{(law)}}{=} F(B, A, X)$ for some functional F, so the distribution of R_t can be represented as a functional of A, B and a squared Bessel process X of index α independent of B.

Theorem 2.14. *If R is a hyperbolic Bessel process of index $\alpha \geq -\frac{1}{2}$, $R_0 = x \geq 0$, then*

$$\cosh(R_t) \stackrel{\text{(law)}}{=} 1 + 2A_{t/4}^{(2\alpha+1)}X_1 \qquad (2.26)$$

for every $t > 0$, where X is a squared Bessel process of index α, independent of a standard Brownian motion B and starting from $\frac{1}{2}(\cosh(x) - 1)e^{2B_{t/4}^{(2\alpha+1)}}/A_{t/4}^{(2\alpha+1)}$. Moreover, we have

$$R_t \stackrel{\text{(law)}}{=} \operatorname{ar cosh}\left(1 + 2A_{t/4}^{(2\alpha+1)}X_1\right). \qquad (2.27)$$

Proof. From Theorem 2.9 and the form of the Laplace transform of a squared Bessel process, that is, (2.25), we have

$$\mathbb{E}\exp(-\lambda\cosh R_t) = e^{-\lambda}\mathbb{E}\left[e^{-\lambda(\cosh(x)-1)\Gamma_t^{(\alpha+1/2)}}\left(1 + \lambda\int_0^t e^{B_u + (\alpha+1/2)u}\,du\right)^{-\alpha-1}\right]$$

$$= e^{-\lambda} \mathbb{E} \exp\left(-\lambda e^{B_t + (\alpha + 1/2)t} \hat{X}_{\int_0^t e^{B_u + (\alpha + 1/2)u} \, du / (2e^{B_t + (\alpha + 1/2)t})}\right)$$

for arbitrary $\lambda > 0$, where $\hat{X}$ is a squared Bessel process of index α, independent of a standard Brownian motion B and starting from $\cosh(x) - 1$. It is now clear that

$$\cosh(R_t) \overset{\text{(law)}}{=} 1 + e^{B_t + (\alpha + 1/2)t} \hat{X}_{\int_0^t e^{B_u + (\alpha + 1/2)u} \, du / (2e^{B_t + (\alpha + 1/2)t})}. \tag{2.28}$$

Using the scaling property of a squared Bessel process (see [19], Chapter XI, Proposition 1.6), we obtain

$$1 + e^{B_t + (\alpha + 1/2)t} \hat{X}_{\int_0^t e^{B_u + (\alpha + 1/2)u} \, du / (2e^{B_t + (\alpha + 1/2)t})} = 1 + \frac{1}{2} \overline{X}_1 \int_0^t e^{B_u + (\alpha + 1/2)u} \, du, \tag{2.29}$$

where $\overline{X}$ is a squared Bessel process of index α independent of B and starting from the stochastic point $\overline{X}_0 = 2(\cosh(x) - 1)e^{B_t + (\alpha + 1/2)t} / \int_0^t e^{B_u + (\alpha + 1/2)u} \, du$. Hence, taking $4t$ instead of t, we infer that

$$\cosh(R_{4t}) \overset{\text{(law)}}{=} 1 + \frac{1}{2} \overline{X}_1 \int_0^{4t} e^{B_u + (\alpha + 1/2)u} \, du \overset{\text{(law)}}{=} 1 + 2\overline{A}_t^{(2\alpha + 1)} X_1,$$

where X is a squared Bessel process of index α independent of a standard Brownian motion $\overline{B}$ and starting from $X_0 = \frac{1}{2}(\cosh(x) - 1)e^{2\overline{B}_t^{(2\alpha + 1)}} / \overline{A}_t^{(2\alpha + 1)}$, where $\overline{A}_t^{(2\alpha + 1)}$ is given by (2.18) with $\overline{B}$ instead of B. As $R_t \geq 0$, the second part of the theorem follows from the first one. The proof is complete. $\qquad \square$

Remark 2.15. Fix $\alpha \geq -1/2$ and $x \geq 0$. For every $t > 0$, from the proof of Theorem 2.14 we deduce that

$$R_t \overset{\text{(law)}}{=} \text{ar} \cosh\left(1 + (1/2)a_t X_1\right), \tag{2.30}$$

where $a_t = \int_0^t e^{B_u + (\alpha + 1/2)u} \, du$, X is a squared Bessel process of index $\alpha \geq -1/2$ independent of a standard Brownian motion B and starting from the random point $X_0 = 2(\cosh(x) - 1)e^{B_t + (\alpha + 1/2)t} / \int_0^t e^{B_u + (\alpha + 1/2)u} \, du$ (see (2.28) and (2.29)).

Theorem 2.14, in the special case $x = 0$, gives

Proposition 2.16. (a) *If $x = 0$ and $\alpha \geq -1/2$, then for every t the density function of $\cosh(R_t)$ on $[1, \infty)$ is*

$$\mathbb{P}\left(\cosh(R_t) \in dz\right) = \frac{1}{4^{\alpha + 1}} \frac{(z - 1)^\alpha}{\Gamma(\alpha + 1)} \mathbb{E}\left[e^{-(z-1)/(4A_{t/4}^{(2\alpha + 1)})} \frac{1}{(A_{t/4}^{(2\alpha + 1)})^{\alpha + 1}}\right] dz. \tag{2.31}$$

(b) *If $x = 0$ and $\alpha \geq -1/2$, then for every t the density function of R_t on $[0, \infty)$ is*

$$\mathbb{P}(R_t \in dz) = \frac{1}{4^{\alpha + 1}} \frac{(\cosh(z) - 1)^\alpha \sinh(z)}{\Gamma(\alpha + 1)} \mathbb{E}\left[e^{-(\cosh(z) - 1)/(4A_{t/4}^{(2\alpha + 1)})} \frac{1}{(A_{t/4}^{(2\alpha + 1)})^{\alpha + 1}}\right] dz.$$

Proof. From Theorem 2.14 for $x = 0$ and $\alpha \geq -1/2$, we obtain

$$\cosh(R_t) \stackrel{\text{(law)}}{=} 1 + 2A_{t/4}^{(2\alpha+1)} X_1, \tag{2.32}$$

where X is a squared Bessel process of index α, independent of B and starting from 0.

(a) (2.32) implies, for $x = 0$ and $\alpha \geq -1/2$, that

$$\mathbb{P}\big(\cosh(R_t) \leq z\big) = \mathbb{P}\left(X_1 \leq \frac{z-1}{2A_{t/4}^{(2\alpha+1)}} \right).$$

Hence and from the form of the density of a squared Bessel process (see [19], Chapter XI, Cor. 1.4), we have for $x = 0$ and $\alpha \geq -1$

$$\mathbb{P}\big(\cosh(R_t) \in dz\big) = \left(\frac{1}{4}\right)^{\alpha+1} \frac{(z-1)^\alpha}{\Gamma(\alpha+1)} \mathbb{E}\left[e^{-(z-1)/(4A_{t/4}^{(2\alpha+1)})} \frac{1}{(A_{t/4}^{(2\alpha+1)})^{\alpha+1}} \right] dz,$$

that is, (2.31).

(b) From the properties of a hyperbolic Bessel process for $\alpha \geq -1/2$ and $x \geq 0$ it follows that $R_t \geq 0$. Thus, by (2.32),

$$R_t \stackrel{\text{(law)}}{=} \operatorname{ar\,cosh}\big(1 + 2A_{t/4}^{(2\alpha+1)} X_1\big),$$

so

$$\mathbb{P}(R_t \leq z) = \mathbb{P}\left(X_1 \leq \frac{\cosh(z) - 1}{2A_{t/4}^{(2\alpha+1)}} \right).$$

Using this equality and again the form of the density of a squared Bessel process, we have, for $x = 0$ and $\alpha \geq -1/2$,

$$\mathbb{P}(R_t \in dz)$$
$$= \left(\frac{1}{4}\right)^{\alpha+1} \frac{(\cosh(z)-1)^\alpha \sinh(z)}{\Gamma(\alpha+1)} \mathbb{E}\left[e^{-(\cosh(z)-1)/A_{t/4}^{(2\alpha+1)}} \frac{1}{(A_{t/4}^{(2\alpha+1)})^{\alpha+1}} \right] dz,$$

and the proof is complete. $\qquad \square$

Remark 2.17. Using the explicit form of the density of $A_t^{(2\alpha+1)}$ and Proposition 2.16, we can obtain the integral form of the density of R_t for $\alpha \geq -1/2$ and $R_0 = 0$. This formula is new and differs significantly from the density obtained by Borodin (see formula (5.4) in [5]).

The next two facts follow immediately from Theorem 2.14 and in slightly different forms can be found in [17].

Corollary 2.18. *For any $z \geq 1$, we have*

$$\mathbb{E}\left[\frac{1}{\sqrt{A_t}} \exp\left(-\frac{(\cosh(z) - 1)}{4A_t}\right)\right] = \sqrt{\frac{2}{t}} \frac{\sqrt{\cosh(z) - 1}}{\sinh(z)} \exp\left(-\frac{z^2}{8t}\right).$$

Proof. This follows from Proposition 2.16 for $x = 0$ and $\alpha = -\frac{1}{2}$ and the form of the density of $\cosh(B_t)$. □

Proposition 2.19. *For every $\lambda > 0$, we have*

$$\mathbb{E}A_t^\lambda = \frac{\mathbb{E}(\cosh(B_{4t}) - 1)^\lambda}{2^\lambda \mathbb{E}(B_1)^{2\lambda}},$$

where B is a standard Brownian motion.

Proof. This follows from Theorem 2.14 for $x = 0$ and $\alpha = -1/2$. □

In the next proposition, we deduce from Theorem 2.14 an alternative purely probabilistic proof of Bougerol's identity (see [7], [3], Proposition 6.1, or [17], Section 3). For completeness, we also present the generalized version of Bougerol's identity which is exactly Proposition 4 in [2]; in our proof we use similar ideas as in [11]. An interesting discussion of Bougerol's identity and its consequences can also be found in [1].

Proposition 2.20. *Let B and W be two independent standard Brownian motions.*
   (a) (Bougerol's identity) *For $t > 0$,*

$$\sinh(B_t) \overset{\text{(law)}}{=} W_{A_t}. \tag{2.33}$$

   (b) ([2]) *If $x \in \mathbb{R}$, then*

$$\sinh(x + B_t) \overset{\text{(law)}}{=} \sinh(x)e^{B_t} + W_{A_t}. \tag{2.34}$$

Proof. (a)

$$(W_{A_t})^2 \overset{\text{(law)}}{=} A_t(W_1)^2 \overset{\text{(law)}}{=} \frac{\cosh(B_{4t}) - 1}{2} = \frac{e^{2\cdot(1/2)B_{4t}} + e^{-2\cdot(1/2)B_{4t}} - 2}{4} \overset{\text{(law)}}{=} \left(\sinh(B_t)\right)^2,$$

where in the second equality we use Theorem 2.14 with $x = 0$ and $\alpha = -1/2$. Hence, using the fact that W_{A_t} and $\sinh(B_t)$ are symmetric random variables we obtain (2.33).
   (b) By the same arguments as in (a), we conclude that

$$\left|\sinh(x + B_t)\right| \overset{\text{(law)}}{=} \left|\sinh(x)e^{B_t} + W_{A_t}\right|$$

and we can expect that we can skip the absolute value signs. Indeed, define

$$M_t = e^{B_t}\left(\sinh(x) + \int_0^t e^{-B_u} \, dW_u\right).$$

Observe, by Proposition 2.1 in [12], that

$$\sinh(x)\mathrm{e}^{B_t} + W_{A_t} \overset{\text{(law)}}{=} \sinh(x)\mathrm{e}^{B_t} + \mathrm{e}^{B_t} \int_0^t \mathrm{e}^{-B_u}\,\mathrm{d}W_u = M_t.$$

It is easy to check that M is a diffusion with the generator

$$\mathcal{A}_M = \frac{1}{2}(x^2 + 1)\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{1}{2}x\frac{\mathrm{d}}{\mathrm{d}x}. \tag{2.35}$$

It is also evident that the SDE corresponding to the above generator has a unique strong solution. To finish the proof it is enough to observe that $\widehat{M}_t := \sinh(x + B_t)$ is a diffusion such that the generators of M and $\widehat{M}$ are equal on C_c^2, so $M \overset{\text{(law)}}{=} \widehat{M}$.   □

A simple but interesting consequence of Proposition 2.20 is the following proposition.

Proposition 2.21. *Let B be a standard Brownian motion. For any $\lambda \geq 0$ and $t \geq 0$ we have*

$$\mathbb{E}(1 + 2\lambda A_t)^{-1/2} = \mathbb{E}\mathrm{e}^{-\lambda \sinh^2(B_t)}. \tag{2.36}$$

Proof. Let B and W be independent standard Brownian motions. Using (2.25), we have

$$\mathbb{E}\mathrm{e}^{-\lambda W_{A_t}^2} = \mathbb{E}\mathbb{E}\big(\mathrm{e}^{-\lambda W_{A_t}^2}|\sigma(B_u, u \leq t)\big) = \mathbb{E}(1 + 2\lambda A_t)^{-1/2}.$$

The assertion follows from Bougerol's identity.   □

In the next proposition, we deduce from Theorem 2.20 a simple form of the characteristic function of the vector $(\mathrm{e}^{B_t}, W_{A_t})$.

Proposition 2.22. *Let B and W be two independent standard Brownian motions. Then, for $u \in \mathbb{R}$ and $v \neq 0$,*

$$\mathbb{E}\mathrm{e}^{iu\mathrm{e}^{B_t} + ivW_{A_t}} = \mathbb{E}\mathrm{e}^{iv\sinh(\operatorname{ar\,sinh}(u/v) + B_t)}. \tag{2.37}$$

Proof. From Theorem 2.20, we infer that

$$\frac{u}{v}\mathrm{e}^{B_t} + W_{A_t} \overset{\text{(law)}}{=} \sinh\big(\operatorname{ar\,sinh}(u/v) + B_t\big),$$

which implies (2.37).   □

Now we establish a representation of the law of the process $\sinh(R)$, where R is a hyperbolic Bessel process of index $\alpha = 0$, in terms of functionals of independent Brownian motions.

Theorem 2.23. *Let $x \geq 0$ and R be a hyperbolic Bessel process of index $\alpha = 0$ starting from x. Then*

$$\big(\sinh(R_t), t \geq 0\big)$$

$$\overset{(\text{law})}{=} \left(e^{-B_t + t/2} \left(\left(\sinh(x) + \int_0^t e^{B_u - u/2} \, dV_u \right)^2 + \left(\int_0^t e^{B_u - u/2} \, dZ_u \right)^2 \right)^{1/2}, t \geq 0 \right), \tag{2.38}$$

where B, V, Z are three independent standard Brownian motions.

Proof. Define $\xi_t = \sinh^2(R_t)$. Then the diffusion ξ satisfies the SDE

$$d\xi_t = 2\sqrt{\xi_t^2 + \xi_t} \, dB_t + (2 + 3\xi_t) \, dt,$$

and the generator of ξ is

$$\mathcal{A}_\xi = 2(x^2 + x) \frac{d^2}{dx^2} + (2 + 3x) \frac{d}{dx}. \tag{2.39}$$

Denote the diffusion and drift coefficients of ξ by $\sigma(x) = 2\sqrt{x^2 + x}$ and $\mu(x) = 2 + 3x$. Observe that for any $x \geq 0$ and $y \in (x - 1, x + 1)$,

$$\big(\sigma(x) - \sigma(y)\big)^2 \leq 4\big(\big|x^2 - y^2\big| + |x - y|\big) = 4|x - y|\big(|x + y| + 1\big) \leq 4|x - y|(2|x| + 2)$$

$$\leq 4|x - y|\big(\sigma(x) + 2\big) \leq 4|x - y|\big(\sigma^2(x)/2 + 5/2\big).$$

The uniqueness of solution for the corresponding SDE follows now from [19], Chapter IX, Ex. 3.14. Thus there is a unique solution of the martingale problem induced by $\mathcal{A}_\xi$.

For a standard Brownian motion B we define, as previously, $Y_t = e^{B_t - t/2}$. We now consider a SDE of the form

$$dX_t = 2\sqrt{X_t} Y_t \, dW_t + 2Y_t^2 \, dt, \tag{2.40}$$

where $X_0 = \sinh^2(x)$ and W is a standard Brownian motion independent of B. Observe, using the Itô lemma, that the process $\psi = \frac{X}{Y^2}$ is a diffusion with the generator $\mathcal{A}_\psi$ having the same form as $\mathcal{A}_\xi$ on C_c^2. Hence, by uniqueness of solution of the martingale problem induced by $\mathcal{A}_\xi$, the processes $|\sinh(R)|$ and $|\sqrt{X}/Y|$ have the same law. To skip the absolute value signs we note that both processes are nonnegative. Moreover, the SDE (2.40) has a unique weak solution. Indeed, using the change of time $\tau_t = \inf\{s \geq 0: \int_0^s Y_u^2 \, du \geq t\}$ we find that the process X_{τ_t} is the square of a 2-dimensional Bessel process, and it is not difficult to see that the process X inherits "good" properties of the square of a 2-dimensional Bessel process. To finish the proof, we observe that the unique weak solution of (2.40) can be written as

$$X_t = \left(\sinh(x) + \int_0^t Y_u \, dV_u \right)^2 + \left(\int_0^t Y_u \, dZ_u \right)^2. \tag{2.41}$$

To verify that (2.41) satisfies (2.40), it is enough to observe that W defined as

$$W_t := \int_0^t 1_{\{\Lambda_s \neq 0\}} \frac{(\sinh(x) + \int_0^s Y_u \, dV_u) \, dV_s + (\int_0^s Y_u \, dZ_u) \, dZ_s}{\sqrt{(\sinh(x) + \int_0^s Y_u \, dV_u)^2 + (\int_0^s Y_u \, dZ_u)^2}}, \tag{2.42}$$

where $\Lambda_s = (\sinh(x) + \int_0^s Y_u \, dV_u)^2 + (\int_0^s Y_u \, dZ_u)^2$, is a standard Brownian motion. A simple use of the Itô lemma finishes the proof. $\square$

It turns out that the methods of the proof of Theorem 2.23 also allow us to find a representation of the law of the hyperbolic sine of a hyperbolic Bessel process.

Theorem 2.24. *Let $x \geq 0$ and R be a hyperbolic Bessel process of index $\alpha \geq -1/2$ starting from x. Then*

$$\big(\sinh(R_t), t \geq 0\big) \stackrel{(\text{law})}{=} (\sqrt{X_t}/Y_t, t \geq 0), \tag{2.43}$$

where $Y_t = e^{B_t - (\alpha+1/2)t}$, X_t satisfies the SDE

$$dX_t = 2\sqrt{X_t} Y_t \, dW_t + 2(1 + \alpha) Y_t^2 \, dt, \tag{2.44}$$

$X_0 = \sinh^2(x)$ and B, W are two independent standard Brownian motions.

Proof. The proof follows the lines of the previous one. $\square$

Theorem 2.25. *Let Y be given by $Y_t = e^{B_t - (\alpha+1/2)t}$, where B is a standard Brownian motion. For any $w \geq 0, t \geq 0$,*

$$\big(\sinh(R_t), t \geq 0\big) \stackrel{(\text{law})}{=} \big((Y_t)^{-1} S_{\int_0^t Y_u^2 \, du}, t \geq 0\big), \tag{2.45}$$

where S is a Bessel process of dimension $2(1 + \alpha)$ independent of B, and $S_0 = \sinh(x)$.

Proof. Consider the process X having dynamics given by (2.44) with B and W independent standard Brownian motions. Let $\tau_t = \inf\{s \geq 0: \int_0^s Y_u^2 \, du \geq t\}$. Observe that the process X_{τ_t} is the square of a $2(1 + \alpha)$-dimensional Bessel process. Observe also that conditionally, under the knowledge of the trajectory of $(Y_s, s \leq t)$, the process X_{τ_t} is still the square of a $2(1 + \alpha)$-dimensional Bessel process. Hence, the assertion follows from Theorem 2.24. $\square$

Proposition 2.26. *Let $k \in \mathbb{N}$, $x \geq 0$, and let Y be a given continuous process such that $\mathbb{E} \int_0^t Y_u^2 \, du < \infty$ for any $t \geq 0$. Then the unique solution of the SDE*

$$dX_t = 2\sqrt{X_t} Y_t \, dW_t + k Y_t^2 \, dt, \tag{2.46}$$

where W is a standard Brownian motion independent of Y and $X_0 = x$, is

$$X_t = \left(\sqrt{x} + \int_0^t Y_u \, dW_u^1\right)^2 + \sum_{i=2}^k \left(\int_0^t Y_u \, dW_u^i\right)^2, \tag{2.47}$$

where W^i, $i = 1, \ldots, k$, are independent standard Brownian motions.

Proof. As previously, the uniqueness of solution of (2.46) follows from the fact that the standard change of time ($\tau_t = \inf\{u : \int_0^u Y_s^2 \, ds > t\}$) results in X becoming the square of a k-dimensional Bessel process. We complete the proof by the direct checking that X defined by (2.47) satisfies (2.46), using analogous arguments to those in the proof of Theorem 2.25. $\qquad\square$

2.1. Hyperbolic Bessel processes with stochastic time

Proposition 2.27. *Let T_δ be a random variable with exponential distribution with parameter $\delta > 0$. Assume that T_δ is independent of a standard Brownian motion B. Then for a hyperbolic Bessel process of the form (2.1) with $\alpha \geq -1/2$, we have*

$$\mathbb{E}\exp(-\lambda \cosh R_{T_\delta}) = \int_0^\infty \int_0^\infty e^{-\lambda u \cosh(x) - (1/2)v} p^\gamma(u, 1, y) \, dy \, du, \qquad (2.48)$$

where for $\gamma = \sqrt{2\delta + (\alpha + \frac{1}{2})^2}$,

$$p^\gamma(u, 1, y) = \frac{\delta}{y^{1/2-\alpha}u} e^{-(y^2+1)/(2u)} I_\gamma\left(\frac{y}{u}\right) \qquad (2.49)$$

and I_γ is a modified Bessel function.

Proof. We use Theorem 2.3 and the result of Matsumoto-Yor [17], Theorem 4.11. $\qquad\square$

Acknowledgements

References

[1] Alili, L., Dufresne, D. and Yor, M. (1997). Sur l'identité de Bougerol pour les fonctionnelles exponentielles du mouvement brownien avec drift. In *Exponential Functionals and Principal Values Related to Brownian Motion* 3–14. Madrid: Revista Matemática Iberoamericana. MR1648654

[2] Alili, L. and Gruet, J. (1997). An explanation of a generalized Bougerol's identity in terms of hyperbolic Brownian motion. In *Exponential Functionals and Principal Values Related to Brownian Motion* 15–33. Madrid: Revista Matemática Iberoamericana.

[3] Alili, L., Matsumoto, H. and Shiraishi, T. (2001). On a triplet of exponential Brownian functionals. In *Séminaire de Probabilités*, *XXXV*. *Lecture Notes in Math.* **1755** 396–415. Berlin: Springer. MR1837300

[4] Barrieu, P., Rouault, A. and Yor, M. (2004). A study of the Hartman–Watson distribution motivated by numerical problems related to the pricing of Asian options. *J. Appl. Probab.* **41** 1049–1058. MR2122799

[5] Borodin, A.N. (2009). Hypergeometric diffusion. *Journal of Mathematical Sciences* **3** 295–304.

[6] Borodin, A.N. and Salminen, P. (2002). *Handbook of Brownian Motion—Facts and Formulae*, 2nd ed. *Probability and Its Applications*. Basel: Birkhäuser. MR1912205

[7] Bougerol, P. (1983). Exemples de théorèmes locaux sur les groupes résolubles. *Ann. Inst. H. Poincaré Sect. B (N.S.)* **19** 369–391. MR0730116

[8] Byczkowski, T., Małecki, J. and Ryznar, M. (2010). Hitting half-spaces by Bessel–Brownian diffusions. *Potential Anal.* **33** 47–83. MR2644214

[9] Carmona, P., Petit, F. and Yor, M. (1997). On the distribution and asymptotic results for exponential functionals of Lévy processes. In *Exponential Functionals and Principal Values Related to Brownian Motion* 73–130. Madrid: Revista Matemática Iberoamericana. MR1648657

[10] Cherny, A.S. and Engelbert, H.J. (2005). *Singular Stochastic Differential Equations*. *Lecture Notes in Math.* **1858**. Berlin: Springer. MR2112227

[11] Comtet, A., Monthus, C. and Yor, M. (1998). Exponential functionals of Brownian motion and disordered systems. *J. Appl. Probab.* **35** 255–271. MR1641852

[12] Donati-Martin, C., Ghomrasni, R. and Yor, M. (2001). On certain Markov processes attached to exponential functionals of Brownian motion; application to Asian options. *Rev. Mat. Iberoam.* **17** 179–193. MR1846094

[13] Gruet, J.C. (1997). Windings of hyperbolic Brownian motion. In *Exponential Functionals and Principal Values Related to Brownian Motion* 35–72. Madrid: Revista Matemática Iberoamericana. MR1648656

[14] Gruet, J.C. (2000). A note on hyperbolic von Mises distributions. *Bernoulli* **6** 1007–1020. MR1809732

[15] Jakubowski, J. and Wiśniewolski, M. (2010). On some Brownian functionals and their applications to moments in lognormal and Stein stochastic volatility models. Available at arXiv:1102.0021v1.

[16] Karatzas, I. and Shreve, S.E. (1991). *Brownian Motion and Stochastic Calculus*, 2nd ed. *Graduate Texts in Mathematics* **113**. New York: Springer. MR1121940

[17] Matsumoto, H. and Yor, M. (2005). Exponential functionals of Brownian motion. I. Probability laws at fixed time. *Probab. Surv.* **2** 312–347. MR2203675

[18] Matsumoto, H. and Yor, M. (2005). Exponential functionals of Brownian motion. II. Some related diffusion processes. *Probab. Surv.* **2** 348–384. MR2203676

[19] Revuz, D. and Yor, M. (2005). *Continuous Martingales and Brownian Motion*, 3rd ed. Berlin: Springer.

[20] Yor, M., ed. (1997). *Exponential Functionals and Principal Values Related to Brownian Motion. A Collection of Research Papers. Biblioteca de la Revista Matemática Iberoamericana.* [*Library of the Revista Matemática Iberoamericana*]. Madrid: Revista Matemática Iberoamericana. MR1648653