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On a class of space–time intrinsic random functions

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Power law generalized covariance functions provide a simple model for describing the local behavior of an isotropic random field. This work seeks to extend this class of covariance functions to spatial-temporal processes for which the degree of smoothness in space and in time may differ while maintaining other desirable properties for the covariance functions, including the availability of explicit convergent and asymptotic series expansions.

Keywords: Fox's H-function; generalized covariance function; Matérn covariance function

1. Introduction

Intrinsic random functions [1,21] provide a popular class of models for spatial processes. These non-stationary random processes are specified by their generalized covariance functions, which determine the variances of certain linear combinations of the process (see Section 2.1 for details). A particularly simple and, therefore, useful class of generalized covariance functions is the power law class, for which the "covariance" between two observations is proportional to the Euclidean distance between the points raised to some power (unless the power is an even integer). More specifically, for $x \geq 0$ indicating interpoint distance, $\zeta > 0$ and $\mathbb{N}$ the set of positive integers, the function γ_ζ

$$\gamma_\zeta(x) = \begin{cases} \Gamma(-\zeta)x^{2\zeta}, & \zeta \notin \mathbb{N}, \\ \dfrac{2(-1)^{\zeta+1}}{\zeta!}x^{2\zeta}\log x, & \zeta \in \mathbb{N} \end{cases} \tag{1}$$

gives a valid generalized covariance function in any number of dimensions. Despite its simplicity, this class of models has the important virtue of admitting a broad range of local behaviors for the process, which is critical when, for example, considering properties of spatial interpolants [30]. Specifically, the larger the value of ζ, the smoother the process, so that, for example, the process is m times mean square differentiable in any direction if and only if $\zeta > m$. A generalized covariance function can be written as the Fourier transform of a positive measure; although, in contrast to the stationary setting, the measure might not have finite total mass. In particular, in d dimensions, the measure corresponding to γ_ζ has density with respect to Lebesgue measure proportional to $|\boldsymbol{\omega}|^{-2\zeta-d}$, where $\boldsymbol{\omega} \in \mathbb{R}^d$ is the spatial frequency.

The goal of this paper is to find a good extension of γ_ζ to the space–time setting. The problem is made difficult by what I will mean by "good." The first requirement is that the class of models includes members allowing any degree of smoothness in space and any (possibly different) degree of smoothness in time. Specifically, for $G(\mathbf{x}, t)$ a generalized covariance function of spatial lag $\mathbf{x}$ and temporal lag t, and any positive ζ_1, ζ_2, C_1 and C_2, the class of generalized covariance functions should include a member that satisfies

$$G(\mathbf{x}, 0) = C_1 \gamma_{\zeta_1}(|\mathbf{x}|) \tag{2}$$

and

$$G(0, t) = C_2 \gamma_{\zeta_2}(|t|) \tag{3}$$

or, failing that, that (2) holds asymptotically as $\mathbf{x} \to \mathbf{0}$, and (3) holds asymptotically as $t \to 0$.

The second requirement is that G be smoother away from the origin than it is at the origin. This requirement is needed to avoid the kinds of anomalies described in [31] for covariance functions that are not smoother away from the origin. As a simple example of this kind of anomaly, consider the covariance function on $\mathbb{R} \times \mathbb{R}$ given by $K(x, t) = \exp(-|x| - |t|)$. Write corr for correlation, and define $\rho(x, t) = \lim_{\varepsilon \downarrow 0} \operatorname{corr}\{Z(0, \varepsilon) - Z(0, 0), Z(x, t + \varepsilon) - Z(x, t)\}$. Then for $x \neq 0$, straightforward calculations show $\rho(x, t) = 0$ for $t \neq 0$ and $\rho(x, 0) = \mathrm{e}^{-|x|}$. The discontinuity in this limiting correlation is due to the fact that $K(x, t)$ has a similar discontinuity in its first derivative in the t direction everywhere along the x axis as it does at the origin. I consider such a discontinuity in ρ as unrealistic for most natural space–time processes. In particular, Stein and Handcock [32] and Stein [29] give examples showing how this lack of continuity away from the origin can lead to optimal (kriging) predictors with undesirable properties. All separable space–time covariance functions, that is, those that factor into a function of space and a function of time such as $\mathrm{e}^{-|x|-|t|}$, have a similar problem unless the process is infinitely differentiable [31], page 311. Furthermore, many non-separable space–time covariance functions proposed in the literature share this problem [31], pages 311–312. For a space–time covariance function with different degrees of smoothness in space and time, that is, satisfying (2) and (3) with $\zeta_1 \neq \zeta_2$, it is not so clear what one should mean by the function being smoother away from the origin than at the origin. This issue is addressed in Section 2.4.

The smoothness of a covariance function away from the origin is closely related to regularity properties of the corresponding spectral density at high frequencies [31]. Indeed, Stein [29] argues that the spectral domain provides a more natural approach for considering the appropriateness of various models for space–time covariance functions. Specifically, [29] gives the following condition as a plausible requirement for the spectral density $f(\boldsymbol{\omega})$ of a natural process in space or space–time: for every $R < \infty$,

$$\lim_{\boldsymbol{\omega} \to \infty} \sup_{|\boldsymbol{\nu}| < R} \left| \frac{f(\boldsymbol{\omega} + \boldsymbol{\nu})}{f(\boldsymbol{\omega})} - 1 \right| = 0. \tag{4}$$

That is, f changes slowly (on a relative scale) at high frequencies. This condition excludes, for example, separable space–time models.

The third and final requirement for the generalized covariance functions is that they can be computed accurately and efficiently via, for example, series expansions, to allow them to be applied routinely to large space–time datasets. In particular, representations of the function as an integral will not be considered an adequate solution to the problem. As best as I am aware, no existing class of generalized covariance functions satisfies all three of these requirements.

It will be convenient to avoid explicitly distinguishing between space and time and consider processes $Z(\mathbf{x}, \mathbf{y})$ on $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ for positive integers d_1 and d_2. Stein [31] proposed the following class of spectral densities (with a different parameterization) as a flexible parametric model for stationary space–time processes:

$$q(\boldsymbol{\tau}, \boldsymbol{\omega}) = \left\{ \left(\frac{\beta_1^2 + |\boldsymbol{\tau}|^2}{\sigma_1^2} \right)^{\alpha_1} + \left(\frac{\beta_2^2 + |\boldsymbol{\omega}|^2}{\sigma_2^2} \right)^{\alpha_2} \right\}^{-\nu} \tag{5}$$

for σ_1, σ_2 positive, $\beta_1^2 + \beta_2^2 > 0$ and $2\nu > d_1/\alpha_1 + d_2/\alpha_2$; this last condition being necessary and sufficient (given the positivity constraints on the other parameters) for q to be integrable. The parameters β_1 and β_2 are inverse range parameters, σ_1 and σ_2 are scale parameters and α_1, α_2 and ν together describe the smoothness of the process in $\mathbf{x}$ and $\mathbf{y}$. More specifically, the process is p times mean square differentiable in each component of $\mathbf{x}$ if and only if $2\nu > (d_1 + 2p)/\alpha_1 + d_2/\alpha_2$ and, similarly, is p times mean square differentiable in components of $\mathbf{y}$ if and only if $2\nu > d_1/\alpha_1 + (d_2 + 2p)/\alpha_2$ [31]. Furthermore, when α_1 and α_2 are integers, the resulting covariance function is infinitely differentiable away from the origin ([31], Proposition 4, although this result also follows from [28], Theorem 1.1). All models in the class (5) satisfy the spectral condition (4). Porcu [22] describes more general approaches to obtaining valid spectral densities that could be useful in the space–time context.

An obstacle to using the class of models (5) is the lack of an explicit expression for the corresponding covariance functions. Except when $\alpha_1 = \alpha_2 = 1$ and some special cases with ν an integer, $\alpha_1 = 1$ and $\alpha_2 = 2$, I am unaware of any cases for which an explicit expression has been written down [31]. For rational spectral densities, which includes the model (5) when α_1, α_2 and ν are all integers as a special case, the covariance function can be expressed as the solution of a certain set of equations that reduce to a partial differential equation when the rational function is just a reciprocal of a polynomial [25,26]. However, writing down an explicit solution for all integers α_1, α_2 and ν is not a simple task. Ma [18] gives explicit expressions for the covariance functions of space–time processes with rational spectral densities in some limited special cases that do not include any instances of (5). For all of the cases treated in [18], the covariance functions are not smoother away from the origin than at the origin in the sense defined in Section 2.3, and the spectral densities do not satisfy (4). Even if one had explicit expressions for the covariance functions of all rational spectral densities, these covariance functions (asymptotically in a neighborhood of the origin) satisfy (2) and (3) only for a countable nowhere dense set of (ζ_1, ζ_2) values. Kelbert, Leonenko and Ruiz-Medina [12] consider the case $d_1 = d_2 = 1, \alpha_1 = 1, \alpha_2 = 2$ and $\beta_1 = 0$ as a stochastic fractional heat equation in some detail, but they only give integral representations for the covariance functions. They consider additionally setting $\beta_2 = 0$ and note that the resulting random field has a self-similarity property when $\nu \in (\frac{3}{4}, \frac{7}{4})$. Christakos [3], page 225, mentions the case with $d_1 = 1, \alpha_1 = \nu = 1$ and $\beta_1 = 0$ as a possible model for spatial-temporal processes, but derives no results for these models.

The parameters β_1 and β_2 are range parameters that do not affect the local behavior of the process, so consider setting $\beta_1 = \beta_2 = 0$ in (5), yielding

$$f(\boldsymbol{\tau}, \boldsymbol{\omega}) = \left\{ \left(\frac{|\boldsymbol{\tau}|}{\sigma_1} \right)^{2\alpha_1} + \left(\frac{|\boldsymbol{\omega}|}{\sigma_2} \right)^{2\alpha_2} \right\}^{-\nu}. \tag{6}$$

These spectral densities are not integrable in a neighborhood of the origin, so they correspond to generalized covariance functions. Section 2.1 gives some background on generalized covariance functions. Theorem 1 in Section 2.2 gives a convergent power series for the generalized covariance function corresponding to (6) when $\alpha_2 = 1, \alpha_1 > 1$ and $\mathbf{y} \neq \mathbf{0}$ and separate explicit formulae to cover the case $\mathbf{y} = \mathbf{0}$. For completeness, the known result [6] for the generalized covariance function when $\alpha_1 = 1$ is also given in Theorem 1.

Section 2.3 shows how the generalized covariance functions in Theorem 1 can, in most cases, be written in terms of the H-function [14], a generalization of the generalized hypergeometric function that is sometimes called Fox's H-function [5]. This result is used to obtain asymptotic series for these generalized covariance functions and to motivate a conjecture extending Theorem 1 to the case $\alpha_1 < 1$. Section 2.4 shows that for any positive C_1, C_2, ζ_1 and ζ_2, one can find a spectral density of this form satisfying (2) and (3). Furthermore, in a sense made precise in Section 2.4, the resulting generalized covariance function is shown to be smoother away from the origin than at the origin. Finally, these results are used to show that at least some of these covariance functions avoid what [13] calls the "dimple" that occurs in some proposed space–time covariance functions, which is a lack of monotonicity in the correlation structure that might often be viewed as unnatural.

Section 3 discusses some limitations and possible extensions of the generalized covariance functions considered in Section 2. This section also touches on some of the difficulties in using the series expansions to compute these functions quickly and accurately. Section A.1 gives a proof of Theorem 1, and Section A.2 collects some needed material on H-functions.

There is a substantial recent literature on the development of space–time covariance functions with explicit representations in terms of well-known special functions. Some references include [3,4,7,9,10,15–19,23,24,27,31,33]. Although these works consider a broad range of models for space–time covariance functions, none of them give a class of covariance functions meeting the criteria set forth in this section of (locally) satisfying (2) and (3) for all positive ζ_1 and ζ_2 as well as being smoother away from the origin than at the origin. As noted in [31], perhaps [16] comes closest to this goal, in that this paper gives a class of models with $d_2 = 1$ including elements satisfying, asymptotically in a neighborhood of the origin, (2) with $\zeta_1 = \frac{1}{2}$ and (3) with $0 < \zeta_2 < \frac{1}{4}$, and the covariance functions are infinitely differentiable away from the origin.

2. Theoretical results

2.1. Generalized space–time covariance functions

Intrinsic random functions and generalized covariance functions have been a standard tool in geostatistics since Matheron's pioneering work [21]. These processes are nearly stationary in the sense that variances of some class of linear combinations of the process are translationally invariant. Specifically, for a random field Z on $\mathbb{R}^d$ (so that $d = d_1 + d_2$ is the total number of dimensions for space–time processes), call $\sum_{j=1}^n \lambda_j Z(\mathbf{z}_j)$ an authorized linear combination of order k, or ALC-k, if $\sum_{j=1}^n \lambda_j P(\mathbf{z}_j) = 0$ for every polynomial P of order at most k. A function G on $\mathbb{R}^d$ is called a generalized covariance function of order k, or GC-k, if, for every ALC-k, $\sum_{\ell,j=1}^n \lambda_\ell \lambda_j G(\mathbf{z}_\ell - \mathbf{z}_j) \geq 0$. A process Z for which $\mathrm{Var}\{\sum_{j=1}^n \lambda_j Z(\mathbf{z}_j)\} = \sum_{\ell,j=1}^n \lambda_\ell \lambda_j G(\mathbf{z}_\ell - \mathbf{z}_j) \geq 0$ and $E\{\sum_{j=1}^n \lambda_j Z(\mathbf{z}_j)\} = 0$ for every ALC-k is said to be an intrinsic random function of order k, or IRF-k, with G as its GC-k. GC-ks are not unique; if G is a GC-k for Z, then so is G plus any even polynomial of degree $2k$ in $\mathbf{x}$. A function f is the spectral density of a GC-k G if

$$\sum_{\ell,j=1}^n \lambda_\ell \lambda_j G(\mathbf{z}_\ell - \mathbf{z}_j) = \int_{\mathbb{R}^d} \left| \sum_{j=1}^n \lambda_j \mathrm{e}^{\mathrm{i}\boldsymbol{\omega}'\mathbf{z}_j} \right|^2 f(\boldsymbol{\omega})\,\mathrm{d}\boldsymbol{\omega}$$

for every ALC-k $\sum_{j=1}^n \lambda_j Z(\mathbf{z}_j)$. A nonnegative even function f is the spectral density for a real-valued GC-k if and only if $f(\boldsymbol{\omega})|\boldsymbol{\omega}|^{2k+2}/(1+|\boldsymbol{\omega}|^{2k+2})$ is integrable [21]. Of course, if Z is an IRF-k, it is also an IRF-k' for all integers $k' \geq k$. Let $\lfloor x \rfloor$ indicate the greatest integer less than or equal to x. For the spectral density (6) and $i = 1, 2$, define $k_i = \lfloor \alpha_i \{\nu - d_1/(2\alpha_1) - d_2/(2\alpha_2)\} \rfloor$ and $k_0 = \max(k_1, k_2)$. Straightforward calculus shows that f in (6) satisfies the integrability condition for an IRF-k if and only if $k \geq k_0$. Christakos [2] considers an extension of the notion of IRFs to the space–time setting in which one essentially allows a different degree of differencing in space and in time, but this concept is not used here.

2.2. Main theorem

This section gives a series expansion for the generalized covariance function corresponding to (6) when $\alpha_2 = 1$, $\alpha_1 > 1$ and $\sigma_1 = \sigma_2 = 1$. Extending the result to other positive values of σ_1 and σ_2 is trivial. Define the function $\mathcal{M}_\nu(y) = y^\nu \mathcal{K}_\nu(y)$, where $\mathcal{K}_\nu$ is a modified Bessel function. When $\nu > 0$, $\mathcal{M}_\nu$ is often called the Matérn covariance function with smoothness parameter ν. Set $\theta = \nu - \frac{1}{2}d_2$ and $\theta' = \theta - d_1/(2\alpha_1)$ so that $k_0 = \lfloor \alpha_1 \theta' \rfloor$ is the (minimal) order of the IRF corresponding to this spectral density. As noted in [31] for the more general model (5), when $\alpha_2 = 1$, the Fourier transform with respect to $\boldsymbol{\omega}$ can be carried out explicitly. Specifically, for $\boldsymbol{\tau} \neq \mathbf{0}$,

$$\int_{\mathbb{R}^{d_2}} (|\boldsymbol{\tau}|^{2\alpha_1} + |\boldsymbol{\omega}|^2)^{-\nu} \mathrm{e}^{\mathrm{i}\boldsymbol{\omega}'\mathbf{y}}\,\mathrm{d}\boldsymbol{\omega} = \frac{\pi^{d_2/2} \mathcal{M}_\theta(|\boldsymbol{\tau}|^{\alpha_1}|\mathbf{y}|)}{2^{\theta-1}\Gamma(\nu)|\boldsymbol{\tau}|^{2\alpha_1\theta}}. \tag{7}$$

Define $r_{\ell j} = |\mathbf{x}_\ell - \mathbf{x}_j|$, $s_{\ell j} = |\mathbf{y}_\ell - \mathbf{y}_j|$ and, for $t > 0$,

$$\Lambda_d(t) = 2^{(d-2)/2}\Gamma\left(\frac{1}{2}d\right) t^{-(d-2)/2} J_{(d-2)/2}(t) = \sum_{m=0}^\infty \frac{(-(1/4)t^2)^m}{m!(d/2)_m}, \tag{8}$$

where, for any real a and positive integer j, $(a)_j = a(a+1)\cdots(a+j-1)$ and $(a)_0 = 1$. For every ALC-k_0 $\sum_{j=1}^n \lambda_j Z(\mathbf{x}_j, \mathbf{y}_j)$, the GC-$k_0$ G corresponding to the spectral density $(|\boldsymbol{\tau}|^{2\alpha_1} +$

$|\boldsymbol{\omega}|^2)^{-\nu}$ satisfies

$$
\begin{aligned}
&\sum_{\ell,j=1}^{n} \lambda_\ell \lambda_j G(\mathbf{x}_\ell - \mathbf{x}_j, \mathbf{y}_\ell - \mathbf{y}_j) \\
&\quad = \int_{\mathbb{R}^{d_1}} \int_{\mathbb{R}^{d_2}} \sum_{\ell,j=1}^{n} \lambda_\ell \lambda_j \mathrm{e}^{\mathrm{i}\boldsymbol{\tau}'(\mathbf{x}_\ell - \mathbf{x}_j) + \mathrm{i}\boldsymbol{\omega}'(\mathbf{y}_\ell - \mathbf{y}_j)} (|\boldsymbol{\tau}|^{2\alpha_1} + |\boldsymbol{\omega}|^2)^{-\nu} \,\mathrm{d}\boldsymbol{\omega}\,\mathrm{d}\boldsymbol{\tau} \\
&\quad = \frac{\pi^{d_2/2}}{2^{\theta-1}\Gamma(\nu)} \int_{\mathbb{R}^{d_1}} \sum_{\ell,j=1}^{n} \lambda_\ell \lambda_j \frac{\mathcal{M}_\theta(|\boldsymbol{\tau}|^{\alpha_1} s_{\ell j})}{|\boldsymbol{\tau}|^{2\alpha_1\theta}} \mathrm{e}^{\mathrm{i}\boldsymbol{\tau}'(\mathbf{x}_\ell - \mathbf{x}_j)} \,\mathrm{d}\boldsymbol{\tau} \\
&\quad = \frac{4\pi^{(d_1+d_2)/2}}{2^\theta \Gamma(\nu)\Gamma((1/2)d_1)} \int_0^\infty \sum_{\ell,j=1}^{n} \lambda_\ell \lambda_j u^{d_1-1-2\alpha_1\theta} \Lambda_{d_1}(r_{\ell j} u) \mathcal{M}_\theta(s_{\ell j} u^{\alpha_1}) \,\mathrm{d}u \\
&\quad = \frac{4\pi^{(d_1+d_2)/2}}{2^\theta \Gamma(\nu)\Gamma((1/2)d_1)\alpha_1} \\
&\qquad \times \int_0^\infty \sum_{\ell,j=1}^{n} \lambda_\ell \lambda_j t^{d_1/\alpha_1 - 2\theta - 1} \Lambda_{d_1}(r_{\ell j} t^{1/\alpha_1}) \mathcal{M}_\theta(s_{\ell j} t) \,\mathrm{d}t,
\end{aligned}
\tag{9}
$$

where the second step uses (7), the third basic results on Fourier transforms of isotropic functions [30], Section 2.10, and the last step the change of variables $t = u^{\alpha_1}$. What one would like to do is substitute (8) into (9) and integrate termwise, but justifying this interchange requires considerable care.

Define

$$
c_m(\alpha_1) = \frac{\pi^{(d_1+d_2)/2}\Gamma((d_1+2m)/(2\alpha_1))}{m!\Gamma(m+(1/2)d_1)}
$$

and the digamma function ψ by $\psi(z) = \frac{\mathrm{d}}{\mathrm{d}z}\log\Gamma(z)$. Equations (10)–(12) are proven in Section A.1. Equations (13) and (14) are taken from [6], Chapter II, Section 3.3, equations (2) and (11).

Theorem 1. *For the spectral density $f(\boldsymbol{\tau},\boldsymbol{\omega}) = (|\boldsymbol{\tau}|^{2\alpha_1} + |\boldsymbol{\omega}|^2)^{-\nu}$ on $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ with $\alpha_1 \geq 1$, $\theta' = \nu - d_1/(2\alpha_1) - d_2/2 > 0$ and $k_0 = \lfloor \alpha_1\theta' \rfloor$, a corresponding GC-$k_0$ is given by $\tilde{G}(|\mathbf{x}|, |\mathbf{y}|)$ for the function $\tilde{G}$, defined by the following equations. First consider $\alpha_1 > 1$. For $s > 0$,*

$$
\tilde{G}(r,s) = \sum_{m=0}^{\infty} \frac{c_m(\alpha_1)}{\alpha_1\Gamma(\nu)} \left\{ -\left(\frac{1}{2}r\right)^2 \right\}^m \gamma_{\theta' - m/\alpha_1}\left(\frac{1}{2}s\right) \tag{10}
$$

with $\gamma(\cdot)$ defined by (1). When $\alpha_1\theta' > k_0$,

$$
\tilde{G}(r,0) = \frac{\pi^{(d_1+d_2)/2}\Gamma(\theta)}{\Gamma(\nu)\Gamma(\alpha_1\theta)} \gamma_{\alpha_1\theta'}\left(\frac{1}{2}r\right), \tag{11}
$$

and when $\alpha_1\theta' = k_0$,

$$\tilde{G}(r,0) = -\frac{c_{k_0}(\alpha_1)}{\Gamma(\nu_0)}\left\{-\left(\frac{1}{2}r\right)^2\right\}^{k_0}\left\{2\log\left(\frac{1}{2}r\right) + \frac{1}{\alpha_1}\psi\left(\frac{2k_0+d_1}{2\alpha_1}\right) + \frac{1}{\alpha_1}\psi(1) - \psi\left(k_0 + \frac{1}{2}d_1\right) - \psi(k_0+1)\right\}. \tag{12}$$

Finally, when $\alpha_1 = 1$ *and* θ' *is not an integer,*

$$\tilde{G}(r,s) = -\frac{\pi^{(d_1+d_2+2)/2}}{\sin(\pi\theta')\Gamma(\theta'+1)\Gamma(\nu)2^{2\theta'}}(r^2+s^2)^{\theta'}, \tag{13}$$

and when $\alpha_1 = 1$ *and* θ' *is an integer,*

$$\tilde{G}(r,s) = \frac{(-1)^{k+1}\pi^{(d_1+d_2)/2}}{\theta'!\Gamma(\nu)2^{2\theta'}}(r^2+s^2)^{\theta'}\log(r^2+s^2). \tag{14}$$

2.3. H-functions

The function $\tilde{G}$ can, in most cases, be written in terms of H-functions [14,20]. Section A.2 gives the definition of H-functions as a contour integral and some other needed information about the functions. If

$$\ell + \frac{m}{\alpha_1} \neq \theta' \qquad \text{for all whole numbers } \ell \text{ and } m, \tag{15}$$

then (45) in Section A.2 is satisfied, and

$$H^{2,1}_{2,2}\left(z \,\middle|\, \begin{matrix} (1,1), \left(\frac{d_1}{2}, 1\right) \\ \left(\frac{d_1}{2\alpha_1}, \frac{1}{\alpha_1}\right), \left(-\theta', \frac{1}{\alpha_1}\right) \end{matrix}\right)$$

is well defined. Since, by (47), $\Delta = \frac{2}{\alpha_1} - 2$, which is negative for $\alpha_1 > 1$, [14], Theorem 1.4, applies, yielding, for $z \neq 0$,

$$H^{2,1}_{2,2}\left(z \,\middle|\, \begin{matrix} (1,1), \left(\frac{d_1}{2}, 1\right) \\ \left(\frac{d_1}{2\alpha_1}, \frac{1}{\alpha_1}\right), \left(-\theta', \frac{1}{\alpha_1}\right) \end{matrix}\right) = \sum_{k=0}^{\infty} h_{1k} z^{-\alpha_1 k}, \tag{16}$$

where

$$h_{1k} = (-1)^k \alpha_1 \Gamma\left(\frac{d_1+2k}{2\alpha_1}\right)\Gamma\left(-\theta' + \frac{k}{\alpha_1}\right)\bigg/\left(k!\Gamma\left(\frac{d_1}{2}+k\right)\right).$$

Comparing this result to (10) yields (assuming (15))

$$\tilde{G}(r,s) = \frac{\pi^{(d_1+d_2)/2}}{\Gamma(\nu)\alpha_1}\left(\frac{1}{2}s\right)^{2\theta'} H_{2,2}^{2,1}\left(\frac{((1/2)s)^{2/\alpha_1}}{((1/2)r)^2}\,\middle|\, \begin{matrix} (1,1),\left(\frac{d_1}{2},1\right) \\ \left(\frac{d_1}{2\alpha_1},\frac{1}{\alpha_1}\right),\left(-\theta',\frac{1}{\alpha_1}\right)\end{matrix}\right), \tag{17}$$

where this result holds for $r = 0$ by continuity. For the parameter values of the H-function in (17), from (46) and (47) in Section 3.2, $\Delta < 0$ and $a^* > 0$, so that by [14], Theorem 1.11, $H_{2,2}^{2,1}(z)$ has an asymptotic expansion as $z \to 0$ for $|\arg z| < \frac{1}{2}a^*\pi$. To avoid complications, let $\mathbb{N}_0$ be the set of nonnegative integers, assume $\theta \notin \mathbb{N}_0$ and that for all $\ell \in \mathbb{N}_0$, $(\theta' - \ell)\alpha_1 \notin \mathbb{N}_0$ and $(\theta - \ell)\alpha_1 \notin \mathbb{N}$. Then, as $z \to 0$,

$$H_{2,2}^{2,1}\left(z\,\middle|\, \begin{matrix} (1,1),\left(\frac{d_1}{2},1\right) \\ \left(\frac{d_1}{2\alpha_1},\frac{1}{\alpha_1}\right),\left(-\theta',\frac{1}{\alpha_1}\right)\end{matrix}\right) \sim \sum_{\ell=0}^{\infty}\left\{h_{1\ell}^* z^{\ell\alpha_1+d_1/2} + h_{2\ell}^* z^{(\ell-\theta')\alpha_1}\right\}, \tag{18}$$

where, using the duplication formula for Γ,

$$\begin{aligned} h_{1\ell}^* &= \frac{(-1)^\ell \alpha_1 \Gamma(-\theta-\ell)\Gamma((1/2)d_1+\ell\alpha_1)}{\ell!\Gamma(-\ell\alpha_1)} \\ &= \frac{\alpha_1 \sin(\pi\ell\alpha_1)\Gamma((1/2)d_1+\ell\alpha_1)\Gamma(\ell\alpha_1+1)}{\sin(\pi\theta)\Gamma(\theta+\ell+1)\ell!} \end{aligned} \tag{19}$$

and

$$\begin{aligned} h_{2\ell}^* &= \frac{(-1)^\ell \alpha_1 \Gamma(\theta-\ell)\Gamma((\ell-\theta')\alpha_1)}{\ell!\Gamma(\alpha_1(\theta-\ell))} \\ &= \frac{\alpha_1 \sin\{\pi\alpha_1(\theta-\ell)\}\Gamma((\ell-\theta')\alpha_1)\Gamma((\ell-\theta)\alpha_1+1)}{\sin(\pi\theta)\Gamma(1-\theta+\ell)\ell!}. \end{aligned}$$

From (19), $h_{10}^* = 0$ and, if α_1 is an integer, $h_{1\ell}^* = 0$ for all ℓ. For $\theta \in \mathbb{N}_0$, [14], (1.8.2) applies, yielding an asymptotic expansion with logarithmic terms, but the result is rather messy and is omitted here.

When $(\frac{1}{2}s)^{2/\alpha_1}/(\frac{1}{2}r)^2$ is small, (10) converges slowly and is numerically unstable. Specifically, for any fixed $m > \alpha_1\theta'$ and $r > 0$, as $s \downarrow 0$, the mth term in (10) tends to $\pm\infty$, even though $\lim_{s\downarrow 0}\tilde{G}(r,s) \to \tilde{G}(r,0)$, which is finite. Thus, there must be a near canceling of large terms of opposite signs in (10) for s small, so that high precision arithmetic would be needed to obtain accurate results for s sufficiently small. Fortunately, the asymptotic expansion (18) can be used to approximate $\tilde{G}(r,s)$ for s small. Substituting (18) into (17) and considering $r > 0$ fixed,

$$\tilde{G}(r,s) \sim \frac{\pi^{(d_1+d_2)/2}}{\Gamma(\nu)\alpha_1}\sum_{\ell=0}^{\infty}\left\{h_{1\ell}^*\left(\frac{1}{2}s\right)^{2\theta}\left(\frac{1}{2}r\right)^{-d_1} + h_{2\ell}^*\left(\frac{1}{2}r\right)^{2\alpha_1\theta'}\right\}\left\{\frac{((1/2)s)^2}{((1/2)r)^{2\alpha_1}}\right\}^\ell \tag{20}$$

as $s \downarrow 0$. Since $h^*_{10} = 0$, when θ is not an integer,

$$\tilde{G}(r, s) = \sum_{\ell=0}^{\lfloor\theta\rfloor+1} A_\ell s^{2\ell} + B\gamma_{\theta+1}(s) + \mathrm{o}(s^{2\theta}) \tag{21}$$

as $s \downarrow 0$ for some constants (depending on r) $A_0, \dots, A_{\lfloor\theta\rfloor+1}, B$, where $B = 0$ if and only if α_1 is an integer. The asymptotic expansion (21) also holds when θ is an integer by [14], (1.8.2) and (1.4.5).

It is apparent that (10) does not give a valid power series expansion for $\tilde{G}$ when $\alpha_1 < 1$, since it is easy to show that the individual terms in the sum do not tend to 0 as $m \to \infty$ for fixed and positive r and s. Nevertheless, the representation in terms of H-functions given by (17) may still be valid for $\alpha_1 \le 1$ when $2\alpha_1\theta > d_1$ (so that $\theta' > 0$). Excluding values of (α_1, ν) for which (15) is not satisfied, a natural conjecture is that (17) holds when $\alpha_1 < 1$ if $2\alpha_1\theta > d_1$. A plausible approach to proving this conjecture would be to use analytic continuation, but this would require at the least extending the definition of H-functions to a strip of complex values of α_1 containing the positive real axis. Note that when $\alpha_1 < 1$, (47) implies $\Delta > 0$ and, by [14], Theorem 1.3, (18) becomes a convergent power series for $H^{2,1}_{2,2}$. On the other hand, now (16) is no longer a convergent power series, but, with a^* defined as in (46), it is a valid asymptotic expansion as $z \to \infty$ for $|\arg z| < \frac{1}{2}a^*\pi$ by [14], Theorem 1.7.

For $\alpha_1 = 1$, (17) can be directly verified when (15) holds. Specifically, for $\alpha_1 = 1$, using [14], Property 2.2, the right-hand side of (17) reduces to

$$\frac{\pi^{(d_1+d_2)/2}}{\Gamma(\nu)} \left(\frac{1}{2}s\right)^{2\theta'} H^{1,1}_{1,1}\left(\frac{s^2}{r^2} \,\middle|\, \begin{matrix}(1, 1)\\(-\theta', 1)\end{matrix}\right). \tag{22}$$

The parameter Δ defined in (47) equals 0 and, thus, one can show that the H-function has a convergent power series given by [14], Theorem 1.3, when $s < r$ and by [14], Theorem 1.4, when $s > r$. Consider $s < r$ and $\theta' \notin \mathbb{N}_0$. Then straightforward calculations yield that (22) equals

$$\begin{aligned}
&\frac{\pi^{(d_1+d_2)/2}}{\Gamma(\nu)2^{2\theta'}} r^{2\theta'} \sum_{\ell=0}^{\infty} \frac{(-1)^\ell \Gamma(-\theta' + \ell)}{\ell!} \left(\frac{s}{r}\right)^{2\ell}\\
&\quad = -\frac{\pi^{(d_1+d_2+2)/2}}{\sin(\pi\theta')\Gamma(\theta'+1)\Gamma(\nu)2^{2\theta'}} r^{2\theta'} \sum_{\ell=0}^{\infty} \binom{\theta'}{\ell} \left(\frac{s}{r}\right)^{2\ell},
\end{aligned}$$

which, using the binomial series, equals (13). A similar argument shows (22) equals (13) for $s > r$, and it additionally holds for $s = r$ by continuity.

2.4. Consequences

One goal of this paper was to find a class of generalized covariance functions that has a member satisfying (2) and (3) for all C_1, C_2, ζ_1 and ζ_2 positive. In fact, the functions of the form $G(b_1\mathbf{x}, b_2\mathbf{y})$ with $\alpha_1 \ge 1$ and b_1 and b_2 positive satisfy this requirement. To prove this, first suppose $\zeta_1 \neq \zeta_2$ and, without loss of generality, take $\zeta_1 > \zeta_2$. Theorem 1 gives explicit expressions for positive constants D_1 and D_2 such that $G(\mathbf{x}, \mathbf{0}) = D_1\gamma_{\alpha_1\theta'}(\frac{1}{2}|\mathbf{x}|)$ and $G(\mathbf{0}, \mathbf{y}) = D_2\gamma_{\theta'}(\frac{1}{2}|\mathbf{y}|)$. Setting $\theta' = \zeta_2$ and $\alpha_1 = \zeta_1/\zeta_2$ achieves the desired degree of smoothness in all directions. From (10), $\tilde{G}(0, b_2 s) = D_2(\frac{1}{2}b_2)^{\zeta_2}\gamma_{\zeta_2}(s)$, so by appropriate choice of b_2, one obtains $\tilde{G}(0, b_2 s) = C_2\gamma_{\zeta_2}(s)$. For $\zeta_1 = \alpha_1\theta'$ not an integer, by (11), $\tilde{G}(b_1 r, 0) = D_1(\frac{1}{2}b_1)^{\zeta_1}\gamma_{\zeta_1}(r)$, so by appropriate choice of b_1, one gets $\tilde{G}(b_1 r, 0) = C_1\gamma_{\zeta_1}(r)$. When $\zeta_1 = \alpha_1\theta'$ is an integer, by (12), $\tilde{G}(b_1 r, 0) = D_1(\frac{1}{2}b_1)^{2\zeta_1}\gamma_{\zeta_1}(r)$ plus some constant times $r^{2\zeta_1}$, so that b_1 can be chosen to make $\tilde{G}(b_1 r, 0) = C_1\gamma_{\zeta_1}(r)$ plus some constant times $r^{2\zeta_1}$. When $\zeta_1 = \zeta_2$, set $\alpha_1 = \alpha_2 = 1$ and $\zeta_1 = \nu - \frac{1}{2}d_1 - \frac{1}{2}d_2$. By (13), there exists $D > 0$ such that $G(b_1\mathbf{x}, b_2\mathbf{y}) = D\gamma_{\zeta_1}(\sqrt{|b_1\mathbf{x}|^2 + |b_2\mathbf{y}|^2})$. As before, one can clearly choose b_1 and b_2 so that (2) and (3) are satisfied as long as one ignores an even polynomial of degree $2\zeta_1$ when ζ_1 is an integer. Since a GC-k is only identified up to even polynomials of degree at most $2k$, it is fair to say that the class of generalized covariance functions corresponding to (6) with $\alpha_2 = 1$ and $\alpha_1 \geq 1$ includes members satisfying (2) and (3) for all $\zeta_1 \geq \zeta_2, C_1$ and C_2.

Now consider in what sense members of G achieve the goal, identified in Section 1, of being smoother away from the origin than they are at the origin. If a function is not infinitely differentiable in any direction at the origin but is infinitely differentiable everywhere but the origin, then one might say without controversy that such a function is smoother away from the origin than at the origin. Thus, when $\alpha_1 \in \mathbb{N}$, the issue is settled. But when α_1 is not an integer, B in (21) is not 0, and G is not infinitely differentiable away from the origin.

One way to describe the smoothness of a function is by its pointwise Hölder exponent s. Consider a function f from $\mathbb{R}^d$ to $\mathbb{R}$, $s > 0$ and $\mathbf{x}_0 \in \mathbb{R}^d$. Then $f \in C^s(\mathbf{x}_0)$ if and only if there exists $\varepsilon > 0$ and a polynomial P of degree less than $\lfloor s \rfloor$ and a constant C such that $|f(\mathbf{x}) - P(\mathbf{x} - \mathbf{x}_0)| \leq C|\mathbf{x} - \mathbf{x}_0|^s$ for all $\mathbf{x}$ satisfying $|\mathbf{x} - \mathbf{x}_0| < \varepsilon$. The Hölder exponent of f at $\mathbf{x}_0$, which I will denote by $\mathrm{HE}(\mathbf{x}_0, f)$, equals $\sup\{s\colon s \in C^s(\mathbf{x}_0)\}$. Because the degree of smoothness at the origin of G varies in different directions, it will not suffice to compare the Hölder exponent at the origin to the Hölder exponent elsewhere. To avoid the kind of anomaly described in Section 1, consider the smoothness of G in each direction separately. Specifically, for vectors $\mathbf{z}_0, \mathbf{z}_1 \in \mathbb{R}^{d_1+d_2}$ and (generalized) covariance function K, consider the function of $t \in \mathbb{R}$ given by $\bar{K}(t; \mathbf{z}_0, \mathbf{z}_1) = K(\mathbf{z}_0 + t\mathbf{z}_1)$. Define $\mathbf{0}$ to be a vector of zeroes whose length is apparent from context. Then I claim that, in the present setting, a useful notion of K being smoother away from the origin than at the origin is

$$\mathrm{HE}(0, \bar{K}(\cdot; \mathbf{z}_0, \mathbf{z}_1)) > \mathrm{HE}(0, \bar{K}(\cdot; \mathbf{0}, \mathbf{z}_1)) \qquad \text{for all } \mathbf{z}_0 \neq \mathbf{0}, \mathbf{z}_1 \neq \mathbf{0}. \tag{23}$$

To see why this definition might be appropriate here, consider the following generalization of the example in the introduction. Suppose K is a covariance function or a generalized covariance function of order 0 and, for nonzero $\mathbf{z}_0$ and $\mathbf{z}_1$ and $0 < \alpha < 2$, $K(t\mathbf{z}_1) = C_0 + C_1|t|^\alpha + o(|t|^\alpha)$ and $K(\mathbf{z}_0 + t\mathbf{z}_1) = D_0 + D_1|t|^\alpha + D_2 t + o(|t|^\alpha)$ as $t \to 0$ for some constants C_0, C_1, D_0, D_1 and D_2 (possibly depending on $\mathbf{z}_0$ and $\mathbf{z}_1$) with C_1 and D_1 nonzero. It follows that $\mathrm{HE}(0, \bar{K}(\cdot; \mathbf{0}, \mathbf{z}_1)) = \mathrm{HE}(0, \bar{K}(\cdot; \mathbf{z}_0, \mathbf{z}_1)) = \alpha$. Furthermore, $\lim_{t\to 0} \operatorname{corr}\{Z(t\mathbf{z}_1) - Z(\mathbf{0}), Z(\mathbf{z}_0 + t\mathbf{z}_1) - Z(\mathbf{z}_0)\} = D_1/C_1 \neq 0$. Now suppose the lack of smoothness of K at $\mathbf{z}_0$ is localized in the sense that there exists $\varepsilon > 0$ such that $\mathrm{HE}(0, \bar{K}(\cdot; \mathbf{z}_0 + \delta\mathbf{z}_1, \mathbf{z}_1)) > \alpha$ for all $0 < |\delta| < \varepsilon$, which, as far as I am aware, holds for any space–time covariance function that has been proposed in the literature. This condition implies $\lim_{t\to 0}\operatorname{corr}\{Z(t\mathbf{z}_1)-Z(\mathbf{0}), Z(\mathbf{z}_0+\delta\mathbf{z}_1+t\mathbf{z}_1)-Z(\mathbf{z}_0+\delta\mathbf{z}_1)\}=0$ for all $0<|\delta|<\varepsilon$. Thus, when K is not smoother in the $\mathbf{z}_1$ direction at $\mathbf{z}_0$ than it is at $\mathbf{0}$, there is a "discontinuity" in correlations of increments. If, instead, $D_1=0$, which will be the case under (23), then this limiting correlation is 0 for all δ in a neighborhood of 0 including $\delta=0$, and no discontinuity occurs.

Define $\mathbf{z}_j=(\mathbf{x}_j,\mathbf{y}_j)$ for $j=0,1$ with $\mathbf{x}_j\in\mathbb{R}^{d_1}$ and $\mathbf{y}_j\in\mathbb{R}^{d_2}$. For G as given in Theorem 1, let $\bar{G}(t;\mathbf{z}_0,\mathbf{z}_1)=G(\mathbf{z}_0+t\mathbf{z}_1)$. For $\mathbf{z}_1\neq\mathbf{0}$, by (10), $\mathrm{HE}(0,\bar{G}(\cdot;\mathbf{0},(\mathbf{x}_1,\mathbf{y}_1))=2\theta'$ if $\mathbf{x}_1\neq\mathbf{0}$ and $\mathrm{HE}(0,\bar{G}(\cdot;\mathbf{0},(\mathbf{0},\mathbf{y}_1))=2\alpha_1\theta'$. Now consider $\mathbf{z}_0\neq\mathbf{0}$. If $\mathbf{y}_0\neq\mathbf{0}$, then it is possible to show (10) can be differentiated termwise and $\mathrm{HE}(0,\bar{G}(\cdot;(\mathbf{x}_0,\mathbf{y}_0),(\mathbf{x}_1,\mathbf{y}_1)))=\infty$. Next, $\mathrm{HE}(0,\bar{G}(\cdot;(\mathbf{x}_0,\mathbf{0}),(\mathbf{0},\mathbf{y}_1)))=\infty$ by (11) or (12). Finally, if $\mathbf{x}_1\neq\mathbf{0}$, then $\mathrm{HE}(0,\bar{G}(\cdot;(\mathbf{x}_0,\mathbf{0}),(\mathbf{x}_1,\mathbf{y}_1)))\geq 2\theta+2$ by (21). Because $\theta>\theta'$, in all cases $\mathrm{HE}(0,\bar{G}(\cdot;\mathbf{z}_0,\mathbf{z}_1))>\mathrm{HE}(0,\bar{G}(\cdot;\mathbf{0},\mathbf{z}_1))+2$ for all nonzero $\mathbf{z}_0$ and $\mathbf{z}_1$, so (23) is more than satisfied.

It is not clear that satisfying (23), or even the stronger condition met by G here, will exclude all possible "discontinuities" or other anomalies in the covariance structure, but it does avoid at least the type considered here. It might be preferable to find covariance functions that satisfy (2) and (3) and are infinitely differentiable away from the origin, but for ζ_1, ζ_2 and ζ_1/ζ_2 all irrational, I am unaware of any generalized covariance functions that satisfy all of these conditions.

Next, consider the problem of the dimple in fully symmetric (even in both its arguments) stationary space–time covariance functions described in [13]. A formal definition of the dimple is given in [13], but the essential point is that the space–time covariance function $K(\mathbf{x},t)$ has a dimple in, say, the time lag t if, for some fixed spatial lag $\mathbf{x}$, $K(\mathbf{x},t)$ has a local minimum in t at $t=0$. This dimple implies that at the spatial lag $\mathbf{x}$, correlation is stronger with both the near future and the near past than with the present, and [13] argues that such a lack of monotonicity in the covariance structure will often be undesirable. A dimple in the spatial lag can be defined similarly.

For a GC-k with $k>0$, it is not clear what one should mean by a dimple, but for $k=0$, the GC-0 that equals 0 at the origin is just minus the semivariogram for the process: $\frac{1}{2}\operatorname{Var}\{Z(\mathbf{x},t)-Z(\mathbf{0},0)\}=-G(\mathbf{x},t)$. If the variogram is expected to increase as one moves "farther away" in space–time, then GC-0s with dimples should be avoided.

Assume $k_0=0$ or, equivalently, $\alpha_1\theta'<1$, so that $\tilde{G}$ is a GC-0. First, if $\alpha_1=1$, $\tilde{G}(r,s)$ is a decreasing function of $\sqrt{r^2+s^2}$, and there is no dimple, so assume $\alpha_1>1$. In addition, assume θ is not an integer so that (20) holds. To show that $\tilde{G}(r,s)$ does not have a dimple, it suffices to show that for every $s\geq 0$, $\tilde{G}(0,s)>\tilde{G}(r,s)$ for all r sufficiently small and, for every $r\geq 0$, $\tilde{G}(r,0)>\tilde{G}(r,s)$ for all s sufficiently small. That $\tilde{G}(0,0)=0$ is greater than $\tilde{G}(r,0)$ and $\tilde{G}(0,s)$ for all positive r and s is immediate from (10) and (11). For fixed $s>0$, from (10),

$$\tilde{G}(0,s)-\tilde{G}(r,s)=\frac{c_1(\alpha_1)}{4\alpha_1\Gamma(\nu)}r^2\gamma_{\theta'-1/\alpha_1}\left(\frac{1}{2}s\right)+\mathrm{O}(r^4)$$

as $r\downarrow 0$. It follows from $c_1(\alpha_1)>0$ and $1/\alpha_1-\theta'>0$ that $\tilde{G}(0,s)>\tilde{G}(r,s)$ for all r sufficiently small. From (20), for fixed $r>0$,

$$\tilde{G}(r,0)-\tilde{G}(r,s)=\frac{\pi^{(d_1+d_2)/2}\Gamma(\theta-1)\Gamma((1-\theta')\alpha_1)}{\Gamma(\nu)\Gamma(\alpha_1(\theta-1))}\left(\frac{1}{2}r\right)^{2\alpha_1(\theta'-1)}\left(\frac{1}{2}s\right)^2+\mathrm{o}(s^2)$$

as $s \downarrow 0$. For $\theta > 1$, $\tilde{G}(r,0) - \tilde{G}(r,s)$ is clearly positive for all s sufficiently small, so now consider $0 < \theta < 1$. In this case, $\Gamma(\theta - 1) < 0$, so $\tilde{G}(r,0) > \tilde{G}(r,s)$ follows for all s sufficiently small if $\Gamma(\alpha_1(\theta - 1)) < 0$, which holds if $\alpha_1(\theta - 1) \in (-2m - 1, -2m)$ for some nonnegative integer m. This condition does not hold for all α_1 and θ' for which $\alpha_1 > 1$ and $\alpha_1\theta' < 1$, but it does always hold when $\theta' \geq \frac{2}{4+d_1}$. To prove this, it suffices to show $\alpha_1(\theta - 1) > -1$, which holds if $\theta' > 1 - (1 + \frac{1}{2}d_1)/\alpha_1$. The curves $\theta' = 1 - (1 + \frac{1}{2}d_1)/\alpha_1$ and $\theta' = 1/\alpha_1$ intersect at $(\alpha_1, \theta') = (2 + \frac{1}{2}d_2, 2/(4 + d_1))$, from which it follows that $\alpha_1(\theta - 1) > -1$ holds for all $\theta' < 1/\alpha_1$ whenever $\theta' \geq \frac{2}{4+d_1}$. This lower bound is $\frac{2}{5}$ for $d_1 = 1$ and is smaller for larger d_1. The lower bound of $\frac{2}{5}$ may not be too restrictive in practice: Brownian motion has generalized covariance function proportional to $\gamma_{1/2}$, and processes less smooth than Brownian motion are somewhat uncommon in applications.

3. Discussion

For G as defined by (9), the series expansions (10) and (20) should, in principle, allow fast and accurate calculation of G, but there do not appear to be any publicly available programs for computing H-functions and writing general purpose code to carry out these calculations would require a major effort. In particular, preliminary investigations suggest that considerable care needs to be taken to piece together the convergent power series (10) and the asymptotic expansion (20) to obtain accurate approximations for all values of the argument of the function. Further work would also be needed to handle those values for (α_1, ν) for which one of the expansions has a singularity or near singularity, including values of α_1 near 1.

Restricting $\alpha_2 = 1$ in (6) was essential to the derivation of series expansions for the resulting generalized covariance functions. Model (6) was, in turn, a simplification of (5), which includes two range parameters. Perhaps H-functions can be used to express, in at least some cases, the (generalized) covariance functions corresponding to these more general models. However, even the richer class of covariance functions given by (5) is inadequate for modeling many natural processes. In particular, these covariance functions all satisfy $G(\mathbf{x}, \mathbf{y}) = G(\mathbf{x}, -\mathbf{y})$ and, hence, are all what [7] calls fully symmetric. Any process with a predominant direction of flow will not be fully symmetric, so this constraint is often inappropriate. Covariance functions that are not fully symmetric can be generated from covariance functions that are [11,31], and these approaches can, in principle, be applied to the models considered here. An easy extension of this model is to allow for geometric anisotropies in either $\mathbf{x}$ or $\mathbf{y}$ by considering $\tilde{G}(|A\mathbf{x}|, |B\mathbf{y}|)$ for any $d_1 \times d_1$ matrix A and any $d_2 \times d_2$ matrix B.

As noted in the Introduction, there has been quite a lot of research in recent years developing new classes of space–time covariance functions. In many of these works, the focus has been on obtaining simple closed form expressions for space–time covariance functions. Having closed form expressions is certainly valuable in applying the models, but it is critical that any such model provides a good description of the spatial-temporal variations of the process to which it is to be applied. Comparing various models in a broad range of applications is one important way to learn about which models will be of most use in practice, but it is also important to consider the theoretical properties of these models, such as their smoothness properties, both at the origin and away from the origin, and the presence of dimples or other possible anomalies. Finding covariance function models for space–time processes that allow for a different degree of smoothness in space and in time, possess certain other desirable properties such as (4) and are accurately computable using series expansions is a major challenge. The results obtained here perhaps provide a first step to show that it may not be necessary to sacrifice desired theoretical properties of space–time models in order to gain computational tractability, although admittedly quite a bit of work on numerical methods would be needed before the generalized covariance functions proposed here could be used routinely (or even not so routinely) in practice.

Appendices

A.1. Proof of Theorem 1

To prove (10) for the process Z with spectral density (6), consider the process $Z_1(x_1, \mathbf{y}) = Z((x_1, 0, \ldots, 0), \mathbf{y})$, which has GC-$k_0$ $G_1(x_1, \mathbf{y}) = G((x_1, 0, \ldots, 0), \mathbf{y}) = \tilde{G}(|x_1|, |\mathbf{y}|)$. Assume for now that $k_0 = \lfloor \alpha_1 \theta' \rfloor$ does not equal $\alpha_1 \theta'$ so that $\alpha_1 \theta' - 1 < k_0 < \alpha_1 \theta'$. Then the process Z_1 is k_0 times mean square differentiable in its first coordinate direction and, for $m \leq k_0$, denote its mth mean square derivative process by $Z_1^m(x_1, \mathbf{y})$. The generalized covariance function for $Z_1^m(x_1, \mathbf{y})$, denoted by $G_1^m(x_1, \mathbf{y})$, can be chosen to satisfy

$$G_1^m(x_1, \mathbf{y}) = (-1)^m \frac{\partial^{2m}}{\partial x_1^{2m}} G_1(x_1, \mathbf{y}). \tag{24}$$

Now $\frac{\partial^{2m+1}}{\partial x_1^{2m+1}} G_1(0, \mathbf{y}) = 0$ for $m < k_0$, so if one knew $\frac{\partial^{2m}}{\partial x_1^{2m}} G_1(0, \mathbf{y})$ for $m \leq k_0$, then $G_1(x_1, \mathbf{y})$ could be recovered from $G_1^{k_0}(x_1, \mathbf{y})$ by integration. Then, since $G(\mathbf{x}, \mathbf{y})$ has a version that only depends on $\mathbf{x}$ through $|\mathbf{x}|$, one can obtain G.

Suppose $d_1 > 1$. The case $d_1 = 1$ requires a slightly different but easier argument. The process $Z_1^{k_0}(x_1, \mathbf{y})$ is an IRF-0, so its GC-0 can be taken to equal negative the semivariogram of the process. Denoting $|\mathbf{y}|$ by s,

$$\begin{aligned}
&-G_1^{k_0}(x_1, \mathbf{y}) \\
&\quad = \int_{\mathbb{R}^{d_1}} \int_{\mathbb{R}^{d_2}} (1 - \mathrm{e}^{\mathrm{i}\tau_1 x_1 + \mathrm{i}\boldsymbol{\omega}'\mathbf{y}}) \tau_1^{2k_0} (|\boldsymbol{\tau}|^{2\alpha_1} + |\boldsymbol{\omega}|^2)^{-\nu} \,\mathrm{d}\boldsymbol{\omega} \,\mathrm{d}\boldsymbol{\tau} \\
&\quad = \frac{\pi^{d_2/2}}{2^{\theta-1}\Gamma(\nu)} \int_{\mathbb{R}^{d_1}} \frac{\tau_1^{2k_0}}{|\boldsymbol{\tau}|^{2\alpha_1\theta}} \{\mathcal{M}_\theta(0) - \cos(\tau_1 x_1) \mathcal{M}_\theta(|\boldsymbol{\tau}|^{\alpha_1} s)\} \,\mathrm{d}\boldsymbol{\tau}.
\end{aligned}$$

Switching to hyperspherical coordinates with $|\boldsymbol{\tau}| = u$, $\tau_1 = u\cos\phi_1$ and integrating over the angles $\phi_2, \dots, \phi_{d_1-1}$ yields

$$
\begin{aligned}
-G_1^{k_0}(x_1, \mathbf{y}) = {} & \frac{\pi^{(d_1+d_2-1)/2}}{2^{\theta-2}\Gamma(\nu)\Gamma((d_1-1)/2)} \\
& \times \int_0^\infty u^{2k_0-2\alpha_1\theta'-1} \int_0^\pi \cos^{2k_0}\phi_1 \sin^{d_1-2}\phi_1 \\
& \qquad \times \{\mathcal{M}_\theta(0) - \cos(x_1 u \cos\phi_1)\mathcal{M}_\theta(u^{\alpha_1}s)\}\,\mathrm{d}\phi_1\,\mathrm{d}u.
\end{aligned} \tag{25}
$$

Making the change of variables $\sigma = \cos\phi_1$ and using [8], 3.251.1 and 3.771.4, and the series expansion for the generalized hypergeometric function ${}_1F_2$ yields

$$
\begin{aligned}
& \int_0^\pi \cos^{2k_0}\phi_1 \sin^{d_1-2}\phi_1\{\mathcal{M}_\theta(0) - \cos(x_1 u\cos\phi_1)\mathcal{M}_\theta(u^{\alpha_1}s)\}\,\mathrm{d}\phi_1 \\
&\quad = 2\int_0^1 \sigma^{2k_0}(1-\sigma^2)^{(d_1-3)/2}\{\mathcal{M}_\theta(0) - \cos(x_1 u\sigma)\mathcal{M}_\theta(u^{\alpha_1}s)\}\,\mathrm{d}\sigma \\
&\quad = B\left(k_0 + \frac{1}{2}, \frac{d_1-1}{2}\right) \\
&\qquad \times \left\{\mathcal{M}_\theta(0) - {}_1F_2\left(k_0+\frac{1}{2}; \frac{1}{2}, k_0+\frac{d_1}{2}; -\frac{x_1^2u^2}{4}\right)\mathcal{M}_\theta(u^{\alpha_1}s)\right\} \\
&\quad = B\left(k_0 + \frac{1}{2}, \frac{d_1-1}{2}\right)\Bigg[\{\mathcal{M}_\theta(0) - \mathcal{M}_\theta(u^{\alpha_1}s)\} \\
&\qquad\qquad - \sum_{\ell=1}^\infty \frac{(k_0+1/2)_\ell}{(k_0+(1/2)d_1)_\ell(2\ell)!}(-x_1^2u^2)^\ell \mathcal{M}_\theta(u^{\alpha_1}s)\Bigg],
\end{aligned} \tag{26}
$$

where B is the beta function.

The following properties of $\mathcal{M}_\theta$ are used in the proof. For any $\theta > 0$, as $t \downarrow 0$,

$$
\mathcal{M}_\theta(t) = \sum_{r=0}^{\lfloor\theta\rfloor} U_r t^{2r} + V\gamma_\theta(t) + \mathrm{o}(t^{2\theta}) \tag{27}
$$

for appropriate values of the U_rs and V [30], Section 2.7. In addition, for any $\theta > 0$, there exist positive constants C and D (depending on θ) such that

$$
0 \le \mathcal{M}_\theta(t) \le C\mathrm{e}^{-Dt} \tag{28}
$$

for all $t \ge 0$. Furthermore, for all real θ and all $t > 0$,

$$
\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{M}_\theta(t) = -t\mathcal{M}_{\theta-1}(t). \tag{29}
$$

Assume $s > 0$ for now. The function $u^{2k_0-2\alpha_1\theta'-1}\{\mathcal{M}_\theta(0) - \mathcal{M}_\theta(u^{\alpha_1}s)\}$ is integrable in u over $(0, \infty)$ since, from (28) and $k_0 < \alpha_1\theta'$, it is integrable over $(0, 1]$, and, from (27) and $k_0 > \alpha_1\theta' - 1$, it is integrable over $(1, \infty)$. Furthermore,

$$\sum_{\ell=1}^{\infty} \frac{(k_0+1/2)_\ell}{(k_0+(1/2)d_1)_\ell(2\ell)!}(x_1^2u^2)^\ell \le \cosh(x_1u)$$

and $\cosh(x_1u)\mathcal{M}_\theta(u^{\alpha_1}s)$ is integrable over $(0, \infty)$ for $\alpha_1 > 1$, so that

$$\sum_{\ell=1}^{\infty} \frac{(k_0+1/2)_\ell}{(k_0+(1/2)d_1)_\ell(2\ell)!}(-x_1^2u^2)^\ell \mathcal{M}_\theta(u^{\alpha_1}s)$$

can be integrated termwise over $(0, \infty)$. Making the change of variables $t = u^{\alpha_1}$, integrating by parts and using the definition of $\mathcal{M}_\theta$ gives

$$\begin{aligned}
&\int_0^\infty u^{2k_0-2\alpha_1\theta'-1}\{\mathcal{M}_\theta(0) - \mathcal{M}_\theta(u^{\alpha_1}s)\}\,\mathrm{d}u \\
&\quad = \frac{1}{\alpha_1}\int_0^\infty t^{2k_0/\alpha_1-\theta'-1}\{\mathcal{M}_\theta(0) - \mathcal{M}_\theta(ts)\}\,\mathrm{d}t \\
&\quad = -\frac{s^2}{2\alpha_1\theta' - 2k_0}\int_0^\infty t^{2k_0/\alpha_1-2\theta'+1}\mathcal{M}_{\theta-1}(ts)\,\mathrm{d}t \\
&\quad = -\frac{s^{\theta+1}}{2\alpha_1\theta' - 2k_0}\int_0^\infty t^{(2k_0+d_1)/\alpha_1-\theta}\mathcal{K}_{\theta-1}(ts)\,\mathrm{d}t.
\end{aligned}$$

Since $(2k_0 + d_1)/\alpha_1 - \theta > |1 - \theta|$, [8], 6.561.16, applies, and one obtains

$$\begin{aligned}
&\int_0^\infty u^{2k_0-2\alpha_1\theta'-1}\{\mathcal{M}_\theta(0) - \mathcal{M}_\theta(u^{\alpha_1}s)\}\,\mathrm{d}u \\
&\quad = -\frac{2^{\theta-2}}{\alpha_1}\Gamma\left(\frac{k_0}{\alpha_1} - \theta'\right)\Gamma\left(\frac{2k_0+d_1}{2\alpha_1}\right)\left(\frac{1}{2}s\right)^{2\theta'-2k_0/\alpha_1}.
\end{aligned} \tag{30}$$

Next, for $\ell \ge 1$, again using the change of variables $t = u^{\alpha_1}$ and [8], 6.561.16, gives

$$\begin{aligned}
&\int_0^\infty u^{2k_0+2\ell-2\alpha_1\theta'-1}\mathcal{M}_\theta(u^{\alpha_1}s)\,\mathrm{d}u \\
&\quad = \frac{2^{\theta-2}}{\alpha_1}\Gamma\left(\frac{k_0+\ell}{\alpha_1} - \theta'\right)\Gamma\left(\frac{2k_0+2\ell+d_1}{2\alpha_1}\right)\left(\frac{1}{2}s\right)^{2\theta'-(2k_0+2\ell)/\alpha_1}.
\end{aligned} \tag{31}$$

For $\mathbf{y} \neq \mathbf{0}$, using (25), (26), (30) and (31),

$$
\begin{aligned}
&G_1^{k_0}(x_1, \mathbf{y}) \\
&\quad = \frac{\pi^{d_1+d_2-1}\Gamma((k_0+1/2)}{\Gamma(\nu)\Gamma(k_0+(1/2)d_1)\alpha_1} \\
&\qquad \times \sum_{\ell=0}^{\infty} \frac{(k_0+1/2)_\ell}{(k_0+(1/2)d_1)_\ell(2\ell)!}\Gamma\left(\frac{2k_0+2\ell+d_1}{2\alpha_1}\right)(-x_1^2)^\ell \gamma_{\theta'-(k_0+\ell)/\alpha_1}\left(\frac{1}{2}s\right) \qquad (32)\\
&\quad = \frac{1}{\alpha_1\Gamma(\nu)}\sum_{\ell=0}^{\infty}\frac{(k_0+\ell)!(1/2)_{k_0+\ell}c_{k_0+\ell}(\alpha_1)}{(2\ell)!}(-x_1^2)^\ell \gamma_{\theta'-(k_0+\ell)/\alpha_1}\left(\frac{1}{2}s\right) \\
&\quad = \frac{1}{\alpha_1\Gamma(\nu)}\sum_{m=k_0}^{\infty}\frac{m!(1/2)_m c_m(\alpha_1)}{\{2(m-k_0)\}!}(-x_1^2)^\ell \gamma_{\theta'-m/\alpha_1}\left(\frac{1}{2}s\right).
\end{aligned}
$$

Next consider $G_1^m(\mathbf{0}, \mathbf{y})$ for $m < k_0$. The process $\frac{\partial^m}{\partial x_1^m} Z(\mathbf{x}, \mathbf{y})$ has spectral density $\tau_1^{2m}(|\boldsymbol{\tau}|^{2\alpha_1} + |\boldsymbol{\omega}|^2)^{-\nu}$, and hence the process $\frac{\partial^m}{\partial x_1^m} Z(\mathbf{0}, \mathbf{y}) = Z_1^m(0, \mathbf{y})$ considered just as a function of $\mathbf{y} \in \mathbb{R}^{d_2}$ has spectral density

$$
\begin{aligned}
&\int_{\mathbb{R}^{d_1}} \frac{\tau_1^{2m}}{(|\boldsymbol{\tau}|^{2\alpha_1} + |\boldsymbol{\omega}|^2)^\nu}\,\mathrm{d}\boldsymbol{\tau} \\
&\quad = \frac{2\pi^{(d_1-1)/2}}{\Gamma((d_1-1)/2)}\int_0^\infty\int_0^\pi \frac{u^{2m+d_1-1}\cos^{2m}\phi_1\sin^{d_1-2}\phi_1}{(u^{2\alpha_1}+|\boldsymbol{\omega}|^2)^\nu}\,\mathrm{d}\phi_1\,\mathrm{d}u \qquad (33)\\
&\quad = \frac{\pi^{(d_1-1)/2}\Gamma(m+1/2)B((2m+d_1)/2\alpha_1, \nu-(2m+d_1)/2\alpha_1)}{\alpha_1\Gamma(m+(1/2)d_1)|\boldsymbol{\omega}|^{2\nu-(2m+d_1)/\alpha_1}}
\end{aligned}
$$

by switching $\boldsymbol{\tau}$ to hyperspherical coordinates and using [8], 3.241.4. The process $Z_1^m(0, \mathbf{y})$ with spectral density (33) is an IRF-$(k_0 - m)$ (it may be an IRF of lower order as well) and its corresponding GC-$(k_0 - m)$ can be taken as [6], Chapter II, Section 3.3, equations (2) and (11),

$$
\begin{aligned}
G_1^m(0, \mathbf{y}) &= \frac{c_m(\alpha_1)m!\Gamma(m+1/2)}{\pi^{1/2}\alpha_1\Gamma(\nu)}\gamma_{\theta'-m/\alpha_1}\left(\frac{1}{2}|\mathbf{y}|\right) \\
&= \frac{c_m(\alpha_1)m!(1/2)_m}{\alpha_1\Gamma(\nu)}\gamma_{\theta'-m/\alpha_1}\left(\frac{1}{2}|\mathbf{y}|\right).
\end{aligned} \qquad (34)
$$

To recover $G_1(x_1, \mathbf{y})$ and hence $G(\mathbf{x}, \mathbf{y})$, repeatedly integrate (32) and use (34) to set the boundary conditions. Specifically, $G_1(x_1, \mathbf{y})$ must be of the form

$$
G_1(x_1, \mathbf{y}) = \sum_{\ell=0}^{k_0-1} x_1^{2\ell}F_\ell(\mathbf{y}) + (-1)^{k_0}\int_0^{x_1}\int_0^{z_1}\cdots\int_0^{z_{2k_0-1}} G_1^{k_0}(z_{2k_0}, \mathbf{y})\,\mathrm{d}z_{2k_0}\cdots\mathrm{d}z_1
$$

for some suitable functions $F_0, \dots, F_{k_0-1}$. Substituting the series for $G_1^{k_0}$ in (32) into the preceding expression and integrating termwise, which is easily justified for $s > 0$ by dominated convergence, yields

$$G_1(x_1, \mathbf{y}) = \sum_{\ell=0}^{k_0-1} x_1^{2\ell} F_\ell(\mathbf{y}) + \frac{1}{\alpha_1 \Gamma(\nu)} \sum_{m=k_0}^{\infty} \frac{c_m(\alpha_1) m! (1/2)_m 2^{2m}}{(2m)!} \left\{ -\left(\frac{1}{2}x_1\right)^2 \right\}^m \gamma_{\theta'-m/\alpha_1}\left(\frac{1}{2}|\mathbf{y}|\right). \tag{35}$$

Elementary calculations demonstrate

$$\frac{m!(1/2)_m 2^{2m}}{(2m)!} = 1 \tag{36}$$

for all $m \in \mathbb{N}_0$. For $0 \le m < k_0$, differentiating (35) $2m$ times, setting $\mathbf{y} = \mathbf{0}$ and using (34) and (36) gives

$$\begin{aligned} F_m(\mathbf{y}) &= \frac{(-1)^m}{(2m)!} G_1^m(0, \mathbf{y}) \\ &= \frac{(-1/4)^m c_m(\alpha_1)}{\alpha_1 \Gamma(\nu)} \gamma_{\theta'-m/\alpha_1}\left(\frac{1}{2}|\mathbf{y}|\right). \end{aligned} \tag{37}$$

Substituting (36) and (37) into (35) yields

$$G_1(x_1, \mathbf{y}) = \frac{1}{\alpha_1 \Gamma(\nu)} \sum_{m=0}^{\infty} \left\{ -\left(\frac{1}{2}x_1\right) \right\}^m c_m(\alpha_1) \gamma_{\theta'-m/\alpha_1}\left(\frac{1}{2}|\mathbf{y}|\right),$$

and (10) follows from $G_1(x_1, \mathbf{y}) = \tilde{G}(|x_1|, |\mathbf{y}|)$.

To obtain an explicit expression for $\tilde{G}(r, 0)$, go back to (25) and change the order of integration. Integrating by parts and using [8], 3.761.4,

$$\begin{aligned} &\int_0^\infty u^{2k_0 - 2\alpha_1\theta' - 1} \{1 - \cos(x_1 u \cos\phi_1)\} \, \mathrm{d}u \\ &\quad = \frac{x_1 \cos\phi_1}{2\alpha_1\theta' - 2k_0} \int_0^\infty u^{2k_0 - 2\alpha_1\theta'} \sin(x_1 u \cos\phi_1) \, \mathrm{d}u \\ &\quad = -\Gamma(2k_0 - 2\alpha_1\theta') \sin\{\pi(k_0 - \alpha_1\theta')\} |x_1 \cos\phi_1|^{2\alpha_1\theta' - 2k_0}. \end{aligned}$$

When $k_0 \neq \alpha_1\theta'$, substituting this result and $\mathcal{M}_\theta(0) = 2^{\theta-1}\Gamma(\theta)$ into (25) and integrating over ϕ_1 yields

$$\begin{aligned} G_1^{k_0}(x_1, \mathbf{0}) = {}& \frac{2\pi^{(d_1+d_2-1)/2} \Gamma(\theta) \Gamma(\alpha_1\theta' + 1/2) \Gamma(2k_0 - 2\alpha_1\theta')}{\Gamma(\nu)\Gamma(\alpha_1\theta)} \\ & \times \cos(\pi\alpha_1\theta') (-1)^{k_0} |x_1|^{2\alpha_1\theta' - 2k_0}. \end{aligned}$$

Now, $\Gamma(2k_0 - 2\alpha_1\theta') = (2\alpha_1\theta' - 2k_0 + 1)_{2k_0}\Gamma(-2\alpha_1\theta')$, so applying the duplication formula for Γ to $\Gamma(-2\alpha_1\theta')$ and then the reflection formula to $\Gamma(-\alpha_1\theta' + \frac{1}{2})$ yields

$$\begin{aligned}
&G_1^{k_0}(x_1, \mathbf{0}) \\
&\quad = \frac{(-1)^{k_0}\pi^{(d_1+d_2)/2}\Gamma(\theta)\Gamma(-\alpha_1\theta')(2\alpha_1\theta' - 2k_0 + 1)_{2k_0}}{\Gamma(\nu)\Gamma(\alpha_1\theta)2^{\alpha_1\theta'}}|x_1|^{2\alpha_1\theta' - 2k_0}.
\end{aligned}$$

Integrating this expression $2k_0$ times and using the boundary condition $G_1^m(0, \mathbf{0}) = 0$ for $m < k_0$ to make $G_1^m(0, \mathbf{y})$ continuous at $\mathbf{y} = \mathbf{0}$ gives (11).

To show that (10) holds when $k_0 = \alpha_1\theta'$, write $\tilde{G}_\nu$ to make the dependence of $\tilde{G}$ on ν explicit (but still suppressing the dependence on α_1, d_1 and d_2). Define $\nu_0 = (k_0 + \frac{1}{2}d_1)/\alpha_1 + \frac{1}{2}d_2$ and view α_1 as fixed. For any given ALC-k_0, the last line of (9) is continuous as $\nu \downarrow \nu_0$, so the first line is as well. Thus,

$$\sum_{\ell,j=1}^{n} \lambda_\ell\lambda_j\tilde{G}_{\nu_0}(r_{\ell j}, s_{\ell j}) = \lim_{\nu\downarrow\nu_0}\sum_{\ell,j=1}^{n}\lambda_\ell\lambda_j\tilde{G}_\nu(r_{\ell j}, s_{\ell j}). \tag{38}$$

Let M_0 be the set of nonnegative integers m for which $h_m = (k_0 - m)/\alpha_1 \in \mathbb{N}_0$. This set is finite and includes k_0 as its largest element. Writing $\nu = \nu_0 + \varepsilon$, define

$$P_\varepsilon(r, s) = \sum_{m\in M_0}\frac{c_m(\alpha_1)\Gamma(-h_m - \varepsilon)}{\alpha_1\Gamma(\nu_0 + \varepsilon)}\left\{-\left(\frac{1}{2}r\right)^2\right\}^m\left(\frac{1}{2}s\right)^{2h_m}.$$

Because $\sum_{\ell=1}^n \lambda_\ell Z(\mathbf{x}_\ell)$ is an ALC-k_0, for $a, b \in \mathbb{N}_0$ and $a + b \le k_0$, subtracting any linear combination of terms like $r_{\ell j}^{2a}s_{\ell j}^{2b}$ from $\tilde{G}_\nu(r_{\ell j}, s_{\ell j})$ on the right-hand side of (38) does not change the result. Now $\alpha_1 > 1$ implies $m + h_m \le k_0$, so $P_\varepsilon(r_{\ell j}, s_{\ell j})$ is of this required form and

$$\sum_{\ell,j=1}^{n}\lambda_\ell\lambda_j\tilde{G}_\nu(r_{\ell j}, s_{\ell j}) = \sum_{\ell,j=1}^{n}\lambda_\ell\lambda_j\{\tilde{G}_\nu(r_{\ell j}, s_{\ell j}) - P_\varepsilon(r_{\ell j}, s_{\ell j})\}.$$

Thus, to prove (10), it suffices to show that for all r, s nonnegative,

$$\tilde{G}_{\nu_0}(r, s) = \lim_{\nu\downarrow\nu_0}\{\tilde{G}_\nu(r, s) - P_\varepsilon(r, s)\}. \tag{39}$$

For $s > 0$, for all ε sufficiently small,

$$\begin{aligned}
\tilde{G}_\nu(r, s) - P_\varepsilon(r, s) = \sum_{m=0}^{\infty}&\frac{c_m(\alpha_1)}{\alpha_1\Gamma(\nu_0 + \varepsilon)}\left\{-\left(\frac{1}{2}r\right)^2\right\}^m \\
&\times\Gamma(-h_m - \varepsilon)\left(\frac{1}{2}s\right)^{2h_m}\left[\left(\frac{1}{2}s\right)^{2\varepsilon} - 1\{m \in M_0\}\right].
\end{aligned}$$

Dominated convergence justifies taking the limit $\nu \downarrow \nu_0$ (equivalently, as $\varepsilon \downarrow 0$) inside this infinite sum. For $m \notin M_0$, the limit is trivial, so consider $m \in M_0$. By the reflection formula for Γ, for $n \in \mathbb{N}_0$,

$$\Gamma(-n-\varepsilon) = \frac{\pi(-1)^{n+1}}{\sin(\pi\varepsilon)\Gamma(n+1+\varepsilon)}. \tag{40}$$

Using this result and straightforward calculus yields

$$\Gamma(-h_m-\varepsilon)\left(\frac{1}{2}s\right)^{2h_m}\left\{\left(\frac{1}{2}s\right)^{2\varepsilon}-1\right\} \to \gamma_{h_m}\left(\frac{1}{2}s\right)$$

as $\nu \downarrow \nu_0$, establishing (39) when $s > 0$.

It remains to establish (39) when $s = 0$ with $\tilde{G}_{\nu_0}(r, 0)$ given by (12). For all ε sufficiently small, applying (40) to $\Gamma(-k_0 - \alpha_1\varepsilon)$ and using $\Gamma(1-\varepsilon) = -\varepsilon\Gamma(-\varepsilon)$, for all ε sufficiently small,

$$\begin{aligned}
&\tilde{G}_\nu(r,0) - P_\varepsilon(r,0) \\
&\quad= \frac{(-1)^{k_0+1}\pi^{(d_1+d_2+2)/2}\Gamma(\theta_\nu)}{\sin(\pi\alpha_1\varepsilon)\Gamma(\nu)\Gamma(\alpha_1\theta_\nu)\Gamma(k_0+1+\alpha_1\varepsilon)}\left(\frac{1}{2}r\right)^{2k_0+2\alpha_1\varepsilon} \\
&\qquad - \frac{c_{k_0}(\alpha_1)\Gamma(-\varepsilon)}{\alpha_1\Gamma(\nu)}\left\{-\left(\frac{1}{2}r\right)^2\right\}^{k_0} \\
&\quad= -\frac{\pi^{(d_1+d_2)/2}}{\Gamma(\nu)}\left\{-\left(\frac{1}{2}r\right)^2\right\}^{k_0} \\
&\qquad\times\left\{\frac{\pi\Gamma(\theta_\nu)}{\sin(\pi\alpha_1\varepsilon)\Gamma(\alpha_1\theta_\nu)\Gamma(k_0+1+\alpha_1\varepsilon)}\left[\left\{\left(\frac{1}{2}r\right)^{2\alpha_1\varepsilon}-1\right\}+1\right]\right. \\
&\qquad\quad \left. - \frac{\Gamma((d_1+2k_0)/2\alpha_1)\Gamma(1-\varepsilon)}{\alpha_1\varepsilon\Gamma(k_0+(1/2)d_1)k_0!}\right\}.
\end{aligned} \tag{41}$$

As $\varepsilon \downarrow 0$,

$$\frac{\pi}{\sin(\pi\alpha_1\varepsilon)}\left\{\left(\frac{1}{2}r\right)^{2\alpha_1\varepsilon}-1\right\} \to 2\log\left(\frac{1}{2}r\right). \tag{42}$$

By the definition of the digamma function ψ, $\Gamma(x+\varepsilon) = \Gamma(x)\{1+\varepsilon\psi(x)+\mathrm{O}(\varepsilon^2)\}$ as $\varepsilon \to 0$ as long as $-x \notin \mathbb{N}_0$. Then

$$\begin{aligned}
&\frac{\pi\Gamma(\theta_\nu)}{\sin(\pi\alpha_1\varepsilon)\Gamma(\alpha_1\theta_\nu)\Gamma(k_0+1+\alpha_1\varepsilon)} - \frac{\Gamma((d_1+2k_0)/(2\alpha_1))\Gamma(1-\varepsilon)}{\alpha_1\varepsilon\Gamma(k_0+(1/2)d_1)k_0!} \\
&\quad= \frac{\Gamma((d_1+2k_0)/(2\alpha_1))}{k_0!\Gamma(k_0+(1/2)d_1)} \\
&\qquad\times\left[\frac{1+\varepsilon\psi((d_1+2k_0)/(2\alpha_1))}{\alpha_1\varepsilon\{1+\alpha_1\varepsilon\psi(k_0+(1/2)d_1)\}\{1+\alpha_1\varepsilon\psi(k_0+1)\}} - \frac{1-\varepsilon\psi(1)}{\alpha_1\varepsilon}\right] + \mathrm{O}(\varepsilon)
\end{aligned} \tag{43}$$

$$= \frac{\Gamma((d_1 + 2k_0)/(2\alpha_1))}{k_0!\Gamma(k_0 + (1/2)d_1)}$$
$$\times \left\{ \frac{1}{\alpha_1} \psi\left(\frac{2k_0 + d_1}{2\alpha_1}\right) + \frac{1}{\alpha_1}\psi(1) - \psi\left(k_0 + \frac{1}{2}d_1\right) - \psi(k_0 + 1) \right\} + \mathrm{O}(\varepsilon).$$

Thus, when $k_0 = \alpha_1\theta'$ and $\tilde{G}_{\nu_0}(r, 0)$ is defined as in (12), (41)–(43) imply (39) holds for $s = 0$.

A.2. Properties of H-functions

This Appendix provides some background material on H-functions and is taken from [14], Section 1.1. Suppose m, n, p and q are integers satisfying $0 \le m \le q, 0 \le n \le p$, $a_1, \dots, a_p, b_1, \dots b_q$ are complex numbers and $\alpha_1, \dots, \alpha_p, \beta_1, \dots, \beta_q$ are positive reals. Then the H-function $H^{m,n}_{p,q}$ is defined by, for complex z,

$$\begin{aligned} &H^{m,n}_{p,q}\left(z \,\middle|\, \begin{matrix} (a_1, \alpha_1), \dots, (a_p, \alpha_p) \\ (b_1, \beta_1), \dots, (b_q, \beta_q) \end{matrix}\right) \\ &\quad = \frac{1}{2\pi \mathrm{i}} \int_{\mathcal{L}} \frac{\prod_{j=1}^{m} \Gamma(b_j + \beta_j s) \prod_{i=1}^{n} \Gamma(1 - a_j + \alpha_j s)}{\prod_{i=n+1}^{p} \Gamma(a_i + \alpha_j s) \prod_{j=m+1}^{q} \Gamma(1 - b_j + \beta_j s)} z^{-s}\, \mathrm{d}s, \end{aligned} \tag{44}$$

where [14], p. 2, gives the form of the contour $\mathcal{L}$, and an empty product is defined to be 1. For this integral to be well defined, none of the poles of the gamma functions in the two products in the numerator of (44) may coincide, or

$$\alpha_i(b_j + \ell) \ne \beta_j(a_i - k - 1) \tag{45}$$

for $1 \le i \le n, 1 \le j \le m$ and all $k, \ell \in \mathbb{N}_0$. The validity of series expansions of $H^{m,n}_{p,q}$ generally depends on the signs of the following two quantities:

$$a^* = \sum_{i=1}^{n} \alpha_i - \sum_{i=n+1}^{p} \alpha_i + \sum_{j=1}^{m} \beta_j - \sum_{j=m+1}^{q} \beta_j \tag{46}$$

and

$$\Delta = \sum_{j=1}^{q} \beta_j - \sum_{i=1}^{p} \alpha_i, \tag{47}$$

where an empty sum is defined to be 0.

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