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On the rate of convergence in the martingale central limit theorem

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Consider a discrete-time martingale, and let V^2 be its normalized quadratic variation. As V^2 approaches 1, and provided that some Lindeberg condition is satisfied, the distribution of the rescaled martingale approaches the Gaussian distribution. For any $p \geq 1$, (*Ann. Probab.* **16** (1988) 275–299) gave a bound on the rate of convergence in this central limit theorem that is the sum of two terms, say $A_p + B_p$, where up to a constant, $A_p = \|V^2 - 1\|_p^{p/(2p+1)}$. Here we discuss the optimality of this term, focusing on the restricted class of martingales with bounded increments. In this context, (*Ann. Probab.* **10** (1982) 672–688) sketched a strategy to prove optimality for $p = 1$. Here we extend this strategy to any $p \geq 1$, thereby justifying the optimality of the term A_p. As a necessary step, we also provide a new bound on the rate of convergence in the central limit theorem for martingales with bounded increments that improves on the term B_p, generalizing another result of (*Ann. Probab.* **10** (1982) 672–688).

Keywords: central limit theorem; martingale; rate of convergence

1. Introduction

Let $\mathbf{X} = (X_1, \ldots, X_n)$ be a sequence of square-integrable random variables such that for any i, X_i satisfies $\mathbb{E}[X_i | \mathcal{F}_{i-1}] = 0$, where $\mathcal{F}_i$ is the σ-algebra generated by $(X_1, \ldots, X_i)$. In other words, $\mathbf{X}$ is a square-integrable martingale difference sequence. Following the notation of [1], we write M_n for the set of all such sequences of length n, and introduce

$$s^2(\mathbf{X}) = \sum_{i=1}^{n} \mathbb{E}[X_i^2],$$

$$V^2(\mathbf{X}) = s^{-2}(\mathbf{X}) \sum_{i=1}^{n} \mathbb{E}[X_i^2 | \mathcal{F}_{i-1}],$$

$$S(\mathbf{X}) = \sum_{i=1}^{n} X_i.$$

$V^2(\mathbf{X})$ can be called the normalized quadratic variation of $\mathbf{X}$. Let $(\mathbf{X}_n)_{n \in \mathbb{N}}$ be such that for any n, $\mathbf{X}_n \in M_n$. It is well known (see, e.g., [2], Section 7.7.a) that if

$$V^2(\mathbf{X}_n) \xrightarrow[n \to +\infty]{\text{(prob.)}} 1 \tag{1.1}$$

and some Lindeberg condition is satisfied, then the rescaled sum $S(\mathbf{X}_n)/s(\mathbf{X}_n)$ converges in distribution to a standard Gaussian random variable, that is,

$$\forall t \in \mathbb{R}, \qquad \mathbb{P}[S(\mathbf{X}_n)/s(\mathbf{X}_n) \le t] \xrightarrow[n\to+\infty]{} \Phi(t), \tag{1.2}$$

where $\Phi(t) = (2\pi)^{-1/2} \int_{-\infty}^{t} \mathrm{e}^{-x^2/2}\,\mathrm{d}x$.

We are interested in bounds on the speed of convergence in this central limit theorem. Several results have been obtained under a variety of additional assumptions. One natural way to strengthen the convergence in probability (1.1) is to change it for a convergence in L^p for some $p \in [1, +\infty]$. Indeed, quantitative estimates in terms of $\|V^2 - 1\|_p$ seem particularly convenient when the aim is to apply the result to practical situations. We write

$$D(\mathbf{X}) = \sup_{t\in\mathbb{R}} |\mathbb{P}[S(\mathbf{X})/s(\mathbf{X}) \le t] - \Phi(t)|$$

and

$$\|\mathbf{X}\|_p = \max_{1\le i\le n} \|X_i\|_p \qquad (p \in [1, +\infty]).$$

[4] proved the following result.

Theorem 1.1 ([4]). *Let $p \in [1, +\infty)$. There exists a constant $C_p > 0$ such that for any $n \ge 1$ and any $\mathbf{X} \in M_n$,*

$$D(\mathbf{X}) \le C_p \left(\|V^2(\mathbf{X}) - 1\|_p^p + s^{-2p}(\mathbf{X}) \sum_{i=1}^{n} \|X_i\|_{2p}^{2p} \right)^{1/(2p+1)}. \tag{1.3}$$

In [7], Theorem 1.1 is generalized to the following.

Theorem 1.2 ([7]). *Let $p \in [1, +\infty]$ and $p' \in [1, +\infty)$. There exists $C_{p,p'} > 0$ such that for any $n \ge 1$ and any $\mathbf{X} \in M_n$,*

$$D(\mathbf{X}) \le C_{p,p'} \left[\|V^2(\mathbf{X}) - 1\|_p^{p/(2p+1)} + \left(s^{-2p'}(\mathbf{X}) \sum_{i=1}^{n} \|X_i\|_{2p'}^{2p'} \right)^{1/(2p'+1)} \right]. \tag{1.4}$$

Here $p/(2p+1) = 1/2$ for $p = +\infty$. In fact, a stronger, nonuniform bound is given; see [7], Theorem 2.2 (or, equivalently, [8]), for details.

The main question addressed here concerns the optimality of the term $\|V^2(\mathbf{X}) - 1\|_p^{p/(2p+1)}$ appearing in the right-hand side of (1.3) or (1.4). About this, [4] constructed a sequence of elements $\mathbf{X}_n \in M_n$ such that

- $s(\mathbf{X}_n) \simeq \sqrt{n}$,
- $D(\mathbf{X}_n) \simeq \log^{-1/2}(n)$,
- $\|V^2(\mathbf{X}) - 1\|_p^p \simeq s^{-2p}(\mathbf{X})\|\mathbf{X}\|_{2p}^{2p} \simeq s^{-2p}(\mathbf{X}) \sum_{i=1}^{n} \|X_i\|_{2p}^{2p} \simeq \log^{-(2p+1)/2}(n)$,

where we write $a_n \simeq b_n$ if there exists $C > 0$, such that $a_n/C \le b_n \le Ca_n$ for all sufficiently large n. This example demonstrates that the exponent $1/(2p+1)$ appearing on the outer bracket of the right-hand side of (1.3) cannot be improved. But because the two terms of the right-hand side of (1.3) are of the same order, no conclusions can be drawn about the optimality of the term $\|V^2(\mathbf{X}) - 1\|_p^{p/(2p+1)}$ alone. Most importantly, it is rather disappointing that in the example, $\|\mathbf{X}\|_{2p}^{2p}$ and $\sum_{i=1}^n \|X_i\|_{2p}^{2p}$ are of the same order, if the typical martingales that one is interested in have increments of roughly the same order.

Using a similar construction, but also imposing the condition that $V^2(\mathbf{X}) = 1$ a.s., [7], Example 2.4, proved the optimality of the exponent $1/(2p' + 1)$ appearing in the second term of the sum in the right-hand side of (1.4). However, the author did not discuss the optimality of the first term $\|V^2(\mathbf{X}) - 1\|_p^{p/(2p+1)}$.

For $1 \le p \le 2$, Theorem 1.1 was in fact already proved by [6]. In [5], Section 3.6, the authors could show only that the bound on $D(\mathbf{X})$ can be no better than $\|V^2(\mathbf{X}) - 1\|_1^{1/2}$.

The proof of Theorem 1.1 given by [4] is inspired by a method introduced by [1], who proved the following results.

Theorem 1.3 ([1]). *Let $\gamma \in (0, +\infty)$. There exists a constant $C_\gamma > 0$ such that for any $n \ge 2$ and any $\mathbf{X} \in M_n$ satisfying $\|\mathbf{X}\|_\infty \le \gamma$ and $V^2(\mathbf{X}) = 1$ a.s.,*

$$D(\mathbf{X}) \le C_\gamma \frac{n \log(n)}{s^3(\mathbf{X})}.$$

Typically, $s(\mathbf{X})$ is of order $\sqrt{n}$ when $\mathbf{X} \in M_n$. Under such circumstances, Theorem 1.3 thus gives a bound of order $\log(n)/\sqrt{n}$. Moreover, [1] provided an example of a sequence of elements $\mathbf{X}_n \in M_n$ satisfying the conditions of Theorem 1.3, such that $s^2(\mathbf{X}_n) = n$ and for which

$$\limsup_{n\to+\infty} \sqrt{n} \log^{-1}(n) D(\mathbf{X}_n) > 0,$$

and so the result is optimal.

Relaxing the condition that $V^2(\mathbf{X}) = 1$ a.s., [1] then showed the following result.

Corollary 1.4 ([1]). *Let $\gamma \in (0, +\infty)$. There exists a constant $\overline{C}_\gamma > 0$ such that for any $n \ge 2$ and any $\mathbf{X} \in M_n$ satisfying $\|\mathbf{X}\|_\infty \le \gamma$,*

$$D(\mathbf{X}) \le \overline{C}_\gamma \left[\frac{n \log(n)}{s^3(\mathbf{X})} + \min\left(\|V^2(\mathbf{X}) - 1\|_1^{1/3}, \|V^2(\mathbf{X}) - 1\|_\infty^{1/2} \right) \right]. \tag{1.5}$$

See [7], Theorem 3.2, for a nonuniform version of this result. A strategy was sketched by [1] to prove that the bound $\|V^2(\mathbf{X}) - 1\|_1^{1/3}$ is indeed optimal, even on the restricted class considered by Corollary 1.4 of martingales with bounded increments. This example provides a satisfactory answer to our question of optimality for $p = 1$. The aim of the present paper is to generalize Corollary 1.4 and the optimality result to any $p \in [1, +\infty)$. We begin by proving the following general result.

Theorem 1.5. *Let $p \in [1, +\infty)$ and $\gamma \in (0, +\infty)$. There exists a constant $C_{p,\gamma} > 0$ such that for any $n \geq 2$ and any $\mathbf{X} \in M_n$ satisfying $\|\mathbf{X}\|_\infty \leq \gamma$,*

$$D(\mathbf{X}) \leq C_{p,\gamma}\left[\frac{n\log(n)}{s^3(\mathbf{X})} + \left(\|V^2(\mathbf{X}) - 1\|_p^p + s^{-2p}(\mathbf{X})\right)^{1/(2p+1)}\right]. \tag{1.6}$$

Note that, somewhat surprisingly, the term $s^{-2p}(\mathbf{X})\sum_{i=1}^n \|X_i\|_{2p}^{2p}$ appearing in inequality (1.3) is no longer present in (1.5), and is changed for the smaller $s^{-2p}(\mathbf{X})$ in (1.6).

Finally, we justify the optimality of the term $\|V^2(\mathbf{X}) - 1\|_p^{p/(2p+1)}$ appearing in the right-hand side of (1.6).

Theorem 1.6. *Let $p \in [1, +\infty)$ and $\alpha \in (1/2, 1)$. There exists a sequence of elements $\mathbf{X}_n \in M_n$ such that*

- $\|\mathbf{X}_n\|_\infty \leq 2$,
- $s(\mathbf{X}_n) \simeq \sqrt{n}$,
- $\|V^2(\mathbf{X}_n) - 1\|_p^{p/(2p+1)} = \mathrm{O}(n^{(\alpha-1)/2})$,
- $\limsup_{n\to+\infty} n^{(1-\alpha)/2} D(\mathbf{X}_n) > 0$.

Our strategy for proving Theorem 1.6 builds on the approach sketched by [1] for the case where $p = 1$. Interestingly, Theorem 1.5 is used in the proof of Theorem 1.6.

The question of optimality of the term $\|V^2(\mathbf{X}) - 1\|_p^{p/(2p+1)}$, now settled by Theorem 1.6, arises naturally in the problem of showing a quantitative central limit theorem for the random walk among random conductances on $\mathbb{Z}^d$ [9]. There, the random walk is approximated by a martingale. The martingale increments are stationary and almost bounded for $d \geq 3$, in the sense that they have bounded L^p norm for every $p < +\infty$. Roughly speaking, for $d \geq 3$, the variance of the rescaled quadratic variation up to time t decays as t^{-1}. This bound is optimal and leads to a Berry–Esseen bound of order $t^{-1/5}$. Thus Theorem 1.6 demonstrates that a better exponent of decay than $1/5$ cannot be obtained when relying solely on information about the variance of the quadratic variation.

Theorem 1.5 is proved in Section 2, and Theorem 1.6 is proved in Section 3.

2. Proof of Theorem 1.5

The proof of Theorem 1.5 is essentially similar to the proof of Corollary 1.4 given by [1], with the additional ingredient of a Burkholder inequality. Let $\mathbf{X} = (X_1, \ldots, X_n) \in M_n$ be such that $\|\mathbf{X}\|_\infty \leq \gamma$. The idea (probably first suggested by [3]) is to augment the sequence to some $\hat{\mathbf{X}} \in M_{2n}$ such that $V^2(\hat{\mathbf{X}}) = 1$ a.s., while preserving the property that $\|\hat{\mathbf{X}}\|_\infty \leq \gamma$, and apply Theorem 1.3 to this enlarged sequence. Let

$$\tau = \sup\left\{k \leq n \colon \sum_{i=1}^k \mathbb{E}[X_i^2 | \mathcal{F}_{i-1}] \leq s^2(\mathbf{X})\right\}.$$

For $i \le \tau$, we define $\hat{X}_i = X_i$. Let r be the largest integer not exceeding

$$\frac{s^2(\mathbf{X}) - \sum_{i=1}^{\tau} \mathbb{E}[X_i^2|\mathcal{F}_{i-1}]}{\gamma^2}.$$

As $\|X\|_\infty \le \gamma$, clearly $r \le n$. Conditional on $\mathcal{F}_\tau$ and for $1 \le i \le r$, we let $\hat{X}_i$ be independent random variables such that $\mathbb{P}[\hat{X}_{\tau+i} = \pm\gamma] = 1/2$. If $\tau + r < 2n$, then we let $\hat{X}_{\tau+r+1}$ be such that

$$\mathbb{P}\left[\hat{X}_{\tau+r+1} = \pm\left(s^2(\mathbf{X}) - \sum_{i=1}^{\tau} \mathbb{E}[X_i^2|\mathcal{F}_{i-1}] - r\gamma^2\right)^{1/2}\right] = \frac{1}{2},$$

with the sign determined independent of everything else. Finally, if $\tau + r + 1 < 2n$, then we let $\hat{X}_{\tau+r+i} = 0$ for $i \ge 2$.

Possibly enlarging the σ-fields, we can assume that $\hat{X}_i$ is $\mathcal{F}_i$-measurable for $i \le n$, and define $\mathcal{F}_i$ to be the σ-field generated by $\mathcal{F}_n$ and $\hat{X}_{n+1}, \dots, \hat{X}_{n+i}$ if $i > n$. By construction, we have

$$\sum_{i=\tau+1}^{2n} \mathbb{E}[\hat{X}_i^2|\mathcal{F}_{i-1}] = s^2(\mathbf{X}) - \sum_{i=1}^{\tau} \mathbb{E}[X_i^2|\mathcal{F}_{i-1}],$$

which can be rewritten as

$$\sum_{i=1}^{2n} \mathbb{E}[\hat{X}_i^2|\mathcal{F}_{i-1}] = s^2(\mathbf{X}).$$

Consequently, $s^2(\hat{\mathbf{X}}) = s^2(\mathbf{X})$ and $V^2(\hat{\mathbf{X}}) = 1$ a.s. The sequence $\hat{\mathbf{X}}$ thus satisfies the assumptions of Theorem 1.3, so

$$D(\hat{\mathbf{X}}) \le 4C_\gamma \frac{n\log(n)}{s^3(\mathbf{X})}. \tag{2.1}$$

For any $x > 0$, we have

$$\begin{aligned}
\mathbb{P}\left[\frac{S(\mathbf{X})}{s(\mathbf{X})} \le t\right] &\le \mathbb{P}\left[\frac{S(\mathbf{X})}{s(\mathbf{X})} \le t, \frac{|S(\mathbf{X}) - S(\hat{\mathbf{X}})|}{s(\mathbf{X})} \le x\right] + \mathbb{P}\left[\frac{|S(\mathbf{X}) - S(\hat{\mathbf{X}})|}{s(\mathbf{X})} \ge x\right] \\
&\le \mathbb{P}\left[\frac{S(\hat{\mathbf{X}})}{s(\mathbf{X})} \le t + x\right] + \frac{1}{x^{2p}}\mathbb{E}\left[\left|\frac{S(\mathbf{X}) - S(\hat{\mathbf{X}})}{s(\mathbf{X})}\right|^{2p}\right].
\end{aligned} \tag{2.2}$$

Due to (2.1), the first term in the right-hand side of (2.2) is smaller than

$$\Phi(t+x) + 4C_\gamma \frac{n\log(n)}{s^3(\mathbf{X})} \le \Phi(t) + \frac{x}{\sqrt{2\pi}} + 4C_\gamma \frac{n\log(n)}{s^3(\mathbf{X})}. \tag{2.3}$$

To control the second term, first note that

$$S(\mathbf{X}) - S(\hat{\mathbf{X}}) = \sum_{i=\tau+1}^{2n} (X_i - \hat{X}_i), \tag{2.4}$$

where we put $X_i = 0$ for $i > n$. Given that $\tau + 1$ is a stopping time, conditional on τ, the $(X_i - \hat{X}_i)_{i \geq \tau+2}$ still forms a martingale difference sequence. Thus we can use Burkholder's inequality (see, e.g., [5], Theorem 2.11), which states that

$$\begin{aligned} &\frac{1}{C}\mathbb{E}\left[\left|\sum_{i=\tau+2}^{2n}(X_i - \hat{X}_i)\right|^{2p}\right] \\ &\quad \leq \mathbb{E}\left[\left(\sum_{i=\tau+2}^{2n}\mathbb{E}[(X_i - \hat{X}_i)^2|\mathcal{F}_{i-1}]\right)^p\right] + \mathbb{E}\Big[\max_{\tau+2\leq i\leq 2n}|X_i - \hat{X}_i|^{2p}\Big], \end{aligned} \tag{2.5}$$

and we can safely discard the summand indexed by $\tau + 1$ appearing in (2.4), which is uniformly bounded. The maximum on the right-hand side of (2.5) is also bounded by $2\gamma^{2p}$. As for the other term, with X_i and $\hat{X}_i$ as orthogonal random variables, we have

$$\begin{aligned} \sum_{i=\tau+1}^{2n}\mathbb{E}[(X_i - \hat{X}_i)^2|\mathcal{F}_{i-1}] &= \sum_{i=\tau+1}^{2n}\mathbb{E}[X_i^2|\mathcal{F}_{i-1}] + \sum_{i=\tau+1}^{2n}\mathbb{E}[\hat{X}_i^2|\mathcal{F}_{i-1}] \\ &= s^2(\mathbf{X})V^2(\mathbf{X}) + s^2(\mathbf{X}) - 2\underbrace{\sum_{i=1}^{\tau}\mathbb{E}[X_i^2|\mathcal{F}_{i-1}]}. \end{aligned} \tag{2.6}$$

Now, if $\tau = n$, then by definition the sum underbraced above is $s^2(\mathbf{X})V^2(\mathbf{X})$. Otherwise, $\sum_{i=1}^{\tau+1}\mathbb{E}[X_i^2|\mathcal{F}_{i-1}]$ exceeds $s^2(\mathbf{X})$, but as the increments are bounded, the sum underbraced is necessarily larger than $s^2(\mathbf{X}) - \gamma^2$. In any case, we thus have

$$\sum_{i=1}^{\tau}\mathbb{E}[X_i^2|\mathcal{F}_{i-1}] \geq \min\big(s^2(\mathbf{X})V^2(\mathbf{X}), s^2(\mathbf{X}) - \gamma^2\big).$$

Consequently, from (2.6), we obtain that

$$\sum_{i=\tau+1}^{2n}\mathbb{E}[(X_i - \hat{X}_i)^2|\mathcal{F}_{i-1}] \leq |s^2(\mathbf{X})V^2(\mathbf{X}) - s^2(\mathbf{X})| + 2\gamma^2.$$

Combining this with equations (2.5), (2.4), (2.3) and (2.2), we finally obtain that

$$\mathbb{P}\left[\frac{S(\mathbf{X})}{s(\mathbf{X})} \leq t\right] - \Phi(t) \leq 4C_\gamma\frac{n\log(n)}{s^3(\mathbf{X})} + \frac{x}{\sqrt{2\pi}} + \frac{C}{x^{2p}}\left(\|V^2(\mathbf{X}) - 1\|_p^p + \frac{\gamma^{2p}}{s^{2p}(\mathbf{X})}\right).$$

Optimizing this over $x > 0$ leads to the correct estimate. The lower bound is obtained in the same way.

3. Proof of Theorem 1.6

Let $p \geq 1$ and $\alpha \in (1/2, 1)$ be fixed. We let $(X_{ni})_{1 \leq i \leq n-n^{\alpha}}$ be independent random variables with $\mathbb{P}[X_{ni} = \pm 1] = 1/2$. The subsequent $(X_{ni})_{n-n^{\alpha} < i \leq n}$ are defined recursively. Let

$$\lambda_{ni} = \sqrt{n - i + \kappa_n^2},$$

where $\kappa_n = n^{1/4}$ (in fact, any n^{β} with $1 - \alpha < 2\beta < \alpha$ would be fine). Assuming that $X_{n,1}, \ldots, X_{n,i-1}$ have been defined, we write $\mathcal{F}_{n,i-1}$ for the σ-algebra that they generate, and let

$$S_{n,i-1} = \sum_{j=1}^{i-1} X_{nj}.$$

For any i such that $n - n^{\alpha} < i \leq n$, we construct X_{ni} such that

$$\mathbb{P}[X_{ni} \in \cdot \,|\mathcal{F}_{n,i-1}] = \begin{vmatrix} \delta_{-\sqrt{3/2}} + \delta_{\sqrt{3/2}} & \text{if } S_{n,i-1} \in [\lambda_{ni}, 2\lambda_{ni}], \\ \delta_{-\sqrt{1/2}} + \delta_{\sqrt{1/2}} & \text{if } S_{n,i-1} \in [-2\lambda_{ni}, -\lambda_{ni}], \\ \delta_{-1} + \delta_{1} & \text{otherwise,} \end{vmatrix} \tag{3.1}$$

where δ_x is the Dirac mass at point x. Here $(S_{ni})_{i \leq n}$ can be viewed as an inhomogeneous Markov chain. We write $\mathbf{X}_n = (X_{n1}, \ldots, X_{nn})$ and $\mathbf{X}_{ni} = (X_{n1}, \ldots, X_{ni})$ for any $i \leq n$. Let

$$\delta(i) = \sup_{n \geq i} D(\mathbf{X}_{ni}). \tag{3.2}$$

Proposition 3.1. *Uniformly over* n,

$$\|V^2(\mathbf{X}_{ni}) - 1\|_p = \mathrm{O}\big(i^{(\alpha-1)(1+1/2p)}\big) \qquad (i \to +\infty) \tag{3.3}$$

and

$$\delta(i) = \mathrm{O}\big(i^{(\alpha-1)/2}\big) \qquad (i \to +\infty). \tag{3.4}$$

The proof goes as follows. First, we bound $\|V^2(\mathbf{X}_{ni}) - 1\|_p$ in terms of $(\delta(j))_{j \leq i}$ in Lemma 3.2. This gives an inequality on the sequence $(\delta(i))_{i \in \mathbb{N}}$ through Theorem 1.5, from which we deduce (3.4), and then (3.3).

Lemma 3.2. *Let* $K_i = \max_{j \leq i} \delta(j) j^{(1-\alpha)/2}$. *For any* n *and* i, *the following inequalities hold*:

$$|\mathbb{E}[X_{ni}^2] - 1| \leq \begin{vmatrix} 0 & \textit{if } i \leq n - n^{\alpha}, \\ 2\delta(i-1) & \textit{if } n - n^{\alpha} < i \leq n, \end{vmatrix} \tag{3.5}$$

$$|s^2(\mathbf{X}_{ni}) - i| \leq \begin{vmatrix} 0 & \textit{if } i \leq n - n^{\alpha}, \\ C i^{(3\alpha-1)/2} K_i \leq C i^{\alpha} & \textit{if } n - n^{\alpha} < i \leq n, \end{vmatrix} \tag{3.6}$$

$$\|V^2(\mathbf{X}_{ni}) - 1\|_p \leq \begin{vmatrix} 0 & \textit{if } i \leq n - n^{\alpha}, \\ C i^{(\alpha-1)(1+1/2p)} (1 + K_i)^{1/p} + C i^{(3\alpha-3)/2} K_i & \textit{otherwise.} \end{vmatrix} \tag{3.7}$$

Proof. Inequality (3.5) is obvious for $i \le n - n^\alpha$. Otherwise, from the definition (3.1), we know that

$$\mathbb{E}[X_{ni}^2] = 1 + \tfrac{1}{2}\mathbb{P}[S_{n,i-1} \in I_{ni}^+] - \tfrac{1}{2}\mathbb{P}[S_{n,i-1} \in I_{ni}^-],$$

where we write

$$I_{ni}^+ = [\lambda_{ni}, 2\lambda_{ni}] \quad \text{and} \quad I_{ni}^- = [-2\lambda_{ni}, -\lambda_{ni}]. \tag{3.8}$$

The random variable $S_{n,i-1}/s(\mathbf{X}_{n,i-1})$ is approximately Gaussian, up to an error controlled by $\delta(i-1)$. More precisely,

$$\left| \mathbb{P}[S_{n,i-1} \in I_{ni}^+] - \int_{I_{ni}^+/s(\mathbf{X}_{n,i-1})} \mathrm{d}\Phi \right| \le 2\delta(i-1).$$

We obtain (3.5) using the fact that

$$\int_{I_{ni}^+/s(\mathbf{X}_{n,i-1})} \mathrm{d}\Phi = \int_{I_{ni}^-/s(\mathbf{X}_{n,i-1})} \mathrm{d}\Phi.$$

As a by-product, we also learn that

$$|s^2(\mathbf{X}_{ni}) - i| \le \begin{vmatrix} 0 & \text{if } i \le n - n^\alpha, \\ 2 \sum_{n-n^\alpha < j \le i} \delta(j-1) & \text{if } n - n^\alpha < i \le n. \end{vmatrix}$$

Recalling that $\alpha < 1$, we obtain (3.6), noting that for $n - n^\alpha < i \le n$,

$$\sum_{n-n^\alpha < j \le i} \delta(j-1) \le n^\alpha (n - n^\alpha)^{(\alpha-1)/2} K_i.$$

In particular, it follows that

$$s^2(\mathbf{X}_{ni}) = i\big(1 + \mathrm{o}(1)\big). \tag{3.9}$$

Turning now to (3.7), $\|V^2(\mathbf{X}_{ni}) - 1\|_p$ is clearly equal to 0 for $i \le n - n^\alpha$, so let us assume the contrary. We have:

$$\begin{aligned} \|V^2(\mathbf{X}_{ni}) - 1\|_p &= s^{-2}(\mathbf{X}_{ni}) \left\| \sum_{j=1}^{i} \mathbb{E}[X_{nj}^2 | \mathcal{F}_{n,j-1}] - s^2(\mathbf{X}_{ni}) \right\|_p \\ &\le \frac{1}{s^2(\mathbf{X}_{ni})} \sum_{j=1}^{i} \|\mathbb{E}[X_{nj}^2 | \mathcal{F}_{n,j-1}] - 1\|_p + \frac{|s^2(\mathbf{X}_{ni}) - i|}{s^2(\mathbf{X}_{ni})} \\ &\le \frac{1}{2s^2(\mathbf{X}_{ni})} \sum_{n-n^\alpha < j \le i} (\mathbb{P}[S_{n,j-1} \in I_{nj}^+ \cup I_{nj}^-])^{1/p} + \frac{|s^2(\mathbf{X}_{ni}) - i|}{s^2(\mathbf{X}_{ni})}. \end{aligned} \tag{3.10}$$

We consider the two terms in (3.10) separately. First, by the definition of δ, we know that

$$\left|\mathbb{P}[S_{n,j-1} \in I_{nj}^+ \cup I_{nj}^-] - \int_{(I_{nj}^+ \cup I_{nj}^-)/s(\mathbf{X}_{n,j-1})} \mathrm{d}\Phi\right| \le 2\delta(j-1).$$

Equation (3.9) implies that, uniformly over $j > n - n^\alpha$,

$$\int_{(I_{nj}^+ \cup I_{nj}^-)/s(\mathbf{X}_{n,j-1})} \mathrm{d}\Phi = (2\pi)^{-1/2} \frac{2\lambda_{nj}}{s(\mathbf{X}_{n,j-1})}\big(1 + \mathrm{o}(1)\big) \le C n^{(\alpha-1)/2},$$

and so the first term of (3.10) is bounded by

$$\begin{aligned} &\frac{C}{i} \sum_{n-n^\alpha < j \le i} \big(n^{(\alpha-1)/2} + 2\delta(j-1)\big)^{1/p} \\ &\quad \le \frac{C}{i} \sum_{n-n^\alpha < j \le i} \big(n^{(\alpha-1)/2} + 2(n-n^\alpha)^{(\alpha-1)/2} K_i\big)^{1/p} \qquad (3.11) \\ &\quad \le C i^{(\alpha-1)(1+1/2p)} (1+K_i)^{1/p}. \end{aligned}$$

The second term in (3.10) is controlled by (3.6), and we obtain inequality (3.7). □

Proof of Proposition 3.1. Applying Theorem 1.5 with the information given by Lemma 3.2, we obtain that, up to a multiplicative constant that does not depend on n and $i \le n$, $D(\mathbf{X}_{ni})$ is bounded by

$$\frac{\log(i)}{\sqrt{i}} + i^{(\alpha-1)/2}(1+K_i)^{1/(2p+1)} + i^{-3(1-\alpha)p/(4p+2)} K_i^{p/(2p+1)} + i^{-p/(2p+1)}. \qquad (3.12)$$

The first term can be disregarded, because it is dominated by $i^{-p/(2p+1)}$. Also note that as $p \ge 1$, we have

$$\frac{3(1-\alpha)p}{4p+2} \ge \frac{1-\alpha}{2},$$

and as $\alpha > 1/2 > 1/(2p+1)$, we also have

$$\frac{p}{2p+1} \ge \frac{1-\alpha}{2}.$$

Multiplying (3.12) by $i^{(1-\alpha)/2}$, we thus obtain

$$K_i \le C(1+K_i)^{1/(2p+1)} + C K_i^{p/(2p+1)},$$

where we recall that the constant C does not depend on i. Observing that the set $\{x \ge 0\colon x \le C(1+x)^{1/(2p+1)} + Cx^{p/(2p+1)}\}$ is bounded, we obtain that K_i is a bounded sequence, so (3.4) is proved. The relation (3.3) then follows from (3.4) and (3.7). □

Proposition 3.3. *We have*

$$\limsup_{i\to+\infty} i^{(1-\alpha)/2}\delta(i) > 0.$$

Proof. Our aim is to contradict, by reductio ad absurdum, the claim that

$$\delta(i) = \mathrm{o}\big(i^{(\alpha-1)/2}\big) \qquad (i \to +\infty). \tag{3.13}$$

Let $Z_1, \dots, Z_n$ be independent standard Gaussian random variables, and let ξ_n be an independent centered Gaussian random variable with variance κ_n^2, all independent of $\mathbf{X}_n$. Assuming (3.13), we contradict the fact that

$$D(\mathbf{X}_n) = \mathrm{o}\big(n^{(\alpha-1)/2}\big). \tag{3.14}$$

Let $W_{ni} = \sum_{j=i+1}^n Z_j + \xi_n$. Noting that $n^{-1/2}\sum_{j=1}^n Z_j$ is a standard Gaussian random variable, and with the aid of [1], Lemma 1, we learn that

$$\left|\mathbb{P}[W_{n0} \le 0] - \frac{1}{2}\right| \le C\frac{\kappa_n}{\sqrt{n}}$$

and, similarly,

$$\left|\mathbb{P}[S_{nn} + \xi_n \le 0] - \frac{1}{2}\right| \le C\left(D(\mathbf{X}_n) + \frac{\kappa_n}{s(\mathbf{X}_n)}\right).$$

Combining these two observations with (3.6), we thus obtain that

$$\mathbb{P}[S_{nn} + \xi_n \le 0] - \mathbb{P}[W_{n0} \le 0] \le C\left(D(\mathbf{X}_n) + \frac{\kappa_n}{\sqrt{n}}\right). \tag{3.15}$$

As $\kappa_n = n^{1/4}$ and $\alpha > 1/2$, we know that $\kappa_n/\sqrt{n} = \mathrm{o}(n^{(\alpha-1)/2})$. We decompose the left-hand side of (3.15) as

$$\sum_{i=1}^n \mathbb{P}[S_{n,i-1} + X_{ni} + W_{ni} \le 0] - \mathbb{P}[S_{n,i-1} + Z_i + W_{ni} \le 0].$$

The random variable W_{ni} is Gaussian with variance $\lambda_{ni}^2 = n - i + \kappa_n^2$ and is independent of $\mathbf{X}_n$; thus the sum can be rewritten as

$$\sum_{i=1}^n \mathbb{E}\left[\Phi\left(-\frac{S_{n,i-1} + X_{ni}}{\lambda_{ni}}\right) - \Phi\left(-\frac{S_{n,i-1} + Z_i}{\lambda_{ni}}\right)\right]. \tag{3.16}$$

Let $\varphi(x) = (2\pi)^{-1/2}\mathrm{e}^{-x^2/2}$. We can replace

$$\Phi\left(-\frac{S_{n,i-1} + X_{ni}}{\lambda_{ni}}\right) \tag{3.17}$$

by its Taylor expansion,

$$\Phi\left(-\frac{S_{n,i-1}}{\lambda_{ni}}\right) - \frac{X_{ni}}{\lambda_{ni}}\varphi\left(-\frac{S_{n,i-1}}{\lambda_{ni}}\right) + \frac{X_{ni}^2}{2\lambda_{ni}^2}\varphi'\left(-\frac{S_{n,i-1}}{\lambda_{ni}}\right), \tag{3.18}$$

up to an error bounded by

$$\frac{|X_{ni}|^3}{6\lambda_{ni}^3}\|\varphi''\|_\infty. \tag{3.19}$$

Step 1. We show that the error term (3.19), after integration and summation over i, is $\mathrm{o}(n^{(\alpha-1)/2})$. Because X_{ni} is uniformly bounded, it suffices to show that

$$\sum_{i=1}^n \frac{1}{\lambda_{ni}^3} = \mathrm{o}\big(n^{(\alpha-1)/2}\big). \tag{3.20}$$

The foregoing sum equals

$$\sum_{i=1}^n \frac{1}{(n-i+\kappa_n^2)^{3/2}} \le n^{-1/2}\int_{(\kappa_n^2-1)/n}^{(\kappa_n^2+n)/n} x^{-3/2}\,\mathrm{d}x = \mathrm{O}(\kappa_n^{-1}).$$

Because we defined κ_n to be $n^{1/4}$ and $\alpha > 1/2$, equation (3.20) is proved.

Step 2. For the second part of the summands in (3.16), the same holds with X_{ni} replaced by Z_i and, similarly,

$$\sum_{i=1}^n \frac{\mathbb{E}[|Z_i|^3]}{\lambda_{ni}^3} = \mathrm{o}\big(n^{(\alpha-1)/2}\big). \tag{3.21}$$

Step 3. Combining the results of the two previous steps, we know that up to a term of order $\mathrm{o}(n^{(\alpha-1)/2})$, the sum in (3.16) can be replaced by

$$\sum_{i=1}^n \mathbb{E}\left[\frac{Z_i - X_{ni}}{\lambda_{ni}}\varphi\left(-\frac{S_{n,i-1}}{\lambda_{ni}}\right) + \frac{X_{ni}^2 - Z_i^2}{2\lambda_{ni}^2}\varphi'\left(-\frac{S_{n,i-1}}{\lambda_{ni}}\right)\right].$$

Conditional on $S_{n,i-1}$, both Z_i and X_{ni} are centered random variables; thus the first part of the summands vanishes, and only the following remains:

$$\sum_{i=1}^n \mathbb{E}\left[\frac{X_{ni}^2 - Z_i^2}{2\lambda_{ni}^2}\varphi'\left(-\frac{S_{n,i-1}}{\lambda_{ni}}\right)\right] = \sum_{i=1}^n \mathbb{E}\left[\frac{\mathbb{E}[X_{ni}^2 - 1|S_{n,i-1}]}{2\lambda_{ni}^2}\varphi'\left(-\frac{S_{n,i-1}}{\lambda_{ni}}\right)\right]. \tag{3.22}$$

From the definition of X_{ni}, we learn that $\mathbb{E}[X_{ni}^2 - 1|S_{n,i-1}]$ is 0 if $i \le n - n^\alpha$ but otherwise equals

$$\left| \begin{array}{ll} 1/2 & \text{if } S_{n,i-1} \in I_{ni}^+, \\ -1/2 & \text{if } S_{n,i-1} \in I_{ni}^-, \\ 0 & \text{otherwise,} \end{array} \right.$$

where I_{ni}^+ and I_{ni}^- are as defined in (3.8). Consequently, it is clear that the contribution of each summand in the right-hand side of (3.22) is positive. Moreover, for $i > n - n^\alpha$ and in the case where $S_{n,i-1} \in I_{ni}^- \cup I_{ni}^+$, we have

$$\mathbb{E}[X_{ni}^2 - 1|S_{n,i-1}]\varphi'\left(-\frac{S_{n,i-1}}{\lambda_{ni}}\right) \ge \frac{1}{2} \inf_{[1,2]} |\varphi'| > 0.$$

Let us assume temporarily that, uniformly over n and i such that $n - n^\alpha < i \le n - (n^\alpha)/2$, we have

$$\mathbb{P}[S_{n,i-1} \in I_{ni}^- \cup I_{ni}^+] \ge C\frac{\lambda_{ni}}{\sqrt{n}}. \tag{3.23}$$

Then the sum in the right-hand side of (3.22) is, up to a constant, bounded from below by

$$\sum_{n-n^\alpha < i \le n-(n^\alpha)/2} \frac{1}{\lambda_{ni}\sqrt{n}} \ge Cn^\alpha \frac{1}{n^{\alpha/2}\sqrt{n}} = Cn^{(\alpha-1)/2}.$$

This contradicts (3.14) via inequality (3.15), and thus completes the proof of the proposition.

Step 4. There remains to show (3.23), for $n - n^\alpha < i \le n - (n^\alpha)/2$. We have

$$\left| \mathbb{P}[S_{n,i-1} \in I_{ni}^+] - \int_{I_{ni}^+/s(\mathbf{X}_{n,i-1})} \mathrm{d}\Phi \right| \le 2\delta(i-1).$$

Using inequality (3.6), it follows that

$$\int_{I_{ni}^+/s(\mathbf{X}_{n,i-1})} \mathrm{d}\Phi \ge C\frac{\lambda_{ni}}{\sqrt{n}}.$$

Because we choose i inside $[n - n^\alpha, n - (n^\alpha)/2]$, λ_{ni} is larger than $Cn^{\alpha/2}$, whereas $\delta(i-1) = \mathrm{o}(i^{(\alpha-1)/2})$ by assumption (3.13). This proves (3.23). □

Remark. To match the example proposed by [1], $\alpha = 1/3$ and $\kappa_n = 1$ should be used in the definition of the sequences $(\mathbf{X}_n)$. In this case, Propositions 3.1 and 3.3 still hold. Although the proof of Proposition 3.1 can be kept unchanged, Proposition 3.3 requires a more subtle analysis. First, ξ_n of variance $\overline{\kappa}_n^2 \neq 1$ must be chosen, which requires changing the λ_{ni} appearing in (3.16) by, say, $\overline{\lambda}_{ni} = \sqrt{n - i + \overline{\kappa}_n^2}$. The sequence $\overline{\kappa}_n^2$ should grow to infinity with n, while remaining

$o(n^\alpha)$. In Step 1, bounding the difference between (3.17) and (3.18) by (3.19) is too crude. Instead, it can be bounded by

$$\frac{C}{\overline{\lambda}_{ni}^3}\Psi\left(-\frac{S_{n,i-1}}{\overline{\lambda}_{ni}}\right),$$

where $\Psi(x) = \sup_{|y|\le 1} |\varphi''(x+y)|$. One can then appeal to [1], Lemma 2, and get through this step, using the fact that $\overline{\kappa}_n$ tends to infinity. Step 2 is similar, but with some additional care required because Z_i is unbounded. The rest of the proof then applies, taking note of the discrepancy between λ_{ni} and $\overline{\lambda}_{ni}$ when necessary.

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