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Further examples of GGC and HCM densities

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We display several examples of generalized gamma convoluted and hyperbolically completely monotone random variables related to positive α-stable laws. We also obtain new factorizations for the latter, refining Kanter's and Pestana–Shanbhag–Sreehari's. These results give stronger credit to Bondesson's hypothesis that positive α-stable densities are hyperbolically completely monotone whenever $\alpha \le 1/2$.

Keywords: generalized Gamma convolution; hyperbolically completely monotone; hyperbolically monotone; positive stable density

1. Introduction

A positive random variable X is called a Generalized Gamma Convolution (GGC) if its Laplace transform reads

$$\mathbb{E}[\mathrm{e}^{-\lambda X}] = \exp -\left[a\lambda + \int_0^\infty (1-\mathrm{e}^{-\lambda x})\frac{\varphi(x)}{x}\,\mathrm{d}x\right], \qquad \lambda \ge 0, \tag{1.1}$$

where $a \ge 0$ and φ is a completely monotone (CM) function over $(0,+\infty)$. The denomination comes from the fact that the above class can be identified as the closure for weak convergence of finite convolutions of Gamma distributions. We refer to [3] and [24] for comprehensive monographs on such random variables. From their definition, GGC random variables are self-decomposable (SD) hence infinitely divisible (ID), absolutely continuous and unimodal – see, for example, [19] for the proofs of the latter properties. We also see from (1.1) that GGC random variables are characterized up to translation by the positive Radon measure on $(0,+\infty)$ uniquely associated to the CM function φ by Bernstein's theorem, which is called the Thorin measure of X and whose total mass, $\varphi(0+)$, might be infinite. As an illustration of this characterization, Theorem 4.1.4 in [3] shows that the density of X vanishes in $a+$ if $\varphi(0+) > 1$, whereas it is infinite in $a+$ if $\varphi(0+) < 1$. We refer to [14] for a recent survey on GGC variables having a finite Thorin measure, dealing in particular with their Wiener–Gamma representations and their relations with Dirichlet processes.

A positive random variable X is said to be hyperbolically completely monotone (HCM) if it has a density f on $(0,+\infty)$ such that for every $u > 0$ the function

$$H_u(w) = f(uv)f(u/v), \qquad w = v + 1/v \ge 2,$$

is CM in the variable w (it is easy to see that H_u is always a function of w). In general, a function $f : (0,+\infty) \to (0,+\infty)$ is said to be HCM when the above CM property holds for H_u, and this extended definition will be important in the sequel. HCM densities turn out to be characterized as pointwise limits of densities of the form

$$x \mapsto C x^{\beta-1} \prod_{i=1}^{N} (x + y_i)^{-\gamma_i}, \tag{1.2}$$

where all above parameters are positive – see Sections 5.2 and 5.3 in [3]. This characterization yields many explicit examples of GGC random variables, since it is also true that HCM random variables are GGC – see Theorem 5.2.1 in [3]. Actually, HCM variables appear as a kind of center for GGC in view of Theorem 6.2.1 in [3] which states that the independent product or quotient of a GGC by a HCM variable is still a GGC. The HCM class is also stable by independent multiplication and power transformations of absolute value greater than one. We refer to [3] for many other properties of HCM densities and functions.

The HCM property is connected to log-concavity in the following way. A positive random variable X is said to be hyperbolically monotone (HM) if it has a density f on $(0, +\infty)$ such that the above function H_u is nonincreasing in the variable w. Similarly as above, one can extend the HM property to all positive functions on $(0, +\infty)$. Obviously, HCM is a subclass of HM. It is easy to see – see [3], pages 101–102 – that X is HM iff its density f is such that $t \mapsto f(\mathrm{e}^t)$ is log-concave on $\mathbb{R}$. This shows that f is a.e. differentiable with $x \mapsto x f'(x)/f(x)$ a nonincreasing function, so that f' has at most one change of sign and X is unimodal. The main theorem in [7] shows that HM variables are actually multiplicatively strong unimodal, viz. their independent product with any unimodal random variable is unimodal. From the log-concavity characterization, the HM property is stable by power transformation of any value, and this entails that the inclusion HCM $\subset$ HM is strict: if $L \sim \mathrm{Exp}(1)$, then $\sqrt{L}$ is HM but not ID, hence not HCM. From Prékopa's theorem, the HM property is also stable by independent multiplication.

For a positive random variable with density, the standard way to derive the GGC property is to read it from the Laplace transform. For example, it is straightforward to see that positive α-stable variables are all GGC – see Example 3.2.1 in [3]. On the other hand, it is easier to study the HCM property from features on the density itself and Laplace transforms are barely helpful. As an illustration of this, we show without much effort in Section 4 of the present paper that the quotient of two positive α-stable variables, whose density is explicit, is HCM iff $\alpha \le 1/2$. Problems become usually more intricate when one searches for GGC without explicit Laplace transform or for HCM without closed expression for the density.

In 1981, Bondesson raised the conjecture that positive α-stable variables should be HCM iff $\alpha \le 1/2$ and this very hard problem (quoting his own recent words, see Remark 3 in [5]) is still unsolved except in the easy case when α is the reciprocal of an integer – see Example 5.6.2 in [3]. Notice, in passing, that the validity of this conjecture is erroneously taken for granted in [14], page 361. We refer to [1], pages 54–55, [3], pages 88–89 and also to the manuscript [4], for several reasons, partly numerical, supporting this hypothesis. Let us also mention the main theorem of [22], which states that positive α-stable random variables are HM iff $\alpha \le 1/2$. Actually, it follows easily from the proofs of Lemmas 1 and 2 in [22] that the pth power of a positive (p/n)-stable variable is HCM for any integers $p, n \ge 2$ such that $p/n \le 1/2$.

In the present paper, we will present several examples of GGC and HCM densities related to the above conjecture. In Section 2, we combine the main results of [18] and [22] to show the GGC property for a large family of negative powers of α-stable variables with $\alpha \le 1/4$. This family is actually a bit larger than the one which would be obtained from the validity of Bondesson's hypothesis. The more difficult case $\alpha \in (1/4, 1/2]$ is also studied, with a partial result. In Section 3, we use Kanter's and Pestana–Shanbhag–Sreehari's factorizations to show that a large class of positive powers of α-stable variables is the product of an HCM variable and an ID variable. The latter turns out to be always a mixture of exponentials (ME), hence very close to a GGC. Along the way, we also obtain an independent proof of Pestana–Shanbhag–Sreehari's factorization. In Section 4, we show the aforementioned HCM result for the quotient of two stable variables, and a similar characterization for Mittag–Leffler variables. Not surprisingly, both yield the same boundary parameter $\alpha = 1/2$.

The results presented in Sections 2 and 3 are probably not optimal and at the end of Section 3 we state another conjecture, where the power exponent $\alpha/(1-\alpha)$ appears naturally. We also hope that the different tools and methods presented here will be helpful to tackle Bondesson's conjecture more deeply, even though we have tried to exploit them to their full extent.

Notations

We will consider real random variables X having a density always denoted by f_X, unless explicitly stated. For the sake of brevity, we will use slightly incorrect expressions like "GGC variable" or "HCM variable" and sometimes even delete the word "variable" (as was actually already done in the present introduction). We will also set "positive (negative) α-stable power" for "positive (negative) power transformation of a positive α-stable random variable".

2. Negative α-stable powers and the GGC property

2.1. Some consequences of the HM property

Let Z_α be a positive α-stable random variable – $\alpha \in (0,1)$ – with density function f_α normalized such that

$$\int_0^\infty \mathrm{e}^{-\lambda t} f_\alpha(t)\,\mathrm{d}t = \mathbb{E}[\mathrm{e}^{-\lambda Z_\alpha}] = \mathrm{e}^{-\lambda^\alpha}, \qquad \lambda \ge 0.$$

In the remainder of this paper, we will use the notation $\beta = 1-\alpha$. We will also set $Z_1 = 1$ by continuity. Recall that when $\alpha = 1/2$, our normalization yields

$$f_{1/2}(x) = \frac{1}{2\sqrt{\pi}x^{3/2}}\mathrm{e}^{-1/4x}\mathbf{1}_{\{x>0\}}. \tag{2.1}$$

Kanter's factorization – see Corollary 4.1 in [15] – reads

$$Z_\alpha \stackrel{d}{=} L^{-\beta/\alpha} \times b_\alpha^{-1/\alpha}(U), \tag{2.2}$$

where $L \sim \text{Exp}(1)$, $U \sim \text{Unif}(0, \pi)$ independent of L, and

$$b_\alpha(u) = \big(\sin u / \sin(\alpha u)\big)^\alpha \big(\sin u / \sin(\beta u)\big)^\beta, \qquad u \in (0, \pi),$$

is a bounded, decreasing and concave function – see Lemma 1 in [23]. Observe that when $\alpha = 1/2$, Kanter's factorization is a particular instance of the so-called Beta–Gamma algebra – see, for example, [3], pages 13–14. Indeed, one has

$$4b_{1/2}^{-2}(U) = \cos^{-2}(U/2) \stackrel{d}{=} \text{Beta}^{-1}(1/2, 1/2)$$

and (2.1) entails $4Z_{1/2} \stackrel{d}{=} \text{Gamma}^{-1}(1/2, 1)$, so that (2.2) amounts when $\alpha = 1/2$ to

$$\text{Gamma}(1/2, 1) \stackrel{d}{=} \text{Beta}(1/2, 1/2) \times \text{Gamma}(1, 1).$$

Put together with Shanbhag–Sreehari's classical factorization of the exponential law – see, for example, Exercise 29.16 in [19], Kanter's factorization also shows that for every $\gamma \geq \alpha/\beta$ the random variable $Z_\alpha^{-\gamma}$ is ME, viz. there exists a positive random variable $U_{\alpha,\gamma}$ such that

$$Z_\alpha^{-\gamma} \stackrel{d}{=} L \times U_{\alpha,\gamma}. \tag{2.3}$$

See, for example, Section 51.1 in [19] for more material on ME random variables. With the help of the HM property, one has the following reinforcement.

Proposition 2.1. *For every $\gamma > 0$, the random variable $Z_\alpha^{-\gamma}$ is ID (with a CM density) iff $\gamma \geq \alpha/\beta$. Moreover, $Z_\alpha^{-\gamma}$ is SD for every $\alpha \leq 1/2$ and every $\gamma \geq \alpha/\beta$.*

Proof. The factorization (2.3) together with Theorem 51.6 and Proposition 51.8 in [19] show that $Z_\alpha^{-\gamma}$ is ID with a CM density if $\gamma \geq \alpha/\beta$. On the other hand, a change of variable and Linnik's asymptotic expansion – see, for example, (14.35) in [19] – yield

$$x^{\alpha/\beta\gamma} \log f_{Z_\alpha^{-\gamma}}(x) \to \kappa_{\alpha,\gamma} \in (-\infty, 0)$$

as $x \to \infty$ for every $\gamma > 0$. Hence, if $\gamma < \alpha/\beta$, Theorem 26.1 in [19] – see also Exercise 29.10 therein – entails that $Z_\alpha^{-\gamma}$ is not ID. When $\alpha \leq 1/2$, the main result in [22] shows that Z_α is HM, so that $\log(Z_\alpha^{-\gamma})$ has a log-concave density. If in addition $\gamma \geq \alpha/\beta$, we have just observed that $Z_\alpha^{-\gamma}$ has a CM density which is hence decreasing and log-convex. This entails that when $\alpha \leq 1/2$ and $\gamma \geq \alpha/\beta$, the random variable $Z_\alpha^{-\gamma}$ belongs to the class mentioned in [3], Remark VI, page 28, and is SD. $\square$

The main result of this section shows that the SD property for $Z_\alpha^{-\gamma}$ can be refined into GGC, in some cases. The proof also relies on the HM property.

Theorem 2.1. *The random variable $Z_\alpha^{-\gamma}$ is GGC for any $\alpha \in (0, 1/4]$, $\gamma \geq 4\alpha$.*

Proof. Fix $\alpha \in (0, 1/4]$, $\gamma \geq 4\alpha$ and set $\delta = 2\alpha/\gamma \in (0, 1/2]$. Bochner's subordination for stable subordinators – see, for example, Example 30.5 in [19] – yields the identity

$$Z_\alpha^{-2\alpha} \stackrel{d}{=} c_\alpha(Z_{1/2}^{-1} \times Z_{2\alpha}^{-2\alpha})$$

for some purposeless constant $c_\alpha > 0$. We hence need to show the GGC property for the random variable

$$\left((4Z_{1/2})^{-1} \times Z_{2\alpha}^{2\alpha}\right)^{1/\delta},$$

whose density is in view of (2.1), the multiplicative convolution formula, and a series of standard changes of variable, expressed as

$$\frac{\delta x^{\delta-3/2}}{\alpha\sqrt{2\pi}} \int_0^\infty \mathrm{e}^{-x^\delta y} f_{2\alpha}(y^{1/2\alpha}) y^{1/2\alpha-1/2} \, \mathrm{d}y.$$

Observe that since

$$\int_0^\infty f_{2\alpha}(y^{1/2\alpha}) y^{1/2\alpha-1/2} \, \mathrm{d}y = 2\alpha \int_0^\infty f_{2\alpha}(z) z^\alpha \, \mathrm{d}z = \frac{2\alpha\sqrt{\pi}}{\Gamma(1-\alpha)} < +\infty$$

(see, e.g., (25.5) in [19] for the second equality), the function $K_\alpha f_{2\alpha}(y^{1/2\alpha}) y^{1/2\alpha-1/2}$ is a probability density on $\mathbb{R}^+$, where we have set $K_\alpha = \Gamma(1-\alpha)/2\alpha\sqrt{\pi}$. Denoting by X_α the corresponding random variable, we have to show that

$$x \mapsto K_{\alpha,\delta} x^{\delta-3/2} \mathbb{E}[\mathrm{e}^{-x^\delta X_\alpha}]$$

is the density of a GGC, with $K_{\alpha,\delta} = \sqrt{2}\delta/\Gamma(1-\alpha)$. Since $\delta \leq 1/2 < 3/2$, we see from Theorem 6.2 in [2] (and the Remark 6.1 thereafter) that this will be done as soon as

$$\mathbb{E}[\mathrm{e}^{-x^\delta X_\alpha}] = \mathbb{E}\left[\mathrm{e}^{-x(Z_\delta \times X_\alpha^{1/\delta})}\right]$$

is, up to normalization, the density of a GGC. We will now obtain this property with the help of Theorem 2 in [18]. On the one hand, it is easy to see that all negative moments of Z_δ and X_α are finite, so that the density of $Z_\delta \times X_\alpha^{1/\delta}$ fulfils (1.1) in [18]. On the other hand, the main result of [22] entails that $Z_{2\alpha}$ is HM because $2\alpha \leq 1/2$, so that the function

$$t \mapsto f_{2\alpha}(\mathrm{e}^{t/2\alpha}) \mathrm{e}^{t/2\alpha - t/2}$$

is log-concave and X_α is HM as well. Also, Z_δ is HM because $\delta \leq 1/2$. Since the HM property is stable by independent multiplication, this shows that $Z_\delta \times X_\alpha^{1/\delta}$ is HM, in other words that it belongs to the class $\mathcal{C}$ defined in [18], page 183, and we can apply Theorem 2 therein to conclude the proof. □

Remarks 2.1. (a) From (14.30) in [19] and a change of variable, one has

$$\sup\left\{u;\ \lim_{x\to 0} f_{Z_\alpha^{-\gamma}}(x)/x^{u-1} = 0\right\} = \alpha/\gamma$$

for every $\gamma > 0$, so that (3.1.4) in [3] shows that under the assumptions of Theorem 2.1, the GGC random variable $Z_\alpha^{-\gamma}$ has a finite Thorin measure whose total mass is α/γ. In other words, there exists a nonnegative random variable $G_{\alpha,\gamma}$ such that

$$\mathbb{E}[\mathrm{e}^{-\lambda Z_\alpha^{-\gamma}}] = \exp -\left[(\alpha/\gamma)\int_0^\infty (1-\mathrm{e}^{-\lambda x})\mathbb{E}[\mathrm{e}^{-xG_{\alpha,\gamma}}]\frac{\mathrm{d}x}{x}\right], \qquad \lambda \geq 0.$$

It would be interesting to get more properties of the random variables $G_{\alpha,\gamma}$.

(b) It is easily seen that the above proof remains unchanged (and is even shorter) if we take $\delta = 1$ viz. $\gamma = 2\alpha$, so that $Z_\alpha^{-2\alpha}$ is GGC as well for any $\alpha \in (0, 1/4]$. In view of Theorem 2.1 and the general conjecture made in [5], Remark 3(ii), it is plausible that $Z_\alpha^{-\gamma}$ is GGC for any $\alpha \in (0, 1/4]$, $\gamma \geq 2\alpha$.

2.2. A certain family of densities on $\mathbb{R}^+$ and a partial result

A drawback of Theorem 2.1 is that it only covers the range $\alpha \in (0, 1/4]$. Indeed, with the same subordination method one should expect to handle the range $\alpha \in (1/4, 1/2]$ as well. Motivated by the key-properties (2.20) and (2.23) in the proof of Theorem 2 in [18], let us define the class $\mathcal{P}$ of probability densities f on $(0, +\infty)$ satisfying

$$f(x)f(c/x) \geq f(1/x)f(cx) \tag{2.4}$$

for all $x, c > 0$ such that $(x-1)(c-1) \geq 0$. With an abuse of notation, we shall say that a random variable X with density f belongs to $\mathcal{P}$ if $f \in \mathcal{P}$. If $X \in \mathcal{P}$, then it is easy to see that $X^\gamma \in \mathcal{P}$ for any $\gamma \neq 0$. Besides, it follows from [18], pages 187–188, that HM $\subset \mathcal{P}$. Notice also that $\mathcal{P} \not\subset$ HM, as the following example shows. Consider the independent quotient $T_\alpha = (Z_\alpha/Z_\alpha)^\alpha$ which has an explicit density g_α given by

$$g_\alpha(x) = \frac{\sin \pi\alpha}{\pi\alpha(x^2 + 2\cos(\pi\alpha)x + 1)}$$

(see, e.g., Exercise 4.21(3) in [6]). A computation yields

$$\frac{(\pi\alpha)^2}{\sin^2\pi\alpha}\left(\frac{1}{g_\alpha(x)g_\alpha(c/x)} - \frac{1}{g_\alpha(cx)g_\alpha(1/x)}\right) = (1-c^2)(x-1/x)(x+1/x+2\cos\pi\alpha),$$

which is clearly nonpositive whenever $(x-1)(c-1) \geq 0$, so that $T_\alpha \in \mathcal{P}$ for any $\alpha \in (0, 1)$. However, it is easy to show – see the proof of Corollary 4.1 below – that T_α is HM iff $\alpha \leq 1/2$. However, we know from (ix), page 68 in [3] that T_α is never HCM since g_α has two poles $\mathrm{e}^{\mathrm{i}\pi\alpha}$ and $\mathrm{e}^{-\mathrm{i}\pi\alpha}$ in $\mathbb{C} \setminus (-\infty, 0]$. Notice also that the variable $T_{1/2}$ is SD but not GGC – see [10] and the references therein. We will come back to this example in Section 4. The following proposition makes the relationship between HM and $\mathcal{P}$ more precise.

Proposition 2.2. *For any nonnegative random variable X having a density, one has*

$$X \textit{ is } \mathrm{HM} \quad \Longleftrightarrow \quad cX \in \mathcal{P} \qquad \forall c > 0.$$

Proof. The direct part is easy since cX is HM for any $c > 0$ whenever X is HM. For the indirect part, setting $g_X(t) = \log f_X(\mathrm{e}^t)$ for any $t \in \mathbb{R}$, the fact that $cX \in \mathcal{P}$ for any $c > 0$ shows that for any $-\infty < a \le b \le c \le d < +\infty$ with $b + c = a + d$, one has $g_X(b) + g_X(c) \ge g_X(a) + g_X(d)$, so that g_X is concave. □

Together with the above example, this proposition entails that $\mathcal{P}$ is not stable by multiplication with positive constants. Since on the other hand $\mathcal{P}$ is clearly stable under weak convergence and since any positive constant can be approximated by a sequence of truncated gaussian variables which all belong to HM $\subset \mathcal{P}$, the instability of $\mathcal{P}$ w.r.t. constant multiplication entails that $\mathcal{P}$ is not – contrary to HM – stable by independent multiplication either, viz. there exist independent $X, Y \in \mathcal{P}$ such that $X \times Y \notin \mathcal{P}$.

Let now Y be a nonnegative random variable with a density of the form $\kappa x^{-a}\mathbb{E}[\mathrm{e}^{-xX}]$ for some $a \ge 0$ and a nonnegative random variable X with finite negative moments such that $c^{-1}X \in \mathcal{P}$ for some $c > 0$. A perusal of the proof of Theorem 2 in [18] – see especially (2.10), (2.20) and (2.23) therein – shows, together with Theorem 6.2 in [2], that $\varphi_Y(x) = \mathbb{E}[\mathrm{e}^{-xY}]$ is such that $H_c(w) = \varphi_Y(cv)\varphi_Y(c/v)$ is CM in the variable $w = v + 1/v$. On the other hand, it is possible to show the following intrinsic property of positive stable densities.

Theorem 2.2. *For every $\alpha \in (0, 1)$, there exists $c_\alpha \ge 0$ such that $cZ_\alpha \in \mathcal{P} \Leftrightarrow c \ge c_\alpha$.*

Though it has independent interest, we prefer not giving the proof of this theorem since it is quite long, relying on the single intersection property for Z_α – see Theorem 4.1 in [15], an extended Yamazato property for f_α which is displayed in (1.4) in [22] and the discussion thereafter, and a detailed analysis. Notice from Proposition 2.2 and the main result in [22] that $c_\alpha = 0$ for any $\alpha \le 1/2$, and that necessarily $c_\alpha > 0$ when $\alpha > 1/2$. Theorem 2.2 and a painless adaptation of the proof of Theorem 2.1 entail the following property of the variable $Z_\alpha^{-2\alpha}$.

Corollary 2.1. *For every $\alpha \in (1/4, 1/2]$, there exists $\tilde{c}_\alpha > 0$ such that for every $c \in [0, \tilde{c}_\alpha]$, the function $H_c^\alpha(w) = \varphi_\alpha(cv)\varphi_\alpha(c/v)$ is CM in the variable $w = v + 1/v$, where $\varphi_\alpha(x) = \mathbb{E}[\mathrm{e}^{-xZ_\alpha^{-2\alpha}}]$, $x \ge 0$.*

If we could show that $\tilde{c}_\alpha = +\infty$, then Theorem 6.1.1 in [3] would entail that $Z_\alpha^{-2\alpha}$ is GGC for every $\alpha \in (1/4, 1/2]$. Notice from Proposition 2.1 that when $\alpha > 1/2$ the random variable $Z_\alpha^{-2\alpha}$ is not ID (since then $2\alpha < \alpha/\beta$), hence not GGC. From Corollary 2.1 and the above Remark 2.1(b), it is very plausible that $Z_\alpha^{-\gamma}$ is GGC for any $\alpha \le 1/2, \gamma \ge 2\alpha$. See also the general Conjecture 3.3 raised at the end of the next section.

3. On the infinite divisibility of Kanter's random variable

In this section, we will deal with positive α-stable powers. From the point of view of the factorization (2.2), we need to study negative powers of L and positive powers of the random variable $b_\alpha^{-1/\alpha}(U)$, which will be referred to as Kanter's variable subsequently. The latter plays an important role in simulation – see [9] where it is called Zolotarev's variable, although the original computation leading to (2.2) is due to Chernine and Ibragimov as explained in [15]. It is interesting to remark that Kanter's variable also appears explicitly in the context of free stable laws – see [8], page 138 and the references therein.

Negative powers of L are not completely well understood from the point of view of infinite divisibility. It follows from (iv) in [3] that L^s is HCM for every $s \le -1$, but it is not even known whether L^s is ID or not for $s \in (-1, 0)$ – see [24], page 521. Here, we will rather focus on positive powers of Kanter's variable. Let us first notice that the above factorization (2.3) was also observed in [21], Theorem 1, where it is actually shown that $U_{\alpha,\gamma} = \exp(-W_{\alpha,\gamma})$ for some ID random variable $W_{\alpha,\gamma}$. In this section, we will investigate (2.3) more thoroughly and give finer properties of the random variable

$$V_\alpha = U_{\alpha,\alpha/\beta}^{-1} = b_\alpha^{-1/\beta}(U).$$

We could actually consider any positive power of Kanter's random variable, but the latter choice is more convenient for our purposes because of the identities

$$Z_\alpha^{\alpha s/\beta} \stackrel{d}{=} L^{-s} \times V_\alpha^s \tag{3.1}$$

for every $s \in \mathbb{R}$. Our purpose is three-fold. First, we provide an alternative proof of Theorem 1 in [21], with the improvement that each positive power V_α^s (in particular, Kanter's variable itself) is ID with a log-convex density. Second, we show that all V_α^s are actually positively translated ME's. Third, we study in some detail the case $\alpha = 1/2$ and propose a general conjecture which is, in some sense, a reinforcement of Bondesson's.

3.1. Another proof of Pestana–Shanbhag–Sreehari's factorization

Let us consider the random variable

$$W_\alpha = \log(V_\alpha) = -(1/\beta)\log(b_\alpha(U))$$

and observe that $W_{\alpha,\gamma} \stackrel{d}{=} \gamma_\alpha^{-1} W_\alpha + \log(Z_{\gamma_\alpha})$ for every $\gamma \ge \alpha/\beta$, with the notation $\gamma_\alpha = \alpha/\beta\gamma$. Since $\log(Z_{\gamma_\alpha})$ is ID – see, for example, Exercise 29.16 and Proposition 15.5 in [19], Theorem 1 in [21] follows as soon as W_α is ID. In view of Theorem 51.2 in [19] and the fact (obvious from the definition of b_α) that the support of W_α is unbounded on the right, this is a consequence of the following, which we prove independently of [21].

Theorem 3.1. *The density of W_α is log-convex.*

This theorem entails that all positive powers of Kanter's random variable $b_\alpha^{-1/\alpha}(U)$ are ID, as shown in the next corollary. In particular, Z_α^γ is the product of a HCM random variable and an ID random variable for every $\gamma \ge \alpha/\beta$.

Corollary 3.1. *For every $s > 0$, the density of V_α^s is decreasing and log-convex.*

Proof. Suppose first that $s = 1$. Since the support of W_α is unbounded on the right, the log-convexity of its density entails that it is also decreasing, so that the function $g_\alpha(t) = f_{V_\alpha}(e^t)$ is also decreasing and log-convex. Since $\log x$ is increasing and concave, this shows that $f_{V_\alpha}(x) = g_\alpha(\log x)$ is decreasing and log-convex. The general case $s > 0$ follows analogously in considering the variable $sW_\alpha = \log(V_\alpha^s)$. □

The proof of Theorem 3.1 relies on the following lemma.

Lemma 3.1. *The function* $h_\alpha = b''_\alpha b_\alpha/(b'_\alpha)^2$ *is increasing on* $(0, \pi)$.

Proof. First, observe that $b_{1/2}(u) = 2\cos(u/2)$, so that $h_{1/2}(u) = -\cot^2(u/2)$, an increasing function on $(0, \pi)$. In the general case, the proof is more involved. Since $b_\alpha = b_\beta$, it is enough to consider the case $\alpha < 1/2$. Set $A_\gamma(u) = \gamma\cot(\gamma u) - \cot(u)$ for every $\gamma \in (0, 1)$, $f = \alpha A_\alpha + \beta A_\beta$ and $g = f' - f^2$. One has $b'_\alpha = -f b_\alpha$ and $b''_\alpha = -g b_\alpha$, so that $h_\alpha = 1 + (1/f)'$ and we need to show that $1/f$ is strictly convex, in other words that

$$2(f')^2 - ff'' = 2gf' - fg' > 0. \tag{3.2}$$

It is shown in Lemma 1 of [23] that $g = \alpha\beta + h + k$, with the further notations $h = \alpha\beta(A_\alpha - A_\beta)^2$ and $k = 2(A_\alpha - A_\beta)(\beta A_\beta - \alpha A_\alpha)$. On the other hand, the Eulerian formula

$$\pi\cot(\pi z) = \frac{1}{z} + 2z\sum_{n\geq 1}\frac{1}{z^2 - n^2}$$

shows that

$$A_\gamma(\pi z) = \frac{2(1-\gamma^2)z}{\pi}\sum_{n\geq 1}\frac{n^2}{(n^2 - z^2)(n^2 - \gamma^2 z^2)}$$

is a strictly absolutely monotonic function on (0, 1) (i.e., all its derivatives are positive) for every $\gamma \in (0, 1)$, so that $f = \alpha A_\alpha + \beta A_\beta$ is absolutely monotonic on $(0, \pi)$, too. In particular, since $\alpha\beta > 0$, (3.2) holds if

$$(2hf' - fh') + (2kf' - fk') \geq 0.$$

A further computation entails $2hf' - fh' = 2(A_\alpha - A_\beta)(A'_\beta A_\alpha - A_\beta A'_\alpha)$ and $2hf' - fh' = 2((\beta A_\beta - \alpha A_\alpha) - 2\alpha\beta(A_\alpha - A_\beta))(A'_\beta A_\alpha - A_\beta A'_\alpha)$, so that we need to prove

$$2(A'_\beta A_\alpha - A_\beta A'_\alpha)\big((\alpha^2 + \beta^2)(A_\alpha - A_\beta) + (\beta A_\beta - \alpha A_\alpha)\big) \geq 0.$$

The above Eulerian formula entails readily that $A_\alpha - A_\beta$ and $\beta A_\beta - \alpha A_\alpha$ are positive (actually, absolutely monotonic) functions on $(0, \pi)$ and we are finally reduced to prove

$$A'_\beta A_\alpha - A'_\alpha A_\beta \geq 0. \tag{3.3}$$

We found no direct argument for the nonnegativity of the above Wronskian. Writing

$$\left(\frac{A_\beta}{A_\alpha}\right)(u) = \frac{\beta\cot(\beta u) - \cot(u)}{\alpha\cot(\alpha u) - \cot(u)} = \left(\frac{\sin(\alpha u)}{\sin(\beta u)}\right)\left(\frac{\beta\cos(\beta u)\sin(u) - \cos(u)\sin(\beta u)}{\alpha\cos(\alpha u)\sin(u) - \cos(u)\sin(\alpha u)}\right)$$

for every $u \in (0, \pi)$ shows that

$$\begin{aligned}\left(\frac{A_\beta}{A_\alpha}\right)'(\pi z) &= \frac{1}{A_\alpha^2}\big((1-\beta^2)A_\alpha - (1-\alpha^2)A_\beta + A_\alpha A_\beta(A_\alpha - A_\beta)\big)(\pi z)\\ &= \frac{8(\beta^2-\alpha^2)(1-\alpha^2)(1-\beta^2)z^3}{\pi^3 A_\alpha^2(\pi z)}\big(S_\alpha(z)S_\beta(z)S(z) - \pi^2 S_{\alpha,\beta}(z)/4\big)\end{aligned}$$

for every $z \in (0, 1)$, with the notations

$$S_\alpha(z) = \sum_{n\ge 1}\frac{n^2}{(n^2-\alpha^2z^2)(n^2-z^2)}, \qquad S_\beta(z) = \sum_{n\ge 1}\frac{n^2}{(n^2-\beta^2z^2)(n^2-z^2)},$$

$$S(z) = \sum_{n\ge 1}\frac{n^2}{(n^2-\alpha^2z^2)(n^2-\beta^2z^2)},$$

$$S_{\alpha,\beta}(z) = \sum_{n\ge 1}\frac{n^2}{(n^2-\alpha^2z^2)(n^2-\beta^2z^2)(n^2-z^2)}.$$

Since $S(z) \ge S(0) = \pi^2/6$, we see that (3.3) is true if

$$S_\alpha(z)S_\beta(z) \ge 3S_{\alpha,\beta}(z)/2$$

for every $z \in (0, 1)$. The latter follows for example, after isolating the first term in each series and using the fact that $\pi^2/6 \ge 3/2$. We leave the details to the reader. Observe that the latter is also true for $z = 0$ because $S_\alpha(0) = S_\beta(0) = \pi^2/6$ and $S_{\alpha,\beta}(0) = \pi^4/90$. □

Proof of Theorem 3.1. We need to show that the density of $\log(b_\alpha^{-1}(U))$ is log-convex, in other words that the function

$$\frac{x f'_{b_\alpha^{-1}(U)}(x)}{f_{b_\alpha^{-1}(U)}(x)}$$

is increasing over its domain of definition which is $[\alpha^\alpha\beta^\beta, +\infty)$. Since $(\log(b_\alpha^{-1}))' = f$ is a strictly absolutely monotonic function, the same holds for b_α^{-1} and we set $\tilde{b}_\alpha$ for its increasing reciprocal function. A computation yields

$$\frac{x f'_{b_\alpha^{-1}(U)}(x)}{f_{b_\alpha^{-1}(U)}(x)} = \frac{x\tilde{b}''_\alpha(x)}{\tilde{b}'_\alpha(x)} = -2 + \left(\frac{b_\alpha b''_\alpha}{(b'_\alpha)^2}\right)(\tilde{b}_\alpha(x)),$$

which is an increasing function by Lemma 3.1. □

3.2. Further properties of W_α and V_α

In [21], the infinite divisibility of W_α is proved together with a closed formula for its Lévy measure. In this paragraph, we use the latter expression to show that W_α is actually a translated ME.

Theorem 3.2. *The density of W_α is CM.*

As a consequence, we obtain the following reinforcement of Corollary 3.1, which entails that Z_α^γ is the product of a HCM random variable and a positively translated ME random variable for every $\gamma \geq \alpha/\beta$:

Corollary 3.2. *For every $s > 0$, there exists $c_{\alpha,s} > 0$ such that $V_\alpha^s - c_{\alpha,s}$ is ME.*

Proof. Suppose first that $s = 1$. Theorem 3.2 shows that $g_\alpha(t) = f_{V_\alpha}(\mathrm{e}^t)$ is CM, with the notation of Corollary 3.1. Since $\log x$ has a CM derivative, the classical Criterion 2 in [12], page 417, entails that $f_{V_\alpha}(x) = g_\alpha(\log x)$ is CM over its domain of definition. It is clear from the definition of b_α that the latter is $[\beta\alpha^{\alpha/\beta}, +\infty)$, so that $V_\alpha - \beta\alpha^{\alpha/\beta}$ is ME, by Proposition 51.8 in [19]. The general case $s > 0$ follows analogously in considering the density of sW_α instead of W_α. □

The proof of Theorem 3.2 relies on the following lemma.

Lemma 3.2. *The Lévy measure of W_α has a CM density given by*

$$w_\alpha(x) = \beta \int_1^\infty \mathrm{e}^{-\beta t x}([t] - [\alpha t] - [\beta t])\,\mathrm{d}t, \qquad x \geq 0,$$

where $[t]$ stands for the integer part of any $t \geq 0$.

Proof. From (3.1) and (3.5), (3.6) and (3.7) in [21] – beware our notation $\beta = 1 - \alpha$, for every $\lambda \geq 0$ one has

$$\begin{aligned}
\mathbb{E}[\mathrm{e}^{-\beta\lambda W_\alpha}] &= \mathbb{E}[Z_\alpha^{-\lambda\alpha}]/\mathbb{E}[L^{\lambda\beta}] \\
&= \frac{\Gamma(1+\lambda)}{\Gamma(1+\lambda\alpha)\Gamma(1+\lambda\beta)} \\
&= \exp -\left[a_\alpha\lambda + \int_0^\infty (1 - \mathrm{e}^{-\lambda x} - \lambda x) g_\alpha(x) \frac{\mathrm{d}x}{x}\right]
\end{aligned}$$

for some constant $a_\alpha \in \mathbb{R}$, where

$$g_\alpha(x) = \frac{\mathrm{e}^{-x}}{1 - \mathrm{e}^{-x}} - \frac{\mathrm{e}^{-x/\alpha}}{1 - \mathrm{e}^{-x/\alpha}} - \frac{\mathrm{e}^{-x/\beta}}{1 - \mathrm{e}^{-x/\beta}}$$

is a nonnegative function – see Lemma 3 in [21] – which is also integrable with

$$\int_0^\infty g_\alpha(x)\,\mathrm{d}x = -(\alpha\log\alpha + \beta\log\beta).$$

Besides, $g_\alpha(x) \to 1$ as $x \to 0$ so that one can rewrite

$$\mathbb{E}[\mathrm{e}^{-\beta\lambda W_\alpha}] = \exp -\left[\tilde{a}_\alpha\lambda + \int_0^\infty (1-\mathrm{e}^{-\lambda x}) g_\alpha(x)\frac{\mathrm{d}x}{x}\right] \tag{3.4}$$

for every $\lambda \geq 0$, where $\tilde{a}_\alpha = \alpha\log\alpha + \beta\log\beta$ is the left-extremity of the support of the variable βW_α (this shows that the above constant a_α is actually zero). Observe that

$$g_\alpha(x) = \sum_{k=0}^\infty \bigl(\mathrm{e}^{-(k+1)x} - \mathrm{e}^{-(k+1)x/\alpha} - \mathrm{e}^{-(k+1)x/\beta}\bigr) = \int_1^\infty \mathrm{e}^{-tx}\mu_\alpha(\mathrm{d}t),$$

where

$$\mu_\alpha = \sum_{k=1}^\infty (\delta_k - \delta_{k/\alpha} - \delta_{k/\beta})$$

is a signed Radon measure over $\mathbb{R}^+$. Integrating by parts, we get

$$g_\alpha(x) = x\int_1^\infty \mathrm{e}^{-tx}\mu_\alpha([1,t))\,\mathrm{d}t = x\int_1^\infty \mathrm{e}^{-tx}([t]-[\alpha t]-[\beta t])\,\mathrm{d}t.$$

Putting everything together shows that the density of the Lévy measure of W_α is given by

$$w_\alpha(x) = \beta\int_1^\infty \mathrm{e}^{-\beta tx}([t]-[\alpha t]-[\beta t])\,\mathrm{d}t, \qquad x \geq 0. \qquad \square$$

Proof of Theorem 3.2. From Lemma 3.2 and Theorem 51.10 in [19], we already know that $W_\alpha - \tilde{a}_\alpha$ belongs to the class B, which is the closure of ME for weak convergence and convolution. Moreover, one has

$$w_\alpha(x) = \int_0^\infty \mathrm{e}^{-tx}\theta_\alpha(t)\,\mathrm{d}t, \qquad x \geq 0,$$

with the notation $\theta_\alpha(t) = ([t/\beta]-[t]-[(\alpha/\beta)t])\mathbf{1}_{\{t\geq\beta\}}$, and it is clear that

$$\int_0^1 \theta_\alpha(t)\frac{\mathrm{d}t}{t} < \infty \quad \text{and} \quad 0 \leq \theta_\alpha(t) \leq 1, \qquad t \geq 0.$$

From Theorem 51.12 in [19], this shows that $W_\alpha - \tilde{a}_\alpha$ belongs to the class ME itself, as required. $\square$

Remarks 3.1. (a) In the terminology of Schilling, Song and Vondraček [20], Lemma 3.2 shows that the Lévy exponent of the ID random variable W_α is, up to translation, a complete Bernstein function. Using Theorems 6.10 and 7.3 in [20] and simple transformations leads to the same conclusion that the density of W_α is CM.

(b) Corollary 3.2 and Theorem 51.10 in [19] show that for every $s > 0$ the Lévy measure of the random variable V_α^s has a density $v_{\alpha,s}$ which is CM. We believe that $x \mapsto x v_{\alpha,s}(x)$ is also CM, in other words, that V_α^s is actually GGC for every $s > 0$. See Conjecture 3.1 below.

(c) The above Radon measure μ_α is signed and not everywhere positive, and we see from (3.4) that W_α is *not* the translation of a GGC random variable. This latter property might have been helpful for a better understanding of the random variables V_α^s, although it is still a conjecture – see Comment (1), page, 101 in [3] – that the transformation $x \mapsto \mathrm{e}^x - 1$ leaves the GGC property invariant.

3.3. The case $\alpha = 1/2$ and three conjectures

Taking $\alpha = 1/2$ in (3.1) yields the factorizations

$$Z_{1/2}^s = L^{-s} \times V_{1/2}^s$$

for every $s \in \mathbb{R}$, where $V_{1/2}$ has the explicit density

$$f_{V_{1/2}}(x) = \frac{1}{2\pi x\sqrt{x - 1/4}} \mathbf{1}_{\{x>1/4\}}.$$

Since $\log V_{1/2}$ has a log-convex density, the random variables $V_{1/2}^s$ are not HM (this fact was already noticed in [15], see the remark before Theorem 4.1 therein) and in particular not HCM. However, the following proposition shows that $V_{1/2}^s$ is GGC at least for $s \in [1/2, 1]$. We set $Y_s = V_{1/2}^s - 4^{-s}$ for every $s > 0$.

Proposition 3.1. *The random variable Y_s is HCM if and only if $s \in [1/2, 1]$.*

Proof. Changing the variable and using the fact that every function f on $\mathbb{R}^+$ is HCM iff $x^\lambda f(1/x)$ is HCM for every $\lambda \in \mathbb{R}$, one sees that the following equivalences hold:

$$Y_s \text{ is HCM} \quad \Leftrightarrow \quad \frac{1}{(x+1)\sqrt{(x+1)^t - 1}} \text{ is HCM} \quad \Leftrightarrow \quad \frac{1}{(x+1)\sqrt{(x+1)^t - x^t}} \text{ is HCM,}$$

with the notation $t = 1/s$. Using the notation

$$f_t(x) = \frac{1}{(x+1)\sqrt{(x+1)^t - x^t}}$$

for every $t > 0$, it is obvious that f_1 and f_2 are HCM. If now $1 < t < 2$, then $x \mapsto (x+1)^t - x^t$ is obviously a Bernstein function (i.e., a positive function with CM derivative), Criterion 2 in [12], page 417, entails that $1/\sqrt{(x+1)^t - x^t}$ is CM and f_t is also CM. Since $f_t(0) = 1$, this shows that f_t is the Laplace transform of some positive random variable and, by Theorem 5.4.1 in [3], f_t will be HCM iff the latter variable is GGC. By the Pick function characterization given in [3], Theorem 3.1.2, setting $g_t(z) = f_t(-z)$ we need to show that g is analytic and zero-free on $\mathbb{C} \setminus [0, \infty)$ and $\operatorname{Im}(g_t'(z)/g_t(z)) \geq 0$ for $\operatorname{Im} z > 0$ (in other words, that g_t'/g_t is a Pick function – see Section 2.4 in [3]). The first point amounts to show that $1 - (z/(z-1))^t$ does not vanish on $\mathbb{C} \setminus [0, \infty)$, which is true because $1 < t < 2$. For the second point, we write

$$\frac{g_t'(z)}{g_t(z)} = \frac{1}{1-z} + \frac{t}{2}\left(\frac{(1-z)^u - (-z)^u}{(1-z)^{u+1} - (-z)^{u+1}}\right)$$

with $u = t - 1 \in (0, 1)$. Since $1/(1-z)$ is obviously Pick, we need to show that

$$h_u(z) = \big((1-z)^u - (-z)^u\big)\big((1-\bar{z})^{u+1} - (-\bar{z})^{u+1}\big)$$

is also Pick for every $u \in (0, 1)$. We compute

$$\begin{aligned}
\operatorname{Im}(h_u(z)) &= \operatorname{Im}(z)(|1-z|^{2u} + |z|^{2u}) - \operatorname{Im}\big((-z)^u(1-\bar{z})^{u+1} + (-\bar{z})^{u+1}(1-z)^u\big)\\
&= \operatorname{Im}(z)\big(|(1-z)^u - (-\bar{z})^u|^2\big) - \operatorname{Im}\big((-z)^u(1-\bar{z})^u\big)\\
&= \operatorname{Im}(z)\big(|(1-z)^u - (-\bar{z})^u|^2\big) + \operatorname{Im}\big((|z|^2 - \bar{z})^u\big)
\end{aligned}$$

which is nonnegative when $\operatorname{Im} z > 0$ because $u \in (0, 1)$. This shows the required property and proves that Y_s is HCM if $s \in [1/2, 1]$. Suppose now that $s > 1$ viz. $t < 1$. If $1/(x+1)\sqrt{(1+x)^t - 1}$ were HCM, then it would also be HM and the function

$$y \mapsto \log(1 + \mathrm{e}^y) + \frac{1}{2}\log\big((1+\mathrm{e}^y)^t - 1\big)$$

would be convex on $\mathbb{R}$. Differentiating the above entails that the function

$$x \mapsto \frac{1}{x} + \frac{t}{2}\left(\frac{x^{t-1} - 1}{x^t - 1}\right) \sim -\frac{t}{2x^t} \qquad \text{as } x \to \infty$$

would be nonincreasing on $[1, +\infty)$, a contradiction. Last, in view of (ix), page 6 in [3] it is easy to see that Y_s is not HCM if $s < 1/2$, since $(1+z)^t - 1$ vanishes at least twice on $\mathbb{C} \setminus (-\infty, 0]$ when $t > 2$. This completes the proof. $\square$

Remarks 3.2. (a) Except in the trivial cases $t = 1$ and $t = 2$, we could not find any series of functions of the type described in (1.2) converging pointwise to f_t when $t \in [1, 2]$.

(b) From the above proposition, one might wonder if Kanter's variable $b_\alpha(U)^{-1/\alpha}$ is not a translated HCM in general. If it were true, then (2.2) would show that Z_α would be the product of a HCM and a translated HCM for every $\alpha \leq 1/2$, so that from Theorem 6.2.2. in [3] we would be quite close to the solution to Bondesson's hypothesis. Nevertheless, to show the above property raises computational difficulties significantly greater than those in Lemma 3.1, and it does not seem that this approach could be simpler than the one suggested in [3], pages 88–89. See also Conjecture 3.1 below.

The following corollary shows that there are GGC random variables related to positive α-stable powers with $\alpha \geq 1/2$.

Corollary 3.3. *For every* $\alpha \in [1/2, 1)$, *the random variable* $(Z_\alpha \times Z_{1/2})^\alpha$ *is GGC.*

Proof. Kanter's and Shanbhag–Sreehari's factorizations entail

$$(Z_\alpha \times Z_{1/2})^\alpha \stackrel{d}{=} (Z_\alpha / L)^\alpha \times V_{1/2}^\alpha \stackrel{d}{=} L^{-1} \times (Y_\alpha + 4^{-\alpha})$$

and from Proposition 3.1 and Theorem 4.3.1 in [3], we know that $Y_\alpha + 4^{-\alpha}$ is GGC because $\alpha \in [1/2, 1)$. Since L^{-1} is HCM, Theorem 6.2.1 in [3] shows that $(Z_\alpha \times Z_{1/2})^\alpha$ is GGC as well. □

As just mentioned, Proposition 3.1 shows that $V_{1/2}^s$ is GGC for every $s \in [1/2, 1]$. From the conjecture made in [5], Remark 3(ii), one may ask if this property does not remain true for every $s > 1$. Considering the variable Y_s instead, this amounts to show that the non-HCM function

$$x \mapsto \frac{t}{\pi(x+1)\sqrt{(x+1)^t - 1}}$$

is still a GGC density for every $t \in (0, 1)$. From Example 15.2.2. in [20], one sees that the Laplace transform of the renewal measure of a tempered t-stable subordinator subordinated through a $(1/2)$-stable subordinator, which is given by

$$x \mapsto \frac{1}{\sqrt{(x+1)^t - 1}},$$

is a factor in this density. However, we could not find any convenient expression for the Stieltjes (i.e., double Laplace) transform of the above renewal measure which would entail that the Laplace transform of Y_s is HCM. If this renewal measure had a log-concave density, we could apply Theorem 4.2.1 in [3], but this property does not seem to be true even in the semi-explicit case $t = 1/2$ (inverse Gaussian distribution).

From the above Remark 3.1(b), we know that for every $s > 0$ and every $\alpha \in (0, 1)$ the Lévy measure of V_α^s has a density $v_{\alpha,s}$ which is CM (more precisely, of the form given in Theorem 51.12 of [19]). Proposition 3.1 yields the reinforcement that $x \mapsto x v_{1/2,s}(x)$ is CM for every $s \in [1/2, 1]$. Even though this is a very particular case, one might wonder if it is not true in general.

Conjecture 3.1. *The random variable* V_α^s *is GGC for every* $s > 0$ *and every* $\alpha \in (0, 1)$.

When $\alpha = 1/2$, a part of this conjecture can be rephrased in terms of a more general question on Beta variables. Since

$$(\mathrm{Beta}(\alpha_1, \alpha_2))^{-1} \stackrel{d}{=} 1 + \frac{\mathrm{Gamma}(\alpha_2, 1)}{\mathrm{Gamma}(\alpha_1, 1)}$$

for every $\alpha_1, \alpha_2 > 0$ – see, for exmple, [3], page 13, the HCM property for Gamma variables and Theorem 5.1.1 in [3] show that $(\text{Beta}(\alpha_1, \alpha_2))^{-1}$ is always a GGC. A particular case of the conjecture made in [5], Remark 3(ii), is hence the following.

Conjecture 3.2. *The random variable* $(\text{Beta}(\alpha_1, \alpha_2))^{-s}$ *is GGC for every* $\alpha_1, \alpha_2 > 0$ *and every* $s \geq 1$.

We could not find in the literature any result on the infinite divisibility of negative powers of Beta variables. Proposition 3.1 shows that $(\text{Beta}(1/2, 1/2))^{-s}$ is a positively translated HCM, hence GGC and ID, when $s \in [1/2, 1]$. The same conclusion holds for other values of the parameters, not covering the full range $\alpha_1, \alpha_2 > 0$. From the identities (3.1) and Theorem 6.2.1 in [3], a positive answer to Conjecture 3.1 would also show that Z_α^s is GGC for every $s \geq \alpha/(1-\alpha)$. In view of our results in the previous section for negative stable powers, it is tantalizing to raise the following general conjecture.

Conjecture 3.3. *The random variable* Z_α^γ *is GGC for every* $\alpha \in (0, 1)$ *and* $|\gamma| \geq \alpha/(1-\alpha)$.

If Bondesson's HCM conjecture is true, then Z_α^s is GGC for every $\alpha \leq 1/2$ if $|s| \geq 1$. The above statement is stronger since it takes values of α which are greater than $1/2$ in consideration, and since $\alpha/(1-\alpha) < 1$ when $\alpha < 1/2$. Notice that if Conjecture 3.3 is true, then from Proposition 2.1 we would also have $Z_\alpha^{-\gamma}$ is GGC $\Leftrightarrow$ $Z_\alpha^{-\gamma}$ is ID for every $\gamma > 0$. Last, since the main theorem of [23] shows that all positive stable powers are unimodal, it would also be interesting to know if Z_α^γ is still a GGC when $\gamma \in (0, \alpha/(1-\alpha))$.

4. Related HCM densities

In this section, we study two families of random variables which are related to Bondesson's hypothesis, and have their independent interest. For every $\alpha \in \mathbb{R}$, introduce the function

$$g_\alpha(u) = \frac{1}{u^{2\alpha} + 2\cos(\pi\alpha)u^\alpha + 1}$$

from $\mathbb{R}^+$ to $\mathbb{R}^+$. The following elementary result might be well known, although we could not trace any reference in the literature. As for Proposition 3.1, we could not find any constructive argument either.

Proposition 4.1. *The function* g_α *is HCM iff* $|\alpha| \leq 1/2$.

Proof. Since $u^{2\alpha} g_\alpha(u)$ HCM $\Leftrightarrow$ $g_\alpha(u)$ HCM and since cosine is an even function, it is enough to consider the case $\alpha \geq 0$. Suppose first $\alpha > 1/2$. Rewriting

$$g_\alpha(u) = \frac{1}{(u^\alpha + \mathrm{e}^{\mathrm{i}\pi\alpha})(u^\alpha + \mathrm{e}^{-\mathrm{i}\pi\alpha})}$$

shows that g_α has two poles in $\mathbb{C}/(-\infty, 0]$ (because $|(\alpha-1)/\alpha| < 1$) and from (ix), page 68 in [3], this entails that g_α is not HCM. Suppose next $0 \le \alpha \le 1/2$. The cases $\alpha = 0$ with $m_0(u) = 1/4$ and $\alpha = 1/2$ with $m_{1/2}(u) = 1/(u+1)$ yield the HCM property explicitly, and we only need to consider the case $0 < \alpha < 1/2$. Rewriting

$$g_\alpha(u) = \frac{u^{-\alpha}}{2\cos(\pi\alpha) + u^{-\alpha} + u^\alpha}$$

we see that it is enough to show that $\rho : u \mapsto 1/(c + u^\alpha + u^{-\alpha})$ is HCM for any $c > 0$. Developing, one obtains

$$\rho(uv)\rho(u/v) = \frac{1}{c^2 + u^{2\alpha} + u^{-2\alpha} + v^{2\alpha} + v^{-2\alpha} + c(u^\alpha + u^{-\alpha})(v^\alpha + v^{-\alpha})}$$

for any $u, v > 0$. Since the function $v^{2\alpha} + v^{-2\alpha} + c(u^\alpha + u^{-\alpha})(v^\alpha + v^{-\alpha})$ has CM derivative in $w = v + 1/v$ for any fixed $u, c > 0$ (see [2], page 183), again Criterion 2 in [12], page 417, entails that the function $\rho(uv)\rho(u/v)$ is CM in w, so that ρ is HCM. $\square$

This proposition has several interesting consequences. Let us first consider the random variable

$$Y_\alpha = T_\alpha^{1/\alpha}$$

with the notation of Section 2.2, and recall that it is the quotient of two independent copies of Z_α. The fact that Y_α has a closed density seems to have been first noticed in [16], in the context of occupation time for certain stochastic processes. If Bondesson's conjecture is true, then Y_α is HCM whenever $\alpha \le 1/2$, as a quotient of two independent HCM random variables – see Theorem 5.1.1 in [3]. The next corollary shows that this is indeed the case.

Corollary 4.1. *The random variable Y_α is HCM iff $\alpha \le 1/2$.*

Proof. From a fractional moment identification, the density of Y_α is explicitly given by

$$f_{Y_\alpha}(x) = \frac{\sin \pi\alpha x^{\alpha-1}}{\pi(x^{2\alpha} + 2x^\alpha \cos \pi\alpha + 1)} = \frac{\sin \pi\alpha}{\pi} x^{\alpha-1} g_\alpha(x)$$

over $\mathbb{R}^+$ – see Exercise 4.21(3) in [6] already mentioned in Section 2. The second derivative of $t \mapsto \log f_{Y_\alpha}(\mathrm{e}^t)$ equals

$$\frac{-4\alpha^2(1 + \cos \pi\alpha \cosh \alpha t)}{(\mathrm{e}^{\alpha t} + 2\cos \pi\alpha + \mathrm{e}^{-\alpha t})^2}$$

and is not everywhere nonpositive whenever $\alpha > 1/2$, so that Y_α is not HM, hence not HCM either. When $\alpha \le 1/2$, the above Proposition 4.1 shows immediately that f_{Y_α} is a HCM function, so that Y_α is HCM as a random variable. $\square$

We now turn our attention to the so-called Mittag–Leffler random variables which were introduced in [17], and appeared since then in a variety of contexts. Let

$$E_\alpha(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(1+\alpha n)}$$

be the classical Mittag–Leffler function with index $\alpha \in (0, 1]$. The Mittag–Leffler random variable M_α has an explicit decreasing density given by

$$f_{M_\alpha}(x) = \alpha x^{\alpha-1} E'_\alpha(-x^\alpha)$$

over $\mathbb{R}^+$. Its Laplace transform is also explicit: for every $\lambda \geq 0$ one has

$$\mathbb{E}[\mathrm{e}^{-\lambda M_\alpha}] = \frac{1}{1+\lambda^\alpha} = \exp\left[-\alpha \int_0^\infty (1-\mathrm{e}^{-\lambda x}) E_\alpha(-x^\alpha) \frac{\mathrm{d}x}{x}\right] \tag{4.1}$$

(see Remark 2.2 in [17] and correct $x^k \to x^{\alpha k}$ therein). From the classical fact that $x \mapsto E_\alpha(-x)$ is the Laplace transform of $Z_\alpha^{-\alpha}$ – see, for example, Exercise 29.18 in [19] – and hence a CM function, this shows that M_α is a GGC (with finite Thorin measure). This latter fact follows also from the factorization

$$M_\alpha \stackrel{d}{=} Z_\alpha \times L^{1/\alpha},$$

where $L \sim \mathrm{Exp}(1)$ (the latter is a direct consequence of the first equality in (4.1) – see also the final remark in [17]), and from Theorem 6.2.1 in [3] since Z_α is GGC and $L^{1/\alpha}$ is HCM. Notice that in Example 3.2.4 of [2], the GGC property for M_α is also obtained in a slightly more general context with the help of Pick functions.

Corollary 4.2. *The random variable M_α is HCM iff $\alpha \leq 1/2$.*

Proof. As a consequence of the above discussions, one has the classical representation

$$E_\alpha(-x^\alpha) = \mathbb{E}[\mathrm{e}^{-x^\alpha Z_\alpha^{-\alpha}}] = \mathbb{E}[\mathrm{e}^{-xY_\alpha}] = \frac{\sin \pi\alpha}{\pi} \int_{\mathbb{R}^+} \frac{u^{\alpha-1}\mathrm{e}^{-xu}}{u^{2\alpha} + 2u^\alpha \cos \pi\alpha + 1} \, \mathrm{d}u$$

so that the density of M_α writes

$$f_{M_\alpha}(x) = \frac{\sin \pi\alpha}{\pi} \int_{\mathbb{R}^+} \frac{u^\alpha \mathrm{e}^{-xu}}{u^{2\alpha} + 2u^\alpha \cos \pi\alpha + 1} \, \mathrm{d}u.$$

From Proposition 4.1, the function

$$u \mapsto \frac{\sin \pi\alpha u^\alpha}{\pi(u^{2\alpha} + 2u^\alpha \cos \pi\alpha + 1)} = u f_{Y_\alpha}(u)$$

is HCM as soon as $\alpha \leq 1/2$, so that it is also a widened GGC density with the notations of Section 3.5 in [2]. By Theorem 5.4.1 in [3], this shows that f_{M_α} is a HCM density function whenever $\alpha \leq 1/2$.

There are two ways to prove that M_α is not HCM for $\alpha > 1/2$. First, again from Theorem 5.4.1 in [2], it suffices to show that $uf_{Y_\alpha}(u)$ is no more a widened GGC density when $\alpha > 1/2$. If it were true, then from Remark 6.1 in [2] the function $u^{1-\delta}f_{Y_\alpha}(u)$ would also be a widened GGC density for every $\delta > 0$. In particular, for every $\delta \in]\alpha, 1+\alpha[$ the function $u^{1-\delta}f_{Y_\alpha}(u)$ would be up to normalization the density of a GGC. The derivative of the above function equals

$$\frac{-u^{\alpha-\delta-1}((\delta+\alpha)u^{2\alpha}+2\delta u^\alpha\cos\pi\alpha+(\delta-\alpha))}{(u^{2\alpha}+2u^\alpha\cos\pi\alpha+1)^2}$$

and is easily seen to vanish twice on $\mathbb{R}^+$ if δ is close enough to $\alpha > 1/2$. This shows that the underlying variable is bimodal and contradicts Theorem 52.1 in [19] since all GGC's are SD.

The second argument shows that M_α is not even HM when $\alpha > 1/2$. From (7), page 207 in [11] one has the asymptotic expansion

$$E_\alpha(-z) = \sum_{n=1}^{N-1}\frac{(-1)^{n+1}z^{-n}}{\Gamma(1-\alpha n)} + \mathrm{O}(z^{-N}), \qquad z \to +\infty, \tag{4.2}$$

for any $N \geq 2$. Besides, by complete monotonicity, one can differentiate this expansion term by term. Taking $N = 3$, one obtains

$$\bigl(zE_\alpha'''(-z) - E_\alpha''(-z)\bigr)E_\alpha'(-z) - z\bigl(E_\alpha''(-z)\bigr)^2 = \frac{-1}{\alpha^2 z^6\Gamma(-\alpha)\Gamma(-2\alpha)} + \mathrm{O}(z^{-7})$$

as $z \to \infty$, and the leading term in the right-hand side is positive when $\alpha > 1/2$: this shows that $t \mapsto E_\alpha'(-\mathrm{e}^t)$ and, a fortiori, $t \mapsto \alpha\mathrm{e}^{(\alpha-1)t}E_\alpha'(-\mathrm{e}^{\alpha t})$ are not log-concave, so that M_α is not HM. □

Remarks 4.1. (a) Since $Z_\alpha^{-\alpha}$ is not ID, the function $E_\alpha(-x) = \mathbb{E}[\mathrm{e}^{-xZ_\alpha^{-\alpha}}]$ is CM but never HCM. However, repeating verbatim the above argument shows that $x \mapsto E_\alpha(-x^\alpha)$ is HCM if and only if $\alpha \leq 1/2$.

(b) From the above proof, one has the equivalence

$$M_\alpha \textit{ is } \mathrm{HM} \iff M_\alpha \textit{ is } \mathrm{HCM} \iff \alpha \leq 1/2.$$

The variable $L^{1/\alpha}$ is always HCM, hence HM, and we know that Z_α is HM $\Leftrightarrow \alpha \leq 1/2$. Since the HM property is closed by independent multiplication, this also proves that M_α is HM as soon as $\alpha \leq 1/2$. On the other hand, the influence of the HM variable $L^{1/\alpha}$ in the product $L^{1/\alpha} \times Z_\alpha$ is not important enough to make M_α HM when $\alpha > 1/2$. From the above equivalence, one can ask if the general identification

$$\mathrm{HM} \cap \mathrm{GGC} = \mathrm{HCM}$$

is true or not, and we could not find any counterexample. Such an identification would show Bondesson's conjecture in view of the main result of [22].

(c) When $\alpha \le 1/2$, the variable M_α^r is clearly HM, hence unimodal, for every $r \neq 0$. On the other hand, when $\alpha > 1/2$, it is possible to find some $r \neq 0$ such that M_α^r is not unimodal. Indeed, it follows easily from the expansion (4.2) that

$$f_{M_\alpha^{-1/\alpha}}(0+) = \frac{-1}{\alpha\Gamma(-\alpha)}, \qquad f'_{M_\alpha^{-1/\alpha}}(0+) = \frac{-1}{\alpha\Gamma(-2\alpha)} > 0$$

and $f_{M_\alpha^s}(0+) = +\infty \ \forall s < -1/\alpha$. By continuity of $s \mapsto f_{M_\alpha^s}(x)$ for every $x > 0$, this shows that M_α^s is not unimodal for some $s < -1/\alpha$ close enough to $-1/\alpha$. Hence, we have shown the further equivalence

$$M_\alpha \textit{ is } \mathrm{HM} \quad \Longleftrightarrow \quad M_\alpha^r \textit{ is unimodal for every } r \neq 0. \tag{4.3}$$

The same equivalence holds for Z_α as a consequence of the main results of [22,23] and we can raise the following natural question: *For which class of positive random variables with density does the equivalence* (4.3) *hold true?* It is easy to see that if X is such that X^r is unimodal for every $r \neq 0$, then X is absolutely continuous.

(d) Reasoning exactly as at the beginning of Paragraph 2.2 in [23], the decomposition

$$M_\alpha^s = (L \times L^{\alpha-1})^{s/\alpha} \times b_\alpha^{-s/\alpha}(U)$$

and the concavity of b_α show that M_α^s is unimodal as soon as $s \ge -\alpha$, for every $\alpha \in (0,1)$. One might ask whether M_a^s is also unimodal for every $-1/\alpha \le s < -\alpha$. Notice that the above reasoning is not valid anymore, at least for $\alpha = 1/2$ because $b_{1/2}^r(U) = 2^r(\cos(U/2))^r$ is bimodal as soon as $r > 1$.

(e) Both random variables Y_α and M_α appear as special instances of the Lamperti-type laws which were introduced in [13], to which we refer for a thorough study. See also the numerous references therein.

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