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Small time Chung-type LIL for Lévy processes

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We prove Chung-type laws of the iterated logarithm for general Lévy processes at zero. In particular, we provide tools to translate small deviation estimates directly into laws of the iterated logarithm.

This reveals laws of the iterated logarithm for Lévy processes at small times in many concrete examples. In some cases, exotic norming functions are derived.

Keywords: law of the Iterated Logarithm; Lévy process; small ball problem; small deviations

1. Introduction

A classical question in stochastic process theory is to understand the asymptotic behavior of a given stochastic process $X = (X_t)_{t\geq 0}$ on the level of paths. In the present work, we consider general Lévy processes and find Chung-type LIL (laws of the iterated logarithm) at zero; that is, given the Lévy process X, we aim at characterizing a norming function b, satisfying

$$\liminf_{t\to 0} \frac{\|X\|_t}{b(t)} = 1, \qquad \text{where } \|X\|_t := \sup_{0\leq s\leq t} |X_s|. \tag{1.1}$$

The topic of large and small time fluctuations of Lévy processes has been studied extensively in the past (see, e.g., Doney [10] for an overview and Bertoin [3], Sato [19], Bertoin, Doney and Maller [5]).

It is well known that, via the Borel–Cantelli lemma, Chung-type LIL for a general stochastic process are connected to the so-called small deviation rate of the process, that is,

$$-\log \mathbb{P}(\|X\|_t \leq \varepsilon), \qquad \text{as } \varepsilon \to 0 \text{ and } t \to 0. \tag{1.2}$$

The main motivation for this paper originates from the recent work Aurzada and Dereich [2], where a framework for obtaining the small deviation rate (1.2) for general Lévy processes (but fixed t) is provided. The difficulty in passing over from the small deviation estimate to the respective LIL concerns circumventing the independence assumption of the Borel–Cantelli lemma.

In this paper we show how the asymptotics of (1.2) imply explicit LIL. We stress that it is not sufficient to have estimates for (1.2) for *fixed* t, which usually are referred to as small deviation estimates.

Small deviation problems are studied independently of LIL and have connections to other fields, such as the approximation of stochastic processes, coding problems, the path regularity of the process, limit laws in statistics and entropy numbers of linear operators. We refer to the surveys Li and Shao [12], Lifshits [15], for an overview of the field, and to Lifshits [14], for a regularly updated list of references, which also includes references to laws of the iterated logarithm of Chung type. The papers of Taylor [25], Mogul'skiĭ [18], Borovkov and Mogul'skiĭ [7], Simon [23,24], Linde and Shi [16], Lifshits and Simon [13], Linde and Zipfel [17], Shmileva [21], Shmileva [22] provide a good source for earlier results on small deviations of Lévy processes.

We now discuss LIL for special Lévy processes that have already appeared in the literature. The norming function $b(t) = \sqrt{\pi^2 t/(8 \log|\log t|)}$ for a standard Brownian motion can be derived from the large time LIL, proved by Chung [9], via time inversion. For any Lévy process with non-trivial Brownian component, the recent result of Buchmann and Maller [8] shows that (1.1) holds with the same norming function as for a standard Brownian motion. If X is an α-stable Lévy process, (1.1) holds with norming function $b(t) = (c_\alpha t/\log|\log t|)^{1/\alpha}$, which goes back to Taylor [25]. The question was studied for subordinators already in [11]; there, the norming function can be obtained from the Laplace transform.

Of course, it is natural to ask for the general structure of the norming function for arbitrary Lévy processes not having the special features of the examples mentioned so far. LIL for more general Lévy processes were obtained in Wee [26]; see Wee [27] for more examples. It was shown that if, for some positive constant θ,

$$\mathbb{P}(X_t > 0) \geq \theta \quad \text{and} \quad \mathbb{P}(X_t < 0) \geq \theta \qquad \text{for all } t \text{ sufficiently small,} \tag{1.3}$$

holds, then upper and lower bounds in the LIL hold in the following sense: for λ_1 sufficiently small and λ_2 sufficiently large,

$$1 \leq \liminf_{t \to 0} \frac{\|X\|_t}{b_{\lambda_1}(t)} \quad \text{and} \quad \liminf_{t \to 0} \frac{\|X\|_t}{b_{\lambda_2}(t)} \leq 1$$

for norming functions b_λ given by

$$b_\lambda(t) := f^{-1}\left(\frac{\log|\log t|}{\lambda t}\right),$$

where f is given by some explicit, but complicated expression depending on the Lévy triplet.

Although the results of Wee are quite general, there are some points which we aim to improve in the present work. First, we try to demonstrate and explain clearly how the LIL follow from small deviation estimates of type (1.2) and which behavior of the process is actually responsible for the correct norming function. Second, we attempt to control the unspecified (and suboptimal) constants λ_1 and λ_2 above, which can influence the norming function essentially (see (3.2) below for an example of influence on the exponential level) in the case when b_λ is not regularly varying at zero. In our approach, we keep track of the appearing constants in an optimal way. This allows us, in the case of known strong small deviation order, to transfer the constant in the strong small deviation order to the limiting constant in the LIL. Third, we provide alternative conditions to (1.3) which are explicit in terms of the Lévy triplet. We believe our conditions to be weaker than (1.3), but, as necessary and sufficient conditions for the latter in terms of the Lévy triplet seem to be unknown in general, it is difficult to verify our claim, although our examples hint at this direction.

This paper is structured as follows. In Section 2, we give the main results that manage the transfer between small deviations and LIL. Several examples of LIL for concrete Lévy processes are collected in Section 3. The proofs are given in Section 4.

Let us finally fix some notation. In this paper we let X be a Lévy process with characteristic triplet (γ, σ^2, Π), where $\gamma \in \mathbb{R}$, $\sigma^2 \geq 0$, and the Lévy measure Π has no atom at zero and satisfies

$$\int (1 \wedge x^2)\Pi(\mathrm{d}x) < \infty.$$

For basic definitions and properties of Lévy processes we refer to Bertoin [3], Sato [19]. As we are interested only in the behavior for small times, we discard all jumps bigger than 1 in absolute value and assume such truncation *throughout* the paper. Hence, the characteristic exponent, $\mathbb{E}\mathrm{e}^{\mathrm{i}zX_t} =: \mathrm{e}^{t\psi(z)}$, has the form

$$\psi(z) = \mathrm{i}\gamma z - \frac{\sigma^2 z^2}{2} + \int_{-1}^{1} (\mathrm{e}^{\mathrm{i}zx} - 1 - \mathrm{i}zx)\Pi(\mathrm{d}x), \qquad z \in \mathbb{R}.$$

For later use we denote by Φ the Laplace exponent of a subordinator A, $\mathbb{E}\mathrm{e}^{-uA_1} = \mathrm{e}^{-\Phi(u)}$,

$$\Phi(u) = u\gamma_A + \int_0^\infty (1 - \mathrm{e}^{-ux})\Pi_A(\mathrm{d}x).$$

Further, we use the standard notation $\bar{\Pi}(\varepsilon) := \Pi([-\varepsilon, \varepsilon]^c)$ for the two-sided tail of the Lévy measure.

In the following, we denote by $f \sim g$ the strong asymptotic equivalence, that is, $\lim f/g = 1$, and by $f \approx g$ the weak asymptotic equivalence, that is, $0 < \liminf f/g \leq \limsup f/g < \infty$.

2. Main results

Our first theorem manages the transfer from small deviation rates to LIL under minimal loss of constants.

Theorem 2.1. *Let X be a Lévy process* (*without loss of generality assume that X has jumps smaller than* 1 *in absolute value*). *Let F be a function increasing to infinity at zero*, *such that with some* $0 < \lambda_1 \leq \lambda_2 < \infty$

$$\lambda_1 F(\varepsilon)t \leq -\log \mathbb{P}(\|X\|_t < \varepsilon) \leq \lambda_2 F(\varepsilon)t \qquad \text{for all } \varepsilon < \varepsilon_0 \text{ and } t < t_0. \tag{2.1}$$

Further, define

$$b_\lambda(t) := F^{-1}\left(\frac{\log|\log t|}{\lambda t}\right)$$

for $\lambda > 0$, *and assume that, as* $n \to \infty$,

$$(n+1)^{-(n+1)^\beta}\left|\int_{|x|>b_{\lambda_2'}(n^{-n^\beta})} x\Pi(\mathrm{d}x) - \gamma\right| \\ = \mathrm{o}(b_{\lambda_2'}(n^{-n^\beta})) \qquad \textit{for all } \beta > 1 \textit{ and } \lambda_2' > \lambda_2. \tag{2.2}$$

Then the LIL

$$1 \le \liminf_{t\to 0} \frac{\|X\|_t}{b_{\lambda_1'}(t)} \quad \textit{and} \quad \liminf_{t\to 0} \frac{\|X\|_t}{b_{\lambda_2'}(t)} \le 1$$

hold almost surely for any $\lambda_1' < \lambda_1$ *and* $\lambda_2' > \lambda_2$.

Remark 2.1. It is important to note the role of (2.2). It ensures that the process does not become too asymmetric when one continues to cut off more and more smaller jumps. Only in this case is it possible to expect an estimate of type (2.1) to follow from the framework given in Aurzada and Dereich [2]. Corollary 2.4 below and, in particular, (2.11) give a sufficient condition when this is the case.

Remark 2.2. Let us relate our condition (2.2) with the condition of Wee [26]. Note that (2.2) is analytic, that is, in terms of the Lévy triplet, whereas Wee's condition (1.3) is probabilistic. It seems that (1.3) cannot always be checked from the Lévy triplet. To understand the difficulty, it may be instructive to look at Theorems 4 and 5 in Andrew [1], which reformulate (1.3) in terms of other probabilistic quantities.

It is crucial that there is almost no loss of constants in the transfer from the small deviations to the LIL as in cases when b_λ is not regularly varying, the constants λ_1', λ_2' may influence the rate function drastically; see (3.2) for an extreme example.

If instead b_λ only depends on λ via a multiplicative constant, our approach allows to strengthen the previous theorem to the optimal limiting constants. Such examples occur, for instance, if the small deviation rate function F is regularly varying.

Corollary 2.1. *In the setting of Theorem 2.1, assume additionally that* F *is regularly varying at zero with non-positive exponent. Then the following LIL hold almost surely*:

$$1 \le \liminf_{t\to 0} \frac{\|X\|_t}{b_{\lambda_1}(t)} \quad \textit{and} \quad \liminf_{t\to 0} \frac{\|X\|_t}{b_{\lambda_2}(t)} \le 1. \tag{2.3}$$

In particular, if there is $\lambda > 0$ *such that* (2.1) *holds for all* $\lambda_1 < \lambda$ *and all* $\lambda_2 > \lambda$, *then*

$$\liminf_{t\to 0} \frac{\|X\|_t}{b_\lambda(t)} = 1 \qquad \textit{a.s.}$$

In the setting of a regularly varying rate function, say F is regularly varying at zero with exponent $-\alpha$, $\alpha > 0$, one can express (2.3) as

$$\liminf_{t\to 0} \frac{\|X\|_t}{b_1(t)} \in [\lambda_1^{1/\alpha}, \lambda_2^{1/\alpha}], \qquad \text{a.s.}$$

This shows that only the quality of the small deviation estimate (2.1) matters in order to obtain the limiting constant in the LIL. Recall that the Blumenthal zero–one law implies that the limit is almost surely equal to a deterministic constant, which in this case can be specified.

Theorem 2.1 reduces the question of the right norming function for the LIL to the question of small deviations which is known precisely for many examples. For general Lévy processes, those have been obtained in Aurzada and Dereich [2] (their results were stated for $t = 1$ only, but hold, in general, as we discuss in Proposition 2.1 below). In particular, for symmetric Lévy processes, their main result states that the rate function is given by

$$F(\varepsilon) = \varepsilon^{-2} U(\varepsilon), \tag{2.4}$$

where $U(\varepsilon)$ is the variance of X with jumps larger than ε replaced by jumps of size ε,

$$U(\varepsilon) := \varepsilon^2 \bar{\Pi}(\varepsilon) + \sigma^2 + \int_{-\varepsilon}^{\varepsilon} x^2 \Pi(\mathrm{d}x). \tag{2.5}$$

From these specific small deviations we can deduce the following corollary for symmetric processes.

Corollary 2.2. *Let X be a symmetric Lévy process; then there are $0 < \lambda_1 \le \lambda_2 < \infty$ such that, almost surely,*

$$1 \le \liminf_{t\to 0} \frac{\|X\|_t}{b_{\lambda_1}(t)} \quad \text{and} \quad \liminf_{t\to 0} \frac{\|X\|_t}{b_{\lambda_2}(t)} \le 1,$$

with

$$b_\lambda(t) := F^{-1}\left(\frac{\log|\log t|}{\lambda t}\right)$$

and F defined in (2.4). If, additionally, F is regularly varying at zero with exponent $-\alpha$, $\alpha > 0$, then the following general bounds hold:

$$\frac{1}{12}\frac{1}{2^\alpha} \le \lambda_1 \le \lambda_2 \le 3^\alpha 10.$$

The loss of constants in the corollary is only due to the general formulation. For some examples we will see below that the small deviations are known in the strong asymptotic sense so that Theorem 2.1 gives the precise law.

In the sequel we call “strongly non-symmetric” Lévy processes the processes for which (2.2) does not hold. Their study requires different assumptions on b_λ; see (2.8). For this case, we provide a different link between small deviation rates and LIL. The next result does not require

(2.2) and thus allows us to study the "strongly non-symmetric" Lévy processes as well as other cases when (2.2) is difficult to verify. The latter is substituted by the seemingly easier (2.8) at the expense of the strength of the result; that is, we manage to keep track of the constants in the norming function in an optimal way, but lose the limiting constant. We have tried unsuccessfully to find a suitable relation between (2.2) and (2.8). We strongly suspect that neither one follows from the other.

Theorem 2.2. *Let X be a Lévy process with jumps smaller than* 1 *in absolute value, and let F be a function increasing to infinity at zero such that for* $0<\lambda_1\le\lambda_2<\infty$

$$\lambda_1 F(\varepsilon)t \le -\log\mathbb{P}(\|X\|_t<\varepsilon)\le \lambda_2 F(\varepsilon)t \qquad \text{for all } \varepsilon<\varepsilon_0 \text{ and } t<t_0. \tag{2.6}$$

Furthermore, set

$$b_\lambda(t):=F^{-1}\left(\frac{\log|\log t|}{\lambda t}\right), \tag{2.7}$$

and suppose that there is a constant $C>0$ such that

$$Cb_\lambda(t)\le b_\lambda(t/2), \qquad 0<t\le t_0, \lambda\in(\lambda_1/2, 2\lambda_2). \tag{2.8}$$

Then the LIL

$$0<\liminf_{t\to 0}\frac{\|X\|_t}{b_{\lambda_1'}(t)} \quad \text{and} \quad \liminf_{t\to 0}\frac{\|X\|_t}{b_{\lambda_2'}(t)}<\infty$$

hold almost surely for all $\lambda_1'<\lambda_1$ and $\lambda_2<\lambda_2'$.

Again, if the rate function F is regularly varying, then we can strengthen the result. Recall that the Lévy processes that appear in the formulation of the next sequence of results have jumps smaller than 1 in absolute value.

Corollary 2.3. *In the setting of Theorem 2.2, assume additionally that F is regularly varying at zero with negative exponent. Then the following LIL holds almost surely*:

$$\liminf_{t\to 0}\frac{\|X\|_t}{b_1(t)}\in(0,\infty).$$

The theorems listed so far manage the transfer between small deviation order and LIL. Similarly to Corollary 2.2, we can combine them with the main results of Aurzada and Dereich [2]. This looks more technical in the present case. We give an explanation of the role of the different terms after stating the result.

Corollary 2.4. *Let X be a Lévy process with triplet (γ,σ^2,Π). Assume that u_ε is the solution of the equation $\Lambda_\varepsilon'(u)=0$, where Λ_ε is the following log Laplace transform*:

$$\Lambda_\varepsilon(u)=\frac{\sigma^2}{2}u^2+\left(\gamma-\int_{[-1,1]\setminus[-\varepsilon,\varepsilon]}x\Pi(\mathrm{d}x)\right)u+\int_{-\varepsilon}^{\varepsilon}(\mathrm{e}^{ux}-1-ux)\Pi(\mathrm{d}x). \tag{2.9}$$

Set

$$F(\varepsilon) := \varepsilon^{-2}U_\varepsilon(\varepsilon) - \Lambda_\varepsilon(u_\varepsilon), \qquad U_\varepsilon(\varepsilon) := \varepsilon^2\bar{\Pi}(\varepsilon) + \sigma^2 + \int_{-\varepsilon}^{\varepsilon} x^2 \mathrm{e}^{-u_\varepsilon x}\,\Pi(\mathrm{d}x), \tag{2.10}$$

and assume F is increasing to infinity as $\varepsilon \to 0$. Define b as in (2.7), *and assume that b satisfies* (2.8). *If, furthermore,*

$$\varepsilon|u_\varepsilon| = \mathrm{o}(\log\log F(\varepsilon)), \qquad \text{as } \varepsilon \to 0, \tag{2.11}$$

is satisfied, then we have, for some $\lambda_1, \lambda_2 > 0$,

$$0 < \liminf_{t\to 0} \frac{\|X\|_t}{b_{\lambda_1}(t)} \quad \text{and} \quad \liminf_{t\to 0} \frac{\|X\|_t}{b_{\lambda_2}(t)} < \infty \qquad \text{a.s.}$$

Let us explain the quantities appearing in Corollary 2.4 in more detail. The main observation is that the proof for the small deviation estimates in Aurzada and Dereich [2] (Theorem 1.5) can be used directly for any $t > 0$ to obtain the following proposition.

Proposition 2.1. *Let Λ_ε be as defined in* (2.9) *and assume that u_ε is the solution of $\Lambda'_\varepsilon(u_\varepsilon) = 0$. Then, with F as in* (2.10), *we have, for all $t > 0$ and all $\varepsilon < 1$,*

$$\frac{1}{12}tF(2\varepsilon) - \varepsilon|u_{2\varepsilon}| - 1 \le -\log \mathbb{P}(\|X\|_t \le \varepsilon) \le 10tF\left(\frac{\varepsilon}{3}\right) + \varepsilon|u_{\varepsilon/3}| + 3. \tag{2.12}$$

The term $\bar{\Pi}(2\varepsilon)$ in (2.12) (included in the F term) comes from the requirement that there should be no jumps larger than 2ε. After removing these jumps, the process may drift out of the interval $[-\varepsilon, \varepsilon]$, which is prevented by applying an Esscher transform to the process, whose "price" is given by the term $-\Lambda_\varepsilon(u_\varepsilon)$. The quantity u_ε is the drift that has to be subtracted in order to make the process a martingale. Then the remaining process is treated as in the symmetric case, and the same term $\varepsilon^{-2}U_\varepsilon(\varepsilon)$ appears as in (2.4), but, this time, with respect to the Lévy measure transformed by the change of measure.

Note that (2.12) is almost the required estimate in (2.6), except for the term $\varepsilon|u_\varepsilon|$, which may spoil the estimate. It is exactly condition (2.11) that ensures that the term $\varepsilon|u_\varepsilon|$ can be neglected.

We stress that in some cases $\varepsilon|u_\varepsilon|$ does give an order that is larger than $tF(\varepsilon)$ so that the function b from (2.7) is not the right norming function. This effect can be observed in some examples below. In particular, this happens for processes of bounded variation with non-zero drift.

Proposition 2.2. *Let X be a Lévy process with bounded variation and non-vanishing effective drift, that is, $\int_{[-1,1]} |x|\Pi(\mathrm{d}x) < \infty$ and $c := \gamma - \int_{-1}^{1} x\Pi(\mathrm{d}x) \neq 0$. Then*

$$\lim_{t\to 0} \frac{\|X\|_t}{t} = |c| \qquad \text{a.s.}$$

The proof of this proposition is based on classical arguments rather than any connection to small deviations.

3. Explicit LIL for Lévy processes

In this section, we collect concrete Lévy processes for which we can transform small deviation results to an LIL. As we have seen, understanding the small deviation rates is crucial.

In this section we keep in mind that our processes in all proofs have no jumps bigger than 1 in absolute value. However, without loss of generality, in some statements we use "stable Lévy processes" and others which presuppose unbounded jumps.

The first corollary gives us a useful variance domination principle for LIL that works for many examples.

Corollary 3.1. *Suppose X^1 and X^2 are independent symmetric Lévy processes, then $X^1 + X^2$ and X^2 fulfill precisely the same LIL if*

$$\lim_{\varepsilon\to 0} \frac{U_{X^1}(\varepsilon)}{U_{X^2}(\varepsilon)} = 0.$$

Proof. This follows directly from Corollary 2.2 noticing that $U_{X^1+X^2} = U_{X^1} + U_{X^2}$. □

In the same spirit, the following corollary (recovering (3.2) in Buchmann and Maller [8]) displays the intuitive fact that a non-zero Brownian component dominates the jumps of a Lévy process.

Corollary 3.2. *If X is a Lévy process with $\sigma \neq 0$, then*

$$\liminf_{t\to 0} \frac{\|X\|_t}{\sqrt{t/\log|\log t|}} = \frac{\pi\sigma}{\sqrt{8}} \qquad a.s.$$

Proof. Following precisely the proof of Corollary 2.6 of Aurzada and Dereich [2], one can show that the small deviation rates of Lévy processes with non-zero Brownian component are given by

$$-\log \mathbb{P}(\|X\|_t < \varepsilon) \sim \frac{\pi^2\sigma^2}{8}\varepsilon^{-2}t, \qquad \text{as } \varepsilon \to 0 \text{ and } t \to 0.$$

Hence, the norming function follows from Theorem 2.1. As the process is not necessarily symmetric, condition (2.2) has to be checked: Since $b(t) = \sqrt{t\pi^2/(8\log|\log t|)}$ and $\int_{|x|>\varepsilon} |x|\Pi(\mathrm{d}x) = \mathrm{o}(\varepsilon^{-1})$, it remains to be seen that

$$a_{n+1} \le cb(a_n)^2 = a_n/\log|\log a_n|$$

for $a_n = n^{-n^\beta}$ and $\beta > 1$. This can be verified by simple computations. □

Similarly to Lévy processes with non-zero Brownian component, symmetric processes of smaller small deviation order (e.g., stable processes of smaller index) are dominated by stable Lévy processes.

Corollary 3.3. *Let X be a symmetric α-stable Lévy process with $\alpha \in (0,2]$, and let Y be symmetric with $U_Y(x) = \mathrm{o}(x^{2-\alpha})$. Then there is a constant $0 < c_\alpha < \infty$ such that*

$$\liminf_{t\to 0} \frac{\|X+Y\|_t}{(t/\log|\log t|)^{1/\alpha}} = \liminf_{t\to 0} \frac{\|X\|_t}{(t/\log|\log t|)^{1/\alpha}} = c_\alpha^{1/\alpha} \qquad a.s.$$

Proof. The small deviation rate is given by

$$-\log \mathbb{P}(\|X\|_t < \varepsilon) \sim c_\alpha \varepsilon^{-\alpha} t, \qquad \text{as } \varepsilon \to 0 \text{ and } t \to 0$$

for some constant $c_\alpha > 0$ (see, e.g., page 220 in Bertoin [3]). Hence, the LIL follows from Corollary 2.1 and Corollary 3.1. □

Remark 3.1. The constant c_α in the LIL of stable Lévy processes is the unknown constant of the small deviations for respective α-stable Lévy processes (see Taylor [25] and Proposition 3 and Theorem 6 in Chapter VIII of Bertoin [3]). The results of Aurzada and Dereich [2] entail the following concrete bounds:

$$\frac{2C}{2^\alpha}\left(\frac{1}{\alpha} + \frac{1}{12(2-\alpha)}\right) < c_\alpha < 3^\alpha \cdot 2C\left(\frac{1}{\alpha} + \frac{10}{2-\alpha}\right),$$

where C is the constant in the Lévy measure: $\Pi(\mathrm{d}x) = C|x|^{-(1+\alpha)}\,\mathrm{d}x$. This implies $c_\alpha \sim 2C/\alpha$, as $\alpha \to 0$. We remark that, contrary to the symmetric case, the constant c_α is known explicitly for completely asymmetric stable Lévy processes; see Bertoin [4].

Let us study the case when Π behaves as a regularly varying function at zero and is symmetric. Then the following LIL are satisfied.

Corollary 3.4. *Let X be a Lévy process with triplet $(0,0,\Pi)$ with Π being symmetric and*

$$\bar{\Pi}(\varepsilon) \approx \varepsilon^{-\alpha}|\log\varepsilon|^{-\gamma}, \qquad as\ \varepsilon \to 0,$$

with $0<\alpha<2$ or $\alpha = 2, \gamma > 1$. Then

$$\liminf_{t\to 0} \frac{\|X\|_t}{b(t)} \in (0,\infty) \qquad a.s.$$

with

$$b(t) = \begin{cases} \left(\dfrac{t|\log t|^{-\gamma}}{\log|\log t|}\right)^{1/\alpha}, & 0<\alpha<2, \\ \left(\dfrac{t|\log t|^{1-\gamma}}{\log|\log t|}\right)^{1/2}, & \alpha = 2, \gamma > 1. \end{cases}$$

Proof. The corollary follows from Theorem 2.1. The required small deviation estimate,

$$-\log \mathbb{P}(\|X\|_t < \varepsilon) \approx \begin{cases} \varepsilon^{-\alpha}|\log\varepsilon|^{-\gamma} t, & 0<\alpha<2, \\ \varepsilon^{-2}|\log\varepsilon|^{1-\gamma} t, & \alpha = 2, \gamma > 1, \end{cases}$$

as $\varepsilon \to 0$ and $t \to 0$, is obtained from Proposition 2.1 (cf. Example 2.2 in Aurzada and Dereich [2] for $t = 1$). Since we deal with a symmetric process, condition (2.11) is trivially satisfied due to $u_\varepsilon = 0$. □

Having discussed the α-stable like cases, we now consider Lévy processes with polynomial tails near zero of *different* exponents. The technique used for this example can be extended to any case with essentially regularly varying Lévy measure at zero. Let X be a Lévy process with triplet $(\gamma, 0, \Pi)$, where Π is given by

$$\frac{\Pi(\mathrm{d}x)}{\mathrm{d}x} = \frac{C_1 \mathbb{1}_{(0,1]}(x)}{x^{1+\alpha_1}} + \frac{C_2 \mathbb{1}_{[-1,0)}(x)}{(-x)^{1+\alpha_2}}, \tag{3.1}$$

with $2 > \alpha_1 \geq \alpha_2$ and $C_1, C_2 \geq 0$, $C_1 + C_2 \neq 0$. We now analyze the pathwise behavior at zero in the cases when $\alpha_1 > 1$, $\alpha_1 = 1$, and $0 < \alpha_1 < 1$, respectively. The second exponent α_2 can be even negative.

Corollary 3.5. *Let X be a Lévy process with triplet $(\gamma, 0, \Pi)$ with Π as in (3.1). Then the following holds:*

1. *If $\alpha_1 \geq \alpha_2$, $C_1 \neq 0$, and $\alpha_1 > 1$, then*

$$\liminf_{t\to 0} \frac{\|X\|_t}{(t/\log|\log t|)^{1/\alpha_1}} \in (0,\infty) \qquad a.s.$$

2. *If $\alpha_1 = \alpha_2 = 1$ and $C_1 = C_2$, then*

$$\liminf_{t\to 0} \frac{\|X\|_t}{t/\log|\log t|} \in (0,\infty) \qquad a.s.$$

3. *If $1 > \alpha_1 \geq \alpha_2$ and the effective drift does not vanish, then*

$$\lim_{t\to 0} \frac{\|X\|_t}{t} = |c| \qquad a.s.$$

Proof. Parts 1 and 2 follow from Theorem 2.1. The required small deviation estimates,

$$-\log \mathbb{P}(\|X\|_t < \varepsilon) \approx \varepsilon^{-\alpha_1} t$$

for $\varepsilon \to 0$ and $t \to 0$, are obtained from Proposition 2.1 (cf. Corollary 2.7, 2.8 and 2.9 of Aurzada and Dereich [2] for $t = 1$; note that $u_\varepsilon \approx \varepsilon^{-1}$ in all cases). One can easily check condition (2.11).

In part 3 the process is of bounded variation, so that the claim is included in Proposition 2.2. □

We now come to Lévy processes obtained from Brownian motion by subordination, that is, $X_t = \sigma B_{A_t}$, where B is a Brownian motion independent of the subordinator A. In this case, the resulting Lévy process is symmetric and the small deviation asymptotics is governed by the truncated variance U from (2.5).

Corollary 3.6. *Let B be a Brownian motion independent of the subordinator A, where A has Laplace exponent Φ. For $\lambda > 0$ we set $b_\lambda(t) := F^{-1}(\frac{\log|\log t|}{\lambda t})$ with*

$$F(\varepsilon) := \Phi(\sigma^2\varepsilon^{-2}) + \gamma_A \sigma^2 \varepsilon^{-2}.$$

Then, for some $\lambda_1, \lambda_2 > 0$,

$$1 \leq \liminf_{t\to 0} \frac{\|X\|_t}{b_{\lambda_1}(t)} \quad \text{and} \quad \liminf_{t\to 0} \frac{\|X\|_t}{b_{\lambda_2}(t)} \leq 1 \qquad a.s.$$

In particular, if $\gamma_A = 0$ and Φ is regularly varying with positive exponent, we have

$$\liminf_{t\to 0} \frac{\|X\|_t}{(\Phi^{-1}(\log|\log t|/t))^{-1/2}} \in (0,\infty) \qquad a.s.$$

Proof. The corollary follows from Theorem 2.1 with the small deviation estimate from Proposition 2.1,

$$-\log \mathbb{P}(\|X\|_t \leq \varepsilon) \approx \big(\Phi(\sigma^2\varepsilon^{-2}) + \gamma_A\sigma^2\varepsilon^{-2}\big)t,$$

as $\varepsilon \to 0$ and $t \to 0$ (cf. Example 2.13 of Aurzada and Dereich [2] for $t = 1$ and note the misprint there). Condition (2.2) is trivially fulfilled as the process is symmetric. □

For a more specific example, in particular, exhibiting exotic small time behavior, we choose the subordinator A to be a Gamma process. Then one defines the so called Variance-Gamma process as

$$X_t = \sigma B_{A_t} + \mu A_t$$

for some constants $\sigma \neq 0$ and $\mu \in \mathbb{R}$.

Corollary 3.7. *Let X be a Variance-Gamma process; then for $\mu = 0$ there are some constants $0 < \lambda_1 \leq \lambda_2 < \infty$ such that*

$$1 \leq \liminf_{t\to 0} \frac{\|X\|_t}{e^{-\lambda_1 \log|\log t|/t}} \quad \text{and} \quad \liminf_{t\to 0} \frac{\|X\|_t}{e^{-\lambda_2 \log|\log t|/t}} \leq 1 \qquad a.s., \tag{3.2}$$

whereas for $\mu \neq 0$

$$\liminf_{t\to 0} \frac{\|X\|_t}{t} = |\mu|\mathbb{E}(A_1) \qquad a.s.$$

Proof. The second part is included in Proposition 2.2, since the process is of bounded variation with non-zero effective drift. In the first part, the effective drift is zero, and the claim follows from Theorem 2.1. The small deviation estimate,

$$-\log \mathbb{P}(\|X\|_t \leq \varepsilon) \approx t|\log \varepsilon|, \qquad \text{as } \varepsilon \to 0 \text{ and } t \to 0,$$

follows from Proposition 2.1 (cf. Example 2.12 of Aurzada and Dereich [2] for $t = 1$). □

In the first case of the previous corollary, the dependence of good small deviation estimates and good LIL becomes transparant. The fact that we cannot specify the constants λ_1, λ_2 in (3.2) is only caused by the weak asymptotics for the small deviation estimate as we do not lose any further constants in the transfer of small deviations to the LIL. If one does not have more control on the constants λ_1, λ_2, the understanding of the precise small time behavior of X is far from optimal as the error enters exponentially.

4. Proofs

We start with a lemma which shows that the small deviation order is at least as large as the term induced by the variance, defined in (2.5).

Lemma 4.1. *Let $\varepsilon > 0$, and let X be a Lévy process with Lévy measure concentrated on* $[-\varepsilon, \varepsilon]$, *then*

$$\mathbb{P}(\|X\|_t \le \varepsilon/2) \le \exp\left(-\varepsilon^{-2}\left(\int_{-\varepsilon}^{\varepsilon} x^2\Pi(\mathrm{d}x) + \sigma^2\right)t\Big/12 + 1\right) \qquad \text{for } t \ge 0.$$

Proof. We proceed similarly to Lemma 4.2 in Aurzada and Dereich [2]. Let τ be the first exit time of X out of $[-\varepsilon, \varepsilon]$. Then, by Wald's identity,

$$\begin{aligned} 4\varepsilon^2 &\ge \limsup_{t\to\infty} \mathbb{E}[X_{t\wedge\tau}^2] \ge \limsup_{t\to\infty} \operatorname{var}[X_{t\wedge\tau}] \\ &= \limsup_{t\to\infty}\left(\int_{-\varepsilon}^{\varepsilon} x^2\Pi(\mathrm{d}x) + \sigma^2\right)\mathbb{E}[t\wedge\tau] = \left(\int_{-\varepsilon}^{\varepsilon} x^2\Pi(\mathrm{d}x) + \sigma^2\right)\mathbb{E}[\tau]. \end{aligned}$$

Therefore,

$$\mathbb{P}\left(\tau \ge 8\varepsilon^2\Big/\left(\int_{-\varepsilon}^{\varepsilon} x^2\Pi(\mathrm{d}x) + \sigma^2\right)\right) \le \frac{(\int_{-\varepsilon}^{\varepsilon} x^2\Pi(\mathrm{d}x) + \sigma^2)\mathbb{E}[\tau]}{8\varepsilon^2} \le \frac{1}{2}.$$

Let $n := \lfloor t(\int_{-\varepsilon}^{\varepsilon} x^2\Pi(\mathrm{d}x) + \sigma^2)/(8\varepsilon^2)\rfloor$, and set $t_i := 8i\varepsilon^2/(\int_{-\varepsilon}^{\varepsilon} x^2\Pi(\mathrm{d}x) + \sigma^2)$, $i = 0, \dots, n$. Then

$$\mathbb{P}(\|X\|_t \le \varepsilon/2) \le \mathbb{P}\left(\forall i = 0, \dots, n-1\colon \sup_{s\in[t_i,t_{i+1})} |X_s - X_{t_i}| \le \varepsilon\right) = \mathbb{P}(\tau \ge t_1)^n \le 2^{-n}. \qquad \square$$

This shows that the small deviation order is always at least as large as the term induced by the truncated variance process. This fact will be needed later on.

Lemma 4.2. *Let F be a function that increases to infinity at zero. If, for some Lévy process X, for $t \le t_0$ and $\varepsilon < \varepsilon_0$,*

$$-\log \mathbb{P}(\|X\|_t \le \varepsilon) \le F(\varepsilon)t,$$

then, for some absolute constant $c > 0$ and all $\varepsilon > 0$ small enough,

$$\varepsilon^{-2}U(\varepsilon) \le c\big(F(\varepsilon)+1\big).$$

Proof. We use the assumption together with the fact that if $\|X\|_t \le \varepsilon$, then X must not have jumps larger than 2ε and the previous lemma,

$$\mathrm{e}^{-F(\varepsilon)t} \le \mathbb{P}(\|X\|_t \le \varepsilon) = \mathrm{e}^{-\bar{\Pi}(2\varepsilon)t}\mathbb{P}(\|X'\|_t \le \varepsilon) \le \mathrm{e}^{-\bar{\Pi}(2\varepsilon)t}\mathrm{e}^{-(2\varepsilon)^{-2}(\int_{-2\varepsilon}^{2\varepsilon} x^2\Pi(\mathrm{d}x)+\sigma^2)t/12+1},$$

where X' has Lévy measure Π restricted to $[-2\varepsilon, 2\varepsilon]$. Noting that Lemma 5.1 of Aurzada and Dereich [2] implies that $U(\varepsilon)/\varepsilon^2 \approx U(2\varepsilon)/(2\varepsilon)^2$, the statement of the lemma is proved. □

The lower bound in the LIL comes from the following lemma.

Lemma 4.3. *Let F be a function that increases to infinity at zero such that, for all $t \le t_0$ and $\varepsilon \le \varepsilon_0$,*

$$\lambda F(\varepsilon)t \le -\log \mathbb{P}(\|X\|_t \le \varepsilon),$$

and, for $\lambda > 0$, we set $b_\lambda(t) := F^{-1}(\frac{\log|\log t|}{\lambda t})$. Then, for any $\lambda' < \lambda$,

$$1 \le \liminf_{t\to 0} \frac{\|X\|_t}{b_{\lambda'}(t)} \qquad a.s.$$

Proof. For any $\lambda' < \lambda$, we can find $0 < r < 1$ such that $1 < \lambda r/\lambda'$. Note that

$$\sum_n \mathbb{P}\big(\|X\|_{r^{n+1}} \le b_{\lambda'}(r^n)\big) < \infty$$

since

$$-\log \mathbb{P}\big(\|X\|_{r^{n+1}} \le b_{\lambda'}(r^n)\big) \ge \lambda F(b_{\lambda'}(r^n))r^n r = \lambda \frac{r}{\lambda'} \log|\log r^n| = \log n^{r\lambda/\lambda'} + \text{const.} \quad (4.1)$$

Hence, by the Borel–Cantelli lemma,

$$\{n\colon \|X\|_{r^{n+1}} \le b_{\lambda'}(r^n)\}$$

is almost surely a finite set. Thus, for each path ω, we have that, for any $n \ge n_0(\omega)$ and any $t \in [r^{n+1}, r^n)$,

$$\frac{\|X\|_t}{b_{\lambda'}(t)} \ge \frac{\|X\|_{r^{n+1}}}{b_{\lambda'}(r^n)} \ge 1,$$

as $b_{\lambda'}$ is an increasing function. We take $\liminf_{t\to 0}$ to obtain the statement. □

The proof of the upper bound in the LIL requires the following lemma.

Lemma 4.4. *Let F be a function that increases to infinity at zero such that for all $t \leq t_0$ and $\varepsilon \leq \varepsilon_0$*

$$-\log \mathbb{P}(\|X\|_t \leq \varepsilon) \leq \lambda F(\varepsilon) t$$

and, for $\lambda > 0$, set $b_\lambda(t) := F^{-1}(\frac{\log|\log t|}{\lambda t})$. Assume that

$$\limsup_{n\to\infty} \frac{\|X\|_{(n+1)^{-(n+1)^\beta}}}{b_\lambda(n^{-n^\beta})} = 0 \qquad a.s. \tag{4.2}$$

for all $\beta > 1$. Then, for any $\lambda' > \lambda$,

$$\liminf_{t\to 0} \frac{\|X\|_t}{b_{\lambda'}(t)} \leq 1 \qquad a.s. \tag{4.3}$$

Proof. For $\lambda' > \lambda$, we choose $\beta > 1$ such that $\lambda' > \lambda\beta$. First note that (4.2) implies

$$\limsup_{n\to\infty} \frac{\|X\|_{(n+1)^{-(n+1)^\beta}}}{b_{\lambda'}(n^{-n^\beta})} = 0 \qquad \text{a.s.}, \tag{4.4}$$

as $b_\lambda(t)$ is an increasing function in λ for fixed $t \geq 0$. Using the Lévy property, we see the following:

$$\begin{aligned}
&\sum_n \mathbb{P}\Big(\sup_{(n+1)^{-(n+1)^\beta} \leq t < n^{-n^\beta}} \big|X_t - X_{(n+1)^{-(n+1)^\beta}}\big| \leq b_{\lambda'}(n^{-n^\beta})\Big) \\
&= \sum_n \mathbb{P}\big(\|X\|_{n^{-n^\beta} - (n+1)^{-(n+1)^\beta}} \leq b_{\lambda'}(n^{-n^\beta})\big) \\
&\geq \sum_n \mathbb{P}\big(\|X\|_{n^{-n^\beta}} \leq b_{\lambda'}(n^{-n^\beta})\big) = \infty.
\end{aligned}$$

The last step follows as in (4.1) since now $\lambda\beta/\lambda' < 1$. The Borel–Cantelli lemma shows that the sequence of independent events

$$A_n = \Big\{\sup_{(n+1)^{-(n+1)^\beta} \leq t < n^{-n^\beta}} \big|X_t - X_{(n+1)^{-(n+1)^\beta}}\big| \leq b_{\lambda'}(n^{-n^\beta})\Big\}$$

satisfies $\mathbb{P}(A_n \text{ i.o.}) = 1$. To reduce to the supremum, note that

$$\frac{\|X\|_{n^{-n^\beta}}}{b_{\lambda'}(n^{-n^\beta})} \leq \frac{\sup_{(n+1)^{-(n+1)^\beta} \leq t < n^{-n^\beta}} |X_t - X_{(n+1)^{-(n+1)^\beta}}|}{b_{\lambda'}(n^{-n^\beta})} + \frac{2\|X\|_{(n+1)^{-(n+1)^\beta}}}{b_{\lambda'}(n^{-n^\beta})},$$

and therefore, by (4.4),

$$\liminf_{n\to\infty} \frac{\|X\|_{n^{-n^\beta}}}{b_{\lambda'}(n^{-n^\beta})} \leq \liminf_{n\to\infty} \frac{\sup_{(n+1)^{-(n+1)^\beta} \leq t < n^{-n^\beta}} |X_t - X_{n^{-(n+1)^\beta}}|}{b_{\lambda'}(n^{-n^\beta})} \leq 1.$$

This shows (4.3). □

Now we are in position to prove Theorem 2.1. For a detailed analysis of the lim sup case, we refer to Savov [20].

Proof of Theorem 2.1. The claim follows from Lemmas 4.3 and 4.4. To verify the use of Lemma 4.4 we still need to check that condition (4.2) holds for all $\beta > 1$.

We fix $\beta > 1$ and $\lambda_2' > \lambda_2$. Since λ_2' is fixed, we set $b := b_{\lambda_2'}$ in order to increase readability. We define the auxiliary function

$$h(t) = b(\phi(t)),$$

where $\phi(t)$ is chosen such that $\phi((\frac{t}{t+1})^{((t+1)/t)^\beta}) = t^{1/t^\beta}$ and $\phi(0) = 0$. Note that ϕ is increasing and that $\phi(s^{-s^\beta}) = (s-1)^{-(s-1)^\beta}$. We also do not record that ϕ and h depend on β and λ_2'.

Step 1: We show that

$$\int_0^{1/2} \bar{\Pi}(h(t))\,\mathrm{d}t < \infty. \tag{4.5}$$

First, by the definition of h and a change of variables, we obtain

$$\begin{aligned}
&\int_0^{1/2} \bar{\Pi}(h(t))\,\mathrm{d}t \\
&= \int_0^{C(\beta)} \bar{\Pi}(b(s^{s^{-\beta}}))\frac{\mathrm{d}(s/(s+1))^{((s+1)/s)^\beta}}{\mathrm{d}s} \\
&= \int_0^{C(\beta)} \bar{\Pi}(b(s^{s^{-\beta}}))\left(\frac{s}{s+1}\right)^{((s+1)/s)^\beta}\left(\frac{s+1}{s}\right)^{\beta-1} s^{-2}\big(1-\beta\log\big(1-(s+1)^{-1}\big)\big)\,\mathrm{d}s,
\end{aligned}$$

which can be estimated from above by

$$\begin{aligned}
&C\int_0^{C(\beta)} \frac{b^2(s^{s^{-\beta}})\bar{\Pi}(b(s^{s^{-\beta}}))}{b^2(s^{s^{-\beta}})}\left(\frac{s}{s+1}\right)^{((s+1)/s)^\beta} s^{-1-\beta}|\log s|\,\mathrm{d}s \\
&\le C\int_0^{C(\beta)} \frac{U(b(s^{s^{-\beta}}))}{b^2(s^{s^{-\beta}})}\left(\frac{s}{s+1}\right)^{((s+1)/s)^\beta} s^{-1-\beta}|\log s|\,\mathrm{d}s \\
&\le C'\int_0^{C(\beta)} F(b(s^{s^{-\beta}}))\left(\frac{s}{s+1}\right)^{((s+1)/s)^\beta} s^{-1-\beta}|\log s|\,\mathrm{d}s \\
&= \frac{C'}{\lambda}\int_0^{C(\beta)} \frac{\log|\log s^{s^{-\beta}}|}{s^{s^{-\beta}}}\left(\frac{s}{s+1}\right)^{((s+1)/s)^\beta} s^{-1-\beta}|\log s|\,\mathrm{d}s \\
&\le \frac{C'}{\lambda}\int_0^{C(\beta)} s^{-1-\beta}(\log|\log s^{s^{-\beta}}|)s^{((s+1)/s)^\beta-1/s^\beta}|\log s|\,\mathrm{d}s < \infty,
\end{aligned}$$

where we have used $x^2\bar{\Pi}(x) \le x^2\bar{\Pi}(x) + \int_{-x}^{x} y^2\Pi(\mathrm{d}y) + \sigma^2 = U(x) \le cx^2F(x)$ for some absolute $c > 0$ by Lemma 4.2 and the definition of b.

Step 2: We denote by

$$A_n := \{\text{there is at least one jump with modulus} > b(n^{-n^\beta}) \text{ up to time } (n+1)^{-(n+1)^\beta}\} \quad (4.6)$$

and show that

$$\sum_n \mathbb{P}(A_n) < \infty. \quad (4.7)$$

This comes from (4.5). Indeed, note that h inherits the monotonicity of b and ϕ, and hence (4.5) implies that

$$\begin{aligned} &\sum_n \big((n+1)^{-(n+1)^\beta} - (n+2)^{-(n+2)^\beta}\big)\bar{\Pi}\big(h\big((n+1)^{-(n+1)^\beta}\big)\big) \\ &\quad \le \sum_n \int_{(n+2)^{-(n+2)^\beta}}^{(n+1)^{-(n+1)^\beta}} \bar{\Pi}(h(t))\,\mathrm{d}t < \infty. \end{aligned} \quad (4.8)$$

Using

$$\begin{aligned} (n+1)^{-(n+1)^\beta} - (n+2)^{-(n+2)^\beta} &\sim (n+1)^{-(n+1)^\beta}, \\ b(n^{-n^\beta}) &= h\big((n+1)^{-(n+1)^\beta}\big), \end{aligned}$$

and that the sequence $(n+1)^{-(n+1)^\beta}\bar{\Pi}(h((n+1)^{-(n+1)^\beta}))$ tends to zero by (4.8), we obtain that

$$\mathbb{P}(A_n) = 1 - \mathrm{e}^{-(n+1)^{-(n+1)^\beta}\bar{\Pi}(b(n^{-n^\beta}))} \sim (n+1)^{-(n+1)^\beta}\bar{\Pi}\big(h\big((n+1)^{-(n+1)^\beta}\big)\big)$$

is summable. Therefore (4.7) is proved.

Step 3: Let us now show how to use (4.7) to deduce (4.2). Obviously, it suffices to show that

$$\limsup_{n\to\infty} \frac{\|X\|_{(n+1)^{-(n+1)^\beta}}}{b(n^{-n^\beta})} < \varepsilon \qquad \text{a.s.}$$

for any $\varepsilon > 0$ and, hence, by the Borel–Cantelli lemma, it suffices to show that

$$\sum_n \mathbb{P}\big(\|X\|_{(n+1)^{-(n+1)^\beta}} > \varepsilon b(n^{-n^\beta})\big) < \infty.$$

Separating jumps of absolute value larger or smaller than $b(n^{-n^\beta})$, and, using the definition of A_n in (4.6), we obtain that

$$\begin{aligned} &\sum_n \mathbb{P}\big(\|X\|_{(n+1)^{-(n+1)^\beta}} > \varepsilon b(n^{-n^\beta})\big) \\ &\quad = \sum_n \mathbb{P}\big(\|X\|_{(n+1)^{-(n+1)^\beta}} > \varepsilon b(n^{-n^\beta}); A_n^c\big) + \sum_n \mathbb{P}\big(\|X\|_{(n+1)^{-(n+1)^\beta}} > \varepsilon b(n^{-n^\beta}); A_n\big), \end{aligned}$$

which is bounded from above by

$$\sum_n \mathbb{P}\big(\|X\|_{(n+1)^{-(n+1)^\beta}} > \varepsilon b(n^{-n^\beta})|A_n^c\big)\cdot\mathbb{P}(A_n^c)+\sum_n \mathbb{P}(A_n).$$

The second term is finite by (4.7); and the first term is bounded by

$$\sum_n \mathbb{P}\big(\|X\|_{(n+1)^{-(n+1)^\beta}} > \varepsilon b(n^{-n^\beta})|A_n^c\big). \tag{4.9}$$

To estimate this sum note that conditionally on A_n^c, $X_t \overset{d}{=} X_t(n)$, where $X(n)$ differs from X only by removing jumps of size larger than $|b(n^{-n^\beta})|$. Clearly, by Wald's identity,

$$\operatorname{var}(X_t(n)) = t\left(\int_{-b(n^{-n^\beta})}^{b(n^{-n^\beta})} y^2\Pi(\mathrm{d}y)+\sigma^2\right) \le tU(b(n^{-n^\beta})). \tag{4.10}$$

Note that

$$\big|\mathbb{E}X_{(n+1)^{-(n+1)^\beta}}(n)\big| = (n+1)^{-(n+1)^\beta}\left|\int_{|x|>b(n^{-n^\beta})} x\Pi(\mathrm{d}x)-\gamma\right|.$$

Therefore, by assumption (2.2), taking also into account that $|\mathbb{E}X_t(n)| = t|\mathbb{E}X_1(n)|$, we obtain

$$\sup_{t\le(n+1)^{-(n+1)^\beta}} |\mathbb{E}X_t(n)| = \big|\mathbb{E}X_{(n+1)^{-(n+1)^\beta}}(n)\big| = \mathrm{o}(b(n^{-n^\beta})).$$

Using the previous relation (first step), Doob's martingale inequality (second step), (4.10) (third step), Lemma 4.2 (fourth step) and the definition of b (fifth step), we are led to the upper bound of the term in (4.9),

$$\begin{aligned}
&\sum_n \mathbb{P}\big(\|X(n)\|_{(n+1)^{-(n+1)^\beta}} > \varepsilon b(n^{-n^\beta})\big)\\
&\quad\le \sum_n \mathbb{P}\left(\|X(n)-\mathbb{E}X(n)\|_{(n+1)^{-(n+1)^\beta}} > \frac{1}{2}\varepsilon b(n^{-n^\beta})\right)\\
&\quad\le \sum_n \frac{4\mathbb{E}|X_{(n+1)^{-(n+1)^\beta}}(n)-\mathbb{E}X_{(n+1)^{-(n+1)^\beta}}(n)|^2}{(\varepsilon/2)^2 b(n^{-n^\beta})^2}\\
&\quad\le \sum_n \frac{4(n+1)^{-(n+1)^\beta}U(b(n^{-n^\beta}))}{(\varepsilon/2)^2 b(n^{-n^\beta})^2}\\
&\quad\le \sum_n \frac{4(n+1)^{-(n+1)^\beta}C\cdot F(b(n^{-n^\beta}))}{(\varepsilon/2)^2}\\
&\quad= \frac{C'}{\lambda\varepsilon^2}\sum_n \frac{(n+1)^{-(n+1)^\beta}\log|\log n^{-n^\beta}|}{n^{-n^\beta}} < \infty,
\end{aligned}$$

where we used the definition of b in the last step. Thus, the term in (4.9) is finite, as required. □

Proof of Corollary 2.1. If F is regularly varying so is b_λ; see Bingham, Goldie and Teugels [6], Proposition 1.5.7. Now note that if F is regularly varying with exponent $-\alpha < 0$, we have

$$\begin{aligned} b_\lambda(t) &= F^{-1}(\log|\log t|/\lambda t) \\ &\sim \lambda^{1/\alpha} F^{-1}(\log|\log t|/t) \\ &= \lambda^{1/\alpha} b_1(t). \end{aligned}$$

Hence, the statement of Theorem 2.1 reads

$$(\lambda_1')^{1/\alpha} \le \liminf_{t\to 0} \frac{\|X\|_t}{b_1(t)} \le (\lambda_2')^{1/\alpha} \qquad \text{a.s.}$$

for all $\lambda_1' < \lambda_1$ and $\lambda_2' > \lambda_2$. Taking the limits on both sides, we obtain

$$(\lambda_1)^{1/\alpha} \le \liminf_{t\to 0} \frac{\|X\|_t}{b_1(t)} \le (\lambda_2)^{1/\alpha} \qquad \text{a.s.}$$

Applying the regular variation argument in the reverse direction yields the claim. □

Proof of Corollary 2.2. This follows directly from Theorem 2.1. The bounds on the constants can be obtained from the absolute constants in Proposition 2.1. □

Proof of Theorem 2.2. Lemma 4.3 gives the lower LIL of the theorem. Unfortunately, the arguments for the proof of Theorem 2.1 do not apply here. Hence, for the reverse direction, we show more directly that the given norming function of the LIL implies the rate function of the small deviations. The following arguments go back to Kesten. The proof is via contradiction, assuming that

$$\liminf_{t\to 0} \frac{\|X\|_t}{b_{\lambda_2'}(t)} > \frac{2}{C} + \delta \tag{4.11}$$

for some $\delta > 0$ and $\lambda_2' > \lambda_2$. We show that under this assumption we can derive, for sufficiently large l, the estimates

$$1 \ge \sum_{n\ge l} \mathbb{P}\left(\frac{\|X\|_{r^j - r^n}}{b_{\lambda_2'}(r^j - r^n)} > \frac{2}{C}; \text{for all } l \le j \le n-1 \right) \mathbb{P}\big(\|X\|_{r^n} \le b_{\lambda_2'}(r^n)\big) \tag{4.12}$$

$$\ge \frac{1}{2} \sum_{n\ge l} \mathbb{P}\big(\|X\|_{r^n} \le b_{\lambda_2'}(r^n)\big) \tag{4.13}$$

which is a contradiction as, by the choice of $b_{\lambda_2'}$ and the small deviation rate (2.6), the sum in (4.13) is infinite. First, let us derive estimate (4.12) for which Assumption (4.11) is not needed.

For any fixed integer l partitioning the probability space, we obtain

$$1 \geq \sum_{n\geq l} \mathbb{P}\big(\|X\|_{r^j} > b_{\lambda_2'}(r^j) \text{ for all } l \leq j \leq n-1;\ \|X\|_{r^n} \leq b_{\lambda_2'}(r^n)\big)$$

$$\geq \sum_{n\geq l} \mathbb{P}\Big(\sup_{r^n \leq s < r^j} |X_s| > b_{\lambda_2'}(r^j) \text{ for all } l \leq j \leq n-1;\ \|X\|_{r^n} \leq b_{\lambda_2'}(r^n)\Big).$$

In order to employ the independence of increments of X we estimate from below by

$$\sum_{n\geq l} \mathbb{P}\Big(\sup_{r^n \leq s < r^j} |X_s - X_{r^n}| > 2b_{\lambda_2'}(r^j) \text{ for all } l \leq j \leq n-1;\ \|X\|_{r^n} \leq b_{\lambda_2'}(r^n)\Big)$$

which equals

$$\sum_{n\geq l} \mathbb{P}\big(\|X\|_{r^j - r^n} > 2b_{\lambda_2'}(r^j) \text{ for all } l \leq j \leq n-1\big)\mathbb{P}\big(\|X\|_{r^n} \leq b_{\lambda_2'}(r^n)\big)$$

$$= \sum_{n\geq l} \mathbb{P}\bigg(\frac{\|X\|_{r^j - r^n}}{b_{\lambda_2'}(r^j - r^n)} > 2\frac{b_{\lambda_2'}(r^j)}{b_{\lambda_2'}(r^j - r^n)} \text{ for all } l \leq j \leq n-1\bigg)\mathbb{P}\big(\|X\|_{r^n} \leq b_{\lambda_2'}(r^n)\big).$$

By the monotonicity of $b_{\lambda_2'}$, this yields the lower bound

$$\sum_{n\geq l} \mathbb{P}\bigg(\frac{\|X\|_{r^j - r^n}}{b_{\lambda_2'}(r^j - r^n)} > 2\frac{b_{\lambda_2'}(r^j)}{b_{\lambda_2'}(r^j - r^{j+1})}; \text{ for all } l \leq j \leq n-1\bigg)\mathbb{P}\big(\|X\|_{r^n} \leq b_{\lambda_2'}(r^n)\big).$$

Finally, we utilize the regularity of $b_{\lambda_2'}$ from (2.8) to obtain the lower bound

$$\sum_{n\geq l} \mathbb{P}\bigg(\frac{\|X\|_{r^j - r^n}}{b_{\lambda_2'}(r^j - r^n)} > \frac{2}{C}; \text{ for all } l \leq j \leq n-1\bigg)\mathbb{P}\big(\|X\|_{r^n} \leq b_{\lambda_2'}(r^n)\big).$$

As required, we derived Estimate (4.12).

Assuming (4.11) we now derive Estimate (4.13). The assumption directly shows that

$$\lim_{t\to 0} \mathbb{P}\bigg(\bigcap_{s\leq t} \{\|X\|_s \geq 2C^{-1} b_{\lambda_2'}(s)\}\bigg) = 1$$

which implies that we may choose l large enough such that

$$\mathbb{P}\bigg(\frac{\|X\|_{r^j - r^n}}{b_{\lambda_2'}(r^j - r^n)} > \frac{2}{C}; \text{ for all } l \leq j \leq n-1\bigg) \geq \mathbb{P}\bigg(\bigcap_{s\leq r^l} \{\|X\|_s \geq 2C^{-1} b_{\lambda_2'}(s)\}\bigg) \geq \frac{1}{2}.$$

Hence, we derived estimate (4.13) so that the proof is complete. □

Proof of Corollary 2.3. This is completely analogous to the proof of Corollary 2.1. □

Proof of Corollary 2.4. We use Proposition 2.1 and Theorem 2.1. In order to do so, we have to see that the term $\varepsilon u_\varepsilon$ in (2.12) has no influence on the order. We apply Lemma 4.3 and follow the proof of Theorem 2.2 with the scaling

$$t = r^n \quad \text{and} \quad \varepsilon = b(r^n)$$

and with the sequence n^{-n^β}, respectively. Therefore, it is sufficient to show that

$$\varepsilon u_\varepsilon = \mathrm{o}(tF(\varepsilon))$$

with the above scalings of t and ε. Since $\varepsilon = b(t)$ and thus $t \sim F(\varepsilon)^{-1} \log\log F(\varepsilon)$, we need to show that

$$\varepsilon u_\varepsilon = \mathrm{o}(\log\log F(\varepsilon)).$$

As this is precisely what we stated in condition (2.11), the proof is complete. □

Proof of Proposition 2.2. As X is of bounded variation, the representation

$$X_t = A_t^1 - A_t^2 + ct$$

holds with two independent pure jump subordinators A^1, A^2. Next, we use the simple observation

$$\frac{|X_t|}{t} \le \frac{\|X\|_t}{t} \le \frac{\|A^1\|_t + \|A^2\|_t + |c|t}{t} = \frac{A_t^1}{t} + \frac{A_t^2}{t} + |c|$$

to conclude the proof. The left-hand side converges to $|c|$, as X has bounded variation (see Theorem 39 of Doney [10]). Finally, the right-hand side converges to $|c|$ as $|A_t^i|/t$ converge at zero almost surely to their drift (see Proposition 5 of Doney [10]). □

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