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Unimodularity for multi-type Galton–Watson trees

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Fix $n \in \mathbb{N}$. Let $\mathbf{T}_n$ be the set of rooted trees (T, o) whose vertices are labeled by elements of $\{1, \dots, n\}$. Let ν be a strongly connected multi-type Galton–Watson measure. We give necessary and sufficient conditions for the existence of a measure μ that is reversible for simple random walk on $\mathbf{T}_n$ and has the property that given the labels of the root and its neighbors, the descendant subtrees rooted at the neighbors of the root are independent multi-type Galton–Watson trees with conditional offspring distributions that are the same as the conditional offspring distributions of ν when the types are ν are ordered pairs of elements of $[n]$. If the types of ν are given by the labels of vertices, then we give an explicit description of such μ.

1. Introduction

Consider simple random walk on a multi-type Galton–Watson tree T with distribution ν. This induces a simple random walk on rooted trees where the root represents the location of the original random walker walking on the unrooted tree T. A stationary measure μ for this random walk can be useful in many ways such as calculating the speed of simple random walk (see [3,4]). If in addition μ is reversible for simple random walk, one has more results through the connection between reversibility of simple random walk and unimodularity (see [1]), which we will explain below.

The case when ν is a multi-type Galton–Watson measure with deterministic conditional offspring distributions is well understood: In [2], we showed that reversible measures for simple random walk on rooted trees whose descendant subtrees are deterministic multi-type Galton–Watson trees exist only for Galton–Watson trees that are descendant subtrees of a cover of a finite, connected, undirected graph. Such Galton–Watson trees are periodic trees, that is, covers of finite, strongly connected, directed graphs. Stationary measures for simple and biased random walk on trees with periodic subtrees were previously studied in [4].

We first define covers of undirected and directed graphs. Let H be a finite connected undirected graph. Label the vertices of H such that any two vertices x, y have the same label if and only if the two rooted graphs H rooted at x and H rooted at y are rooted isomorphic. Fix a vertex x of H and consider the unrooted tree T_x whose vertices are finite paths on H that start from x and do not backtrack, where two vertices are connected by an edge if one is the extension of the other by one edge. It is easy to see that for all $x, y \in V(H)$, we have T_x is isomorphic to T_y (when both are unrooted). Any tree that is isomorphic to T_x for some $x \in V(H)$ is called a *cover* of H. The labeling on H lifts to a labeling of the vertices of the covers of H by labeling each vertex of a cover of H by the label of the last vertex of the path on H that represents that vertex.

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Similarly one can define covers of a finite, connected, directed graph G. Label the vertices of G such that any two vertices x, y have the same label if and only if the two rooted graphs G rooted at x and G rooted at y are rooted isomorphic. For each vertex x of G, we get a cover T_x whose vertices are finite directed paths in G that start at x and two vertices of T_x are connected by an (undirected) edge if the path corresponding to one vertex is the extension of the path corresponding to the other vertex by a directed edge. The covers of G are not necessarily isomorphic.

Now, we explore a connection between covers of undirected and directed graphs, namely, we show that the descendant subtrees of covers of an undirected graph are covers of a directed graph. Let H and T_x for $x \in V(H)$ be as above. Since the vertices of T_x are finite paths in H starting at $x \in V(H)$, there is a vertex of T_x that corresponds to the path consisting of just the vertex $x \in V(H)$. Call this vertex of T_x, x, as well. Consider T_x as rooted at $x \in V(T_x)$. Delete $x \in V(T_x)$, hence the edges incident to it, in T_x. The remaining trees, the descendant subtrees of the neighbors of $x \in V(T_x)$, are covers of the same directed graph. To see this remember that the vertices of T_x are labeled by the label of the last vertex of the corresponding path on H. Thus, if $y \in V(T_x)$ has label i and the parent of y, the neighbor of y on the path to x, has label j, it means that the last edge of the path in H corresponding to y has endpoints labeled j and i with i being the label of the last vertex of this path. Since the paths do not backtrack any vertex of T_x that has label i and whose parent has label j, has the same descendant tree. Now, we can describe the directed graph G, whose covers are exactly the descendant subtrees of T_x. Vertices of G are ordered pairs of labels of H and for any labels i, j, k, (j, i) is connected to (i, k) by m directed edges if a vertex of H labeled i has a neighbor labeled j and m other neighbors that are labeled k.

Let (T, o) denote the tree T rooted at $o \in V(T)$. Then for any rooted cover (T, o) of a finite connected undirected graph H, when o is deleted, the remaining trees are deterministic multi-type Galton–Watson trees, as explained in the previous paragraph, where the type of a vertex is determined by a one-to-one function of the label of the vertex and of its parent. The uniform measure on the vertices of H biased by the degree of the vertex lifts to a measure μ on rooted covers of H that is reversible for simple random walk. This also establishes the existence of a unimodular measure with the same support since reversibility of μ is equivalent to the unimodularity of the probability measure obtained by biasing μ by the reciprocal of the degree of the root. We will obtain our results in the setting of reversible measures on rooted trees.

This relationship for (single-type, non-deterministic) Galton–Watson trees was established in [3]. Let ν be a Galton–Watson measure with offspring distribution $\{p_k\}_{k \in \mathbb{N}}$. Start with one vertex, the root, that has $k + 1$ neighbors with probability p_k, each of which has independent Galton–Watson descendant subtrees with offspring distribution ν. This is the Augmented Galton–Watson measure and is reversible for simple random walk on rooted trees. Biasing this measure by the reciprocal of the degree of the root, one gets the unimodular Galton–Watson measure.

Now, let ν be a multi-type Galton–Watson measure. Motivated by the results on deterministic multi-type Galton–Watson trees and non-deterministic single-type Galton–Watson trees, we try to find reversible measures μ for simple random walk on labeled rooted trees with the following property: Conditioned on the labels of the root and its neighbors the subtrees rooted at the neighbors of the root are independent (non-deterministic) multi-type Galton–Watson subtrees whose conditional offspring distributions are the same as the conditional offspring distributions of ν.

The deterministic case shows that if μ satisfies the property above, then the types of ν are not necessarily given by the labels of vertices. Therefore, we introduce relabeling functions on rooted labeled trees.

For $n \in \mathbb{N}$, let $\mathbf{T}_n$ be the set of rooted trees (T, o) whose vertices are labeled by elements of $\{1, \dots, n\}$. Let f be a function from $\mathbf{T}_n$ to rooted trees whose vertices are labeled except for the root, with the following properties: $V(f(T, o)) = V(T, o)$, $E(f(T, o)) = E(T, o)$ and $f(T, o)$ is rooted at o. Let μ be a measure on $\mathbf{T}_n$. Suppose that conditioned on the labels of the root and its neighbors, after relabeling the vertices by f the descendant subtrees of the neighbors of the root are independent Galton–Watson trees with conditional offspring distributions the same as those of ν. In this case, we write $\mu \sim_f \nu$ and the label of a vertex after relabeling gives the ν-type of that vertex.

We consider two cases: In the first case, the new label of a vertex is given by a one-to-one function of the label of the vertex and of its parent. That is, if the labels of $x \in V(f(T_1, o_1))$ and $y \in V(f(T_2, o_2))$ are the same, then $x \in V(T_1, o_1)$ and $y \in V(T_2, o_2)$ have the same label and their parents have the same label as well. For a given ν, we give necessary and sufficient conditions on the conditional offspring distributions of ν for the existence of a measure μ such that $\mu \sim_f \nu$ where f is as above and μ is reversible for simple random walk (see Theorem 3.5 and Example 3.6). We show that among all the measures that have a given conditional offspring distribution, there is at most one measure μ such that $\mu \sim_f \nu$ for some strongly connected Galton–Watson measure ν and μ is reversible and we parametrize all such μ (see Proposition 3.9 and Example 3.10).

We also study the case when there is no relabeling, that is, for all (T, o) and all $x \in V(T, o)$, the label of x in (T, o) is the same as the label of x in $f(T, o)$. In this case, given the degree and the type of the root, the conditional offspring distributions of the root with respect to μ and ν are multinomial and the parameters for the conditional offspring distributions do not depend on the degree of the root (see Theorem 4.1 and Example 4.3).

2. Definitions and notations

A rooted tree (T, o) is a connected graph T with no cycles that has a distinguished vertex o. For any $x \in V(T, o)$ other than o, we call the neighbor of x on the unique simple path between x and o the *parent* of x. The remaining neighbors of x are called *children* of x. The neighbors of the root are called children of the root.

We shall work with rooted isomorphism classes of rooted labeled trees: For all $n \in \mathbb{N}$ let $[n] := \{1, \dots, n\}$. A labeled rooted tree is a rooted tree whose vertices are labeled by elements of some fixed $[n]$. Two rooted labeled trees (T, o), (T', o') are called *rooted isomorphic* if there is a bijection between the vertices of the two trees that maps o to o', preserves adjacency and any two vertices of T that have the same label are mapped to vertices that have same label and conversely. We shall use (T, o) to denote the rooted isomorphism class of the labeled rooted tree (T, o) as well. For all n, let $\mathbf{T}_n$ denote the set of rooted isomorphism classes of rooted trees labeled by elements of $[n]$.

Fix $n \in \mathbb{N}$. Let ν be a Galton–Watson measure whose types are elements of $[n]$. We say that ν is *strongly connected* if for all $i, j \in [n]$, the root is of type i with positive probability and a tree has a vertex of type j with positive probability given that the root has type i.

Let $\mathbf{S}_n := \{(c_1, \dots, c_n)\colon\ c_i \in \mathbb{N} \cup 0\}$. For $\mathbf{c} = (c_1, \dots, c_n) \in \mathbf{S}_n$, let $\mathbf{c}_j := (c_1, \dots, c_j - 1, \dots, c_n)$, $\mathbf{c}^k := (c_1, \dots, c_k + 1, \dots, c_n)$, $\mathbf{c}_j^k := (\mathbf{c}_j)^k$ and $|\mathbf{c}| := \sum_i c_i$. We use numbers in parenthesis to index a set of vectors. If $\mathbf{c}(1), \dots, \mathbf{c}(m)$ are m vectors, then $c(i)_p$ denotes the pth coordinate of the vector $\mathbf{c}(i)$.

We say that a vertex has $\mathbf{c}$ neighbors if it has exactly c_j neighbors labeled j for every $j \in [n]$. For all $i \in \mathbb{N}$, let N_i be the set of trees rooted at a vertex labeled i. Let $N_i(\mathbf{c})$ be the set of trees whose roots are labeled i and have $\mathbf{c}$ neighbors.

For a probability measure σ on $\mathbf{T}_n$ and $i \in [n]$, let $\sigma(i) := \sigma(N_i)$.

For any measure σ on $\mathbf{S}_n$, we say that σ has the *multinomial* distribution with *parameters* $p_1, \dots, p_n \in [0, 1]$ satisfying $\sum_{i=1}^n p_i = 1$ if for some $l \in \mathbb{N}$, we have $\sigma(\mathbf{c}) = \frac{|\mathbf{c}|!}{\prod_{k\in[n]} c_k!} \prod_{k\in[n]} p_k^{c_k}$ for all $\mathbf{c} \in \mathbf{S}_n$ with $|\mathbf{c}| = l$.

For all $k \in \mathbb{N}$, $\mathbf{c}, \mathbf{d} \in \mathbf{S}_n$ and $i_1, i_2, \dots, i_k \in [n]$, define $B_{\mathbf{c},i_1,i_2,\dots,i_k,\mathbf{d}}$ to be the set of trees whose roots are labeled i_1 and whose roots have $\mathbf{c}^{i_2}$ children and at least one child of the root that is labeled i_2 has at least one child labeled i_3, which has at least one child labeled i_4 and so on up to a vertex labeled i_k, which has $\mathbf{d}$ children. We also define

$$B_{\mathbf{c},i_1,i_2,\dots,i_k} := \bigcup_{\mathbf{d}} B_{\mathbf{c},i_1,i_2,\dots,i_k,\mathbf{d}},$$

$$B_{i_1,i_2,\dots,i_k,\mathbf{d}} := \bigcup_{\mathbf{c}} B_{\mathbf{c},i_1,i_2,\dots,i_k,\mathbf{d}},$$

$$B_{i_1,i_2,\dots,i_k} := \bigcup_{\mathbf{c},\mathbf{d}} B_{\mathbf{c},i_1,i_2,\dots,i_k,\mathbf{d}}.$$

For a probability measure ρ on $\mathbf{S}_n$, we define the following sets:

$$A(\rho) := \{j \in [n]\colon\ \exists \mathbf{c} \text{ such that } c_j > 0, \rho(\mathbf{c}) > 0\},$$
$$D(\rho) := \{d\colon\ \exists \mathbf{c} \text{ with } |\mathbf{c}| = d, \rho(\mathbf{c}) > 0\}.$$

For a probability measure σ on $\mathbf{T}_n$ and $i \in [n]$, let σ_i denote the conditioning of σ on N_i. Let $\sigma_i(\mathbf{c}) := \sigma_i(N_i(\mathbf{c}))$. This way we interpret σ_i as a probability measure on $\mathbf{S}_n$.

Thus, the neighbors of a vertex x of $(T, o) \in \mathbf{T}_n$ whose label is i are labeled by elements of $A(\mu_i)$ μ-a.s. and the number of neighbors of x is in $D(\mu_i)$ μ-a.s. We write $\mathbf{c} \in A(\rho)$ if $c_j = 0$ for every $j \notin A(\rho)$. Hence, $\rho(\mathbf{c}) > 0$ implies $\mathbf{c} \in A(\rho)$.

3. Main results

Remember that the relabeling function f is a function from rooted labeled trees to rooted trees whose vertices are labeled except for the root such that the vertices, edges and the root of $f(T, o)$ is the same as those of (T, o). So if we forget about the labels of (T, o) and $f(T, o)$, they are the same rooted tree. Fix f such that for all $(T, o), (T', o') \in \mathbf{T}_n$ and $x \in V(T, o)$, $y \in V(T', o')$ the labels of x in $f(T, o)$ and y in $f(T', o')$ are the same if and only if the labels of x in (T, o) and y in (T', o') are the same and the parent of x in (T, o) has the same label as the parent of y in (T', o'). Thus, if $x \in V(T, o)$ has label j and its parent has label i, then we denote the label of x in $f(T, o)$ by (i, j). We write $\nu_{i,j}$ for $\nu_{(i,j)}$ and $\nu_{i,j}(\mathbf{c})$ for $\nu_{i,j}((j, c_1), \ldots, (j, c_n)) = \nu_{i,j}(N_{(i,j)}((j, c_1), \ldots, (j, c_n)))$.

For this section, μ denotes a measure on $\mathbf{T}_n$ such that $\mu(i) > 0$ for all $i \in [n]$ and ν denotes a Galton–Watson measure whose types are ordered pairs of elements of $[n]$. The relabeling function f is fixed as above and we write $\mu \sim \nu$ instead of $\mu \sim_f \nu$.

Our first goal is to find necessary and sufficient conditions on ν for the existence of a reversible measure μ for simple random walk such that $\mu \sim \nu$. We start by studying properties of μ and ν when $\mu \sim \nu$.

Remark 3.1. Assume $\mu \sim \nu$. For all $i_1, \ldots, i_k \in [n]$ and for all $\mathbf{c} \in \mathbf{S}_n$, if $\mu(B_{i_1,\ldots,i_k,\mathbf{c}}) > 0$, then the following two properties hold:

(i) for all $j \in [2, n-1]$, $(i_j, i_{j+1}) \in A(\nu_{i_{j-1},i_j})$,
(ii) $\nu_{i_{k-1},i_k}(\mathbf{c}) > 0$.

Lemma 3.2. *Assume that $\mu \sim \nu$ and that μ is stationary for simple random walk. Then for all $\mathbf{c} \in \mathbf{S}_n$ and $i, j \in [n]$ such that $c_j > 0$,*

$$\mu\big(N_i(\mathbf{c})\big) > 0 \ (\textit{or equivalently } \mu_i(\mathbf{c}) > 0) \quad \iff \quad \nu_{j,i}(\mathbf{c}_j) > 0. \tag{3.1}$$

In this case, we have

$$\forall i, j, k \in [n] \textit{ and } \mathbf{c} \in \mathbf{S}_n \textit{ with } c_j, c_k > 0 \qquad \nu_{k,i}(\mathbf{c}_k) > 0 \quad \iff \quad \nu_{j,i}(\mathbf{c}_j) > 0. \tag{3.2}$$

Proof. Fix i, j and $\mathbf{c}$ with $c_j > 0$.

Assume $\mu(N_i(\mathbf{c})) > 0$. Then $\mu(B_{\mathbf{c}_j,i,j}) > 0$. A random walker at $(T, o) \in B_{\mathbf{c}_j,i,j}$ can walk to a tree in $B_{j,i,\mathbf{c}_j}$ with positive probability. Since μ is stationary and $\mu(B_{\mathbf{c}_j,i,j}) > 0$, we have $\mu(B_{j,i,\mathbf{c}_j}) > 0$. By Remark 3.1 we have $\nu_{j,i}(\mathbf{c}_j) > 0$.

Conversely, assume $\nu_{j,i}(\mathbf{c}_j) > 0$. Since ν is strongly connected, there exist $i_1, \ldots, i_{k-1} = j, i_k = i$ such that $\mu(B_{i_1,i_2,\ldots,j,i,\mathbf{c}_j}) > 0$. A random walker at $(T, o) \in B_{i_1,i_2,\ldots,j,i,\mathbf{c}_j}$ can walk to a tree in $B_{\mathbf{c}_j,i,j,\ldots,i_2,i_1}$ with positive probability. Since μ is stationary, this gives $\mu(B_{\mathbf{c}_j,i,j,\ldots,i_2,i_1}) > 0$, which implies $\mu(N_i(\mathbf{c})) = \mu(B_{\mathbf{c}_j,i,j}) > 0$.

The statement in (3.2) is a direct corollary of the first statement of the lemma. □

Let μ be a measure on $\mathbf{T}_n$ and ν be a multi-type Galton–Watson measure with $\mu \sim \nu$. For $A \in \mathbf{T}_n$, let $\mathbf{P}((T, o) \to A)$ denote the probability that a simple random walker that starts at (T, o) moves to some $(T, o') \in A$ in one step. Then μ is reversible if and only if for all measurable sets $A, B \subset \mathbf{T}_n$

$$\int_A \mathbf{P}\big((T, o) \to B\big)\, \mathrm{d}\mu(T, o) = \int_B \mathbf{P}\big((T, o) \to A\big)\, \mathrm{d}\mu(T, o). \tag{3.3}$$

We first prove a lemma that says μ is reversible if and only if the reversibility equations (3.3) hold for a special family of measurable sets.

Lemma 3.3. *Let μ be a measure on $\mathbf{T}_n$. Let ν be a multi-type Galton–Watson measure. Assume that $\mu \sim \nu$ and (3.1) holds. Then the following statements are equivalent*:

(i) *μ is reversible*,

(ii) *for all $i, j \in [n]$ and $\mathbf{c}, \mathbf{d} \in \mathbf{S}_n$ with $c_j, d_i > 0$ we have*

$$\mu(i)\mu_i(\mathbf{c})\nu_{i,j}(\mathbf{d}_i)\frac{c_j}{|\mathbf{c}|} = \mu(j)\mu_j(\mathbf{d})\nu_{j,i}(\mathbf{c}_j)\frac{d_i}{|\mathbf{d}|}, \tag{3.4}$$

(iii) *for all $i, j \in [n]$ and $\mathbf{c}, \mathbf{d} \in \mathbf{S}_n$*

$$\int_{N_i(\mathbf{c})} \mathbf{P}\big((T,o) \to N_j(\mathbf{d})\big)\,\mathrm{d}\mu(T,o) = \int_{N_j(\mathbf{d})} \mathbf{P}\big((T,o) \to N_i(\mathbf{c})\big)\,\mathrm{d}\mu(T,o). \tag{3.5}$$

Proof. Statement (i) is equivalent to (3.3), so our task is to show that (3.3), (3.4) and (3.5) are equivalent.

If ν is a deterministic multi-type Galton–Watson measure, then any measurable set is a finite disjoint union of the sets $N_i(\mathbf{c}) = N_i$ up to sets of measure zero. Hence, (3.3) and (3.5) are equivalent. On the other hand when μ is deterministic, (3.5) is equivalent to

$$\mu\big(N_i(\mathbf{c})\big)\frac{c_j}{|\mathbf{c}|} = \mu\big(N_j(\mathbf{d})\big)\frac{d_i}{|\mathbf{d}|}$$

since $\mathbf{P}((T,o) \to N_j(\mathbf{d})) = \frac{c_j}{|\mathbf{c}|}$ for all (actually the unique) $(T,o) \in N_i(\mathbf{c})$. Both sides of (3.4) and (3.5) are zero unless the root has label i and has neighbors $\mathbf{c}$ with $c_j > 0$ with positive probability and the root has label j and has neighbors $\mathbf{d}$ with positive probability. Since $\mu(i)\mu_i(\mathbf{c}) = \mu(N_i(\mathbf{c}))$, $\mu(j)\mu_j(\mathbf{d}) = \mu(N_j(\mathbf{d}))$ and $\nu_{i,j}(\mathbf{d}_i) = \nu_{j,i}(\mathbf{c}_j) = 1$, we conclude that (3.5) is equivalent to (3.4).

Let ν be a non-deterministic multi-type Galton–Watson measure. We first show (3.3) is equivalent to condition (ii). We emulate the proof of Theorem 3.1 in [3].

Given two rooted trees $(T,o), (T',o') \in \mathbf{T}_n$, define $[(T,o) \bullet\!\!-\!(T',o')]$ to be the tree formed by joining disjoint copies of (T,o) and (T',o') by an edge between o and o' and rooting the new tree at o. For sets $F, F' \subset \mathbf{T}_n$, write

$$[F \bullet\!\!-\!F'] := \big\{\big[(T,o) \bullet\!\!-\!\big(T',o'\big)\big] \colon (T,o) \in F, \big(T',o'\big) \in F'\big\}.$$

For trees $(T_1,o_1), \dots, (T_m,o_m) \in \mathbf{T}_n$ and $i \in [n]$, let $[i \bullet\!\!-\! \bigvee_{s=1}^m (T_s,o_s)]$ denote the tree formed by joining the vertices o_s by single edges to a new vertex labeled i, the latter being the root of the new tree. Furthermore, for sets $F_1, \dots, F_m$, write

$$\left[i \bullet\!\!-\! \bigvee_{s=1}^m F_s\right] := \left\{\left[i \bullet\!\!-\! \bigvee_{s=1}^m (T_s,o_s)\right] \colon (T_s,o_s) \in F_s\right\}.$$

For any rooted tree (T,o) and $x \in V(T)$, the *distance* of x to o is the number of edges on the simple path between x and o. For any rooted tree (T,o) and for a positive integer h, the restriction of (T,o) to the first h levels is the finite tree induced by the vertices whose distance to o is less than or equal to h. The *height* of a finite rooted tree (T, o) is the maximum of the distances of the vertices of T to o.

For a finite tree (T, o) of height $h \in \mathbb{N}$, let $B_{(T,o)}$ be the set of rooted trees whose restriction to the first h levels is isomorphic to (T, o). The sets $B_{(T,o)}$ generate the σ-field. Let K be the set of all $B_{(T,o)}$. Then any element of K is of the form $[B_{(T,o)} \bullet\!- B_{(T',o')}] = B_{[(T,o)\bullet\!-(T',o')]}$ where

$$(T, o) := \left[i \bullet\!- \bigvee_{s=1}^{l} (T_s, o_s) \right] \quad \text{and} \quad \left(T', o'\right) := \left[j \bullet\!- \bigvee_{s=1}^{m} \left(T'_s, o'_s\right) \right] \tag{3.6}$$

for finite trees $(T_1, o_1), \dots, (T_l, o_l)$ and $(T'_1, o'_1), \dots, (T'_m, o'_m)$.

We now show that (3.3) holds for all sets of the form $A = B_{[(T,o)\bullet\!-(T',o')]} \in K$ and $B = B_{[(T',o')\bullet\!-(T,o)]}$ if and only if (3.4) holds. Let $l(x)$ denote the label of a vertex x. Fix $[(T, o) \bullet\!-(T', o')]$. Let h be the height of $[(T, o) \bullet\!-(T', o')]$. Without loss of generality, assume (T, o) has height h. Let $\mathbf{c}, \mathbf{d} \in \mathbf{S}_n$ be such that for all k we have

$$c_k = \left|\left\{x \in V\left(\left[(T, o) \bullet\!-\left(T', o'\right)\right]\right):\ x \sim o, l(x) = k\right\}\right|$$

and

$$d_k = \left|\left\{x \in V\left(\left[\left(T', o'\right) \bullet\!-(T, o)\right]\right):\ x \sim o', l(x) = k\right\}\right|.$$

Let $j = l(o')$ and $i = l(o)$. Then we have $c_j > 0$ and $d_i > 0$. Define c^*_j by

$$c^*_j := \left|\left\{(T_s, o_s):\ (T_s, o_s) \cong \left(T', o'\right), s \in [l]\right\}\right| + 1.$$

Then the probability that the chain moves from an element of A to an element of B is $\frac{c^*_j}{|\mathbf{c}|}$.

Let (T^*, o) be the restriction of (T, o) to the first $h - 2$ levels. Let d^*_i be 1 plus the number of $o'_k \sim o' \in V(T')$ for $k = 1, \dots, m$ such that (T^*, o) is isomorphic to (T'_k, o'_k).

Let D, D' be such that

$$\nu_{j,i}(B_{(T,o)}) = \nu_{j,i}(\mathbf{c}_j) \binom{c_j - 1}{c^*_j - 1} \nu_{i,j}(B_{(T',o')})^{c^*_j - 1} D \tag{3.7}$$

and

$$\nu_{i,j}(B_{(T',o')}) = \nu_{i,j}(\mathbf{d}_i) \binom{d_i - 1}{d^*_i - 1} \nu_{j,i}(B_{(T^*,o)})^{d^*_i - 1} D'. \tag{3.8}$$

We first calculate $\mu(B_{[(T,o)\bullet\!-(T',o')]})$. In (3.7), the term D takes care of all the descendant subtrees of the neighbors of o that are not labeled j and also the descendant subtrees of the $c_j - 1 - (c^*_j - 1) = c_j - c^*_j$ neighbors of o that are labeled j and have descendant subtrees whose restriction to the first $h - 1$ levels are not (T', o'). To calculate $\mu(B_{[(T,o)\bullet\!-(T',o')]})$, we choose $c_j - c^*_j$ of the c_j neighbors of the root that are labeled j to have descendant subtrees whose restrictions to the first $h - 1$ levels are not (T', o'). We have

$$\mu(B_{[(T,o)\bullet\!-(T',o')]}) = \mu(i)\mu_i(\mathbf{c}) \binom{c_j}{c_j - c^*_j} \nu_{i,j}(B_{(T',o')})^{c^*_j} D,$$

which by (3.8) is equal to

$$\mu(i)\mu_i(\mathbf{c})\binom{c_j}{c_j^*}\nu_{i,j}(B_{T',o'})^{c_j^*-1}D\nu_{i,j}(\mathbf{d}_i)\binom{d_i-1}{d_i^*-1}\nu_{j,i}(B_{(T^*,o)})^{d_i^*-1}D'.$$

Then the left-hand side of (3.3) for $A = B_{[(T,o)\bullet\!-(T',o')]}$ and $B = B_{[(T',o')\bullet\!-(T,o)]}$ is equal to $\mu(B_{[(T,o)\bullet\!-(T',o')]})\frac{c_j^*}{|\mathbf{c}|}$, which in turn is equal to

$$\mu(i)\mu_i(\mathbf{c})\nu_{i,j}(\mathbf{d}_i)\frac{c_j}{|\mathbf{c}|}\binom{c_j-1}{c_j^*-1}\nu_{i,j}(B_{(T',o')})^{c_j^*-1}\binom{d_i-1}{d_i^*-1}\nu_{j,i}(B_{(T^*,o)})^{d_i^*-1}DD'. \tag{3.9}$$

Similarly, we calculate the right-hand side of (3.3). For $k \in [d_i^*]$, let $B^k_{(T',o')}$ be the set of all trees (T'',o') such that

(i) the restriction of (T'',o') to the first $h-2$ levels is isomorphic to (T',o'),

(ii) when $o' \in V(T'',o')$ is deleted, there are exactly $k-1$ trees (out of the $|\mathbf{d}_i| = m$) whose restrictions to the first h levels are isomorphic to (T,o).

So for any $(T'',o') \in B^k_{(T',o')}$, we have that o' has d_i^*-1 neighbors whose descendant trees' restrictions to the first $h-2$ levels are isomorphic to (T^*,o) and exactly $k-1$ of them have descendant trees whose restrictions to the first h levels are isomorphic to (T,o). Let E be defined by $\nu_{j,i}(B_{(T,o)}) = \nu_{j,i}(B_{(T^*,o)})E$. Now we have

$$\mu\big([B^k_{(T',o')}\bullet\!-B_{(T,o)}]\big) = \mu(j)\mu_j(\mathbf{d})\binom{d_i}{d_i^*}\nu_{j,i}(B_{(T^*,o)})^{d_i^*}\binom{d_i^*}{k}E^k(1-E)^{d_i^*-k}D'.$$

Then the right-hand side of (3.3) for $A = B_{[(T,o)\bullet\!-(T',o')]}$ and $B = [B^k_{(T',o')}\bullet\!-B_{(T,o)}]$ is equal to $\mu([B^k_{(T',o')}\bullet\!-B_{(T,o)}])\frac{k}{|\mathbf{d}|}$. Since $B_{[(T',o')\bullet\!-(T,o)]} = \bigcup_{k=1}^{d_i^*}[B^k_{(T',o')}\bullet\!-B_{(T,o)}]$ and the sets $[B^k_{(T',o')}\bullet\!-B_{(T,o)}]$ are disjoint, the right-hand side of (3.3) for $A = B_{[(T,o)\bullet\!-(T',o')]}$ and $B = B_{[(T',o')\bullet\!-(T,o)]}$ is given by

$$\begin{aligned}&\sum_{k=1}^{d_i^*}\mu(j)\mu_j(\mathbf{d})\binom{d_i}{d_i^*}\nu_{j,i}(B_{T^*,o})^{d_i^*}\binom{d_i^*}{k}E^k(1-E)^{d_i^*-k}D'\frac{k}{|\mathbf{d}|}\\ &\quad= \mu(j)\mu_j(\mathbf{d})\binom{d_i}{d_i^*}\nu_{j,i}(B_{(T^*,o)})^{d_i^*}d_i^*D'\frac{1}{|\mathbf{d}|}E\sum_{k=1}^{d_i^*}\binom{d_i^*-1}{k-1}E^{k-1}(1-E)^{d_i^*-1-(k-1)}.\end{aligned}$$

The sum on the right is equal to 1. Since $\nu_{j,i}(B_{(T^*,o)})E = \nu_{j,i}(B_{T,o})$, the above expression reduces to

$$\mu(j)\mu_j(\mathbf{d})\frac{d_i}{|\mathbf{d}|}\binom{d_i-1}{d_i^*-1}\nu_{j,i}(B_{(T^*,o)})^{d_i^*-1}D'\nu_{j,i}(B_{(T,o)}).$$

Using (3.7) this is equal to

$$\mu(j)\mu_j(\mathbf{d})\frac{d_i}{|\mathbf{d}|}\binom{d_i-1}{d_i^*-1}\nu_{j,i}(B_{(T^*,o)})^{d_i^*-1}D'\nu_{j,i}(\mathbf{c}_j)\binom{c_j-1}{c_j^*-1}\nu_{i,j}(B_{T',o'})^{c_j^*-1}D. \tag{3.10}$$

Setting (3.9) equal to (3.10) we see that when (3.1) holds and $A = B_{[(T,o)\bullet\!-(T',o')]}$ and $B = B_{[(T',o')\bullet\!-(T,o)]}$, (3.3) is equivalent to (3.4).

Since K is a π-system, by the $\pi - \lambda$ theorem (3.3) holds for all $[A \bullet\!- B]$ if and only if it holds for all $[B_{(T,o)} \bullet\!- B_{(T',o')}] \in K$. Thus, our first claim is proved.

Next, we show that (3.5) is equivalent to (3.4). If $(T,o) \in N_i(\mathbf{c})$, then a random walker at (T,o) moves to a tree $(T',o') \in N_j(\mathbf{d})$ with probability $\frac{k}{|\mathbf{c}|}$, where k is the number of children of o of type j that have $\mathbf{d}_i$ children. Hence, we have

$$\begin{aligned}
\text{LHS of (3.5)} &= \sum_{k=1}^{c_j}\mu\big(N_i(\mathbf{c})\big)\big(\nu_{i,j}(\mathbf{d}_i)\big)^k\big(1-\nu_{i,j}(\mathbf{d}_i)\big)^{c_j-k}\binom{c_j}{k}\frac{k}{|\mathbf{c}|}\\
&= \mu\big(N_i(\mathbf{c})\big)\nu_{i,j}(\mathbf{d}_i)\frac{c_j}{|\mathbf{c}|}\sum_{k=1}^{c_j}\big(\nu_{i,j}(\mathbf{d}_i)\big)^{k-1}\big(1-\nu_{i,j}(\mathbf{d}_i)\big)^{c_j-k}\binom{c_j-1}{k-1}\\
&= \mu\big(N_i(\mathbf{c})\big)\nu_{i,j}(\mathbf{d}_i)\frac{c_j}{|\mathbf{c}|} = \mu(i)\mu_i(\mathbf{c})\nu_{i,j}(\mathbf{d}_i)\frac{c_j}{|\mathbf{c}|}.
\end{aligned}$$

Making the same calculation for the right-hand side of (3.5), we have that (3.5) is equivalent to (3.4). □

Now assume that μ is reversible for simple random walk with $\mu \sim \nu$. Let $\mathbf{c},\mathbf{d},\mathbf{e} \in \mathbf{S}_n$ be such that $c_j, d_i, e_i > 0$ and $\mu_i(\mathbf{c}), \mu_j(\mathbf{d}), \mu_j(\mathbf{e}) > 0$. By Lemma 3.2, we have that (3.2) holds. Then (3.5) holds for $A = N_i(\mathbf{c})$ and $B = N_j(\mathbf{d})$ which reduces to (3.4). Similarly (3.5) holds for $A = N_i(\mathbf{c})$ and $B = N_j(\mathbf{e})$ which reduces to

$$\mu(i)\mu_i(\mathbf{c})\nu_{i,j}(\mathbf{e}_i)\frac{c_j}{|\mathbf{c}|} = \mu(j)\mu_j(\mathbf{e})\nu_{j,i}(\mathbf{c}_j)\frac{e_i}{|\mathbf{e}|}. \tag{3.11}$$

Combining (3.4) and (3.11), we get

$$\frac{\nu_{i,j}(\mathbf{d}_i)}{\nu_{i,j}(\mathbf{e}_i)} = \frac{\mu_j(\mathbf{d})d_i/|\mathbf{d}|}{\mu_j(\mathbf{e})e_i/|\mathbf{e}|} \tag{3.12}$$

holds for all $N_j(\mathbf{d}), N_j(\mathbf{e})$ for which $\mu_j(\mathbf{d}), \mu_j(\mathbf{e}) > 0$ with $d_i, e_i > 0$.

Now for all k, $i_1,\dots,i_k = i_1 \in [n]$ and $\mathbf{c}(1),\dots,\mathbf{c}(k) = \mathbf{c}(1)$ such that $\mu_{i_p}(\mathbf{c}(p)) > 0$ with $c(p)_{i_{p-1}}, c(p)_{i_{p+1}} > 0$ for $p = 1,\dots,k-1$, we have

$$\prod_{p=1}^{k-1}\frac{\mu_{i_p}(\mathbf{c}(i))}{\mu_{i_{p+1}}(\mathbf{c}(i+1))} = 1.$$

Using (3.12), we have

$$\prod_{p=1}^{k-1} \frac{\nu_{i_p,i_{p+1}}(\mathbf{c}(p+1)_{i_p})}{\nu_{i_{p+1},i_p}(\mathbf{c}(p)_{i_{p+1}})} \frac{c(p)_{i_{p+1}}}{c(p+1)_{i_p}} = 1. \tag{3.13}$$

In Lemma 3.4, we prove that (3.13) holds for all $i_1, \dots, i_k$ and $\mathbf{c}(1), \dots, \mathbf{c}(k)$ as above if it holds only for a special family. Then in Theorem 3.5, we prove that if the conditional offspring distributions of ν satisfy (3.13) for this special family, as well as (3.2), then there exists a measure μ such that μ is reversible for simple random walk and $\mu \sim \nu$.

Lemma 3.4. *Let $n \in \mathbb{N}$. Let ν be the distribution of a multi-type Galton–Watson tree whose types are ordered pairs of elements of $[n]$. Then (3.13) holds for all k, $i_1, \dots, i_k = i_1 \in [n]$ and $\mathbf{c}(1), \dots, \mathbf{c}(k) = \mathbf{c}(1) \in \mathbf{S}_n$ such that $\nu_{i_p,i_{p+1}}(\mathbf{c}(p+1)_{i_p}) > 0$ for $p = 1, \dots, k-1$ if and only if (3.13) holds under the restriction that $\mathbf{c}(s) \neq \mathbf{c}(t)$ whenever $i_s = i_t$ for $s, t \in [k-1]$.*

Proof. The latter condition is a special case of the former so we prove it implies the former statement. Let $i_1, \dots, i_m = i_1 \in [n]$ and $\mathbf{d}(1), \dots, \mathbf{d}(m) = \mathbf{d}(1)$ such that $\nu_{i_p,i_{p+1}}(\mathbf{d}(p+1)_{i_p}) > 0$ for $p = 1, \dots, m-1$.

If for some $l, u \in [m]$, $\mathbf{d}(l), \dots, \mathbf{d}(u) = \mathbf{d}(l) \in \mathbf{S}_n$ satisfy the latter condition in the statement of the lemma, then

$$\begin{aligned}
&\prod_{p=1}^{m-1} \frac{\nu_{i_p,i_{p+1}}(\mathbf{d}(p+1)_{i_p})}{\nu_{i_{p+1},i_p}(\mathbf{d}(p)_{i_{p+1}})} \frac{d(p)_{i_{p+1}}}{d(p+1)_{i_p}} \\
&\quad = \prod_{p=l}^{u-1} \frac{\nu_{i_p,i_{p+1}}(\mathbf{d}(p+1)_{i_p})}{\nu_{i_{p+1},i_p}(\mathbf{d}(p)_{i_{p+1}})} \frac{d(p)_{i_{p+1}}}{d(p+1)_{i_p}} \prod_{p\in[1,l-1]\cup[u,m-1]} \frac{\nu_{i_p,i_{p+1}}(\mathbf{d}(p+1)_{i_p})}{\nu_{i_{p+1},i_p}(\mathbf{d}(p)_{i_{p+1}})} \frac{d(p)_{i_{p+1}}}{d(p+1)_{i_p}} \\
&\quad = \prod_{p\in[1,l-1]\cup[u,m-1]} \frac{\nu_{i_p,i_{p+1}}(\mathbf{d}(p+1)_{i_p})}{\nu_{i_{p+1},i_p}(\mathbf{d}(p)_{i_{p+1}})} \frac{d(p)_{i_{p+1}}}{d(p+1)_{i_p}}
\end{aligned}$$

since the first product in the middle line is equal to 1. Now $i_1, \dots, i_{l-1}, i_u, \dots, i_m$ and $\mathbf{d}(1), \dots, \mathbf{d}(l-1), \mathbf{d}(u), \dots, \mathbf{d}(m) = \mathbf{d}(1)$ satisfy the former condition. We can repeat the same process until $\mathbf{d}(s) \neq \mathbf{d}(t)$ whenever $i_s = i_t$ for $s, t \in [k-1]$. Since at each step, the product is preserved, the result follows. □

Theorem 3.5. *Let $n \in \mathbb{N}$. Let ν be a strongly connected multi-type Galton–Watson measure whose types are ordered pairs of elements of $[n]$. Then there exists a measure μ on $\mathbf{T}_n$ such that μ is reversible for simple random walk and $\mu \sim \nu$ if and only if the following two conditions hold.*

(i) *For all $i, j, k \in [n]$, $\mathbf{c} \in \mathbf{S}_n$ such that $\nu_{j,i}(\mathbf{c}_j) > 0$ and $c_k > 0$, we have $\nu_{k,i}(\mathbf{c}_k) > 0$,*

(ii) (3.13) *holds for all k, $i_1, \dots, i_k = i_1 \in [n]$ and $\mathbf{c}(1), \dots, \mathbf{c}(k) = \mathbf{c}(1) \in \mathbf{S}_n$ such that $\nu_{i_p,i_{p+1}}(\mathbf{c}(p+1)_{i_p}) > 0$ for $p = 1, \dots, k-1$ and for all $s, t \in [k-1]$, if $i_s = i_t$, then $\mathbf{c}(s) \neq \mathbf{c}(t)$.*

Proof. By Lemma 3.4, it is enough to prove the theorem when in (ii), (3.13) holds for all cycles.

($\Leftarrow$) Assume (i) and (ii) hold. Since the μ-conditional distributions of the descendant trees of the neighbors of the root are determined by ν, in order to define μ, it's enough to specify the values $\mu(i)$ and $\mu_i(\mathbf{c})$ for all $i \in [n]$ and $\mathbf{c} \in \mathbf{S}_n$.

For all $i \in [n]$ and $\mathbf{c} \in \mathbf{S}_n$, let $\mu_i(\mathbf{c}) = 0$ if $\nu_{j,i}(\mathbf{c}_j) = 0$ for some (equivalently for all by assumption (i)) $j \in [n]$ such that $c_j > 0$. Let $\mu_i(\mathbf{c}) = 1$ if for all $j \in [n]$ and $\mathbf{d} \in \mathbf{S}_n$ such that $\nu_{j,i}(\mathbf{d}_j) > 0$, we have $\mathbf{d} = \mathbf{c}$.

For all $i \in [n]$ let $F_i := \{\mathbf{c} \in \mathbf{S}_n\colon \exists j, \nu_{j,i}(\mathbf{c}_j) > 0\}$. For each $i \in [n]$ define the probabilities $\{\mu_i(\mathbf{c})\colon \mathbf{c} \in F_i\}$ to be the unique solution of the equations

$$\sum_{\mathbf{c} \in F_i} \mu_i(\mathbf{c}) = 1 \qquad \text{for } i \in [n] \tag{3.14}$$

and

$$\frac{\mu_i(\mathbf{c})}{\mu_i(\mathbf{d})} = \prod_{p=1}^{m-1} \frac{\nu_{i_{p+1},i_p}(\mathbf{c}(p)_{i_{p+1}})}{\nu_{i_p,i_{p+1}}(\mathbf{c}(p+1)_{i_p})} \frac{|\mathbf{c}(p)|}{|\mathbf{c}(p+1)|} \frac{c(p+1)_{i_p}}{c(p)_{i_{p+1}}} \tag{3.15}$$

for $i_1 = i, \dots, i_m = i \in [n]$ and $\mathbf{c}(1) = \mathbf{c}, \dots, \mathbf{c}(m) = \mathbf{d}$ such that $\nu_{i_p,i_{p+1}}(\mathbf{c}(p+1)_{i_p}) > 0$ for $p = 1, \dots, m-1$.

Define $\{\mu(i)\colon i \in [n]\}$ to be the unique solution of the equations $\sum_{i \in [n]} \mu(i) = 1$ for $i \in [n]$ and

$$\frac{\mu(i)}{\mu(j)} = \frac{\mu_j(\mathbf{d})\nu_{j,i}(\mathbf{e}_j) d_i / |\mathbf{d}|}{\mu_i(\mathbf{e})\nu_{i,j}(\mathbf{d}_i) e_j / |\mathbf{e}|} \tag{3.16}$$

for $\mathbf{d} \in F_j, \mathbf{e} \in F_i$ with $d_i, e_j > 0$.

In (3.15) since $\nu_{i_p,i_{p+1}}(\mathbf{c}(p+1)_{i_p}) > 0$ for $p = 1, \dots, m-1$, condition (i) in Theorem 3.5 implies that $\nu_{i_{p+1},i_p}(\mathbf{c}(p)_{i_{p+1}}) > 0$ for $p = 2, \dots, m-1$. Since $\mathbf{c} \in F_i$, condition (i) also implies that $\nu_{i_2,i_1}(\mathbf{c}(1)_{i_2}) > 0$.

The right-hand side of (3.15) does not depend on $i_1 = i, \dots, i_m = i \in [n]$ and $\mathbf{c}(1) = \mathbf{c}, \dots, \mathbf{c}(m) = \mathbf{d}$: If $l_1 = i, \dots, l_t = i \in [n]$ and $\mathbf{e}(1) = \mathbf{c}, \dots, \mathbf{e}(t) = \mathbf{d} \in \mathbf{S}_n$ are such that $\nu_{l_p,l_{p+1}}(\mathbf{e}(p+1)_{l_p}) > 0$ for $p = 1, \dots, t-1$, then by Lemma 3.4 and condition (ii) of the theorem $i_1, \dots, i_m = i = l_t, l_{t-1}, \dots, l_1$ and $\mathbf{c}(1), \dots, \mathbf{c}(m) = \mathbf{d} = \mathbf{e}(t), \dots, \mathbf{e}(1) = \mathbf{c} = \mathbf{c}(1)$ satisfy

$$\prod_{p=1}^{m-1} \frac{\nu_{i_p,i_{p+1}}(\mathbf{c}(p+1)_{i_p})}{\nu_{i_{p+1},i_p}(\mathbf{c}(p)_{i_{p+1}})} \frac{c(p)_{i_{p+1}}}{c(p+1)_{i_p}} \prod_{p=1}^{t-1} \frac{\nu_{l_{p+1},l_p}(\mathbf{e}(p)_{l_{p+1}})}{\nu_{l_p,l_{p+1}}(\mathbf{e}(p+1)_{l_p})} \frac{e(p+1)_{l_p}}{e(p)_{l_{p+1}}} = 1,$$

which gives

$$\prod_{p=1}^{m-1} \frac{\nu_{i_p,i_{p+1}}(\mathbf{c}(p+1)_{i_p})}{\nu_{i_{p+1},i_p}(\mathbf{c}(p)_{i_{p+1}})} \frac{c(p)_{i_{p+1}}}{c(p+1)_{i_p}} \frac{|\mathbf{c}(p+1)|}{|\mathbf{c}(p)|} \times \prod_{p=1}^{t-1} \frac{\nu_{l_{p+1},l_p}(\mathbf{e}(p)_{l_{p+1}})}{\nu_{l_p,l_{p+1}}(\mathbf{e}(p+1)_{l_p})} \frac{e(p+1)_{l_p}}{e(p)_{l_{p+1}}} \frac{|\mathbf{e}(p)|}{|\mathbf{e}(p+1)|} = 1.$$

Therefore, we have

$$\prod_{p=1}^{m-1} \frac{\nu_{i_{p+1},i_p}(\mathbf{c}(p)_{i_{p+1}})}{\nu_{i_p,i_{p+1}}(\mathbf{c}(p+1)_{i_p})} \frac{c(p+1)_{i_p}}{c(p)_{i_{p+1}}} \frac{|\mathbf{c}(p)|}{|\mathbf{c}(p+1)|} = \prod_{p=1}^{t-1} \frac{\nu_{l_{p+1},l_p}(\mathbf{e}(p)_{l_{p+1}})}{\nu_{l_p,l_{p+1}}(\mathbf{e}(p+1)_{l_p})} \frac{e(p+1)_{l_p}}{e(p)_{l_{p+1}}} \frac{|\mathbf{e}(p)|}{|\mathbf{e}(p+1)|}.$$

Now, (3.15) gives the relative weights of the numbers $\mu_i(\mathbf{c})$ which add to 1 and therefore $\mu_i(\mathbf{e})$ for $\mathbf{e} \in F_i$ are uniquely determined by (3.15) and (3.14). If $\mathbf{e} \notin F_i$, then $\mu_i(\mathbf{e}) = 0$. Since i is arbitrary, $\mu_i(\mathbf{e})$ is defined for all i and $\mathbf{e}$.

The right-hand side of (3.16) does not depend on $\mathbf{e}$ and $\mathbf{d}$. We need to show that

$$\frac{\mu_j(\mathbf{d})\nu_{j,i}(\mathbf{e}_j)d_i/|\mathbf{d}|}{\mu_i(\mathbf{e})\nu_{i,j}(\mathbf{d}_i)e_j/|\mathbf{e}|} = \frac{\mu_j(\mathbf{f})\nu_{j,i}(\mathbf{c}_j)f_i/|\mathbf{f}|}{\mu_i(\mathbf{c})\nu_{i,j}(\mathbf{f}_i)c_j/|\mathbf{c}|} \tag{3.17}$$

whenever $\mu_j(\mathbf{d}), \mu_j(\mathbf{f}), \mu_i(\mathbf{c}), \mu_j(\mathbf{e}) > 0$ with $d_i, f_i, c_j, e_j > 0$, in which case the remaining terms are all positive. In fact it is enough to show (3.17) when $\mathbf{d} = \mathbf{f}$ since repeated application of (3.17) in that case will prove (3.17) in the general case. When $\mathbf{d} = \mathbf{f}$, (3.17) reduces to

$$\frac{\nu_{j,i}(\mathbf{c}_j)}{\nu_{j,i}(\mathbf{e}_j)} = \frac{\mu_i(\mathbf{c})c_j/|\mathbf{c}|}{\mu_i(\mathbf{e})e_j/|\mathbf{e}|} \tag{3.18}$$

which is equivalent to (3.12) and a special case of (3.15): If $\nu_{j,i}(\mathbf{c}), \nu_{j,i}(\mathbf{e}) > 0$, then let $\mathbf{d} \in \mathbf{S}_n$ such that $d_i > 0$ and $\nu_{i,j}(\mathbf{d}_i) > 0$. Then for $\mathbf{c}(1) = \mathbf{c}, \mathbf{c}(2) = \mathbf{d}, \mathbf{c}(3) = \mathbf{e}$ and $i_1 = i_3 = i, i_2 = j$, (3.15) reduces to (3.18).

Since $\sum_{i\in[n]} \mu(i) = 1$, the probabilities $\mu(i)$ for $i \in [n]$ are uniquely determined. We have that (3.16) is equivalent to (3.4). Since (3.12) holds, condition (i) of the theorem implies (3.1). By Lemma 3.3, μ is reversible.

($\Rightarrow$) This direction has been proved by Lemma 3.2, Lemma 3.4 and the discussion before Lemma 3.4. □

Example 3.6. Let $\mathbf{a} = (1, 1), \mathbf{b} = (2, 1)$ and $\mathbf{d} = (0, 3)$. Let ν be a multi-type Galton–Watson measure whose types are $\{(i, j) \colon i, j \in \{1, 2\}\}$ with offspring distribution as follows

$$\nu_{1,1}(0,1) = \nu_{1,1}(\mathbf{a}_1) = \tfrac{3}{7}, \qquad \nu_{1,1}(1,1) = \nu_{1,1}(\mathbf{b}_1) = \tfrac{4}{7},$$
$$\nu_{2,1}(1,0) = \nu_{2,1}(\mathbf{a}_2) = \tfrac{3}{5}, \qquad \nu_{2,1}(2,0) = \nu_{2,1}(\mathbf{b}_2) = \tfrac{2}{5},$$
$$\nu_{2,2}(0,2) = \nu_{2,2}(\mathbf{d}_2) = \tfrac{2}{3}, \qquad \nu_{2,2}(1,0) = \nu_{2,2}(\mathbf{a}_2) = \tfrac{1}{3},$$
$$\nu_{1,2}(0,1) = \nu_{1,2}(\mathbf{a}_1) = 1.$$

Let's check that ν satisfies the two conditions of Theorem 3.5. We have $\nu_{1,1}(\mathbf{a}_1)$ and $\nu_{2,1}(\mathbf{a}_2)$ are both positive, exactly as condition (i) asks for: In order to have $\mu \sim \nu$ and $\nu_{1,1}(\mathbf{a}_1)$ together, one has to have $\mu(N_1(\mathbf{a})) > 0$ which in turn implies $\nu_{2,1}(\mathbf{a}_2) > 0$. Similarly $\nu_{1,1}(\mathbf{b}_1)$ and $\nu_{2,1}(\mathbf{b}_2)$ are both positive and so are $\nu_{2,2}(\mathbf{a}_2)$ and $\nu_{1,2}(\mathbf{a}_1)$. Since $\mathbf{d}_1 = 0$, condition (i) says $\nu_{2,2}(\mathbf{d}_2) > 0$ implies $\nu_{2,2}(\mathbf{d}_2) > 0$, which certainly holds.

For condition (ii), let us verify (3.13) for a particular sequence. One actually has to check 4 such sequences in total. Let $i_1 = 1, i_2 = 2, i_3 = 2, i_4 = 1, i_5 = 1$ and $\mathbf{c}(1) = \mathbf{a}, \mathbf{c}(2) = \mathbf{a}, \mathbf{c}(3) = \mathbf{a}, \mathbf{c}(4) = \mathbf{b}, \mathbf{c}(5) = \mathbf{a}$. It might be easier to imagine a cycle of 4 vertices labeled $i_1 = i_5, i_2, i_3, i_4$, and the vertex labeled i_s has $\mathbf{c}(s)$ neighbors. Starting the cycle at a different vertex, we can see this is equivalent to showing that (3.13) holds for, $i_1 = 1, i_2 = 1, i_3 = 2, i_4 = 2, i_5 = 1$ and $\mathbf{c}(1) = \mathbf{b}, \mathbf{c}(2) = \mathbf{a}, \mathbf{c}(3) = \mathbf{a}, \mathbf{c}(4) = \mathbf{a}, \mathbf{c}(5) = \mathbf{b}$.

One then can determine $\mu_1(\mathbf{a}), \mu_1(\mathbf{b}), \mu_2(\mathbf{d}), \mu_2(\mathbf{a})$ by (3.18). In fact they are all equal to $\frac{1}{2}$. Finally, one can determine $\mu(1)$ and $\mu(2)$ by (3.16), which are $\frac{3}{8}$ and $\frac{5}{8}$ respectively. Since $\mu \sim \nu$, we have completely determined μ.

Next, we parametrize all reversible measures μ for simple random walk such that $\mu \sim \nu$ for some strongly connected Galton–Watson measure ν. To state our result, Proposition 3.9, we first prove two lemmas.

Lemma 3.7. *Assume that $\mu \sim \nu$ and that μ is stationary for simple random walk. Then for all $i, j \in [n]$ we have $i \in A(\mu_j)$ if and only if $j \in A(\mu_i)$.*

Proof. It is enough to prove one direction since i and j are interchangeable. Let $i, j \in [n]$ and assume $j \in A(\mu_i)$. Then there exists $\mathbf{c} \in \mathbf{S}_n$ such that $c_j > 0$ and $\mu_i(\mathbf{c}) > 0$. Since $\mu(i) > 0$, we have $\mu(N_i(\mathbf{c})) > 0$. The argument in the first paragraph of the proof of Lemma 3.2 gives $\mu(B_{j,i,\mathbf{c}_j}) > 0$ which implies $i \in A(\mu_j)$. □

Lemma 3.8. *Assume that $\mu \sim \nu$ and μ is stationary for simple random walk. Then ν is strongly connected if and only if $\forall i, j \in [n]$, there exist $i = i_1, \dots, i_m = j$ and $\mathbf{c}(2), \dots, \mathbf{c}(m-1) \in \mathbf{S}_n$ such that*

$$c(k)_{i_{k-1}}, c(k)_{i_{k+1}} > 0 \quad \textit{and} \quad \mu_{i_k}\big(\mathbf{c}(k)\big) > 0 \qquad \forall k \in \{2, \dots, m-1\}. \tag{3.19}$$

Proof. Assume ν is strongly connected. Fix $i, j \in [n]$. Since $\mu(i), \mu(j) > 0$ and ν is strongly connected, there exists $i_1 = i, \dots, i_m = j$ such that $(i_k, i_{k+1}) \in A(\nu_{i_{k-1}, i_k})$. Then for each $k = 2, \dots, m-1$ there exists $\mathbf{c}(k) \in \mathbf{S}_n$ with $c_{i_{k+1}}, c_{i_{k-1}} > 0$ such that $\nu_{i_{k-1}, i_k}(\mathbf{c}(k)_{i_{k-1}}) > 0$. Lemma 3.2 gives that $\mu_{i_k}(\mathbf{c}) > 0$.

The other direction can be proved in a similar fashion. □

Let M be a subset of $\{N_i(\mathbf{c})\colon i \in [n], \mathbf{c} \in \mathbf{S}_n\}$. If μ is a reversible measure for simple random walk on $\mathbf{T}_n$ such that $\mu \sim \nu$ for some strongly connected Galton–Watson measure ν and such that the following holds

$$\forall i \in [n] \text{ and } \mathbf{c} \in \mathbf{S}_n \qquad \mu_i(\mathbf{c}) > 0 \quad \Longleftrightarrow \quad N_i(\mathbf{c}) \in M, \tag{3.20}$$

then by Lemma 3.7 and Lemma 3.8 the elements of M satisfy the following two conditions.

(i) $\forall i, j \in [n]$ we have $\exists N_i(\mathbf{c}) \in M$ with $c_j > 0 \Longleftrightarrow \exists N_j(\mathbf{d}) \in M$ with $d_i > 0$, (3.21)

(ii) $\forall i, j \in [n] \exists N_{i_1}(\mathbf{c}(1)), \dots, N_{i_k}(\mathbf{c}(k)) \in M$ such that $i_1 = i, i_k = j$ and $\mathbf{c}(s)_{i_{s-1}} > 0$ and $\mathbf{c}(s)_{i_{s+1}} > 0$ for all $s \in [k]$. (3.22)

Let M_r be the set of reversible measures μ for simple random walk on $\mathbf{T}_n$ that satisfy (3.20) and $\mu \sim \nu$ for some strongly connected Galton–Watson measure ν.

Proposition 3.9. *Reversible measures μ which satisfy $\mu \sim \nu$ for some strongly connected Galton–Watson measure ν can be parametrized in the following way. Let M be a subset of $\{N_i(\mathbf{c})\colon i \in [n], \mathbf{c} \in \mathbf{S}_n\}$ that satisfies (3.21) and (3.22) above. Let $\{\mu_i'(\mathbf{c}) \in (0,1)\colon i \in [n], \mathbf{c} \in \mathbf{S}_n\}$ satisfy (3.20) and $\sum_{\mathbf{c} \in \mathbf{S}_n} \mu_i'(\mathbf{c}) = 1$ for each $i \in [n]$. In terms of all the preceding parameters, define μ by*

(iii) $\mu_i(\mathbf{c}) = \mu_i'(\mathbf{c})$ *for all* $i \in [n], \mathbf{c} \in \mathbf{S}_n$,

(iv) *conditioned on the labels of the root and its neighbors the descendant subtrees of the neighbors of the root are independent multi-type Galton–Watson trees with conditional offspring distributions $\{\nu_{i,j}(\mathbf{c})\colon i, j \in [n], \mathbf{c} \in \mathbf{S}_n\}$ that are the unique solutions of the equations*

$$\frac{\nu_{i,j}(\mathbf{d}_i)}{\nu_{i,j}(\mathbf{e}_i)} = \frac{\mu'_j(\mathbf{d}) d_i/|\mathbf{d}|}{\mu'_j(\mathbf{e}) e_i/|\mathbf{e}|} \tag{3.23}$$

for $N_j(\mathbf{d}), N_j(\mathbf{e}) \in M$ with $d_i, e_i > 0$ and $\sum_{N_j(\mathbf{d}) \in M} \nu_{i,j}(\mathbf{d}_i) = 1$,

(v) *$\{\mu(i)\colon i \in [n]\}$ are the unique solutions of the equations*

$$\frac{\mu(i)}{\mu(j)} = \frac{\mu'_j(\mathbf{d}) \nu_{j,i}(\mathbf{e}_j) d_i/|\mathbf{d}|}{\mu'_i(\mathbf{e}) \nu_{i,j}(\mathbf{d}_i) e_j/|\mathbf{e}|} \tag{3.24}$$

for $N_j(\mathbf{d}), N_i(\mathbf{e}) \in M$ with $d_i, e_j > 0$.

Proof. We first note that since $\mu \sim \nu$, we have that $\{\mu_i(\mathbf{c})\colon N_i(\mathbf{c}) \in M\}, \{\mu(i)\colon i \in [n]\}$ and $\{\nu_{i,j}(\mathbf{c})\colon N_i(\mathbf{c}_j) \in M, c_j > 0\}$ determine μ uniquely.

Assume that $\{\mu_i'(\mathbf{c})\colon i \in [n], \mathbf{c} \in \mathbf{S}_n\}$ are as in Proposition 3.9 and μ is defined by (iii), (iv) and (v). Now by (iii) of Proposition 3.9 μ satisfies (3.20) and by (3.24) we have that $\mu(i)$, $\mu_i(\mathbf{c})$ and $\nu_{i,j}(\mathbf{c})$ are solutions to (3.4). By (ii) and (iii), (3.1) holds. Thus, by Lemma 3.3, μ is reversible for simple random walk. By Lemma 3.8, ν is strongly connected.

On the other hand for $\mu \in M_r$, we already showed that (3.24) and (3.23) hold. Therefore, our parametrization is complete. □

We illustrate Proposition 3.9 with the following example.

Example 3.10. Let $\mathbf{a}, \mathbf{b}, \mathbf{d}$ be as in Example 3.6. Let

$$M = \{N_1(\mathbf{a}), N_1(\mathbf{b}), N_2(\mathbf{d}), N_2(\mathbf{a})\}.$$

It is easy to check that M satisfies (3.21) and (3.22). Let $s, t \in (0,1)$ and

$$\mu'_1(\mathbf{a}) = s = 1 - \mu'_1(\mathbf{b}) \quad \text{and} \quad \mu'_2(\mathbf{a}) = t = 1 - \mu'_2(\mathbf{d}).$$

Now, (3.20) holds. So by Proposition 3.9, using (3.23), we have

$$\frac{\nu_{1,1}(0,1)}{\nu_{1,1}(1,1)} = \frac{s}{1-s}\frac{1/2}{2/3}.$$

Since $\nu_{1,1}(0,1) + \nu_{1,1}(1,1) = 1$, we get

$$\nu_{1,1}(0,1) = \frac{3s}{4-s} \quad \text{and} \quad \nu_{1,1}(1,1) = \frac{4-4s}{4-s}.$$

Similarly one can calculate

$$\nu_{2,1}(2,0) = \frac{2s}{3-s} \quad \text{and} \quad \nu_{2,1}(1,0) = \frac{3-3s}{3-s},$$
$$\nu_{2,2}(0,2) = \frac{2t}{2-t} \quad \text{and} \quad \nu_{2,2}(1,0) = \frac{2-3t}{2-t}.$$

Since M contains $N_2(\mathbf{a})$ and $N_2(\mathbf{d})$ but $d_1 = 0$, by the last equation in (iv) of Proposition 3.9, we have

$$\nu_{1,2}(0,1) = \nu_{1,2}(\mathbf{a}_1) = 1.$$

The probabilities $\mu_i(\mathbf{c})$ for $N_i(\mathbf{c})$ are determined by (iii) and $\mu(1)$, $\mu(2)$ are determined by (3.24). They turn out to be

$$\mu(1) = \frac{3-3s}{6-4s} \quad \text{and} \quad \mu(2) = \frac{3-s}{6-4s}.$$

When $s = t = \frac{1}{2}$, then we get the measure of Example 3.6.

Remark 3.11. Let $n \in \mathbb{N}$ and M be as in Proposition 3.9. Then any $\mu \in M_r$ is determined by the probabilities $\mu_i(\mathbf{c})$. Since $\sum_{N_i(\mathbf{c})\in M} \mu_i(N_i(\mathbf{c})) = 1$ for each $i \in [n]$ we have M_r is an $(|M| - n)-$dimensional affine set.

4. A special case

In this section, we study the case when there is no relabeling: Let g be the function which maps $(T, o) \in \mathbf{T}_n$ to the tree obtained by removing the label of o so that the root has no label and all the other vertices have the same label as before. We give a complete description of measures μ that are reversible and $\mu \sim_g \nu$ for some strongly connected Galton–Watson measure ν. We prove the following theorem.

Theorem 4.1. *Let ν be a strongly connected multi-type Galton–Watson measure whose types are elements of $[n]$. Then there exists a reversible measure μ for simple random walk on $\mathbf{T}_n$ such that $\mu \sim_g \nu$ if and only if for all $i \in [n]$ and $d \in D(\nu_i)$, given that the number of children of the* *root is d, the ν_i-offspring distribution of the root is multinomial with parameters $\{p_{i,j}\colon j \in [n]\}$ that do not depend on d and that satisfy*

$$\prod_{s=1}^{m-1} \frac{p_{i_s,i_{s+1}}}{p_{i_{s+1},i_s}} = 1 \tag{4.25}$$

for all $i_1, \dots, i_m = i_1$ such that $p_{i_s,i_{s+1}} > 0$.

In this case, the reversible measure μ is unique, and $d+1 \in D(\mu_i)$ if and only if $d \in D(\nu_i)$ for all $i \in [n]$, and given the root has label i, the μ_i-probability that the root has degree $d+1$ and the ν_i-probability that the root has degree d are equal. The probabilities of the label of the root are given by the solution of

$$\frac{\mu(i)}{\mu(j)} = \frac{p_{j,i}}{p_{i,j}} \tag{4.26}$$

and $\sum_{k=1}^{n} \mu(i) = 1$.

Corollary 4.2. *A probability measure $\mu \in \mathbf{T}_n$ is reversible for simple random walk and $\mu \sim \nu$ for some multi-type Galton–Watson measure ν if and only if μ and ν are as described in Theorem 4.1.*

First, we show how Theorem 4.1 can be used to construct a class of measures μ on $\mathbf{T}_3$ such that μ is reversible for simple random walk and $\mu \sim_g \nu$ for some multi-type Galton–Watson measure ν. The process is the analog of Proposition 3.9.

Example 4.3. By Theorem 4.1, if μ is reversible for simple random walk and $\mu \sim_g \nu$ for some Galton–Watson measure ν, then given the label of the root is i and the degree of the root, each neighbor of the root has label j, with probability, say $p_{i,j}$, independent of the labels of the other neighbors of the root. Let $p_{1,1}, p_{1,2}, p_{1,3}, p_{2,1}, p_{2,3}, p_{3,1}, p_{3,2} \in (0,1)$ with $p_{1,1} + p_{1,2} + p_{1,3} = 1$ and $p_{2,1} + p_{2,3} = 1$ and $p_{3,1} + p_{3,2} = 1$. So if the root has label 1, then it can have neighbors labeled 1, 2, 3 and if the root has label 2, then it can have neighbors labeled 1, 3 and if the root has label 3, then it can have neighbors labeled 1, 2 with positive probability.

Suppose that (4.25) holds. The degrees for the root conditioned on the label of the root and their probabilities can be chosen arbitrarily. For simplicity, let $D(\mu_2) = D(\mu_3) = \{5\}$ and $D(\mu_1) = \{2, 3\}$, that is, almost surely when the label or the root is 2 or 3, the root has 5 neighbors, otherwise has 2 or 3 neighbors, say with equal probability.

By Theorem 4.1, the μ-probability that the root has label i can be determined by (4.26). Now $\mu(N_i(\mathbf{c}))$ can be determined for any $i \in \{1,2,3\}$ and $\mathbf{c} \in \mathbf{S}_n$. The descendant subtrees of the neighbors of the root, conditioned on their labels, are independent multi-type Galton–Watson trees, whose offspring distributions are given by Theorem 4.1, namely, a vertex labeled 2 or 3 will have 4 children with probability 1. Each child of a vertex labeled 2 has label 1 or 3 with probabilities $p_{2,1}, p_{2,3}$, respectively, independent of the labels of the other children. Similarly each child of a vertex labeled 3 has label 2 or 3 with probabilities $p_{3,1}, p_{3,2}$ respectively, independent of the labels of the other children. A vertex labeled 1 has 1 child or 2 children, with equal probability and given the number of children, each child has label 1, 2 or 3 with probabilities $p_{1,1}, p_{1,2}, p_{1,3}$, respectively, independent of the labels of the other children.

Let us see what restriction (4.25) imposes on the $p_{i,j}$s. Suppose we have chosen $p_{1,1}$, $p_{1,2}$ and $p_{2,1}$ arbitrarily. Letting $i_1 = 1, i_2 = 2, i_3 = 3, i_4 = 1$ and using (4.25) together with $p_{3,2} + p_{3,1} = 1$, one can solve for $p_{3,2}$ and $p_{3,1}$. Thus the multinomial parameters for the neighbors of a vertex labeled 3 are determined by the multinomial parameters for the neighbors of vertices of types 1 and 2.

It is easy to see that this is the only restriction. Just like the case in the previous section, it will be enough to check (4.25) for sequences where no label other than the first label is repeated. The sequences of length 3 do not impose any restrictions and the other sequences of length four are actually the same as the sequence above.

We devote the rest of this section to proving Theorem 4.1. Assume that μ is a reversible measure for simple random walk on $\mathbf{T}_n$ and $\mu \sim_g \nu$. Since $\mu \sim_g \nu$, by stationarity of μ we have $d \in D(\nu_i)$ if and only if $d + 1 \in D(\mu_i)$. We first prove the following claims in order. For all $i \in [n]$ and $d \in D(\nu_i)$ given the degree of the root is d, we have that (the offspring distribution of) ν_i is multinomial. Given the degree of the root is $d + 1$, μ_i is multinomial with the same parameters as ν_i (given the root has degree d). For each $i \in [n]$ and $d \in A(\nu_i)$, the multinomial parameters for ν_i given the root has degree d do not depend on d. For all $i \in [n]$ and $d \in A(\nu_i)$, the μ_i-probability that the root has degree $d + 1$ and the ν_i-probability that the root has degree d are equal. Our main tool is Lemma 4.5. Before proving that, we make a basic observation.

Lemma 4.4. *Assume that μ is reversible and $\mu \sim_g \nu$. Then for all $i, j \in [n]$ we have*

$$i \in A(\nu_j) \iff i \in A(\mu_j) \iff j \in A(\mu_i) \iff j \in A(\nu_i). \tag{4.27}$$

Proof. The first equivalence (and the last) is the analog of Lemma 3.2 and can be proven exactly in the same way. For the second equivalence since μ is stationary, we have $\mu(B_{i,j}) > 0$ if and only if $\mu(B_{j,i}) > 0$. By definition of $B_{i,j}$ we have $j \in A(\mu_i)$ if and only if $\mu(B_{i,j}) > 0$. Thus, (4.27) holds. □

Lemma 4.5. *Let ρ be a probability measure on $\mathbf{S}^n$. Let $d \in D(\rho)$. Assume that the following two conditions hold*:

(i) $\rho(\mathbf{c}) > 0$ *for all* $\mathbf{c} \in A(\rho)$ *with* $|\mathbf{c}| = d$,
(ii) $\forall \mathbf{c}$ *such that* $|\mathbf{c}| = d$, $\rho(\mathbf{c}) > 0, c_j, c_k > 0$, *we have*

$$\frac{c_j + \delta_{j,k}}{c_j + 1} \frac{\rho(\mathbf{c})}{\rho(\mathbf{c}_j^k)} = \frac{c_k + 1}{c_k + \delta_{j,k}} \frac{\rho(\mathbf{c}_k^j)}{\rho(\mathbf{c})}. \tag{4.28}$$

Then $\rho(\mathbf{c} \mid |\mathbf{c}| = d)$ is multinomial.

Proof. We write (4.28) in a different form, which will be used later in the proof. Setting $j = s$ and $k = t$ in (4.28) and rearranging the terms, we have

$$\rho(\mathbf{c}) = \sqrt{\frac{(c_s + 1)(c_t + 1)}{(c_s + \delta_{s,t})(c_t + \delta_{s,t})}} \sqrt{\rho(\mathbf{c}_t^s)\rho(\mathbf{c}_s^t)}. \tag{4.29}$$

Let P be a measure on finite lists labeled by elements of $[n]$ with the following property: If B and C are two lists whose elements have labels $\mathbf{c}$ with $|\mathbf{c}| = d \in D(\rho)$, then $P(B) = P(C) = \rho(\mathbf{c})/(|\mathbf{c}|/\prod_{k\in[n]} c_k!)$. Then $P(B) > 0$ if and only if $\mathbf{c} \in A(\rho)$. Fix $d \in D(\rho)$. Let $P(j|\mathbf{c})$ denote the P-probability that the first element of a list of size d is labeled j given that the other elements have labels $\mathbf{c}$. We shall show that

$$P(j|\mathbf{c}) = P(j|\mathbf{d}) \tag{4.30}$$

for all $j, \mathbf{c}, \mathbf{d} \in A(\rho)$ with $|\mathbf{c}| = |\mathbf{d}| = d - 1$. It is enough to show that

$$P(j|\mathbf{c}) = P\big(j|\mathbf{c}_l^m\big) \tag{4.31}$$

for all $l, m, \mathbf{c} \in A(\rho)$ such that $c_l > 0$ and $|\mathbf{c}| = d - 1$. In (4.30), we may assume without loss of generality that $c_j \ge d_j$. Thus, in (4.31), we may assume that $m \ne j$, $m \ne l$. We can also assume that $c_j > 0$ since if we want to show (4.30) for $\mathbf{c}$ with $c_j = 0$ (consequently $d_j = 0$), we can pick $\mathbf{e}$ with $e_j > 0$ and show that (4.30) holds for $\mathbf{e}$ and $\mathbf{d}$ and then for $\mathbf{e}$ and $\mathbf{c}$, which together imply that (4.30) holds for $\mathbf{c}$ and $\mathbf{d}$.

All possible assignments of the labels $\mathbf{c}^j$ to a list of $|\mathbf{c}| + 1$ elements have the same probability, which is $\rho(\mathbf{c}^j)/((|\mathbf{c}| + 1)!/\prod_{k\in[n]}(c_k + \delta_{k,j})!)$. Hence, the P-probability that a list of $|\mathbf{c}| + 1$ elements are labeled by $\mathbf{c}^j$ and the first element of the list has label j is given by

$$\rho\big(\mathbf{c}^j\big)\left(\frac{|\mathbf{c}|!}{\prod_{k\in[n]} c_k!}\Big/\frac{(|\mathbf{c}|+1)!}{\prod_{k\in[n]}(c_k+\delta_{k,j})!}\right) = \rho\big(\mathbf{c}^j\big)\left(\frac{c_j+1}{|\mathbf{c}|+1}\right).$$

Using this equation, we have

$$P(j|\mathbf{c}) = \frac{(c_j+1)/(|\mathbf{c}|+1)\rho(\mathbf{c}^j)}{\sum_{k\in A}(c_k+1)/(|\mathbf{c}|+1)\rho(\mathbf{c}^k)} = \frac{(c_j+1)\rho(\mathbf{c}^j)}{\sum_{k\in A}(c_k+1)\rho(\mathbf{c}^k)},$$

where $A := A(\rho)$. Similarly

$$P\big(j|\mathbf{c}_l^m\big) = \frac{(c_j+1-\delta_{j,l})\rho(\mathbf{c}_l^{m,j})}{\sum_{k\in A}(c_k+1+\delta_{k,m}-\delta_{k,l})\rho(\mathbf{c}_l^{m,k})}.$$

Hence,

$$P(j|\mathbf{c}) = P\big(j|\mathbf{c}_l^m\big) \quad\Longleftrightarrow\quad \frac{(c_j+1)\rho(\mathbf{c}^j)}{\sum_{k\in A}(c_k+1)\rho(\mathbf{c}^k)} = \frac{(c_j+1-\delta_{j,l})\rho(\mathbf{c}_l^{m,j})}{\sum_{k\in A}(c_k+1+\delta_{k,m}-\delta_{k,l})\rho(\mathbf{c}_l^{m,k})},$$

which is equivalent to

$$\begin{aligned}(c_j+1)\rho\big(\mathbf{c}^j\big)&\sum_{k\in A}(c_k+1+\delta_{k,m}-\delta_{k,l})\rho\big(\mathbf{c}_l^{m,k}\big)\\ &= (c_j+1-\delta_{j,l})\rho\big(\mathbf{c}_l^{m,j}\big)\sum_{k\in A}(c_k+1)\rho\big(\mathbf{c}^k\big).\end{aligned} \tag{4.32}$$

We shall show that $\forall k \in A$

$$(c_j+1)\rho(\mathbf{c}^j)(c_k+1+\delta_{k,m}-\delta_{k,l})\rho(\mathbf{c}_l^{m,k}) = (c_j+1-\delta_{j,l})\rho(\mathbf{c}_l^{m,j})(c_k+1)\rho(\mathbf{c}^k). \quad (4.33)$$

This implies (4.32), which completes the proof.

Since $c_l > 0$ and $c_j > 0$, by (4.29), we have the following equations

$$\begin{aligned}
\rho(\mathbf{c}^j) &= \sqrt{\frac{(c_j+2)(c_l+1+\delta_{l,j})}{(c_j+1+\delta_{l,j})(c_l+2\delta_{l,j})}}\sqrt{\rho(\mathbf{c}_l^{j,j})\rho(\mathbf{c}^l)} \\
&= \sqrt{\frac{(c_j+2-\delta_{l,j})(c_l+1)}{(c_j+1)(c_l+\delta_{l,j})}}\sqrt{\rho(\mathbf{c}_l^{j,j})\rho(\mathbf{c}^l)}, \\
\rho(\mathbf{c}_l^{m,k}) &= \sqrt{\frac{(c_m+2+\delta_{m,k})(c_k+2+\delta_{m,k}-\delta_{l,k})}{(c_m+1+2\delta_{m,k})(c_k+1+2\delta_{m,k}-\delta_{l,k})}}\sqrt{\rho(\mathbf{c}_l^{m,m})\rho(\mathbf{c}_l^{k,k})} \\
&= \sqrt{\frac{(c_m+2)(c_k+2-\delta_{l,k})}{(c_m+1+\delta_{m,k})(c_k+1+\delta_{m,k}-\delta_{l,k})}}\sqrt{\rho(\mathbf{c}_l^{m,m})\rho(\mathbf{c}_l^{k,k})}, \\
\rho(\mathbf{c}^k) &= \sqrt{\frac{(c_k+2)(c_l+1+\delta_{l,k})}{(c_k+1+\delta_{l,k})(c_l+2\delta_{l,k})}}\sqrt{\rho(\mathbf{c}_l^{k,k})\rho(\mathbf{c}^l)} \\
&= \sqrt{\frac{(c_k+2-\delta_{l,k})(c_l+1)}{(c_k+1)(c_l+\delta_{l,k})}}\sqrt{\rho(\mathbf{c}_l^{k,k})\rho(\mathbf{c}^l)}, \\
\rho(\mathbf{c}_l^{m,j}) &= \sqrt{\frac{(c_m+2+\delta_{m,j})(c_j+2+\delta_{m,j}-\delta_{l,j})}{(c_m+1+2\delta_{m,j})(c_j+1+2\delta_{m,j}-\delta_{l,j})}}\sqrt{\rho(\mathbf{c}_l^{m,m})\rho(\mathbf{c}_l^{j,j})} \\
&= \sqrt{\frac{(c_m+2)(c_j+2-\delta_{l,j})}{(c_m+1)(c_j+1-\delta_{l,j})}}\sqrt{\rho(\mathbf{c}_l^{m,m})\rho(\mathbf{c}_l^{j,j})}.
\end{aligned}$$

Substituting these into (4.33), after cancelations and squaring the equation, we get

$$(c_j+1)(c_k+1+\delta_{k,m}-\delta_{k,l})(c_l+\delta_{l,k})(c_m+1) = (c_k+1)(c_j+1-\delta_{j,l})(c_l+\delta_{l,j})(c_m+1+\delta_{m,k}).$$

Suppose $k=j$, then $\delta_{k,m}=\delta_{j,m}=0$. Substituting j for k and using that $\delta_{j,m}=0$, we see that the equation above holds when $j=k$ and so does (4.33).

So assume $k \neq j$.

If $k=l$, then $l \neq j$ and $m \neq k$ and we can evaluate all the δ's to see that the equation above holds.

If $k \neq l$, the equation above reduces to

$$(c_j+1)(c_l) = (c_j+1-\delta_{j,l})(c_l+\delta_{l,j}),$$

which holds whether $j=l$ or $j \neq l$. □

We shall apply Lemma 4.5 with $\rho := \nu_i$ conditioned on the number of children (for each i). We now show that for each i, ν_i satisfies condition (i) of Lemma 4.5.

Lemma 4.6. *Assume μ is stationary for simple random walk and $\mu \sim_g \nu$. Suppose $i, l \in [n]$ and $\mathbf{d} \in \mathbf{S}_n$ are such that $\nu_i(\mathbf{d}) > 0$ with $d_l > 0$. Then $\nu_i(\mathbf{d}_l^m) > 0$ for every $m \in A(\nu_i)$.*

Proof. Fix $\mathbf{d}, i, l, m$ as in the lemma. Since $m \in A(\nu_i)$, (4.27) gives $\mu(B_{m,i}) > 0$. Since $\mu \sim \nu$, $\mu(B_{m,i}) > 0$ and $\nu_i(\mathbf{d}) > 0$ together imply $\mu(B_{m,i,\mathbf{d}}) > 0$. For every $(T, o) \in B_{m,i,\mathbf{d}}$, a random walker starting at (T, o) moves to a tree $(T, o') \in B_{l,i,\mathbf{d}_l^m}$ in two steps with positive probability. Since μ is stationary and $\mu(B_{m,i,\mathbf{d}}) > 0$, we have $\mu(B_{l,i,\mathbf{d}_l^m}) > 0$. Since $\mu \sim_g \nu$, this implies $\nu_i(\mathbf{d}_l^m) > 0$. □

Now if μ is reversible for simple random walk and $\mu \sim_g \nu$ we have that for all $i \in [n]$ condition (i) of Lemma 4.5 for $\rho = \nu_i$ is a corollary of Lemma 4.6. Next, we show that under the same hypothesis condition (ii) of Lemma 4.5 holds for $\rho = \nu_i$ for all $i \in [n]$.

Assume that $\nu_i(\mathbf{c}) > 0$ with $c_j, c_k > 0$. Then $j, k \in A(\nu_i)$, which implies $i \in A(\nu_j) \cap A(\nu_k)$ by Lemma 4.4. Let $\mathbf{d}, \mathbf{e}$ be such that $d_i, \nu_i(\mathbf{d}) > 0$ and $\mu_i(\mathbf{e}), e_i > 0$. Let $i_1 = i, i_2 = j, i_3 = i, i_4 = k, i_5 = i$ and $\mathbf{c}(1) = \mathbf{c}(5) = \mathbf{c}^k, \mathbf{c}(2) = \mathbf{d}, \mathbf{c}(3) = \mathbf{c}^j, \mathbf{c}(4) = \mathbf{e}$. Then since μ is reversible, (3.13) holds for $i_1, \ldots, i_5$ and $\mathbf{c}(1), \ldots, \mathbf{c}(5)$ where $\nu_{s,t} = \nu_t$ for all s, t (since $f = g$),

$$\frac{\nu_j(\mathbf{d}_i)}{\nu_i(\mathbf{c}_j^k)} \frac{c_j + \delta_{j,k}}{d_i} \frac{\nu_i(\mathbf{c})}{\nu_j(\mathbf{d}_i)} \frac{d_i}{c_j + 1} \frac{\nu_k(\mathbf{e}_i)}{\nu_i(\mathbf{c}_k^j)} \frac{c_k + \delta_{j,k}}{e_i} \frac{\nu_i(\mathbf{c})}{\nu_k(\mathbf{e}_i)} \frac{e_i}{c_k + 1} = 1,$$

which is equivalent to (4.28).

Therefore, we can apply Lemma 4.5 to ν_i for $i \in [n]$ to deduce that conditioned on the degree of the root and the type of the root is i, the ν_i offspring distribution of the root is multinomial.

We next show that Lemma 4.5 can be applied to μ_i for each $i \in [n]$.

Lemma 4.7. *Assume μ is reversible for simple random walk and $\mu \sim_g \nu$. Suppose $i, l \in [n]$ and $\mathbf{c} \in \mathbf{S}_n$ are such that $\mu_i(\mathbf{c}) > 0$ with $c_l > 0$. Then $\mu_i(\mathbf{c}_l^m) > 0$ for every $m \in A(\mu_i)$.*

Proof. Assume $|\mathbf{c}| = 1$. Then $\mu(B_{,i,l}) > 0$, which implies $\mu(B_{l,i,}) > 0$. Hence $\nu_i(\mathbf{0}) > 0$. Since $m \in A(\mu_i)$, $i \in A(\mu_m)$. Consequently the set of trees whose root has label m and has a neighbor labeled i that does not have any other neighbor has positive μ-probability. By stationarity of μ, $\mu(B_{,i,m}) > 0$. Therefore, we have $\mu_i(\mathbf{c}_l^m) > 0$.

If $|\mathbf{c}| > 1$, let $\mathbf{d} := \mathbf{c}_j$ for some j such that $d_l > 0$. We have $\mu(B_{\mathbf{d},i,j}) = \mu(N_i(\mathbf{c})) > 0$. Then stationarity of μ implies that $\mu(B_{j,i,\mathbf{d}}) > 0$. Since $\mu \sim \nu$, we have $\nu_i(\mathbf{d}) > 0$. By Lemma 4.6, $\nu_i(\mathbf{d}_l^m) > 0$ for all $m \in A(\nu_i)$. We also have $A(\nu_i) = A(\mu_i)$. Hence for all $m \in A(\mu_i)$, we have $\nu_i(\mathbf{d}_l^m) > 0$, which implies $\mu(B_{j,i,\mathbf{d}_l^m}) > 0$. Since μ is stationary, we get $\mu(N_i(\mathbf{c}_l^m)) = \mu(B_{\mathbf{d}_l^m,i,j}) > 0$. □

As a corollary to Lemma 4.7, if μ is reversible for simple random walk and $\mu \sim_g \nu$, then μ_i satisfies condition (i) of Lemma 4.5. We now show that in this case μ_i also satisfies condition (ii) in the statement of Lemma 4.5.

Since the ν_i conditional offspring distribution of the root is multinomial, for all $i \in [n]$ and for all $d \in D(\nu_i)$ there exist probabilities $\{p^d_{i,j}\colon\ j \in [n]\}$ such that for all $\mathbf{c} \in A(\nu_i)$ with $|\mathbf{c}| = d$, we have $\nu_i(\mathbf{c}) = \frac{|\mathbf{c}|!}{\prod_{j\in[n]} c_j!} \prod_{j\in[n]} (p^d_{i,j})^{c_j}$.

Fix $\mathbf{c} \in \mathbf{S}_n$ and $i, j, k \in [n]$ such that $\mu_i(\mathbf{c}) > 0$ with $c_j, c_k > 0$ and $j \neq k$. Since $j \in A(\mu_i)$, we have $i \in A(\mu_j)$ by Lemma 4.4. Let $\mathbf{d}$ be such that $\mu_j(\mathbf{d}) > 0$ and $d_i > 0$. Then since μ is reversible, (3.4) holds and is equivalent to

$$\mu(i)\mu_i(\mathbf{c})\nu_j(\mathbf{d}_i)\frac{c_j}{|\mathbf{c}|} = \mu(j)\mu_j(\mathbf{d})\nu_i(\mathbf{c}_j)\frac{d_i}{|\mathbf{d}|}. \tag{4.34}$$

Similarly we have

$$\mu(i)\mu_i\bigl(\mathbf{c}^j_k\bigr)\nu_j(\mathbf{d}_i)\frac{c_j+1}{|\mathbf{c}^j_k|} = \mu(j)\mu_j(\mathbf{d})\nu_i(\mathbf{c}_k)\frac{d_i}{|\mathbf{d}|}.$$

Combining the two equations, we get

$$\frac{\mu_i(\mathbf{c})}{\mu_i(\mathbf{c}^j_k)} = \frac{c_j+1}{c_j}\frac{\nu_i(\mathbf{c}_j)}{\nu_i(\mathbf{c}_k)}. \tag{4.35}$$

Since ν_i is multinomial given the number of children and $d := |\mathbf{c}_j| = |\mathbf{c}_k|$, we have

$$\frac{\mu_i(\mathbf{c})}{\mu_i(\mathbf{c}^j_k)} = \frac{c_j+1}{c_j}\frac{\nu_i(\mathbf{c}_j)}{\nu_i(\mathbf{c}_k)} = \frac{c_j+1}{c_j}\frac{c_j}{c_k}\frac{p^d_{i,k}}{p^d_{i,j}} = \frac{c_j+1}{c_k}\frac{p^d_{i,k}}{p^d_{i,j}}. \tag{4.36}$$

Interchanging j and k in (4.36), we get

$$\frac{\mu_i(\mathbf{c})}{\mu_i(\mathbf{c}^k_j)} = \frac{c_k+1}{c_j}\frac{p^d_{i,j}}{p^d_{i,k}}. \tag{4.37}$$

Combining (4.36) and (4.37), we obtain

$$\frac{c_k}{c_{j+1}}\frac{\mu_i(\mathbf{c})}{\mu_i(\mathbf{c}^j_k)} = \frac{p^d_{i,k}}{p^d_{i,j}} = \frac{c_k+1}{c_j}\frac{\mu_i(\mathbf{c}^k_j)}{\mu_i(\mathbf{c})},$$

which is equivalent to (4.28) when $k \neq j$. When $j = k$, (4.28) holds trivially.

Thus for all $i \in [n]$ and for all $d+1 \in D(\mu_i)$, given the degree of the root is $d+1$ and the type of the root is i, the μ_i-offspring distribution of the root is multinomial. By (4.37) the multinomial parameters for μ_i given that the degree of the root is $d+1$ and for ν_i given that the degree of the root is d are identical.

Lemma 4.8. *Assume μ is reversible for simple random walk and $\mu \sim_g \nu$. Then $p^d_{i,j} = p^{d'}_{i,j}$ for all $i \in [n]$, $j \in A(\nu_i)$ and $d, d' \in D(\nu_i)$.*

Proof. Let i, j, d, d' be as in the lemma and $\mathbf{c}, \mathbf{e}, \mathbf{d}$ be such that $\mu_i(\mathbf{c}), \mu_i(\mathbf{e}), \mu_j(\mathbf{d}) > 0$ with $c_j, e_j, d_i > 0$ and $|\mathbf{c}| = d + 1$, $|\mathbf{e}| = d' + 1$. Writing (4.34) for $\mathbf{e}$ and $\mathbf{d}$, we get

$$\mu(i)\mu_i(\mathbf{e})\nu_j(\mathbf{d}_i)\frac{e_j}{|\mathbf{e}|} = \mu(j)\mu_j(\mathbf{d})\nu_i(\mathbf{e}_j)\frac{d_i}{|\mathbf{d}|}. \tag{4.38}$$

Combining (4.34) and (4.38), we get

$$\frac{\mu_i(\mathbf{c})c_j}{\nu_i(\mathbf{c}_j)|\mathbf{c}|} = \frac{\mu_i(\mathbf{e})e_j}{\nu_i(\mathbf{e}_j)|\mathbf{e}|}. \tag{4.39}$$

Since the μ-probability that the root has degree $d+1$ given that it has label i is equal to $\sum_{|\mathbf{f}|=d+1}\mu_i(\mathbf{f})$, we can rewrite the left-hand side of (4.39) as

$$\frac{\sum_{|\mathbf{f}|=d+1}\mu_i(\mathbf{f})\prod_{k\in[n]}(p^d_{i,k})^{c_k}|\mathbf{c}|!/(\prod_k c_k!)c_j}{\sum_{|\mathbf{f}|=d}\nu_i(\mathbf{f})\prod_{k\in[n]}(p^d_{i,k})^{c_k-\delta_{k,j}}(|\mathbf{c}|-1)!/(\prod_k(c_k-\delta_{k,j})!)|\mathbf{c}|} = \frac{\sum_{|\mathbf{f}|=d+1}\mu_i(\mathbf{f})}{\sum_{|\mathbf{f}|=d}\nu_i(\mathbf{f})}p^d_{i,j}.$$

The right-hand side of (4.39) can be calculated in a similar way. Then (4.39) reduces to

$$\frac{\sum_{|\mathbf{f}|=d+1}\mu_i(\mathbf{f})}{\sum_{|\mathbf{f}|=d}\nu_i(\mathbf{f})}p^d_{i,j} = \frac{\sum_{|\mathbf{f}|=d'+1}\mu_i(\mathbf{f})}{\sum_{|\mathbf{f}|=d'}\nu_i(\mathbf{f})}p^{d'}_{i,j} \tag{4.40}$$

and (4.40) holds for all $j \in A(\mu_i)$. Since $\sum_{j\in A(\mu_i)} p^d_{i,j} = \sum_{j\in A(\mu_i)} p^{d'}_{i,j} = 1$, (4.40) implies that $p^d_{i,j} = p^{d'}_{i,j} := p_{i,j}$ for all $j \in A(\mu_i)$. □

We now finish proving the claims in the first paragraph. Since

$$\sum_{d+1\in D(\mu_i)}\sum_{|\mathbf{f}|=d+1}\mu_i(\mathbf{f}) = \sum_{d\in D(\nu_i)}\sum_{|\mathbf{f}|=d}\nu_i(\mathbf{f}) = 1,$$

(4.40) implies that $\sum_{|\mathbf{f}|=d+1}\mu_i(\mathbf{f}) = \sum_{|\mathbf{f}|=d}\nu_i(\mathbf{f}) =: p^d_i$ for all $d \in A(\nu_i)$.

Next, we show (4.25). The left-hand side of (4.34) is equal to

$$\mu(i)p^d_i\prod_{k\in[n]}(p^d_{i,k})^{c_k}\frac{|\mathbf{c}|!}{\prod_{k\in[n]}c_k!}p^{d'}_j\prod_{k\in[n]}(p^{d'}_{j,k})^{d_k-\delta_{i,k}}\frac{d'!}{\prod_k(e_k-\delta_{i,k})!}\frac{c_j}{|\mathbf{c}|}.$$

Similarly the right-hand side is equal to

$$\mu(j)p^{d'}_j\prod_{k\in[n]}(p^{d'}_{j,k})^{d_k}\frac{|\mathbf{d}|!}{\prod_{k\in[n]}d_k!}p^d_i\prod_{k\in[n]}(p^d_{i,k})^{c_k-\delta_{k,j}}\frac{d!}{\prod_k(c_k-\delta_{k,j})!}\frac{d_i}{|\mathbf{d}|}.$$

Setting the two equal to each other we get (4.26). Since for any $i_1, \dots, i_m = i_1$ we have $\prod_{s=1}^{m-1}\frac{\mu(i_s)}{\mu(i_{s+1})} = 1$, (4.26) gives (4.25).

Conversely assume that given the degree of the root the ν_i-offspring distributions of the root are multinomial with parameters $\{p_{i,j}\colon j \in [n]\}$ that do not depend on d and that (4.25) holds. For all $i \in [n]$ and $d \in D(\nu_i)$, let p_i^d be the ν_i-probability that the root has degree d.

Define μ in the following way. Let $\{\mu(i)\colon i \in [n]\}$ be the unique solution of the equations (4.26) and $\sum_{i=1}^n \mu(i) = 1$. For all $i \in [n]$ let the μ_i-probability that the root has degree $d+1$ given the root has label i be p_i^d. Given the labels of the root and its neighbors the descendant subtrees of the neighbors of the root are independent multi-type Galton–Watson trees with conditional offspring distributions same as conditional offspring distributions of ν.

Since (4.25) holds, there is a solution to (4.26) such that μ is a probability measure on $[n]$. The equations (4.26) give the relative weights of $\mu(i)$ hence the solution is unique. It is easy to see that (4.38) holds and therefore μ is reversible.

This completes the proof of Theorem 4.1. We finish with two open questions.

What can be said for more general relabeling functions? In other words are there other Galton–Watson measures ν for which there exist a reversible measure μ for simple random walk and a relabeling function f (other than the ones we have studied so far) such that $\mu \sim_f \nu$?

Suppose that when conditioned on the labels of the root and its neighbors, the descendant subtrees of the root are multi-type Galton–Watson trees (not necessarily independent). In this case, if the labels of the root and the neighbors of the root are given, then for any neighbor of the root, the descendant subtrees of the neighbors of that neighbor of the root are independent multi-type Galton–Watson trees. We ask whether reversibility of μ implies the conditional independence of the descendant subtrees of the neighbors of the root.

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