Bernoulli **19**(5A), 2013, 1750–1775
DOI: 10.3150/12-BEJ428

Penalization methods for the Skorokhod problem and reflecting SDEs with jumps

WERONIKA ŁAUKAJTYS* and LESZEK SŁOMIŃSKI**

Faculty of Mathematics and Computer Science, Nicolaus Copernicus University, ul. Chopina 12/18 87–100 Toruń, Poland. E-mail: *vera@mat.uni.torun.pl; **leszeks@mat.uni.torun.pl

We study the problem of approximation of solutions of the Skorokhod problem and reflecting stochastic differential equations (SDEs) with jumps by sequences of solutions of equations with penalization terms. Applications to discrete approximation of weak and strong solutions of reflecting SDEs are given. Our proofs are based on new estimates for solutions of equations with penalization terms and the theory of convergence in the Jakubowski S-topology.

Keywords: penalization methods; reflecting stochastic differential equation; S-topology; Skorokhod problem

1. Introduction

Let D be a convex open set in $\mathbb{R}^d$. Consider a d-dimensional reflecting stochastic differential equation (SDE),

$$X_t = H_t + \int_0^t f(X_{s-})\,\mathrm{d}Z_s + K_t, \qquad t \in \mathbb{R}^+, \tag{1.1}$$

where Z is a d-dimensional semimartingale with $Z_0 = 0$, H is an adapted process with $H_0 \in \bar{D} = D \cup \partial D$, and $f : \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d$ is a continuous function such that

$$\|f(x)\| \le L(1 + |x|), \qquad x \in \mathbb{R}^d \tag{1.2}$$

(for the precise definition, see Section 3). Our main purpose is to study the problem of approximation of weak solution of (1.1) by solutions of nonreflecting SDEs of the form

$$X_t^n = H_t^n + \int_0^t f(X_{s-}^n)\,\mathrm{d}Z_s^n - n\int_0^t (X_s^n - \Pi(X_s^n))\,\mathrm{d}s, \qquad t \in \mathbb{R}^+, n \in \mathbb{N}, \tag{1.3}$$

where H^n and Z^n are perturbations of H and Z, respectively, and $\Pi(x)$ denotes projection of x on $\bar{D}$. Because for large $n \in \mathbb{N}$, the drift term $-n\int_0^t (X_s^n - \Pi(X_s^n))\,\mathrm{d}s$ forces X^n to stay close to $\bar{D}$, it is called the penalization term, and the SDE (1.3) is called the SDE with penalization term.

The foregoing problem was intensively investigated in the case where f is a Lipschitz continuous function, $H = X_0$, and Z is a continuous semimartingale. In particular, Lions *et al.* [16] and Menaldi [20] have proven that $E\sup_{t \le q} |X_t^n - X_t|^2 \longrightarrow 0$ for $q \in \mathbb{R}^+$, provided that Z is

1350-7265

a d-dimensional standard Wiener process. In the case where Z has jumps, to the best of our knowledge, such a problem has been considered previously only by Menaldi and Robin [21] and Łaukajtys and Słomiński [15]. Menaldi and Robin studied the case where Z is a diffusion with Poissonian jumps and $H = X_0$. However, they imposed a very restrictive condition on the Poissonian measure coefficient, and consequently, K is a process with continuous trajectories. In this case, earlier methods of approximation remain in force. In earlier work, we considered in detail the case where $H = X_0$ and Z is a general semimartingale. Because the approximating sequence $\{X^n\}$ might not be relatively compact in the Skorokhod topology J_1, we proved our convergence results in the S-topology introduced by Jakubowski [10]. It is worth pointing out that in both of the aforementioned papers, the initial process H is constant (i.e., $H = X_0$), and f is a Lipschitz continuous function.

The purpose of the present paper is to investigate the problem of approximation of X by $\{X^n\}$ in the case of arbitrary initial process H and arbitrary continuous coefficient f satisfying the linear growth condition (1.2). Our proofs are based on new estimates for solutions of equations with penalization terms.

The paper is organized as follows. In Section 2 we consider a deterministic problem of approximating a solution of the Skorokhod problem $x_t = y_t + k_t$, $t \in \mathbb{R}^+$ on domain D associated with a given function $y \in \mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$ such that $y_0 \in \bar{D}$ (for precise definition, see Section 2). The penalization method involves approximating x by solutions of equations of the form

$$x_t^n = y_t^n - n \int_0^t \left(x_s^n - \Pi\left(x_s^n\right)\right) \mathrm{d}s, \qquad t \in \mathbb{R}^+, \tag{1.4}$$

where $y^n \longrightarrow y$ in the Skorokhod topology J_1. Lions and Sznitman [17] and Cépa [5] proved that $\{x^n\}$ tends to x if y is continuous. We omit the latter assumption and consider arbitrary function $y \in \mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$. In this general case, we prove that the variation of the penalization term of the SDE (1.4) is locally uniformly bounded and for fixed $t \in \mathbb{R}^+$ $x_t^n \longrightarrow x_t$, provided that $\Delta y_t = 0$, which implies in particular that x^n tends toward x in the S-topology. It is noteworthy that, similar to [5], here we do not assume that the domain D satisfies the so-called condition (β) introduced by Tanaka [29].

In Section 3 we present new estimates on solutions X^n of equations with penalization terms associated with a given process Y^n such that $Y_0^n \in \bar{D}$, that is, solutions of SDEs,

$$X_t^n = Y_t^n - n \int_0^t \left(X_s^n - \Pi\left(X_s^n\right)\right) \mathrm{d}s, \qquad t \in \mathbb{R}^+. \tag{1.5}$$

In particular, we prove that if Y^n is a process admitting the decomposition $Y^n = H^n + M^n + V^n$, where H^n is an $(\mathcal{F}_t^n)$ adapted process, M^n is an $(\mathcal{F}_t^n)$ adapted local martingale with $M_0^n = 0$ and V^n is an $(\mathcal{F}_t^n)$ adapted processes of bounded variation with $V_0^n = 0$, then, for every $\delta, q > 0$, $a \in D$ there exist constants $C_1, C_2 > 0$ such that for every $\eta > 0$,

$$P\Big(\sup_{t \le q} \left|X_t^n - a\right| \ge \eta\Big) \le P\left(\omega'_{H^n}(\delta, q) \ge d_a/2\right) + P\Big(\sup_{t \le q} \left|H_t^n - a\right| \ge C_1 \eta\Big) + C_2 \eta^{-2} E\left(\left[M^n\right]_q + \left|V^n\right|_q^2\right)$$

and

$$P\left(n\int_0^q \left|X_s^n - \Pi\left(X_s^n\right)\right| \mathrm{d}s \geq \eta^2\right) \leq P\left(\omega'_{H^n}(\delta, q) \geq d_a/2\right) + 7P\left(\sup_{t\leq q}\left|H_t^n - a\right| \geq C_1\eta\right)$$
$$+ C_2\eta^{-2}E\left(\left[M^n\right]_q + \left|V^n\right|_q^2\right),$$

where ω' denotes the usual modulus of continuity and $d_a = \mathrm{dist}(a, \partial D)$.

In Section 4, we use estimates derived in Section 3 to prove our main results on the approximation of X by $\{X^n\}$. We assume that $\{Z^n\}$ is a sequence of semimartingales satisfying the so-called condition (UT), and we prove that if (H^n, Z^n) converges weakly to (H, Z) in the J_1 topology, then $\{X^n\}$ converges weakly in the S topology to X. Moreover, we prove convergence of finite-dimensional distributions of $\{X^n\}$ to the corresponding finite-dimensional distributions of X outside the set of discontinuity points of Z and H. Consequently, using discrete approximations constructed in a manner analogous to Euler's formula, we prove the existence of a weak solution of the SDE (1.1), provided that f is continuous and satisfies (1.2). Moreover, if the SDE (1.1) has the weak uniqueness property, then our approximations computed by simple recurrent formulas allows us to obtain numerical solution of the SDE (1.1). In the case of reflected diffusion processes, similar approximation schemes have been considered previously (see, e.g., Liu [18], Pettersson [23], Słomiński [26]). In this section we also present some natural conditions ensuring convergence of $\{X^n\}$ to X in probability provided that (1.1) has the so-called pathwise uniqueness property. Related results concerning diffusion processes have been given by, for instance, Kaneko and Nakao [12], Gyöngy and Krylov [6], Bahlali, Mezerdi and Ouknine [3], Alibert and Bahlali [1] and Słomiński [26].

We note that we consider the space $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$ equipped with two different topologies, J_1 and S. Definitions and required results for the Skorokhod topology J_1 have been given by, for example, Billingsley [4] and Jacod and Shiryayev [8]. For the convenience of the reader, we have collected basic definitions and properties of the S-topology in the Appendix. More details have been provided in Jakubowski [10].

In this paper, we use the following notation. Every process X appearing in the sequel is assumed to have trajectories in the space $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$. If $X = (X^1, \dots, X^d)$ is a semimartingale, then $[X]_t$ represents $\sum_{i=1}^d [X^i]_t$ and $[X^i]$ represents the quadratic variation process of X^i, $i = 1, \dots, d$. Similarly, $\langle X\rangle_t = \sum_{i=1}^d \langle X^i\rangle_t$, and $\langle X^i\rangle$ represents the predictable compensator of $[X^i]$, $i = 1, \dots, d$. If $K = (K^1, \dots, K^d)$ is the process with locally finite variation, then $|K|_t = \sum_{i=1}^d |K^i|_t$, where $|K^i|_t$ is a total variation of K^i on $[0, t]$. In general, we let $\underset{\mathcal{D}}{\longrightarrow}$ and $\underset{\mathcal{P}}{\longrightarrow}$ denote convergence in law and in probability, respectively. To avoid ambiguity, we write $X^n \underset{\mathcal{D}}{\longrightarrow} X$ ($X^n \underset{\mathcal{P}}{\longrightarrow} X$) in $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$ if $\{X^n\}$ converges weakly (in probability) to X in the space $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$ equipped with J_1. Following [10], we write $X^n \overset{*}{\underset{\mathcal{D}}{\longrightarrow}} X$ ($X^n \overset{*}{\underset{\mathcal{P}}{\longrightarrow}} X$) in $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$ when we consider the S topology. For $x \in \mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$, $\delta > 0$, $q \in \mathbb{R}^+$, we let $\omega'_x(\delta, q)$ and $\omega''_x(\delta, q)$ denote classical moduli of continuity of x on $[0, q]$, that is, $\omega'_x(\delta, q) = \inf\{\max_{i\leq r} \omega_x([t_{i-1}, t_i)); 0 = t_0 < \cdots < t_r = q, \inf_{i<r}(t_i - t_{i-1}) \geq \delta\}$, and $\omega''_x(\delta, q) = \sup\{\min(|x_u - x_s|, |x_t - x_u|), 0 \leq s < u < t \leq q, t - s < \delta\}$, where $\omega_x(I) = \sup_{s,t\in I} |x_s - x_t|$. We also use the modulus $\bar{\omega}''_{(x,y)}(\delta, q)$ introduced in Jakubowski [10]. We recall that for $x, y \in \mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$, $\delta > 0$, $q \in \mathbb{R}^+$ $\bar{\omega}''_{(x,y)}(\delta, q) = \sup\{\min(|x_u - x_s|, |y_t - y_u|), 0 \le s < u < t \le q, t - s < \delta\}$.

2. A deterministic case

Let D be a nonempty convex (possibly unbounded) open set in $\mathbb{R}^d$, and let $\mathcal{N}_x$ denote the set of inward normal unit vectors at $x \in \partial D$ ($\mathbf{n} \in \mathcal{N}_x$ if and only if $\langle y - x, \mathbf{n}\rangle \ge 0$ for every $y \in \bar{D}$, where $\langle \cdot, \cdot \rangle$ denotes the usual inner product in $\mathbb{R}^d$). The following remark also can be found in Menaldi [20] or Storm [27].

Remark 2.1. (i) If $\operatorname{dist}(x, \bar{D}) > 0$, then there exists a unique $\Pi(x) \in \partial D$ such that $|\Pi(x) - x| = \operatorname{dist}(x, \bar{D})$. Moreover, $(\Pi(x) - x)/|\Pi(x) - x| \in \mathcal{N}_{\Pi(x)}$.

(ii) For every $a \in D$,

$$\bigl|x - \Pi(x)\bigr| \le d_a^{-1}\bigl\langle x - a, x - \Pi(x)\bigr\rangle, \qquad x \in \mathbb{R}^d,$$

where $d_a = \operatorname{dist}(a, \partial D)$.

Let $y \in \mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$ be a function with initial value in $\bar{D}$. We recall that a pair of functions $(x, k) \in \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{2d})$ is called a solution of the Skorokhod problem associated with y if

- $x_t = y_t + k_t,\ t \in \mathbb{R}^+$,
- x is $\bar{D}$-valued,
- k is a function with locally bounded variation such that $k_0 = 0$ and

$$k_t = \int_0^t \mathbf{n}_s \,\mathrm{d}|k|_s, \qquad |k|_t = \int_0^t \mathbf{1}_{\{x_s \in \partial D\}} \,\mathrm{d}|k|_s, \qquad t \in \mathbb{R}^+,$$

  where $\mathbf{n}_s \in \mathcal{N}_{x_s}$ if $x_s \in \partial D$.

The problem of existence of solutions of the Skorokhod problem and its approximation by solutions of equations with penalization terms has been discussed by many authors. Tanaka [29] proved existence and uniqueness of solutions in the case of continuous y and domains also satisfying the following condition:

(β) there exist constants $\epsilon > 0$ and $\delta > 0$ such that for every $x \in \partial D$, we can find $x_0 \in D$ such that $B(x_0, \epsilon) \subset D$ and $|x - x_0| \le \delta$.

Tanaka also observed that (β) holds true in dimensions 1 and 2 or if D is a bounded set. On the other hand, in dimension >2, one can construct examples of nonbounded convex domains not satisfying (β). For instance, the cone with the basis $\{(x, y, 0) \in \mathbb{R}^3; y \ge x^2\}$ and peak at $(0, 0, 1)$, that is, the set

$$C = \bigl\{(\lambda x, \lambda y, 1 - \lambda) \in \mathbb{R}^3; x^2 \le y, 0 \le \lambda \le 1\bigr\}, \tag{2.1}$$

does not satisfy (β). Cépa [5] omitted the assumption (β) and proved the existence and uniqueness of the solution to the Skorokhod problem in the case of continuous function y. In addition, Cépa proved convergence $\sup_{t\le q}|x^n_t - x_t| \longrightarrow 0$, $q\in\mathbb{R}^+$ of solutions of equations (1.4) for every sequence $\{y^n\}$ such that $\sup_{t\le q}|y^n_t - y_t| \longrightarrow 0$, $q\in\mathbb{R}^+$.

The case of functions with jumps was considered for the first time by Anulova and Liptser [2], who proved the existence and uniqueness of solutions under condition (β). Their result was generalized to the case of arbitrary convex D by Łaukajtys [14]. In an earlier work [15], we considered the problem of approximating noncontinuous x by solutions of equations with penalization terms only in this very special case. We now consider the problem of approximating noncontinuous x by solutions of equations with penalization terms in the general case of arbitrary sequences $\{y^n\}$ such that $y^n \longrightarrow y$ in $\mathbb{D}(\mathbb{R}^+,\mathbb{R}^d)$. Our main tools are the following estimates on the solution of (1.4):

Lemma 2.2. *Let $y^n\in\mathbb{D}(\mathbb{R}^+,\mathbb{R}^d)$, $y^n_0\in\bar{D}$ and let x^n be a solution of the equation* (1.4). *Then for any $q,\delta>0$ and $a\in D$ such that*

$$\omega'_{y^n}(\delta,q) < d_a/2, \tag{2.2}$$

we have

(i) $\sup_{t\le q}|x^n_t - a| \le 2\sqrt{7}([q/\delta]+1)\sup_{t\le q}|y^n_t - a|$,
(ii) $|k^n|_q \le 55([q/\delta]+1)^3 d_a^{-1}\sup_{t\le q}|y^n_t - a|^2$,

where $k^n_t = -n\int_0^t (x^n_s - \Pi(x^n_s))\,\mathrm{d}s$, $t\in\mathbb{R}^+$ and $[q/\delta]$ denotes the largest integer less or equal to q/δ.

Proof. We follow the proof of Theorem 3.2 in [5]. Let $0\le t\le q$. Because k^n is a continuous function such that $k^n_0 = 0$,

$$\begin{aligned}\left|x^n_t - a\right|^2 &= \left|y^n_t - a\right|^2 + \langle k^n_t, k^n_t\rangle + 2\int_0^t \langle y^n_t - a, \mathrm{d}k^n_u\rangle\\ &= \left|y^n_t - a\right|^2 + 2\int_0^t \langle x^n_u - a, \mathrm{d}k^n_u\rangle + 2\int_0^t \langle y^n_t - y^n_u, \mathrm{d}k^n_u\rangle.\end{aligned}$$

Therefore, for any $0\le s\le t\le q$,

$$\begin{aligned}\left|x^n_t - a\right|^2 - \left|x^n_s - a\right|^2 &= \left|y^n_t - a\right|^2 - \left|y^n_s - a\right|^2 + 2\int_s^t \langle x^n_u - a, \mathrm{d}k^n_u\rangle\\ &\quad - 2\int_s^t \langle y^n_u - y^n_s, \mathrm{d}k^n_u\rangle + 2\langle k^n_t, y^n_t - y^n_s\rangle.\end{aligned}$$

By Remark 2.1(ii),

$$\begin{aligned}2\int_s^t \langle x^n_u - a, \mathrm{d}k^n_u\rangle &= -2n\int_s^t \langle x^n_u - a, x^n_u - \Pi(x^n_u)\rangle\,\mathrm{d}u\\ &\le -2d_a n\int_s^t \left|x^n_u - \Pi(x^n_u)\right|\mathrm{d}u = -2d_a\left|k^n\right|_s^t,\end{aligned}$$

and, consequently,

$$\begin{aligned}
|x_t^n - a|^2 - |x_s^n - a|^2 &\le |y_t^n - a|^2 - |y_s^n - a|^2 - 2d_a|k^n|_s^t - 2\int_s^t \langle y_u^n - y_s^n, \mathrm{d}k_u^n\rangle \\
&\quad - 2\langle y_t^n - a, y_t^n - y_s^n\rangle - 2\langle a - x_t^n, y_t^n - y_s^n\rangle \\
&\le 5\sup_{t\le q}|y_t^n - a|^2 + 4\sup_{t\le q}|y_t^n - a|\cdot\sup_{t\le q}|x_t^n - a| - 2d_a|k^n|_s^t \\
&\quad - 2\int_s^t \langle y_u^n - y_s^n, \mathrm{d}k_u^n\rangle.
\end{aligned}$$

By (2.2), there exists a subdivision (s_k) of $[0,q]$ such that $0 = s_0 < s_1 < \cdots < s_r = q$, $\delta \le s_k - s_{k-1}$, $k = 1,\ldots,r-1$, where $r = [q/\delta] + 1$ and $\omega_{y^n}([s_{k-1}, s_k)) < d_a/2$. Thus, in particular,

$$\int_{s_{k-1}}^{s_k} \langle y_u^n - y_{s_{k-1}}^n, \mathrm{d}k_u\rangle \le \left|\int_{(s_{k-1},s_k)} \langle y_u^n - y_{s_{k-1}}^n, \mathrm{d}k_u^n\rangle\right| \le \frac{d_a}{2}|k^n|_{s_{k-1}}^{s_k}.$$

Therefore,

$$2\left(-\int_{s_{k-1}}^{s_k}\langle y_u^n - y_{s_{k-1}}^n, \mathrm{d}k_u^n\rangle - d_a|k^n|_{s_{k-1}}^{s_k}\right) \le 2\left(\frac{d_a}{2}|k^n|_{s_{k-1}}^{s_k} - d_a|k^n|_{s_{k-1}}^{s_k}\right) = -d_a|k^n|_{s_{k-1}}^{s_k},$$

which implies that

$$|x_{s_k}^n - a|^2 - |x_{s_{k-1}}^n - a|^2 \le 5\sup_{t\le q}|y_t^n - a|^2 + 4\sup_{t\le q}|y_t^n - a|\cdot\sup_{t\le q}|x_t^n - a| - d_a|k^n|_{s_{k-1}}^{s_k}. \tag{2.3}$$

From (2.3), it follows immediately that

$$|x_{s_k}^n - a|^2 - |x_{s_{k-1}}^n - a|^2 \le 5\sup_{t\le q}|y_t^n - a|^2 + 4\sup_{t\le q}|y_t^n - a|\cdot\sup_{t\le q}|x_t^n - a|.$$

Set $k_0 = \max\{k, s_k \le t\}$. Then

$$\begin{aligned}
|x_t^n - a|^2 &= \sum_{k=1}^{k_0}\left(|x_{s_k}^n - a|^2 - |x_{s_{k-1}}^n - a|^2\right) \\
&\quad + |x_t^n - a|^2 - |x_{s_{k_0}}^n - a|^2 + |x_0^n - a|^2 \\
&\le r\left(5\sup_{t\le q}|y_t^n - a|^2 + 4\sup_{t\le q}|y_t^n - a|\cdot\sup_{t\le q}|x_t^n - a|\right) + \sup_{t\le q}|y_t^n - a|^2,
\end{aligned}$$

which implies that

$$\sup_{t\le q}|x_t^n - a|^2 \le 14r^2\sup_{t\le q}|y_t^n - a|^2 + \sup_{t\le q}|x_t^n - a|^2/2.$$

Thus, $\sup_{t\le q}|x_t^n-a|^2\le 28r^2\sup_{t\le q}|y_t^n-a|^2$, and the proof of (i) is complete.

(ii) Using (2.3) and (i) gives

$$\begin{aligned} d_a\left|k^n\right|_{s_{k-1}}^{s_k} &\le 5\sup_{t\le q}\left|y_t^n-a\right|^2+4\sup_{t\le q}\left|y_t^n-a\right|\cdot\sup_{t\le q}\left|x_t^n-a\right| \\ &\quad+\left|x_{s_{k-1}}^n-a\right|^2-\left|x_{s_k}^n-a\right|^2 \\ &\le 13\sup_{t\le q}\left|y_t^n-a\right|^2+\frac{3}{2}\sup_{t\le q}\left|x_t^n-a\right|^2\le 55r^2\sup_{t\le q}\left|y_t^n-a\right|^2 \end{aligned}$$

for $k=1,\dots,r$. Thus, $|k^n|_q\le\sum_{k=1}^r|k^n|_{s_{k-1}}^{s_k}\le 55r^3d_a^{-1}\sup_{t\le q}|y_t^n-a|^2$, which completes the proof. □

Theorem 2.3. *Assume that $\{y^n\}\subset\mathbb{D}(\mathbb{R}^+,\mathbb{R}^d)$, $y_0^n\in\bar{D}$, and let x^n denote the solution of the equation (1.4), $n\in\mathbb{N}$. If $y^n\longrightarrow y$ in $\mathbb{D}(\mathbb{R}^+,\mathbb{R}^d)$, then*

(i) $\sup_{n\in\mathbb{N}}\sup_{t\le q}|x_t^n|<+\infty$, $q\in\mathbb{R}^+$ *and* $\sup_{n\in\mathbb{N}}|k^n|_q<+\infty$, $q\in\mathbb{R}^+$,

(ii) $x_t^n\longrightarrow x_t$, *provided that* $|\Delta y_t|=0$,

(iii) $x^n\xrightarrow[S]{*}x$,

where x denotes the solution of the Skorokhod problem associated with y.

Proof. (i) Because $\{y^n\}$ is relatively compact in $\mathbb{D}(\mathbb{R}^+,\mathbb{R}^d)$, $\sup_n\sup_{t\le q}|y_t^n|<\infty$ for any $q\in\mathbb{R}^+$, and for any $a\in D$, $q>0$, there exists $\delta>0$ such that $\sup_n\omega'_{y^n}(\delta,q)<d_a/2$. Therefore, the first conclusion follows from Lemma 2.2.

(ii) Let $\{\delta^i\}$ be a sequence of constants such that $\delta^i\downarrow 0$ and $|\Delta y_t|\ne\delta^i$, $t\in\mathbb{R}^+$. Set $t_{n,0}^i=0$, $t_{n,k+1}^i=\min(t_{n,k}^i+\delta_k^i,\inf\{t>t_{n,k}^i,|\Delta y_t^n|>\delta^i\})$, $t_0^i=0$, $t_{k+1}^i=\min(t_k^i+\delta_k^i,\inf\{t>t_k^i,|\Delta y_t|>\delta^i\})$, where $\{\{\delta_k^i\}\}$ is an array of constants satisfying $\delta^i/2\le\delta_k^i\le\delta^i$ and $|\Delta y_{t_k^i+\delta_k^i}|=0$. Now, for every $i\in\mathbb{N}$, set $y_t^{n,(i)}=y_{t_{n,k}^i}^n$, $t\in[t_{n,k}^i,t_{n,k+1}^i)$, $y_t^{(i)}=y_{t_k^i}$, $t\in[t_k^i,t_{k+1}^i)$, $n,k\in\mathbb{N}\cup\{0\}$. Observe that for every $q\in\mathbb{R}^+$,

$$\lim_{i\to\infty}\sup_{t\le q}\left|y_t^{(i)}-y_t\right|=0\quad\text{and}\quad\lim_{i\to\infty}\limsup_{n\to\infty}\sup_{t\le q}\left|y_t^{n,(i)}-y_t^n\right|=0.$$

Let $x^{n,(i)}$ be a solution of an equation with a penalization term of the form

$$x_t^{n,(i)}=y_t^{n,(i)}-n\int_0^t\left(x_s^{n,(i)}-\Pi\left(x_s^{n,(i)}\right)\right)\mathrm{d}s=y_t^{n,(i)}+k_t^{n,(i)},\qquad t\in\mathbb{R}^+.$$

Fix $t\in\mathbb{R}^+$ and consider the decomposition $x_t^n-x_t=x_t^n-x_t^{n,(i)}+x_t^{n,(i)}-x_t^{(i)}+x_t^{(i)}-x_t$, where $x^{(i)}$ denotes a solution of the Skorokhod problem associated with $y^{(i)}$. Due to [15], Lemma 2.2(i),

$$\left|x_t^n-x_t^{n,(i)}\right|^2\le\left|y_t^n-y_t^{n,(i)}\right|^2+4\sup_{s\le t}\left|y_s^n-y_s^{n,(i)}\right|\left(\left|k^n\right|_t+\left|k^{n,(i)}\right|_t\right),$$

where variations $|k^n|_t$, $|k^{n,(i)}|_t$ are bounded uniformly by Lemma 2.2(ii). Therefore,

$$\lim_{i\to\infty}\limsup_{n\to\infty}\left|x_t^n - x_t^{n,(i)}\right| = 0.$$

Moreover, if $|\Delta y_t| = 0$, then $\lim_{i\to\infty}|\Delta y_t^{(i)}| = 0$. Because, by [15], Lemma 3.3, $\limsup_{n\to\infty}|x_t^{n,(i)} - x_t^{(i)}| \le |\Delta y_t^{(i)}|$, $i\in\mathbb{N}$, it follows that $\lim_{i\to\infty}\limsup_{n\to\infty}|x_t^{n,(i)} - x_t^{(i)}| \le \lim_{i\to\infty}|\Delta y_t^{(i)}| = 0$. On the other hand, by [29], Lemma 2.2, $\sup_{t\le q}|x_t^{(i)} - x_t| \longrightarrow 0$, $q\in\mathbb{R}^+$, and (ii) follows.

(iii) The sequence $\{y^n\}$ is relatively compact in J_1, and consequently it is relatively S compact. Because by part (i), $\sup_{n\in\mathbb{N}}|k^n|_q < +\infty$, $q\in\mathbb{R}^+$, the sequence $\{k^n\}$ is also relatively S compact, and thus $\{x^n\}$ is relatively S compact as well. In view of Corollary A.2 (Appendix), this proves (iii). □

Recall that if $y^n \longrightarrow y$ in $\mathbb{D}(\mathbb{R}^+,\mathbb{R}^d)$, then for every $t\in\mathbb{R}^+$ there exists a sequence $t_n \longrightarrow t$ such that

$$y_{t_n}^n \longrightarrow y_t, \qquad y_{t_n-}^n \longrightarrow y_{t-} \quad\text{and}\quad \Delta y_{t_n}^n \longrightarrow \Delta y_t. \tag{2.4}$$

Moreover, for arbitrary sequences $\{t_n'\}$, $\{t_n''\}$ such that $t_n' < t_n \le t_n''$, $n\in\mathbb{N}$, and $\lim_{n\to\infty} t_n' = \lim_{n\to\infty} t_n'' = t$, we have

$$y_{t_n'}^n \longrightarrow y_{t-} \quad\text{and}\quad y_{t_n''}^n \longrightarrow y_t \tag{2.5}$$

(see, e.g., [8], Chapter VI, Proposition 2.1).

Corollary 2.4. *Under the assumptions of Theorem 2.3,*

(i) *for every $t\in\mathbb{R}^+$, if $t_n \longrightarrow t$ is a sequence satisfying (2.4), then*

$$x_{t_n}^n \longrightarrow x_{t-} + \Delta y_t$$

and for arbitrary sequences $\{t_n'\}$, $\{t_n''\}$ such that $t_n' < t_n < t_n''$, $n\in\mathbb{N}$ and $\lim_{n\to\infty} t_n' = \lim_{n\to\infty} t_n'' = t$, we have

$$x_{t_n'}^n \longrightarrow x_{t-} \quad\textit{and}\quad x_{t_n''}^n \longrightarrow x_t,$$

(ii) $(\Pi(x^n), y^n) \longrightarrow (x, y)$ *in* $\mathbb{D}(\mathbb{R}^+,\mathbb{R}^{2d})$,

(iii) *moreover, if y is continuous, then*

$$\sup_{t\le q}\left|x_t^n - x_t\right| \longrightarrow 0, \qquad q\in\mathbb{R}^+.$$

Proof. (i) If y, y^n are step functions, then the result follows from [15], Lemma 3.3. In the general case, it is sufficient to use (2.4), (2.5) and repeat the approximation procedure from the proof of Theorem 2.3.

(ii) By Theorem 2.3(ii), $\Pi(x_t^n) \longrightarrow x_t$, provided that $\Delta y_t = 0$. Therefore, it suffices to prove that $\{(\Pi(x^n), y^n)\}$ is relatively compact in $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^{2d})$. Because $\sup_{n\in\mathbb{N}} \sup_{t\le q} |\Pi(x_t^n)| < +\infty$, $\sup_{n\in\mathbb{N}} \sup_{t\le q} |y_t^n| < +\infty$, $q \in \mathbb{R}^+$, it is sufficient to show that

$$\lim_{\delta\downarrow 0} \limsup_{n\to\infty} \omega_{(\Pi(x^n), y^n)}\big([0, \delta]\big) = 0 \tag{2.6}$$

and

$$\lim_{\delta\downarrow 0} \limsup_{n\to\infty} \omega''_{(\Pi(x^n), y^n)}(\delta, q) = 0, \qquad q \in \mathbb{R}^+. \tag{2.7}$$

To prove (2.6), first observe that $x_0^n \longrightarrow x_0$ and $y_0^n \longrightarrow 0$, which implies that (2.6) is equivalent to the following:

- for every sequence $\{s_0^n\}$ such that $0 \le s_0^n \longrightarrow 0$,

$$\Pi\big(x^n_{s_0^n}\big) \longrightarrow x_0, \qquad y^n_{s_0^n} \longrightarrow y_0. \tag{2.8}$$

Next, note that (2.8) is implied by (i) (it is sufficient to put $t = 0$ and observe that in this case, $t_n = 0$).

Similarly, (2.7) is equivalent to the condition

- for every $t \le q$ and every sequence $\{s_i^n\}$, $i = 1, 2, 3$ such that $s_1^n \le s_2^n \le s_3^n$, $n \in \mathbb{N}$, if $\lim_{n\to\infty} s_i^n = t$, $\Pi(x^n_{s_i^n}) \to a_i$, $y^n_{s_i^n} \to b_i$, $i = 1, 2, 3$, then

$$a_1 = a_2 \quad \text{and} \quad b_1 = b_2 \quad \text{or} \quad a_2 = a_3 \quad \text{and} \quad b_2 = b_3. \tag{2.9}$$

Because Π is continuous and $x_t = \Pi(x_{t-} + \Delta y_t)$, it follows from (i) that for arbitrary sequences $\{t'_n\}$, $\{t''_n\}$ such that $t'_n < t_n \le t''_n$ and $\lim_{n\to\infty} t'_n = \lim_{n\to\infty} t''_n = t$, we have

$$\Pi\big(x^n_{t'_n}\big) \longrightarrow x_{t-} \quad \text{and} \quad \Pi\big(x^n_{t''_n}\big) \longrightarrow x_t. \tag{2.10}$$

Combining (2.5) with (2.10), we see that there are only four possibilities:

$$\begin{aligned}
&a_1 = a_2 = a_3 = x_{t-} \quad \text{and} \quad b_1 = b_2 = b_3 = y_{t-},\\
&a_1 = a_2 = x_{t-}, \qquad a_3 = x_t \quad \text{and} \quad b_1 = b_2 = y_{t-}, \qquad b_3 = y_t,\\
&a_1 = x_{t-}, \qquad a_2 = a_3 = x_t \quad \text{and} \quad b_1 = y_{t-}, \qquad b_2 = b_3 = y_t,\\
&a_1 = a_2 = a_3 = x_t \quad \text{and} \quad b_1 = b_2 = b_3 = y_t.
\end{aligned}$$

Thus, in each case (2.9) is satisfied.

(iii) In this case, x is continuous as well. Moreover, (2.4) is satisfied for every $t \in \mathbb{R}^+$ and every sequence $t_n \longrightarrow t$. Consequently, by part (i), for every $t \in \mathbb{R}^+$ and every sequence $t_n \longrightarrow t$,

$$x^n_{t_n} \longrightarrow x_{t-} + \Delta y_t = x_t,$$

which is equivalent to (iii). □

Corollary 2.5. *Let $\{(y^n, z^n)\}$ be relatively compact in $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^{2d})$, $y_0^n \in \bar{D}$, $n \in \mathbb{N}$. If x^n denotes the solution of* (1.4), *$n \in \mathbb{N}$, then, for every $q \in \mathbb{R}^+$,*

$$\lim_{\delta \downarrow 0} \limsup_{n\to\infty} \bar{\omega}''_{(x^n, z^n)}(\delta, q) = 0.$$

Proof. Without loss of generality, we may and will assume that $(y^n, z^n) \longrightarrow (y, z)$ in $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^{2d})$. Then, for every $t \in \mathbb{R}^+$, there is a sequence $t_n \longrightarrow t$ such that

$$y^n_{t'_n} \longrightarrow y_{t-}, \qquad z^n_{t'_n} \longrightarrow z_{t-} \quad \text{and} \quad y^n_{t''_n} \longrightarrow y_t, \qquad z^n_{t''_n} \longrightarrow z_t \tag{2.11}$$

for arbitrary sequences $t'_n < t_n \le t''_n$ such that $\lim_{n\to\infty} t'_n = \lim_{n\to\infty} t''_n = t$. Because $\sup_{n\in\mathbb{N}} \sup_{t\le q} |x_t^n| < +\infty$ and $\sup_{n\in\mathbb{N}} \sup_{t\le q} |y_t^n| < +\infty$, $q \in \mathbb{R}^+$, the proof is completed by showing that

- for every $t \le q$ and every sequence $\{s_i^n\}$ such that $\lim_{n\to\infty} s_i^n = t$, $i = 1, 2, 3$ and $s_1^n \le s_2^n \le s_3^n$, $n \in \mathbb{N}$, if there exists limits $x^n_{s_1^n} \to a_1$, $x^n_{s_2^n} \to a_2$, and $z^n_{s_2^n} \to b_1$, $z^n_{s_3^n} \to b_2$, then

$$a_1 = a_2 \quad \text{or} \quad b_1 = b_2. \tag{2.12}$$

From (2.5), (2.11), and Theorem 2.4(i), we conclude that there are only the following cases:

$$\begin{aligned}
&a_1 = a_2 = x_{t-} \quad \text{and} \quad b_1 = b_2 = z_{t-} \quad \text{or} \quad b_1 = z_{t-} \quad \text{and} \quad b_2 = z_t,\\
&a_1 = x_{t-}, \qquad a_2 = x_{t-} + \Delta y_t \quad \text{and} \quad b_1 = b_2 = z_t,\\
&a_1 = x_{t-}, \qquad a_2 = x_t \quad \text{and} \quad b_1 = b_2 = z_t,\\
&a_1 = a_2 = x_{t-} + \Delta y_t \quad \text{and} \quad b_1 = b_2 = z_t,\\
&a_1 = x_{t-} + \Delta y_t, \qquad a_2 = x_t \quad \text{and} \quad b_1 = b_2 = z_t,\\
&a_1 = a_2 = x_t \quad \text{and} \quad b_1 = b_2 = z_t;
\end{aligned}$$

that is, (2.12) is satisfied. □

3. Applications to stochastic processes

Let Y^n be an $(\mathcal{F}_t^n)$ adapted process with $Y_0^n \in \bar{D}$ and let X^n be a solution of the equation (1.5). We will consider processes Y^n admitting the decomposition

$$Y_t^n = H_t^n + M_t^n + V_t^n, \qquad t \in \mathbb{R}^+, \tag{3.1}$$

where H^n is an $(\mathcal{F}_t^n)$ adapted process, M^n is an $(\mathcal{F}_t^n)$ adapted local martingale with $M_0^n = 0$, and V^n is an $(\mathcal{F}_t^n)$ adapted processes of bounded variation with $V_0^n = 0$.

Remark 3.1 ([15]). Let $\widehat{Y}^n$ be another $(\mathcal{F}_t^n)$ adapted processes of the form $\widehat{Y}^n = H^n + \widehat{M}^n + \widehat{V}^n$, where $\widehat{M}^n$ is a local martingale, $\widehat{V}^n$ is a process with locally bounded variation, and $\widehat{M}_0^n = \widehat{V}_0^n = 0$. Assume that $\widehat{Y}_0^n \in \bar{D}$, and let $\widehat{X}^n$ be a solution of the equation

$$\widehat{X}_t^n = \widehat{Y}_t^n - n\int_0^t \big(\widehat{X}_s^n - \Pi\big(\widehat{X}_s^n\big)\big)\,\mathrm{d}s, \qquad t \in \mathbb{R}^+. \tag{3.2}$$

For every $p \in \mathbb{N}$, there exists $C(p)$ such that

$$E\sup_{t\le\tau}\big|X_t^n - \widehat{X}_t^n\big|^{2p} \le C(p)E\big(\big[M^n - \widehat{M}^n\big]_\tau^p + \big|V^n - \widehat{V}^n\big|_\tau^{2p}\big) \tag{3.3}$$

and

$$E\sup_{t<\tau}\big|X_t^n - \widehat{X}_t^n\big|^{2p} \le C(p)E\big(\big[M^n - \widehat{M}^n\big]_{\tau-}^p + \big|V^n - \widehat{V}^n\big|_{\tau-}^{2p} + \big\langle M^n - \widehat{M}^n\big\rangle_{\tau-}^p\big) \tag{3.4}$$

for every stopping time τ.

Theorem 3.2. *Let Y^n be a process of the form* (3.1), *and let X^n denote the solution of* (1.5). *For every $\delta, q, a \in D$, there exist constants $C_1, C_2 > 0$ such that for every $\eta > 0$,*

$$\begin{aligned} P\Big(\sup_{t\le q}\big|X_t^n - a\big| \ge \eta\Big) &\le P\big(\omega'_{H^n}(\delta,q) \ge d_a/2\big) + P\Big(\sup_{t\le q}\big|H_t^n - a\big| \ge C_1\eta\Big) \\ &\quad + C_2\eta^{-2}E\big(\big[M^n\big]_q + \big|V^n\big|_q^2\big) \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} P\bigg(n\int_0^q \big|X_s^n - \Pi\big(X_s^n\big)\big|\,\mathrm{d}s \ge \eta^2\bigg) &\le P\big(\omega'_{H^n}(\delta,q) \ge d_a/2\big) + 7P\Big(\sup_{t\le q}\big|H_t^n - a\big| \ge C_1\eta\Big) \\ &\quad + C_2\eta^{-2}E\big(\big[M^n\big]_q + \big|V^n\big|_q^2\big). \end{aligned} \tag{3.6}$$

Proof. Let $C' = 2\sqrt{7}([T/\delta] + 1)$ be a constant from Lemma 2.2(i), and let $\widehat{X}^n$ be a solution of equation with penalization term (3.2) associated with $\widehat{Y}^n = H^n$. Then

$$\begin{aligned} P\Big(\sup_{t\le q}\big|\widehat{X}_t^n - a\big| \ge \eta/2\Big) &= P\Big(\sup_{t\le q}\big|\widehat{X}_t^n - a\big| \ge \eta/2, \omega'_{H^n}(\delta,q) < d_a/2\Big) \\ &\quad + P\big(\omega'_{H^n}(\delta,q) \ge d_a/2\big) \\ &\le P\Big(\sup_{t\le q}\big|H_t^n - a\big| \ge C'^{-1}\eta/2\Big) + P\big(\omega'_{H^n}(\delta,q) \ge d_a/2\big). \end{aligned}$$

By the foregoing and by (3.3),

$$\begin{aligned} P\Big(\sup_{t\le q}\big|X_t^n - a\big| \ge \eta\Big) &\le P\Big(\sup_{t\le q}\big|\widehat{X}_t^n - a\big| \ge \eta/2\Big) + P\Big(\sup_{t\le q}\big|X_t^n - \widehat{X}_t^n\big| \ge \eta/2\Big) \\ &\le P\Big(\sup_{t\le q}\big|H_t^n - a\big| \ge \eta\big(2C'\big)^{-1}\Big) + P\big(\omega'_{H^n}(\delta,q) \ge d_a/2\big) \\ &\quad + 4C(1)\eta^{-2}E\big(\big[M^n\big]_q + \big|V^n\big|_q^2\big), \end{aligned}$$

which completes the proof of (3.5) with $C_1 = 2C'$ and $C_2 = 4C(1)$.

Now, for simplicity of notation, set $K_t^n = -n\int_0^t (X_s^n - \Pi(X_s^n))\,\mathrm{d}s$. Our proof of (3.6) starts with the observation that the estimates similar to (3.5) is true for $\sup_{t\le q}|K_t^n|$ as well. Indeed, there exist constants $C_1, C_2 > 0$ such that for every $\eta > 0$,

$$\begin{aligned} P\Bigl(\sup_{t\le q}\bigl|K_t^n\bigr| \ge \eta\Bigr) &\le P\Bigl(\sup_{t\le q}\bigl|X_t^n - a\bigr| \ge \eta/3\Bigr) + P\Bigl(\sup_{t\le q}\bigl|H_t^n - a\bigr| \ge \eta/3\Bigr) \\ &\quad + P\Bigl(\sup_{t\le q}\bigl|M_t^n + V_t^n\bigr| \ge \eta/3\Bigr) \\ &\le 2P\Bigl(\sup_{t\le q}\bigl|H_t^n - a\bigr| > C_1\eta\Bigr) + P\bigl(\omega'_{H^n}(\delta, q) \ge d_a/2\bigr) \\ &\quad + C_2\eta^{-2}E\bigl(\bigl[M^n\bigr]_q + \bigl|V^n\bigr|_q^2\bigr). \end{aligned} \tag{3.7}$$

On the other hand, by Remark 2.1(ii),

$$\begin{aligned} \bigl|K^n\bigr|_q &\le -\frac{1}{d_a}\int_0^q \bigl\langle X_s^n - a, \mathrm{d}K_s^n\bigr\rangle \\ &= -\frac{1}{d_a}\biggl(\int_0^q \bigl\langle H_s^n - a, \mathrm{d}K_s^n\bigr\rangle + \int_0^q \bigl\langle Z_s^n, \mathrm{d}K_s^n\bigr\rangle + \int_0^q \bigl\langle K_s^n, \mathrm{d}K_s^n\bigr\rangle\biggr), \end{aligned}$$

where $Z^n = M^n + V^n$. Because K^n has continuous trajectories, it follows by the integration by parts formula that $\int_0^q \langle Z_s^n, \mathrm{d}K_s^n\rangle = \langle Z_q^n, K_q^n\rangle - \int_0^q \langle K_s^n, \mathrm{d}Z_s^n\rangle$, and $\int_0^q \langle K_s^n, \mathrm{d}K_s^n\rangle = \frac{1}{2}|K_q|^2 \ge 0$. Therefore,

$$\bigl|K^n\bigr|_q \le -\frac{1}{d_a}\biggl(\int_0^q \bigl\langle H_s^n - a, \mathrm{d}K_s^n\bigr\rangle + \bigl\langle Z_q^n, K_q^n\bigr\rangle - \int_0^q \bigl\langle K_s^n, \mathrm{d}Z_s^n\bigr\rangle\biggr). \tag{3.8}$$

Fix $\omega \in \{\omega'_{H^n}(\delta, q) < d_a/2\}$. There exists a subdivision (s_k) of $[0, q]$ such that $\delta \le s_k - s_{k-1}$, $k = 1, 2, \ldots, r-1$, where $r = [q/\delta] + 1$ and $\omega_{H^n(\omega)}([s_{k-1}, s_k)) < d_a/2$. Set $H_t^{n*}(\omega) = H_{s_{k-1}}^n(\omega)$, for $t \in [s_{k-1}, s_k)$. Then $\sup_{t<q}|H_t^n(\omega) - H_t^{n*}(\omega)| < d_a/2$ and, consequently,

$$\begin{aligned} \biggl|\int_0^q \bigl\langle H_s^n(\omega) - a, \mathrm{d}K_s^n(\omega)\bigr\rangle\biggr| &\le \biggl|\int_0^q \bigl\langle H_s^n(\omega) - H^{n*}(\omega), \mathrm{d}K_s^n(\omega)\bigr\rangle\biggr| \\ &\quad + \biggl|\int_0^q \bigl\langle H_s^{n*}(\omega) - a, \mathrm{d}K_s^n(\omega)\bigr\rangle\biggr| \\ &\le \frac{d_a}{2}\bigl|K^n\bigr|_q(\omega) + 2r\sup_{t\le q}\bigl|H_t^n(\omega) - a\bigr|\bigl|K_t^n(\omega)\bigr|. \end{aligned}$$

Combining this inequality with (3.8), we see that on the set $\{\omega'_{H^n}(\delta, q) < d_a/2\}$, we have

$$\bigl|K^n\bigr|_q \le 2d_a^{-1}\biggl(2r\sup_{t\le q}\bigl|H_t^n - a\bigr|\bigl|K_t^n\bigr| + \sup_{t\le q}\bigl|Z_t^n\bigr|\bigl|K_t^n\bigr| + \biggl|\int_0^q \bigl\langle K_s^n, \mathrm{d}Z_s^n\bigr\rangle\biggr|\biggr).$$

Thus, there is a constant $C > 0$ such that

$$P(|K^n|_q \geq \eta^2) \leq P(|K^n|_q \geq \eta^2, \omega'_{H^n}(\delta, q) < d_a/2) + P(\omega'_{H^n}(\delta, q) \geq d_a/2)$$
$$\leq P\Big(\sup_{t \leq q} |H^n_t - a||K^n_t| \geq C\eta^2\Big) + P\Big(\sup_{t \leq q} |Z^n_t||K^n_t| \geq C\eta^2\Big) \tag{3.9}$$
$$+ P\bigg(\bigg|\int_0^q \langle K^n_s, \mathrm{d}Z^n_s\rangle\bigg| \geq C\eta^2\bigg) + P(\omega'_{H^n}(\delta, q) \geq d_a/2).$$

Clearly,

$$P\Big(\sup_{t \leq q} |H^n_t - a||K^n_t| \geq C\eta^2\Big) \leq P\Big(\sup_{t \leq q} |K^n_t| \geq \eta\Big) + P\Big(\sup_{t \leq q} |H^n_t - a| \geq C\eta\Big)$$

and

$$P\Big(\sup_{t \leq q} |Z^n_t||K^n_t| \geq C\eta^2\Big) \leq P\Big(\sup_{t \leq q} |K^n_t| \geq \eta\Big) + P\Big(\sup_{t \leq q} |Z^n_t| \geq C\eta\Big)$$
$$\leq P\Big(\sup_{t \leq q} |K^n_t| \geq \eta\Big) + C_2\eta^{-2} E([M^n]_q + |V^n|^2_q).$$

Moreover, if we set $\tau^n = \inf\{t, |K^n_t| \geq \eta\} \wedge q$, then, obviously, $\sup_{t \leq \tau^n} |K^n_t| \leq \eta$ and

$$P\bigg(\bigg|\int_0^q \langle K^n_s, \mathrm{d}Z^n_s\rangle\bigg| \geq C\eta^2\bigg) \leq P(q > \tau^n) + P\bigg(\bigg|\int_0^{\tau^n} \langle K^n_s, \mathrm{d}Z^n_s\rangle\bigg| \geq C\eta^2\bigg)$$
$$\leq P\Big(\sup_{t \leq q} |K^n_t| \geq \eta\Big) + C_2\eta^{-4} E\bigg(\int_0^{\tau^n} \langle K^n_s, \mathrm{d}Z^n_s\rangle\bigg)^2$$
$$\leq P\Big(\sup_{t \leq q} |K^n_t| \geq \eta\Big) + C_2\eta^{-2} E([M^n]_q + |V^n|^2_q).$$

Combining the last three inequalities with (3.7) and (3.9) yields (3.6). □

Now, for $n \in \mathbb{N}$, let Z^n be a semimartingale adapted to some filtration $(\mathcal{F}^n_t)$. We assume that $\{Z^n\}$ satisfies the following condition (UT) introduced by Stricker [28]:

(UT) For every $q \in \mathbb{R}^+$, the family of random variables

$$\bigg\{\int_{[0,q]} U^n_s \,\mathrm{d}Z^n_s; n \in \mathbb{N}, U^n \in \mathbf{U}^n_q\bigg\}$$

is bounded in probability. Here $\mathbf{U}^n_q$ is the class of discrete predictable processes of the form $U^n_s = U^n_0 + \sum_{i=0}^k U^n_i \mathbf{1}_{\{t_i < s \leq t_{i+1}\}}$, where $0 = t_0 < t_1 < \cdots < t_k = q$ and U^n_i is $\mathcal{F}^n_{t_i}$ measurable, $|U^n_i| \leq 1$ for $i \in \{0, \ldots, k\}$, $n, k \in \mathbb{N}$.

The condition (UT) proved to be very useful in the theory of limit theorems for stochastic integrals and for solutions of SDEs (see, e.g., [11,13,19,24,25]).

Corollary 3.3. *Let $\{Y^n\}$ be a sequence of $(\mathcal{F}_t^n)$ adapted processes, and let $\{X^n\}$ be a sequence of solutions of equations with penalization terms* (1.5). *Assume that every Y^n is of the form $Y^n = H^n + Z^n$ with $H_0^n \in \bar{D}$, $Z_0^n = 0$, where $\{H^n\}$ is tight in the $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$ sequence of $(\mathcal{F}_t^n)$ adapted processes and $\{Z^n\}$ is a sequence of $(\mathcal{F}_t^n)$-adapted semimartingales satisfying* (UT). *Then, for every $q \in \mathbb{R}^+$, the sequence $\{n \int_0^q |X_s^n - \Pi(X_s^n)|\,\mathrm{d}s\}$ is bounded in probability.*

Proof. Define $\tau_k^n = \inf\{t; |H_t^n| \vee |Z_t^n| > k\}$, $k, n \in \mathbb{N}$. Because (UT) implies that $\{\sup_{t\le q} |Z_t^n|\}$ is bounded in probability and $\{\sup_{t\le q} |H_t^n|\}$ is bounded in probability by the tightness of $\{H^n\}$ in $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$, we have

$$\lim_{k\to\infty} \limsup_{n\to\infty} P\left(\tau_k^n \le q\right) = 0, \qquad q \in \mathbb{R}^+. \tag{3.10}$$

Furthermore, by simple calculations for every $\delta > 0$ and $q \in \mathbb{R}^+$ $\omega'_{H^{n,\tau_k^n-}}(\delta, q) \le \omega'_{H^n(\delta,q)}$ and $\sup_{t\le q} |H_t^{n,\tau_k^n-}| \le \sup_{t\le q} |H_t^n|$, and thus the sequence $\{H^{n,\tau_k^n-}\}$ also is tight in $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$. On the other hand, from the definition of (UT), we see that $\{Z^{n,\tau_k^n}\}$ satisfies (UT) as well. Moreover, because $Z_\cdot^{n,\tau_k^n-} = Z_\cdot^{n,\tau_k^n} - \Delta Z_{\tau_k^n}^n \mathbf{1}_{\{\cdot\ge\tau_k^n\}}$ and $|\Delta Z_\cdot^n| \le 2|Z_\cdot^n|$, it follows from the definition of (UT) that $\{Z^{n,\tau_k^n-}\}$ satisfies (UT) as well. Therefore, in view of (3.10), without loss of generality, we can and will assume that $H^n = H^{n,\tau_k^n-}$ and $Z^n = Z^{n,\tau_k^n-}$ for some $k \in \mathbb{R}^+$. Then Z^n admits the decomposition $Z^n = M^n + V^n$ with $|\Delta M^n|, |\Delta V^n| \le 4k$, where $\{[M^n]_q\}$, $\{|V^n|_q\}$ are bounded in probability for each $q \in \mathbb{R}^+$ (see, e.g., [19]). Set $\gamma_b^n = \inf\{t; |H^n|_t \vee [M^n]_t \vee |V^n|_t > b\}$ for $n \in \mathbb{N}$, $b \in \mathbb{R}^+$. Then

$$\lim_{b\to\infty} \limsup_{n\to\infty} P\left(\gamma_b^n \le q\right) = 0, \qquad q \in \mathbb{R}^+, \tag{3.11}$$

so as before, we can assume that $H^n = H^{n,\gamma_b^n}$, $M^n = M^{n,\gamma_b^n}$, $V^n = V^{n,\gamma_b^n}$ for some $b \in \mathbb{R}^+$, and thus that $[M^n]_\infty \le b + 16k^2$, $|V^n|_\infty^2 \le b^2 + 16k^2$.

To complete the proof, it suffices to use Theorem 3.2. Fix $\varepsilon > 0$, $q \in \mathbb{R}^+$, and $a \in D$, and let $\delta > 0$, $n_0 \in \mathbb{N}$ be such that $P(\omega'_{H^n}(\delta, q) \ge d_a/2) \le \varepsilon/2$ for every $n \ge n_0$. If we put $\eta = \max((k + |a| + 1)/C_1, \sqrt{2C_2(b^2 + b + 32k^2)/\varepsilon})$ in (3.6) then

$$P\left(n \int_0^q \left|X_s^n - \Pi\left(X_s^n\right)\right| \mathrm{d}s \ge \eta^2\right) \le \varepsilon$$

for any $n \ge n_0$, which completes the proof. □

Corollary 3.4. *For $n \in \mathbb{N}$, let Y^n and $\widehat{Y}^n$ be processes adapted to filtrations $(\mathcal{F}_t^n)$ and $(\widehat{\mathcal{F}}_t^n)$, respectively, and let X^n be a solution of* (1.5) *and $\widehat{X}^n$ be a solution of* (3.2). *If $\{Y^n = H^n + Z^n\}$, $\{\widehat{Y}^n = \widehat{H}^n + \widehat{Z}^n\}$ with $H_0^n, \widehat{H}_0^n \in \bar{D}$ and $Z_0^n = \widehat{Z}_0^n = 0$, and $\{H^n\}$, $\{\widehat{H}^n\}$ are tight in $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$, $\{Z^n\}$, $\{\widehat{Z}^n\}$ satisfy* (UT) *and*

$$\sup_{t\le q} \left|Y_t^n - \widehat{Y}_t^n\right| \xrightarrow[\mathcal{P}]{} 0, \qquad q \in \mathbb{R}^+$$

then

$$\sup_{t\le q}\left|X^n_t-\widehat{X}^n_t\right|\underset{\mathcal{P}}{\longrightarrow}0,\qquad q\in\mathbb{R}^+.$$

Proof. The proof follows immediately from [15], Lemma 2.2(i), and Corollary 3.3. □

Corollary 3.5. *Let $\{Y^n\}$ be a given sequence of processes, $Y^n_0\in\bar{D}$, $n\in\mathbb{N}$, and let $\{X^n\}$ be a sequence of solutions of equations with penalization terms* (1.5):

(i) *For any sequence of processes $\{Z^n\}$, if*

$$\{(Y^n,Z^n)\}\qquad \text{is tight in } \mathbb{D}(\mathbb{R}^+,\mathbb{R}^{2d})$$

then for every $\varepsilon>0$, $q\in\mathbb{R}^+$

$$\lim_{\delta\downarrow 0}\limsup_{n\to\infty}P^n\left(\bar{\omega}''_{(X^n,Z^n)}(\delta,q)>\varepsilon\right)=0. \tag{3.12}$$

(ii) *For any sequences of processes $\{Z^n\}$, $\{H^n\}$, if*

$$\left(Y^n,H^n,Z^n\right)\underset{\mathcal{D}}{\longrightarrow}(Y,H,Z)\qquad \text{in } \mathbb{D}(\mathbb{R}^+,\mathbb{R}^{3d})$$

then

$$\left(\Pi(X^n),H^n,Z^n\right)\underset{\mathcal{D}}{\longrightarrow}(X,H,Z)\qquad \text{in } \mathbb{D}(\mathbb{R}^+,\mathbb{R}^{3d})$$

and

$$\left(X^n_{t_1},\dots,X^n_{t_m},H^n,Z^n\right)\underset{\mathcal{D}}{\longrightarrow}(X_{t_1},\dots,X_{t_m},H,Z)\quad \text{in } \mathbb{R}^m\times\mathbb{D}(\mathbb{R}^+,\mathbb{R}^{2d})$$

for any $m\in\mathbb{N}$, any $t_1,\dots,t_m\in\mathbb{R}^+$ such that $P(|\Delta Y_{t_i}|=0)=1$, $i=1,\dots,m$, where $X=Y+K$ is a solution of the Skorokhod problem associated with a process Y.

Proof. In the proof, it suffices to make the observation that (3.12) is equivalent to the fact that $\bar{\omega}''_{(X^n,Z^n)}(\delta_n,)\underset{\mathcal{P}}{\longrightarrow}0$ for every sequence $\{\delta_n\}$ such that $\delta_n\downarrow 0$ and to combine the deterministic results given in Theorem 2.3, Corollary 2.4 and Corollary 2.5 with the Skorokhod representation theorem. □

4. Penalization methods for reflecting SDEs

Let $(\Omega,\mathcal{F},(\mathcal{F}_t),P)$ be a filtered probability space, let H be an $(\mathcal{F}_t)$-adapted process, and let Z be an $(\mathcal{F}_t)$ adapted semimartingale such that $H_0\in\bar{D}$, $Z_0=0$. Now recall that a pair (X,K) of $(\mathcal{F}_t)$-adapted processes is called a solution of the reflecting SDE (1.1) if (X,K) is a solution of the Skorokhod problem associated with Y defined by

$$Y_t=H_t+\int_0^t f(X_{s-})\,\mathrm{d}Z_s,\qquad t\in\mathbb{R}^+.$$

We say that the SDE (1.1) has a weak solution if there exists a probability space $(\widehat{\Omega}, \widehat{\mathcal{F}}, (\widehat{\mathcal{F}}_t), \widehat{P})$ and $(\widehat{\mathcal{F}}_t)$-adapted processes $\widehat{H}$,$\widehat{Z}$ and $(\widehat{X}, \widehat{K})$ such that $\mathcal{L}(\widehat{H}, \widehat{Z}) = \mathcal{L}(H, Z)$ and $(\widehat{X}, \widehat{K})$ is a solution of the Skorokhod problem associated with $\widehat{Y}_t = \widehat{H}_t + \int_0^t f(\widehat{X}_{s-})\,\mathrm{d}\widehat{Z}_s$, $t \in \mathbb{R}^+$. If any two weak solutions $(\widehat{X}, \widehat{K})$, $(\widehat{X}', \widehat{K}')$ of the SDE (1.1), possibly defined on two different probability spaces, are such that $\mathcal{L}(\widehat{X}, \widehat{K}) = \mathcal{L}(\widehat{X}', \widehat{K}')$, we say that the weak uniqueness for the SDE (1.1) holds.

In this section we prove general results on weak and strong approximations of X. We begin with two technical lemma. In the first lemma, which is a simple consequence of Corollary A.7 (Appendix), the sequence $\{X^n\}$ need not consist of solutions of penalized equations, and the process X need not be a solution of (1.1).

Lemma 4.1. *Let $\{X^n\}$ be a sequence of $(\mathcal{F}_t^n)$ adapted processes, and let $\{Z^n\}$ be a sequence of $(\mathcal{F}_t^n)$-adapted semimartingales satisfying* (UT) *such that $Z_0^n = 0$, $n \in \mathbb{N}$, $\{H^n\}$ be a sequence of processes. If $\{X^n\}$ is S-tight and there exist processes X, H, and Z such that*

$$\left(X_{t_1}^n, \dots, X_{t_m}^n, H^n, Z^n\right) \underset{\mathcal{D}}{\longrightarrow} (X_{t_1}, \dots, X_{t_m}, H, Z) \qquad \text{in } \mathbb{R}^{md} \times \mathbb{D}\left(\mathbb{R}^+, \mathbb{R}^{2d}\right)$$

for any $m \in \mathbb{N}$ and any $t_1, t_2, \dots, t_m$ from a dense subset $\mathbb{Q}$ of $\mathbb{R}^+$ and

$$\lim_{\delta\downarrow 0} \limsup_{n\to\infty} P\left(\bar{\omega}''_{(X^n, Z^n)}(\delta, q) > \varepsilon\right) = 0, \qquad \varepsilon > 0, q \in \mathbb{R}^+, \tag{4.1}$$

then for every continuous function $g : \mathbb{R}^d \longrightarrow \mathbb{R}^d \otimes \mathbb{R}^d$

$$\left(X_{t_1}^n, \dots, X_{t_m}^n, H^n, Z^n, \int_0^{\cdot} \left\langle g\left(X_{s-}^n\right), \mathrm{d}Z_s^n\right\rangle\right) \underset{\mathcal{D}}{\longrightarrow} \left(X_{t_1}, \dots, X_{t_m}, H, Z, \int_0^{\cdot} \left\langle g(X_{s-}), \mathrm{d}Z_s\right\rangle\right)$$

in $\mathbb{R}^{md} \times \mathbb{D}(\mathbb{R}^+, \mathbb{R}^3)$ for any $m \in \mathbb{N}$ and $t_1, t_2, \dots, t_m \in \mathbb{Q}$.

Proof. Clearly,

$$\begin{aligned} &\left(g\left(X_{t_1}^n\right), \dots, g\left(X_{t_m}^n\right), X_{t_1}^n, \dots, X_{t_m}^n, H^n, Z^n\right) \\ &\quad \underset{\mathcal{D}}{\longrightarrow} \left(g(X_{t_1}), \dots, g(X_{t_m}), X_{t_1}, \dots, X_{t_m}, H, Z\right) \qquad \text{in } \mathbb{R}^{2md} \times \mathbb{D}\left(\mathbb{R}^+, \mathbb{R}^{2d}\right) \end{aligned} \tag{4.2}$$

for any $m \in \mathbb{N}$ and any $t_1, t_2, \dots, t_m \in \mathbb{Q}$. Because $\{X^n\}$ is S-tight, it follows that $\{g(X^n)\}$ is S-tight. Similarly, (4.1) implies that

$$\lim_{\delta\downarrow 0} \limsup_{n\to\infty} P\left(\bar{\omega}''_{(g(X^n), Z^n)}(\delta, T) > \varepsilon\right) = 0, \qquad \varepsilon > 0. \tag{4.3}$$

Combining (4.2), (4.3) and putting $Y^n = g(X^n)$, $l = m$, $K_i^n = X_{t_i}^n$, $i = 1, \dots, m$, in Corollary A.7 (Appendix) we complete the proof. □

Lemma 4.2. *Let $\{H^n\}$ be a sequence of $(\mathcal{F}_t^n)$ adapted processes, $H_0^n \in \bar{D}$, $n \in \mathbb{N}$, and let $\{Z^n\}$ be a sequence of $(\mathcal{F}_t^n)$-adapted semimartingales satisfying* (UT), *$Z_0^n = 0$, $n \in \mathbb{N}$. Let $\{X^n\}$ be a* *sequence of solutions of the SDE* (1.3). *If* $f:\mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d$ *satisfies* (1.2) *and* $\{H^n\}$ *is tight in* $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$, *then* $\{Y^n = X^n - H^n\}$ *satisfies* (UT).

Proof. First, we show that for every $q \in \mathbb{R}^+$,

$$\Big\{\sup_{t \le q} |X_t^n|\Big\} \text{ is bounded in probability.} \tag{4.4}$$

Let $\widehat{X}^n$ denote the solution of the equation with penalization term (3.2) with $\widehat{Y}^n = H^n$, $n \in \mathbb{N}$. Because $\{H^n\}$ is tight in $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$, it follows by Corollary 3.3 that for every $q \in \mathbb{R}^+$, $\{\sup_{t\le q} |\widehat{X}_t^n|\}$ is bounded in probability. On the other hand, $\{Z^n\}$ satisfies (UT), and thus we may and will assume that $Z_t^n = M_t^n + V_t^n$ and $M_0^n = V_0^n = 0$, where $\{[M^n]_q\}$, $\{|V^n|_q\}$ are bounded in probability and $|\Delta M^n| \le c$ for some $c > 0$. In this case, $\{\langle M^n\rangle_q\}$ is bounded in probability as well. Define $\tau_k^n = \inf\{t; |\widehat{X}_t^n| \vee |V^n|_t \vee [M^n]_t \vee \langle M^n\rangle_t > k\} \wedge k$, $n, k \in \mathbb{N}$. It is clear that

$$\lim_{k\to+\infty} \limsup_{n\to+\infty} P\big(\tau_k^n \le q\big) = 0, \qquad q \in \mathbb{R}^+. \tag{4.5}$$

By (3.4) with $p = 1$ and by (1.2) for every stopping time σ^n,

$$\begin{aligned}
&E \sup_{t<\sigma^n\wedge\tau_k^n} \big|X_t^n - \widehat{X}_t^n\big|^2 \\
&\quad\le C(1)\bigg\{E\int_0^{(\sigma^n\wedge\tau_k^n)-} \big\|f\big(X_{s-}^n\big)\big\|^2 \,\mathrm{d}\big[M^n\big]_s + E\int_0^{(\sigma^n\wedge\tau_k^n)-} \big\|f\big(X_{s-}^n\big)\big\|^2 \,\mathrm{d}\big\langle M^n\big\rangle_s \\
&\qquad + kE\int_0^{(\sigma^n\wedge\tau_k^n)-} \big\|f\big(X_{s-}^n\big)\big\|^2 \,\mathrm{d}\big|V^n\big|_s\bigg\} \\
&\quad\le C(k,L)\bigg\{1 + E\int_0^{(\sigma^n\wedge\tau_k^n)-} \sup_{u\le s}\big|X_{u-}^n - \widehat{X}_{u-}^n\big|^2 \,\mathrm{d}\big(\big|V^n\big| + \big[M^n\big] + \big\langle M^n\big\rangle\big)_s\bigg\}.
\end{aligned}$$

Therefore, for every stopping time σ^n,

$$\begin{aligned}
&E \sup_{t<\sigma^n} \big|X_t^{n,\tau_k^n-} - \widehat{X}_t^{n,\tau_k^n-}\big|^2 \\
&\quad\le C(k,L)\bigg\{1 + E\int_0^{\sigma^n-} \sup_{u\le s}\big|X_{u-}^{n,\tau_k^n-} - \widehat{X}_{u-}^{n,\tau_k^n-}\big|^2 \,\mathrm{d}\big(\big|V^{n,\tau_k^n-}\big| + \big[M^{n,\tau_k^n-}\big] + \big\langle M^{n,\tau_k^n-}\big\rangle\big)_s\bigg\}.
\end{aligned}$$

Thus, by Gronwall's lemma (see, e.g., [24], Lemma 3),

$$E \sup_{t<\tau_k^n} \big|X_t^n - \widehat{X}_t^n\big|^2 \le C(k,L)\exp\big\{3kC(k,L)\big\},$$

and using (4.5) gives (4.4). Combining (4.4) with (1.2) shows that $\{\sup_{t\le q} \|f(X_{t-}^n)\|\}$ is also bounded in probability, $q \in \mathbb{R}^+$. Thus, the sequence of stochastic integrals $\{\int_0^\cdot \langle f(X_{s-}^n), \mathrm{d}Z_s^n\rangle\}$ satisfies (UT). Because of Corollary 3.3, the proof is complete. □

We can now formulate our main theorem.

Theorem 4.3. *Let $\{H^n\}$ be a sequence of $(\mathcal{F}_t^n)$ adapted processes, $H_0^n \in \bar{D}$, $n \in \mathbb{N}$, and let $\{Z^n\}$ be a sequence of $(\mathcal{F}_t^n)$ adapted semimartingales satisfying* (UT), $Z_0^n = 0$, $n \in \mathbb{N}$. *Let $\{X^n\}$ be a sequence of solutions of the SDE* (1.3). *If $f : \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d$ is continuous, satisfies* (1.2), *and $(H^n, Z^n) \underset{\mathcal{D}}{\longrightarrow} (H, Z)$ in $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^{2d})$, then, for every continuous function $g : \mathbb{R}^d \longrightarrow \mathbb{R}^d \otimes \mathbb{R}^d$,*

(i) $\{(X_{t_1}^n, \ldots, X_{t_m}^n, H^n, Z^n, \int_0^\cdot \langle g(X_{s-}^n), \mathrm{d}Z_s^n \rangle)\}$ *is tight in $\mathbb{R}^{md} \times \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{3d})$ and its every limit point has the form*

$$\left(X_{t_1}, \ldots, X_{t_m}, H, Z, \int_0^\cdot \langle g(X_{s-}), \mathrm{d}Z_s \rangle\right)$$

for any $m \in \mathbb{N}$ and any $t_1, \ldots, t_m \in \mathbb{R}^+$ such that $P(|\Delta Z_{t_i}| = 0) = P(|\Delta H_{t_i}| = 0) = 1$, $i = 1, \ldots, m$, where X is weak solution of the SDE (1.1),

(ii) *if* (1.1) *has a unique weak solution X, then*

$$\left(X_{t_1}^n, \ldots, X_{t_m}^n, H^n, Z^n, \int_0^\cdot \langle g(X_{s-}^n), \mathrm{d}Z_s^n \rangle\right) \underset{\mathcal{D}}{\longrightarrow} \left(X_{t_1}, \ldots, X_{t_m}, H, Z, \int_0^\cdot \langle g(X_{s-}), \mathrm{d}Z_s \rangle\right)$$

in $\mathbb{R}^{md} \times \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{3d})$, for any $m \in \mathbb{N}$ and any $t_1, \ldots, t_m \in \mathbb{R}^+$ such that $P(|\Delta Z_{t_i}| = 0) = P(|\Delta H_{t_i}| = 0) = 1$, $i = 1, \ldots, m$,

(iii) *if X has a unique weak solution, then $X^n \underset{\mathcal{D}}{\overset{*}{\longrightarrow}} X$.*

Proof. (i) First, note that by Lemma 4.2, X^n has the form

$$X_t^n = H_t^n + W_t^n, \qquad t \in \mathbb{R}^+, \tag{4.6}$$

where $\{W^n\}$ is a sequence of semimartingales satisfying (UT). We set $\gamma_0^i = 0$, $\gamma_{k+1}^i = \min(\gamma_k^i + \delta_k^i, \inf\{t > \gamma_k^i; |\Delta H_t| > \delta^i\})$ and $\gamma_0^{ni} = 0$, $\gamma_{k+1}^{ni} = \min(\gamma_k^{ni} + \delta_k^i, \inf\{t > \gamma_k^{ni}; |\Delta H_t^n| > \delta^i\})$, where $\{\delta^i\}$, $\{\{\delta_k^i\}\}$ are families of positive constants such that $\delta^i \downarrow 0$, $\delta^i/2 \le \delta_k^i \le \delta^i$ $P(|\Delta H_t| = \delta^i, t \in \mathbb{R}^+) = 0$, $P(|\Delta H_{\gamma_k^i + \delta_k^i}| = 0) = 1$. For every $i \in \mathbb{N}$, define a new sequence, $\{H^{ni}\}$, of processes by putting $H_t^{ni} = H_{\gamma_k^{ni}}^n$, $t \in [\gamma_k^{ni}, \gamma_{k+1}^{ni})$, $k \in \mathbb{N} \cup \{0\}$, $n \in \mathbb{N}$. Then, using the continuous mapping theorem, we have $(H^n, H^{ni}) \underset{\mathcal{D}}{\longrightarrow} (H, H^i)$ in $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^{2d})$, which implies that

$$\lim_{i\to\infty} \limsup_{n\to\infty} P\left(\sup_{t\le q} |H_t^{ni} - H_t^n| \ge \varepsilon\right) = 0, \qquad \varepsilon > 0, q \in \mathbb{R}^+. \tag{4.7}$$

If X^{ni} denotes the solution of the equation

$$X_t^{ni} = H_t^{ni} + \int_0^t \langle f(X_{s-}^n), \mathrm{d}Z_s^n \rangle + K_t^{ni}, \qquad t \in \mathbb{R}^+, n, i \in \mathbb{N},$$

then, by (4.7) and Corollary 3.4, $\lim_{i\to\infty}\limsup_{n\to\infty} P(\sup_{t\le q}|X_t^{ni}-X_t^n|\ge\varepsilon)=0$, $\varepsilon>0$, $q\in\mathbb{R}^+$. Furthermore, it is well known that for continuous $f:\mathbb{R}^d\longrightarrow\mathbb{R}^d\otimes\mathbb{R}^d$, one can construct a sequence $\{f^i\}$ of functions such that $f^i\in\mathcal{C}^2$, $i\in\mathbb{N}$ and $\sup_{x\in K}\|f^i(x)-f(x)\|\longrightarrow 0$ for any compact subset $K\subset\mathbb{R}^d$. If we set

$$Y_t^{ni}=H_t^n+\int_0^t\langle f^i(X_{s-}^{ni}),\mathrm{d}Z_s^n\rangle,\qquad t\in\mathbb{R}^+,n\in\mathbb{N},$$

$$Y_t^{n}=H_t^n+\int_0^t\langle f(X_{s-}^{n}),\mathrm{d}Z_s^n\rangle,\qquad t\in\mathbb{R}^+,n\in\mathbb{N},$$

then $\lim_{i\to\infty}\limsup_{n\to\infty} P(\sup_{t\le q}|Y_t^{ni}-Y_t^n|\ge\varepsilon)=0$, $\varepsilon>0$, $q\in\mathbb{R}^+$. Because $f^i\in\mathcal{C}^2$ and $\{X^{ni}\}$ satisfies (UT), $\{f^i(X^{ni})\}$ satisfies (UT) as well. By [25], Lemma 4.3, the sequences $\{Y^{ni}\}$ and $\{Y^n\}$ are tight in $\mathbb{D}(\mathbb{R}^+,\mathbb{R}^d)$. Moreover, we can see that $\{(Y^n,H^n,Z^n)\}$ is tight in $\mathbb{D}(\mathbb{R}^+,\mathbb{R}^{3d})$. Because X^n is a solution of the equation with a penalization term of the form (1.5), it follows by Corollary 3.3, Proposition A.4 (Appendix), and Corollary 3.5(i) that

$$\{X^n\}\text{ is }S\text{-tight}\tag{4.8}$$

and

$$\lim_{\delta\downarrow 0}\limsup_{n\to\infty}P(\bar{\omega}''_{(X^n,Z^n)}(\delta,q)>\varepsilon)=0,\qquad \varepsilon>0,q\in\mathbb{R}^+.\tag{4.9}$$

Assume that there exists a subsequence $\{n'\}\subset\{n\}$ such that $(Y^{n'},H^{n'},Z^{n'})\underset{\mathcal{D}}{\longrightarrow}(\widehat{Y},\widehat{H},\widehat{Z})$ in $\mathbb{D}(\mathbb{R}^+,\mathbb{R}^{3d})$, where $\mathcal{L}(\widehat{H},\widehat{Z})=\mathcal{L}(H,Z)$. Then, by Corollary 3.5(ii),

$$(X_{t_1}^{n'},\dots,X_{t_m}^{n'},H^{n'},Z^{n'})\underset{\mathcal{D}}{\longrightarrow}(\widehat{X}_{t_1},\dots,\widehat{X}_{t_m},\widehat{H},\widehat{Z})\qquad\text{in }\mathbb{R}^{dm}\times\mathbb{D}(\mathbb{R}^+,\mathbb{R}^{2d})\tag{4.10}$$

for any $m\in\mathbb{N}$ and any $t_1,\dots,t_m\in\mathbb{R}^+$ such that $\widehat{P}(|\Delta\widehat{Y}_{t_i}|=0)=1$, $i=1,\dots,m$, where $\widehat{X}$ is a solution of the Skorokhod problem associated with $\widehat{Y}$. Combining (4.8)–(4.10) with Lemma 4.1 yields

$$\Bigl(X_{t_1}^{n'},\dots,X_{t_m}^{n'},H^{n'},Z^{n'},\int_0^\cdot\langle g(X_{s-}^{n'}),\mathrm{d}Z_s^{n'}\rangle\Bigr)\underset{\mathcal{D}}{\longrightarrow}\Bigl(\widehat{X}_{t_1},\dots,\widehat{X}_{t_m},\widehat{H},\widehat{Z},\int_0^\cdot\langle g(\widehat{X}_{s-}),\mathrm{d}\widehat{Z}_s\rangle\Bigr)$$

in $\mathbb{R}^{md}\times\mathbb{D}(\mathbb{R}^+,\mathbb{R}^{3d})$ for any $m\in\mathbb{N}$ and any $t_1,\dots,t_m\in\mathbb{R}^+$ such that $\widehat{P}(|\Delta\widehat{Y}_{t_i}|=0)=1$, $i=1,\dots,m$. Thus, in particular, putting $g=f$, we obtain that $\widehat{Y}=\widehat{H}+\int_0^\cdot f(\widehat{X}_{s-})\,\mathrm{d}\widehat{Z}_s$, which implies that $\widehat{X}$ is a weak solution of the SDE (1.1). Because $\{t;\widehat{P}(|\Delta\widehat{Y}_t|=0)=1\}\subset\{t;\widehat{P}(|\Delta\widehat{H}_t|=0)=1$ and $\widehat{P}(|\Delta\widehat{Z}_t|=0)=1\}$, the proof of (i) is complete. (ii) Follows immediately from (i).

(iii) Because $\{X^n\}$ is S-tight, the desired result follows from the convergence of finite-dimensional distributions of X^n to these of X proven in part (ii) and from Corollary A.5 (Appendix). □

We now consider an array $\{\{t_{n,k}\}\}$ of nonnegative numbers such that the nth row $T_n = \{t_{n,k}\}$ forms a partition of $\mathbb{R}^+$ such that $0 = t_{n,0} < t_{n,1} < \cdots$, $\lim_{k\to\infty} t_{n,k} = +\infty$ and $\max_k(t_{n,k} - t_{n,k-1}) \longrightarrow 0$ as $n \to +\infty$. For the array $\{\{t_{n,k}\}\}$, we define a sequence of summation rules $\{\rho^n\}$, $\rho^n : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ by $\rho^n_t = \max\{t_{n,k}; t_{n,k} \le t\}$, and then for a fixed adapted process H and a semimartingale Z, we define $H^{\rho^n} = H_{\rho^n_\cdot}$, $Z^{\rho^n} = Z_{\rho^n_\cdot}$, that is,

$$H^{\rho^n}_t = H_{t_{n,k}}, \qquad Z^{\rho^n}_t = Z_{t_{n,k}} \qquad \text{for } t \in [t_{n,k}, t_{n,k+1}), k \in \mathbb{N} \cup \{0\}, n \in \mathbb{N}.$$

Let $\{\bar{X}^n\}$ be a sequence of solutions to equations with penalization terms driven by $\{Z^{\rho^n}\}$, that is,

$$\bar{X}^n_t = H^{\rho^n}_t + \int_0^t f(\bar{X}^n_{s-})\,\mathrm{d}Z^{\rho^n}_s - n\int_0^t \big(\bar{X}^n_s - \Pi(\bar{X}^n_s)\big)\,\mathrm{d}s, \qquad t \in \mathbb{R}^+, n \in \mathbb{N}. \tag{4.11}$$

The special form of Z^{ρ^n} implies that

$$\bar{X}^n_t = \begin{cases} H_0, & t = 0,\\ \Pi(\bar{X}^n_{t_{n,k}}) + (\bar{X}^n_{t_{n,k}} - \Pi(\bar{X}^n_{t_{n,k}}))\mathrm{e}^{-n(t-t_{n,k})}, & t \in (t_{n,k}, t_{n,k+1}), k \in \mathbb{N}\cup\{0\},\\ \bar{X}^n_{(t_{n,k+1})-} + (H_{t_{n,k+1}} - H_{t_{n,k}}) & \\ \quad + f(\bar{X}^n_{(t_{n,k+1})-})(Z_{t_{n,k+1}} - Z_{t_{n,k}}), & t = t_{n,k+1}, k \in \mathbb{N}\cup\{0\}. \end{cases}$$

Corollary 4.4. *Let $\{\bar{X}^n\}$ be a sequence of solutions of* (4.11). *If f is continuous and satisfies* (1.2), *then*

(i) *$\{\bar{X}^n\}$ is S-tight, and its every limit point X is a weak solution of the SDE* (1.1),

(ii) *if the SDE* (1.1) *has a unique weak solution, then $\bar{X}^n \xrightarrow[\mathcal{D}]{*} X$.*

Proof. It is easily seen that $(H^{\rho^n}, Z^{\rho^n}) \to (H, Z)$ almost surely in $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^2)$. On the other hand, based on the theorem of Bichteler, Dellacherie and Mokobodzki, the sequence $\{Z^{\rho^n}\}$ of discrete semimartingales satisfies (UT). Therefore, the assumptions of Theorem 4.3 are satisfied. Thus, (i) follows from (4.8), whereas (ii) follows from Theorem 4.3(iii). □

We note that Corollary 4.4(i) implies the existence of a weak solution of the SDE (1.1).

In the sequel, we consider convergence in probability of solutions of equations with penalization terms. We assume that the SDE (1.1) has the pathwise uniqueness property; that is, for any two solutions $\widehat{X}$, $\widehat{X}'$ of the SDE (1.1) corresponding to processes $(\widehat{H}, \widehat{Z})$, $(\widehat{H}', \widehat{Z}')$ and defined on a probability space $(\widehat{\Omega}, \widehat{\mathcal{F}}, (\widehat{\mathcal{F}}_t), \widehat{P})$ with filtration $(\widehat{\mathcal{F}}_t)$, the following implication holds:

$$\widehat{P}\big((\widehat{H}_t, \widehat{Z}_t) = (\widehat{H}'_t, \widehat{Z}'_t); t \in \mathbb{R}^+\big) = 1 \quad \Rightarrow \quad \widehat{P}\big(\widehat{X}_t = \widehat{X}'_t; t \in \mathbb{R}^+\big) = 1.$$

It is well known that the existence of weak solutions and the pathwise uniqueness property implies the existence of a unique strong solution on arbitrary probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ with given adapted process H and the semimartingale Z (see, e.g., Yamada and Watanabe [31] and Jacod and Mémin [7] for the case of general semimartingales). The classical example of equation with the pathwise uniqueness property and non-Lipschitz coefficient f was given by Yamada and Watanabe [31] in the case of diffusion equations. Tudor [30] proved that this example works for SDEs driven by general semimartingales. Using, for example, [25], Lemma C.3, we can give the following version of his result:

Example 4.5. Assume that f is continuous, satisfies (1.2) and

$$\|f(x)-f(y)\|^2 \le \rho\big(|x-y|^2\big), \qquad x,y\in \bar{D},$$

where $\rho:\mathbb{R}^+\to\mathbb{R}^+$ is strictly increasing and concave, $\rho(0)=0$ and $\int_{0+}\frac{\mathrm{d}u}{\rho(u)}=+\infty$. Then the SDE (1.1) has the pathwise uniqueness property.

Corollary 4.6. *Moreover, under the assumptions of Theorem 4.3, if* $(H^n,Z^n)\underset{\mathcal{P}}{\longrightarrow}(H,Z)$ *in* $\mathbb{D}(\mathbb{R}^+,\mathbb{R}^{2d})$ *and the SDE* (1.1) *has the pathwise uniqueness property, then for any continuous function* $g:\mathbb{R}^d\longrightarrow\mathbb{R}^d\otimes\mathbb{R}^d$,

(i)

$$\left(X^n_{t_1},\dots,X^n_{t_m},H^n,Z^n,\int_0^\cdot \big\langle g\big(X^n_{s-}\big),\mathrm{d}Z^n_s\big\rangle\right)\underset{\mathcal{P}}{\longrightarrow}\left(X_{t_1},\dots,X_{t_m},H,Z,\int_0^\cdot \big\langle g(X_{s-}),\mathrm{d}Z_s\big\rangle\right)$$

in $\mathbb{R}^{md}\times\mathbb{D}(\mathbb{R}^+,\mathbb{R}^{3d})$, *for any* $m\in\mathbb{N}$ *and any* $t_1,\dots,t_m\in\mathbb{R}^+$ *such that* $P(|\Delta Z_{t_i}|=0)=P(|\Delta H_{t_i}|=0)=1$, $i=1,\dots,m$,

(ii) $X^n \underset{\mathcal{P}}{\overset{*}{\longrightarrow}} X$,

where X *is a unique strong solution of the SDE* (1.1).

Proof. In the proof, it suffices to use Theorem 4.3 and repeat arguments from the proof of Theorem 1(ii) in [24]. Fix $B\in\mathcal{F}$, $P(B)>0$ and define $\mathcal{Q}_B=P(A|B)$ for every $A\in\mathcal{F}$. Obviously, $\mathcal{Q}_B\ll P$ and $(\frac{\mathrm{d}\mathcal{Q}}{\mathrm{d}P})=\frac{\mathbf{1}_B}{P(B)}$. Let $\{H^n\}$, $\{Z^n\}$ be sequences of processes satisfying the assumptions of Corollary 4.6. Then $\{Z^n\}$ is a sequence of semimartingales on $(\Omega,\mathcal{F},\mathcal{Q}_B)$ satisfying (UT) (see, e.g., [24], Lemma 4) and $(H^n,Z^n)\longrightarrow_{\mathcal{D}(\mathcal{Q}_B)}(H,Z)$ in $\mathbb{D}(\mathbb{R}^+,\mathbb{R}^{2d})$. Moreover, the stochastic integral $\int f(X^n_{s-})\,\mathrm{d}Z^n_s$, calculated with respect to P is, for $\mathcal{Q}_B$, almost all $\omega\in\Omega$ equal to the integral calculated with respect to $\mathcal{Q}_B$ and

$$X^n_t=H^n_t+\int_0^t f\big(X^n_{s-}\big)\,\mathrm{d}Z^n_s-n\int_0^t\big(X^n_s-\Pi\big(X^n_s\big)\big)\,\mathrm{d}s,\qquad t\in\mathbb{R}^+,\ \mathcal{Q}_B\text{-a.e.}, n\in\mathbb{N}.$$

Consequently, by Theorem 4.3(ii), $(X^n_1,\dots,X^n_{t_m},H^n,Z^n,Y^n)\longrightarrow_{\mathcal{D}(\mathcal{Q}_B)}(X,H,Z,Y)$, in $\mathbb{R}^m\times\mathbb{D}(\mathbb{R}^+,\mathbb{R}^{3d})$, where $Y^n=\int_0^\cdot g(X^n_{s-})\,\mathrm{d}Z^n_s$, $Y=\int_0^\cdot g(X_{s-})\,\mathrm{d}Z_s$. Thus, for all bounded and continuous mappings Φ, $\Phi:\mathbb{R}^m\times\mathbb{D}(\mathbb{R}^+,\mathbb{R}^{2d})\longrightarrow\mathbb{R}$,

$$\lim_{n\to+\infty}\int_\Omega \Phi\big(X^n_{t_1},\dots,X^n_{t_m},H^n,Z^n,Y^n\big)\,\mathrm{d}\mathcal{Q}_B=\int_\Omega \Phi(X_{t_1},\dots,X_{t_m},H,Z,Y)\,\mathrm{d}\mathcal{Q}_B,$$

or, equivalently,

$$\lim_{n\to+\infty}\int_B \Phi\big(X^n_{t_1},\dots,X^n_{t_m},H^n,Z^n,Y^n\big)\,\mathrm{d}P=\int_B \Phi(X_{t_1},\dots,X_{t_m},H,Z,Y)\,\mathrm{d}P. \tag{4.12}$$

Because (4.12) holds for all $B\in\mathcal{F},\mathcal{P}(B)>0$ and all bounded continuous mappings $\Phi:\mathbb{R}^m\times\mathbb{D}(\mathbb{R}^+,\mathbb{R}^{3d})\longrightarrow\mathbb{R}$, the proof of (i) is complete. Using (i), the claim (ii) follows readily. □

Corollary 4.7. *Assume that f is continuous and satisfies* (1.2), *and that* (1.1) *has the pathwise uniqueness property. Let X be a strong solution of* (1.1), *and let $\{\bar{X}^n\}$ be a sequence of solutions of* (4.11). *Then*

(i) $\bar{X}^n_t \underset{\mathcal{P}}{\longrightarrow} X_t$ *for every* $t\in\mathbb{R}^+$ *such that* $P(\Delta H_t=0)=P(\Delta Z_t=0)=1$,

(ii) *for any continuous* $g:\mathbb{R}^d\to\mathbb{R}^d\otimes\mathbb{R}^d$,

$$\sup_{t\le q,t\in T_n}\left|\int_0^t g\big(\bar{X}^n_{s-}\big)\,\mathrm{d}Z^{\rho^n}_s-\int_0^t g(X_{s-})\,\mathrm{d}Z_s\right|\underset{\mathcal{P}}{\longrightarrow}0,\qquad q\in\mathbb{R}^+,$$

(iii) $\bar{X}^n \overset{*}{\underset{\mathcal{P}}{\longrightarrow}} X$ *in* $\mathbb{D}(\mathbb{R}^+,\mathbb{R}^d)$,

(iv) *if, moreover, H, Z are processes with continuous trajectories, then*

$$\sup_{t\le q}\big|\bar{X}^n_t-X_t\big|\underset{\mathcal{P}}{\longrightarrow}0,\qquad q\in\mathbb{R}^+.$$

Proof. We can see that $(H^{\rho^n},Z^{\rho^n})\to(H,Z)$ almost surely in $\mathbb{D}(\mathbb{R}^+,\mathbb{R}^{2d})$, and, given the theorem of Bichteler, Dellacherie, and Mokobodzki, the sequence $\{Z^{\rho^n}\}$ of discrete semimartingales satisfies (UT). Therefore, by Corollary 4.6, the conclusions (i) and (iii) follow. Moreover, by Corollary 4.6(i), for every continuous g,

$$\left(\int_0^\cdot g\big(\bar{X}^n_{s-}\big)\,\mathrm{d}Z^{\rho^n}_s,Z^{\rho^n}\right)\underset{\mathcal{P}}{\longrightarrow}\left(\int_0^\cdot g(X_{s-})\,\mathrm{d}Z_s,Z\right),\qquad \text{in } \mathbb{R}^m\times\mathbb{D}\big(\mathbb{R}^+,\mathbb{R}^{2d}\big).$$

Using this and arguing as in the proof of Theorem 3 in [24], the conclusion (ii) also follows.

To prove (iv), we first observe that by Corollary 4.6(i),

$$Y^n=H^{\rho^n}+\int_0^\cdot f\big(\bar{X}^n_{s-}\big)\,\mathrm{d}Z^{\rho^n}_s\underset{\mathcal{P}}{\longrightarrow}H+\int_0^\cdot f(X_{s-})\,\mathrm{d}Z_s=Y\qquad\text{in }\mathbb{D}\big(\mathbb{R}^+,\mathbb{R}^d\big),$$

where Y is a process with continuous trajectories. From this, and by Corollary 2.4(iii), the result follows readily. □

Appendix: The topology *S*

The S topology on the space $\mathbb{D}(\mathbb{R}^+,\mathbb{R}^d)$ of $\mathbb{R}^d$-valued functions that are right-continuous and have left-hand limits was introduced by Jakubowski [10]. It is weaker than the Skorokhod topology J_1 but stronger than the Meyer–Zheng topology considered in [22]. We collect here only basic definitions and properties of the S topology; more details can be found in Jakubowski [9,10].

Proposition A.1. (i) $K \subset \mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$ *is relatively S-compact if and only if*

$$\sup_{x\in K} \sup_{t\le q} |x_t| < +\infty, \qquad q \in \mathbb{R}^+ \tag{A.1}$$

and for all $a < b$, $a, b \in \mathbb{R}$

$$\sup_{x\in K} N^{a,b}(x, q) < +\infty, \qquad q \in \mathbb{R}^+, \tag{A.2}$$

where $N^{a,b}$ is the usual number of up-crossings given levels $a < b$, that is, $N^{a,b}(x, q) \ge k$ if one can find numbers $0 \le t_1 < t_2 < \cdots < t_{2k-1} < t_{2k} \le q$ such that $x_{t_{2i-1}} < a$ and $x_{t_{2i}} > b$, $i = 1, 2, \ldots, k$.

(ii) *x^n converges to x in the S-topology if and only if $\{x^n\}$ satisfies* (A.1), (A.2) *and in every subsequence $\{n_k\}$, one can find a further subsequence $\{n_{k_l}\}$ and a dense subset $\mathbb{Q} \subset \mathbb{R}^+$ such that $x_t^{n_{k_l}} \to x_t$, $t \in \mathbb{Q}$.*

Corollary A.2. *If $\{x^n\}$ is relatively S-compact and there exists a dense subset $\mathbb{Q}$ such that for every $t \in \mathbb{Q}$, $x_t^n \to x_t$, then $\{x^n\}$ converges to x.*

Recall that the sequence of processes $\{X^n\}$ converges weakly to X in the S topology ($X^n \xrightarrow[\mathcal{D}]{*} X$) if in every subsequence $\{X^{n_k}\}$, we can find a further subsequence $\{X^{n_{k_l}}\}$ and stochastic processes $\{Y_l\}$ defined on $([0,1], \mathcal{B}_{[0,1]}, \ell)$, such that the laws of Y_l and $X^{n_{k_l}}$ are the same, $l \in \mathbb{N}$, for each $\omega \in [0,1]$ $Y_l(\omega)$ converges to $Y(\omega)$ in the S topology, and for each $\varepsilon > 0$, there exists an S-compact subset $K_\varepsilon \subset \mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$ such that

$$P\big(\big\{\omega \in [0,1]\colon\ Y_l(\omega) \in K_\varepsilon, l = 1, 2, \ldots\big\}\big) > 1 - \varepsilon.$$

Proposition A.3. *The following two conditions are equivalent*:

(i) *$\{X^n\}$ is S-tight.*

(ii) *$\{X^n\}$ is relatively compact with respect to the convergence "$\xrightarrow[\mathcal{D}]{*}$".*

Proposition A.4. *Let $\{X^n\}$ be a sequence of processes of the form $X^n = H^n + Z^n$, $n \in \mathbb{N}$, where $\{H^n\}$ is tight in $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$ and $\{Z^n\}$ is a sequence of semimartingales satisfying* (UT). *Then $\{X^n\}$ is S-tight.*

Corollary A.5. *If $\{X^n\}$ is S-tight and there exists a dense subset $\mathbb{Q} \subset \mathbb{R}^+$ such that for every $m \in \mathbb{N}$ and every $t_1, t_2, \ldots, t_m \in \mathbb{Q}$*

$$\big(X_{t_1}^n, X_{t_2}^n, \ldots, X_{t_m}^n\big) \xrightarrow[\mathcal{D}]{} \big(X_{t_1}^n, X_{t_2}^n, \ldots, X_{t_m}^n\big) \qquad \text{in } \mathbb{R}^d$$

then $X^n \xrightarrow[\mathcal{D}]{*} X$.

Theorem A.6. *Suppose* $\{Z^n\}$ *satisfies* (UT) *and is tight in* $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$ *and* $\{Y^n\}$ *is S-tight. If there exists a dense subset* $\mathbb{Q} \subset \mathbb{R}^+$ *such that for any* $m \in \mathbb{N}$, *any* $t_1, \dots, t_m, t_j \in \mathbb{Q}$, $j = 1, \dots, m$,

$$\left(Y^n_{t_1}, Z^n_{t_1}, \dots, Y^n_{t_m}, Z^n_{t_m}\right) \xrightarrow[\mathcal{D}]{} (Y_{t_1}, Z_{t_1}, \dots, Y_{t_m}, Z_{t_m}) \qquad \text{in } \mathbb{R}^{2md},$$

where both processes Y *and* Z *have trajectories in* $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$ *and there are no oscillations of* Y^n *preceding oscillations of* Z^n *i.e.*

$$\lim_{\delta \to 0} \limsup_{n \to \infty} P\left(\bar{\omega}''_{(Y^n, Z^n)}(\delta, q) > \varepsilon\right) = 0, \qquad \varepsilon > 0, q \in \mathbb{R}^+ \tag{A.3}$$

then we have $(\int_0^\cdot Y^n_{s-} \, \mathrm{d}Z^n_s, Z^n) \xrightarrow[\mathcal{D}]{} (\int_0^\cdot Y_{s-} \, \mathrm{d}Z_s, Z)$ *in* $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^{d+1})$.

Corollary A.7. *Suppose* $\{Z^n\}$ *satisfies* (UT) *and is tight in* J_1 *and* $\{Y^n\}$ *is S-tight. If there exist sequences of random vectors* $\{K^n_1\}, \dots, \{K^n_l\}$ *and a dense subset* $\mathbb{Q} \subset \mathbb{R}^+$ *such that for any* $m \in \mathbb{N}$, *any* $t_1, \dots, t_m, t_j \in \mathbb{Q}$, $j = 1, \dots, m$,

$$\left(Y^n_{t_1}, \dots, Y^n_{t_m}, K^n_1, \dots, K^n_l, H^n, Z^n\right) \xrightarrow[\mathcal{D}]{} (Y_{t_1}, \dots, Y_{t_m}, K_1, \dots, K_l, H, Z)$$

in $\mathbb{R}^{(m+l)d} \times \mathbb{D}(\mathbb{R}^+, \mathbb{R}^2)$, *where both processes* Y *and* Z *have trajectories in* $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$ *and* (A.3) *holds true then*

$$\left(K^n_1, \dots, K^n_l, H^n, Z^n, \int_0^\cdot \langle Y^n_{s-}, \mathrm{d}Z^n_s\rangle\right) \xrightarrow[\mathcal{D}]{} \left(K_1, \dots, K_l, H, Z, \int_0^\cdot \langle Y_{s-}, \mathrm{d}Z_s\rangle\right)$$

in $\mathbb{R}^{ld} \times \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{2d+1})$.

Acknowledgements

We thank the referee for careful reading of the paper and many valuable remarks. We also thank prof. Stanisław Kasjan from Nicolaus Copernicus University for communicating us example (2.1). The second author was supported by the Polish Ministry of Science and Higher Education Grant N N201372436.

References

[1] Alibert, J.J. and Bahlali, K. (2001). Genericity in deterministic and stochastic differential equations. In *Séminaire de Probabilités*, *XXXV*. *Lecture Notes in Math*. **1755** 220–240. Berlin: Springer. MR1837290

[2] Anulova, S.V. and Liptser, R.S. (1990). Diffusion approximation for processes with normal reflection. *Theory Probab. Appl*. **35** 411–423. MR1091198

[3] Bahlali, K., Mezerdi, B. and Ouknine, Y. (1998). Pathwise uniqueness and approximation of solutions of stochastic differential equations. In *Séminaire de Probabilités*, *XXXII*. *Lecture Notes in Math*. **1686** 166–187. Berlin: Springer. MR1655150

[4] Billingsley, P. (1968). *Convergence of Probability Measures*. New York: Wiley. MR0233396

[5] Cépa, E. (1998). Problème de Skorohod multivoque. *Ann. Probab.* **26** 500–532. MR1626174

[6] Gyöngy, I. and Krylov, N. (1996). Existence of strong solutions for Itô's stochastic equations via approximations. *Probab. Theory Related Fields* **105** 143–158. MR1392450

[7] Jacod, J. and Mémin, J. (1981). Weak and strong solutions of stochastic differential equations: Existence and stability. In *Stochastic Integrals* (*Proc. Sympos., Univ. Durham, Durham*, 1980). *Lecture Notes in Math.* **851** 169–212. Berlin: Springer. MR0620991

[8] Jacod, J. and Shiryaev, A.N. (1987). *Limit Theorems for Stochastic Processes*. *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **288**. Berlin: Springer. MR0959133

[9] Jakubowski, A. (1996). Convergence in various topologies for stochastic integrals driven by semimartingales. *Ann. Probab.* **24** 2141–2153. MR1415245

[10] Jakubowski, A. (1997). A non-Skorohod topology on the Skorohod space. *Electron. J. Probab.* **2** 1–21. MR1475862

[11] Jakubowski, A., Mémin, J. and Pagès, G. (1989). Convergence en loi des suites d'intégrales stochastiques sur l'espace $\mathbf{D}^1$ de Skorokhod. *Probab. Theory Related Fields* **81** 111–137. MR0981569

[12] Kaneko, H. and Nakao, S. (1988). A note on approximation for stochastic differential equations. In *Séminaire de Probabilités, XXII. Lecture Notes in Math.* **1321** 155–162. Berlin: Springer. MR0960522

[13] Kurtz, T.G. and Protter, P. (1991). Weak limit theorems for stochastic integrals and stochastic differential equations. *Ann. Probab.* **19** 1035–1070. MR1112406

[14] Łaukajtys, W. (2004). On stochastic differential equations with reflecting boundary condition in convex domains. *Bull. Pol. Acad. Sci. Math.* **52** 445–455. MR2128281

[15] Łaukajtys, W. and Słomiński, L. (2003). Penalization methods for reflecting stochastic differential equations with jumps. *Stoch. Stoch. Rep.* **75** 275–293. MR2017780

[16] Lions, P.L., Menaldi, J.L. and Sznitman, A.S. (1981). Construction de processus de diffusion réfléchis par pénalisation du domaine. *C. R. Acad. Sci. Paris Sér. I Math.* **292** 559–562. MR0614669

[17] Lions, P.L. and Sznitman, A.S. (1984). Stochastic differential equations with reflecting boundary conditions. *Comm. Pure Appl. Math.* **37** 511–537. MR0745330

[18] Liu, Y. (1993). Numerical approaches to stochastic differential equations with boundary conditions. Ph.D. thesis. Purdue Univ. MR2689982

[19] Mémin, J. and Słomiński, L. (1991). Condition UT et stabilité en loi des solutions d'équations différentielles stochastiques. In *Séminaire de Probabilités, XXV. Lecture Notes in Math.* **1485** 162–177. Berlin: Springer. MR1187779

[20] Menaldi, J.L. (1983). Stochastic variational inequality for reflected diffusion. *Indiana Univ. Math. J.* **32** 733–744. MR0711864

[21] Menaldi, J.L. and Robin, M. (1985). Reflected diffusion processes with jumps. *Ann. Probab.* **13** 319–341. MR0781408

[22] Meyer, P.A. and Zheng, W.A. (1984). Tightness criteria for laws of semimartingales. *Ann. Inst. Henri Poincaré Probab. Stat.* **20** 353–372. MR0771895

[23] Pettersson, R. (1997). Penalization schemes for reflecting stochastic differential equations. *Bernoulli* **3** 403–414. MR1483695

[24] Słomiński, L. (1989). Stability of strong solutions of stochastic differential equations. *Stochastic Process. Appl.* **31** 173–202. MR0998112

[25] Słomiński, L. (1996). Stability of stochastic differential equations driven by general semimartingales. *Dissertationes Math.* (*Rozprawy Mat.*) **349** 1–113. MR1377600

[26] Słomiński, L. (2001). Euler's approximations of solutions of SDEs with reflecting boundary. *Stochastic Process. Appl.* **94** 317–337. MR1840835

[27] Storm, A. (1995). Stochastic differential equations with a convex constraint. *Stoch*. *Stoch*. *Rep*. **53** 241–274. MR1381680
[28] Stricker, C. (1985). Lois de semimartingales et critères de compacité. In *Séminaire de Probabilités*, *XIX*, 1983/84. *Lecture Notes in Math*. **1123** 209–217. Berlin: Springer. MR0889478
[29] Tanaka, H. (1979). Stochastic differential equations with reflecting boundary condition in convex regions. *Hiroshima Math*. *J*. **9** 163–177. MR0529332
[30] Tudor, C. (1984). Sur les solutions fortes des équations différentielles stochastiques. *C*. *R*. *Acad*. *Sci*. *Paris Sér. I Math*. **299** 117–120. MR0756534
[31] Yamada, T. and Watanabe, S. (1971). On the uniqueness of solutions of stochastic differential equations. *J*. *Math*. *Kyoto Univ*. **11** 155–167. MR0278420

Received May 2011 and revised October 2011