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A numerical scheme for backward doubly stochastic differential equations

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In this paper we propose a numerical scheme for the class of backward doubly stochastic differential equations (BDSDEs) with possible path-dependent terminal values. We prove that our scheme converges in the strong L^2-sense and derives its rate of convergence. As an intermediate step we derive an L^2-type regularity of the solution to such BDSDEs. Such a notion of regularity, which can be thought of as the modulus of continuity of the paths in an L^2-sense, is new.

Keywords: backward doubly SDEs; L^∞-Lipschitz functionals; L^2-regularity; numerical scheme; regression estimation

1. Introduction

In this paper we are interested in the following backward doubly stochastic differential equations (BDSDEs, in short):

$$Y_t = \xi + \int_t^T f(s, X_s, Y_s, Z_s)\,\mathrm{d}s + \int_t^T g(s, X_s, Y_s)\,\overleftarrow{\mathrm{d}B_s} - \int_t^T Z_s\,\mathrm{d}W_s, \tag{1.1}$$

where W and B are two independent Brownian motions defined on $(\Omega, \mathcal{F}, \mathbf{P})$, a complete probability space. This kind of equation has two different directions of stochastic integrals: standard (forward) stochastic integrals, driven by W, and backward stochastic ones, driven by B. Initiated by Pardoux and Peng [18], BDSDEs are connected to quasi-linear stochastic partial differential equations (SPDEs, in short) in order to derive the Feynman–Kac formula for SPDEs. In this setting, BDSDEs have been extensively studied in the past decade. We refer the readers to the papers of Buckdahn and Ma [7,8], Aman *et al.* [2], Aman [1], Bahlali and Gherbal [3] and references therein for more information on both theory and applications, especially in mathematical finance and stochastic control, for such equations. In contrast, there was little progress made in the direction of the numerical implementation of BDSDEs. In special case of BDSDEs ($g \equiv 0$) called BSDEs, many efforts have been made in this direction as well.

Up to now basically two types of schemes have been considered. Based on the theoretical four-step scheme from Ma *et al.* [15], the first type of numerical algorithms for BSDEs have been developed by Douglas *et al.* [10] and more recently by Milstein and Trekyakov [17]. The main focus of these algorithms is the numerical solution of parabolic PDEs which is related to BSDEs.

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A second type of algorithm works backward through time and tries to tackle the stochastic problem directly. Bally [4] and Chevance [9] were the first to study this type of algorithm with a (hardly implementable) random time partition under strong regularity assumptions. The works of Ma *et al.* [14] and Briand *et al.* [6] are in the same spirit, replacing, however, the Brownian motion by a binary random walk. Recently, Zhang [20] proved a new notion of L^2-regularity on the control part Z of the solution which allowed proof of convergence of this backward approach with deterministic partitions under rather weak regularity assumptions (see Zhang [20], Bouchard and Touzi [5] and Lemor *et al.* [13]) for different algorithms. All numerical schemes provide alternatives to construct algorithms for PDEs. To the best of our knowledge, to date, there has been no discussion in the literature concerning numerical algorithms in the spirit of the last three works cited above in the general case, that is, $g \neq 0$. This constitutes an insufficiency when we know that almost all the deterministic problems in these applied fields (PDEs) have their stochastic counterparts (SPDEs).

In this paper, to fill this void, our goal is to build a numerical scheme following the idea used by Bouchard and Touzi [5] and study its convergence. These results are important from a purely mathematical point of view, and also in application to the world of finance. Particularly, this numerical scheme clears the way for a possible algorithm for determining the price of options on financial assets whose dynamics are a solution of SPDEs.

Similarly to the special case $g \equiv 0$, the main difficulty lies in the approximation of the "martingale integrand" Z. In fact, in a sense the problem often comes down to the path regularity of Z. However, in case $g \neq 0$, this regularity becomes a natural question to ask. Therefore, the first main result in this paper is to derive the path regularity called L^2-regularity for BDSDEs with the terminal value ξ is the form $\Phi(X)$, where X and $\Phi(\cdot)$ are, respectively, a diffusion process and an L^∞-Lipschitz functional (see Section 3 for precise definition). The proof is heavily related to Girsanov's Transformation which exists in the BDSDEs case, only if g does not depend on z.

The above L^2-regularity result allows us to provide the rate of convergence of our numerical scheme which is different from the one constructed in Bouchard and Touzi [5]. Indeed, since BDSDEs have two directions of integrals, our numerical scheme needs, at each step, the conditional expectation with respect to the filtration $\mathcal{F}^\pi_{t_i}$ defined by $\mathcal{F}^\pi_{t_i} = \sigma(X^\pi_{t_j}, j \leq i) \vee \sigma(B_{t_j}, j \leq i)$. However, we obtain the same convergence rate.

The rest of this paper is organized as follows. In Section 2, we introduce some fundamental knowledge and assumptions of BDSDEs. Section 3 is devoted to L^2-regularity results. In Section 4, we built our numerical scheme and prove the rate of convergence. Finally in Section 5, we focus some ideas for the regression approximation and give it a convergence rate.

2. Preliminaries

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a complete probability space and $T > 0$ be fixed throughout this paper. Let $\{W_t, 0 \leq t \leq T\}$ and $\{B_t, 0 \leq t \leq T\}$ be two mutually independent standard Brownian motion processes, with values, respectively, in $\mathbf{R}^d$ and $\mathbf{R}^\ell$, defined on $(\Omega, \mathcal{F}, \mathbf{P})$. Let $\mathcal{N}$ denote the class of $\mathbf{P}$-null sets of $\mathcal{F}$. For each $t \in [0, T]$, we define

$$\widetilde{\mathcal{F}}_t = \mathcal{F}^W_t \vee \mathcal{F}^B_{t,T},$$

where for any process $(\eta_s\colon 0 \le s \le T)$, $\mathcal{F}^\eta_{s,t} = \sigma\{\eta_r - \eta_s, s \le r \le t\} \vee \mathcal{N}$, $\mathcal{F}^\eta_t = \mathcal{F}^\eta_{0,t}$.

We note that since the collection $(\widetilde{\mathcal{F}}_t)_{t\ge 0}$ is neither increasing nor decreasing, it does not constitute a filtration. Therefore, we define the filtration $(\mathcal{F}_t)_{t\ge 0}$ by

$$\mathcal{F}_t = \mathcal{F}^W_t \vee \mathcal{F}^B_T,$$

which contains $\widetilde{\mathcal{F}}_t$, and play a key role in the building of our numerical scheme.

For any real $p \ge 2$ and $k \in \mathbf{N}^*$, let $\mathcal{S}^p(\mathbf{R}^k)$ denote the set of jointly measurable processes $\{X_t\}_{t\in[0,T]}$, taking values in $\mathbf{R}^k$, which satisfy:

(i) $\|X\|_{\mathcal{S}^p} = \mathbf{E}(\sup_{0\le t\le T} |X_t|^p)^{1/p} < +\infty$;
(ii) X_t is $\widetilde{\mathcal{F}}_t$-measurable, for any $t \in [0, T]$.

We denote similarly by $\mathcal{M}^p(\mathbf{R}^k)$ the set of (classes of $\mathrm{d}\mathbf{P} \otimes \mathrm{d}t$ a.e. equal) k-dimensional jointly measurable processes which satisfy:

(i) $\|X\|_{\mathcal{M}^p} = \mathbf{E}[(\int_0^T |X_t|^2 \,\mathrm{d}t)^{p/2}]^{1/p} < +\infty$;
(ii) X_t is $\widetilde{\mathcal{F}}_t$-measurable, for a.e. $t \in [0, T]$.

We denote by:

- $W^{1,\infty}(\mathbf{R}^k)$ the space of all measurable functions $\psi : \mathbf{R}^k \to \mathbf{R}$, such that for some constant $K > 0$ it holds that $|\psi(x) - \psi(y)| \le K|x - y|$, $\forall x, y \in \mathbf{R}^k$;
- $\mathbf{D}$ the space of all càdàg functions defined on $[0, T]$;
- $C^m_b([0, T] \times \mathbf{R}^k)$ the space of all continuous functions (not necessarily bounded) $\psi : [0, T] \times \mathbf{R}^k \to \mathbf{R}$, such that ψ has uniformly bounded derivatives with respect to the spatial variables up to order m. We often denote $C^m_b = C^m_b([0, T] \times \mathbf{R}^k)$ for simplicity, when the context is clear.

Let

$$\begin{aligned}
b &: [0, T] \times \mathbf{R}^d \to \mathbf{R}^d,\\
\sigma &: [0, T] \times \mathbf{R}^d \to \mathbf{R}^{d\times d},\\
f &: [0, T] \times \mathbf{R}^d \times \mathbf{R} \times \mathbf{R}^d \to \mathbf{R},\\
g &: [0, T] \times \mathbf{R}^d \times \mathbf{R} \to \mathbf{R}^\ell
\end{aligned}$$

be the functions satisfying the following assumptions: there exists constant $K > 0$ such that for all $s, s' \in [0, T]$, $x, x' \in \mathbf{R}^d$, $y, y' \in \mathbf{R}$, $z, z' \in \mathbf{R}^d$,

(H1) $|b(s, x) - b(s, x')| + \|\sigma(s, x) - \sigma(s, x')\| \le K|x - x'|$.
(H2) (i) $|f(s, x, y, z) - f(s', x', y', z')|^2 \le K(|s - s'|^2 + |x - x'|^2 + |y - y'|^2 + |z - z'|^2)$,
(ii) $|g(s, x, y) - g(s, x', y')|^2 \le K(|s - s'|^2 + |x - x'|^2 + |y - y'|^2)$.
(H3) $\sup_{0\le t\le T}\{|b(t, 0)| + |\sigma(t, 0)| + |f(t, 0, 0, 0)| + |g(t, 0, 0)|\} \le K$.

Given $\xi \in L^2(\Omega, \widetilde{\mathcal{F}}_T, \mathbf{P}; \mathbf{R}^d)$, denote (X, Y, Z) be the solution to the following FBDSDE:

$$X_t = x + \int_0^t b(s, X_s)\,\mathrm{d}s + \int_0^t \sigma(s, X_s)\,\mathrm{d}W_s, \tag{2.1}$$

$$Y_t = \xi + \int_t^T f(s, X_s, Y_s, Z_s)\,\mathrm{d}s + \int_t^T g(s, X_s, Y_s)\,\overleftarrow{\mathrm{d}B_s} - \int_t^T Z_s\,\mathrm{d}W_s. \tag{2.2}$$

Let us now recall some standard results that appear in the SDEs and BDSDEs literature.

Proposition 2.1 (Karatzas and Shreve [11]). *Assume* (H1) *holds. Then for any initial condition* $x \in \mathbf{R}^d$, *FSDE* (2.1) *has a unique solution* $(X_t)_{0\le t\le T}$ *that belongs to* $\mathcal{S}^p(\mathbf{R}^d)$.

Moreover, for any $p \ge 2$, *there exists a constant* $C_p > 0$, *depending only on* T, K *and* p, *such that*

$$\mathbf{E}\Big(\sup_{0\le t\le T} |X_t|^p\Big) \le C_p\bigg(|x|^p + \int_0^T [|b(t,0)|^p + |\sigma(t,0)|^p]\,\mathrm{d}t\bigg)$$

and

$$\mathbf{E}[|X_t - X_s|^p] \le C_p\mathbf{E}\Big(|x|^p + \sup_{0\le t\le T}|b(t,0)|^p + \sup_{0\le t\le T}|\sigma(t,0)|^p\Big)|t-s|^{p/2}.$$

Proposition 2.2 (Pardoux and Peng [18]). *Under assumption* (H2), *BDSDE* (2.2) *has a unique solution* $(Y_t, Z_t)_{0\le t\le T}$ *in* $\mathcal{S}^p(\mathbf{R}) \times \mathcal{M}^p(\mathbf{R}^d)$.

Moreover, for any $p \ge 2$, *there exists a constant* $C_p > 0$, *depending only on* T, K *and* p, *such that*

$$\mathbf{E}\bigg[\sup_{0\le t\le T}|Y_t|^p + \bigg(\int_0^T |Z_s|^2\,\mathrm{d}s\bigg)^{p/2}\bigg] \le C_p\mathbf{E}\bigg(|\xi|^p + \int_0^T [|f(t,0,0,0)|^p + |g(t,0,0)|^p]\,\mathrm{d}t\bigg)$$

and

$$\begin{aligned}\mathbf{E}[|Y_t - Y_s|^p] \le C_p\mathbf{E}\bigg\{&\Big[|\xi|^p + \sup_{0\le t\le T}|f(t,0,0,0)|^p + \sup_{0\le t\le T}|g(t,0,0)|^p\Big]|t-s|^{p-1}\\ &+ \bigg(\int_s^t |Z_s|^2\,\mathrm{d}s\bigg)^{p/2}\bigg\}.\end{aligned}$$

Remark 2.1. In Proposition 2.2, the existence and uniqueness result needs only a Lipschitz condition on f and g, with respect to variables y and z, uniformly in t and x.

Proposition 2.3 (Stability). *Let* $(X^\varepsilon, Y^\varepsilon, Z^\varepsilon)$ *be the solution to the perturbed FBDSDE* (2.1) *and* (2.2) *in which the coefficients are replaced by* $b^\varepsilon, \sigma^\varepsilon, f^\varepsilon, g^\varepsilon$, *with initial state* x^ε *an terminal value* ξ^ε. *Assume that the assumption* (H1) *and* (H2) *hold for all coefficients* $b^\varepsilon, \sigma^\varepsilon, f^\varepsilon, g^\varepsilon$ *and assume that* $\lim_{\varepsilon\to 0} x^\varepsilon = x$, *and for fixed* (x, y, z),

$$\lim_{\varepsilon\to 0}\mathbf{E}\bigg\{\int_0^T [|b^\varepsilon(t,x) - b(t,x)|^2 + |\sigma^\varepsilon(t,x) - \sigma(t,x)|^2]\,\mathrm{d}t\bigg\} = 0,$$

$$\lim_{\varepsilon\to 0}\mathbf{E}\bigg\{|\xi^\varepsilon - \xi|^2 + \int_0^T [|f^\varepsilon(t,x,y,z) - f(t,x,y,z)|^2 + |g^\varepsilon(t,x,y) - g(t,x,y)|^2]\,\mathrm{d}t\bigg\} = 0.$$

Then, we have

$$\lim_{\varepsilon\to 0}\mathbf{E}\Big\{\sup_{0\le t\le T}|X_t^\varepsilon - X_t|^2 + \sup_{0\le t\le T}|Y_t^\varepsilon - Y_t|^2 + \sup_{0\le t\le T}|Z_t^\varepsilon - Z_t|^2\Big\} = 0.$$

3. L^2-regularity result for BDSDEs

In this section we establish the first main result of this paper, which is L^2-regularity of the martingale integrand Z, and which can be thought of as the modulus of continuity of the paths in an L^2 sense. Such a regularity, combined with the estimate for X and Y, plays a key role for deriving the rate of convergence of our numerical scheme in Section 4. We shall consider a class of BDSDEs with terminal values which are path dependent, that is, of the form $\xi = \Phi(X)$, where a deterministic functional $\Phi : \mathbf{D} \to \mathbf{R}$ satisfies:

(H4) (L^∞-Lipschitz condition). There exists a constant K such that

$$|\Phi(X_1) - \Phi(X_2)| \le K \sup_{0\le t\le T}|X_1(t) - X_2(t)| \qquad \forall X_1, X_2 \in \mathbf{D}. \tag{3.1}$$

(H5) $\Phi(\mathbf{0})$ is bounded by K, where $\mathbf{0}$ denotes the constant function taking value 0 on $[0, T]$.

This approximation, due to Ma and Zhang [16], for L^∞-Lipschitz functional, will be useful in the sequel.

Lemma 3.1. *Suppose* (H4) *and* (H5) *hold. Let* $\Pi = \{\pi\}$ *be a family of partitions of* $[0, T]$. *Then there exists a family of discrete functionals* $\{h^\pi : \pi \in \Pi\}$ *such that*:

(i) *for each* $\pi \in \Pi$, *assuming* π: $0 = t_0 < t_1 < t_2 < \cdots < t_n = T$, *we have that* $h^\pi \in C_b^1(\mathbf{R}^{d(n+1)})$, *and satisfies*

$$\sum_{i=1}^{n}|\partial_{x_i} h^\pi(x)| \le K \qquad \forall x = (x_0, x_1, \ldots, x_n) \in \mathbf{R}^{d(n+1)}, \tag{3.2}$$

where K *is the same constant as that in* (3.1);

(ii) *for any* $X \in \mathbf{D}$, *it holds that*

$$\lim_{|\pi|\to 0}|h^\pi(X_{t_0}, X_{t_1}, \ldots, X_{t_n}) - \Phi(X)| = 0, \tag{3.3}$$

where $|\pi| = \max_{1\le i\le n}|t_i - t_{i-1}|$.

Our main result in this section is the following theorem.

Theorem 3.1. *Assume* (H1)–(H5). *Let* π_0: $s_0 < \ldots, s_m$ *be any partition of* $[0, T]$, *and for each* $1 \le i \le m$, *let us define*

$$\tilde{Z}_{s_{i-1}}^{\pi_0} = \frac{1}{s_i - s_{i-1}}\mathbf{E}\bigg[\int_{s_{i-1}}^{s_i} Z_s \, \mathrm{d}s \Big| \mathcal{F}_{s_{i-1}}\bigg]. \tag{3.4}$$

Then there exists a constant C depending only on T and K, such that

$$\mathbf{E}\left[\max_{1\le i\le m}\sup_{s_{i-1}\le t\le s_i}|Y_t-Y_{s_{i-1}}|^2+\sum_{i=1}^m\int_{s_{i-1}}^{s_i}|Z_s-\tilde{Z}^\pi_{s_{i-1}}|^2\,\mathrm{d}s\right]\le C|\pi_0|. \tag{3.5}$$

In the sequel, let $\pi\colon 0=t_0,\dots,t_n=T$ be any partition of $[0,T]$, finer than Π_0, and, without loss of generality, we assume $s_i=t_{l_i}$ for $i=1,\dots,m$. Since Φ satisfies the L^∞-Lipschitz condition (3.1), by virtue of Lemma 3.1 one can find $h^\pi\in C^1(\mathbf{R}^{d(n+1)})$ satisfying (3.2) and (3.3). Let (Y^π,Z^π) be the solution to the following BDSDE:

$$\begin{aligned}Y^\pi_t&=h^\pi(X_{t_0},\dots,X_{t_n})+\int_t^T f(s,X_s,Y^\pi_s,Z^\pi_s)\,\mathrm{d}s\\&\quad+\int_t^T g(s,X_s,Y^\pi_s)\,\overleftarrow{\mathrm{d}B_s}-\int_t^T Z^\pi_s\,\mathrm{d}W_s.\end{aligned}\tag{3.6}$$

Moreover, setting $\Theta^\pi=(\Xi^\pi,Z^\pi)$, with $\Xi^\pi=(X,Y^\pi)$, let $(\nabla X,\nabla^iY^\pi,\nabla^iZ^\pi)$ be the unique solution of the following FBDSDE:

$$\begin{aligned}\nabla X_t&=I_d+\int_0^t b_x(r,X_r)\nabla X_r\,\mathrm{d}r+\int_0^t\sigma_x(r,X_r)\nabla X_r\,\mathrm{d}W_r,\\ \nabla^iY^\pi_t&=\sum_{j\ge i}^n\frac{\partial h^\pi}{\partial x_j}(X_{t_0},\dots,X_{t_n})\nabla X_{t_j}\\&\quad+\int_t^T[f_x(\Theta^\pi_r)\nabla X_r+f_y(\Theta^\pi_r)\nabla^iY^\pi_r+f_z(\Theta^\pi_r)\nabla^iZ^\pi_r]\,\mathrm{d}r\\&\quad+\int_t^T[g_x(\Xi^\pi_r)\nabla X_r+g_y(\Xi^\pi_r)\nabla^iY^\pi_r)]\,\overleftarrow{\mathrm{d}B}_r\\&\quad-\int_t^T\nabla^iZ^\pi_r\,\mathrm{d}W_r,\qquad t\in[t_i,T],i=0,\dots,n-1.\end{aligned}\tag{3.7}$$

We denote

$$\begin{aligned}\xi^0&=\int_0^T f_x(\Theta^\pi_r)\nabla X_rN_r^{-1}\,\mathrm{d}r+\int_0^T g_x(\Xi^\pi_r)\nabla X_rN_r^{-1}\,\overleftarrow{\mathrm{d}B}_r;\\ \xi^i&=h^\pi(X_{t_0},\dots,X_{t_n})\nabla X_{t_i}N_T^{-1},\qquad i=1,\dots,n,\end{aligned}$$

where

$$\begin{aligned}N_t&=\exp\biggl(\int_0^t f_y(\Theta^\pi_r)\,\mathrm{d}r+\int_0^t g_y(\Xi^\pi_r)\,\overleftarrow{\mathrm{d}B}_r-\frac12\int_0^t|g_y(\Xi^\pi_r)|^2\,\mathrm{d}r\biggr),\\ M_t&=\exp\biggl\{\int_0^t f_z(\Theta^\pi_r)\,\mathrm{d}W_r-\frac12\int_0^t|f_z(\Theta^\pi_r)|^2\,\mathrm{d}r\biggr\}.\end{aligned}\tag{3.8}$$

The following technical lemma is the building block of the proof of Theorem 3.1.

Lemma 3.2. *Let us consider the partition π defined above and h^π given by Lemma 3.1, and assume $\sigma, b, f, g, \in C_b^1$. Then for all $i = 1, \dots, n$*

$$\nabla^i Y_t^\pi = \Big(\xi_t^0 + \sum_{j \ge i} \xi_t^j\Big) M_t^{-1} N_t - \int_0^t f_x(\Theta_r^\pi) \nabla X_r N_r^{-1}\, \mathrm{d}r N_t - \int_0^t g_x(\Xi_r^\pi) \nabla X_r N_r^{-1}\, \overleftarrow{\mathrm{d}B}_r N_t,$$

where $\xi_t^j = \mathbf{E}(M_T \xi^j | \mathcal{F}_t)$, $j = 0, \dots, n$.

Proof. For each $0 \le i \le n$, we recall $(\nabla X, \nabla^i Y^\pi, \nabla^i Z^\pi)$, the solution of the linear FBDSDE (3.7). Let (γ^0, ζ^0) and (γ^j, ζ^j), $j = 1, \dots, n$, be the solution of the BDSDEs

$$\begin{aligned}
\gamma_t^0 &= \int_t^T [f_x(\Theta_r^\pi)\nabla X_r + f_y(\Theta_r^\pi)\gamma_r^0 + f_z(\Theta_r^\pi)\zeta_r^0]\,\mathrm{d}r \\
&\quad + \int_t^T [g_x(\Xi_r^\pi)\nabla X_r + g_y(\Xi_r^\pi)\gamma_r^0]\,\overleftarrow{\mathrm{d}B}_r - \int_t^T \zeta_r^0\,\mathrm{d}W_r; \\
\gamma_t^j &= \frac{\partial h^\pi}{\partial x_j}(X_{t_0}, \dots, X_{t_n})\nabla X_{t_j} + \int_t^T [f_y(\Theta_r^\pi)\gamma_r^j + f_z(\Theta_r^\pi)\zeta_r^j]\,\mathrm{d}r \\
&\quad + \int_t^T g_y(\Xi_r^\pi)\gamma_r^j\,\overleftarrow{\mathrm{d}B}_r - \int_t^T \zeta_r^j\,\mathrm{d}W_r,
\end{aligned} \tag{3.9}$$

respectively; then we have the following decomposition:

$$\nabla^i Y_s^\pi = \gamma_s^0 + \sum_{j=i}^n \gamma_s^j, \qquad s \in [t_{i-1}, t_i). \tag{3.10}$$

Recall (3.8), and, since f_y, f_z and g_y are uniformly bounded, by Girsanov's Theorem (see, e.g., Karatzas and Shreve [11]), we know that M is a $\mathbf{P}$-martingale on $[0, T]$, and $\widetilde{W}_t = W_t - \int_0^t f_z(\Theta_r^\pi)\,\mathrm{d}r$, $t \in [0, T]$, is an $\mathcal{F}_t$-Brownian motion on the new probability space $(\Omega, \mathcal{F}, \widetilde{\mathbf{P}})$, where $\widetilde{\mathbf{P}}$ is defined by $\frac{\mathrm{d}\widetilde{\mathbf{P}}}{\mathrm{d}\mathbf{P}} = M_T$.

Now for $0 \le i \le n$, define

$$\widetilde{\gamma}_t^i = \gamma_t^i N_t^{-1}, \qquad \widetilde{\zeta}_t^i = \zeta_t^i N_t^{-1}, \qquad t \in [0, T].$$

Then, using integration by parts and equation (3.9) we have

$$\widetilde{\gamma}_t^i = \xi^i - \int_t^T \widetilde{\zeta}_r^j\,\mathrm{d}\widetilde{W}_r, \qquad t \in [0, T].$$

Therefore, by the Bayes rule (see, e.g., Karatzas and Shreve [11], Lemma 3.5.3) we have, for $t \in [0, T]$,

$$\gamma_t^i = \widetilde{\gamma}_t^i N_t = \mathbf{E}(M_T \xi^i | \mathcal{F}_t) M_t^{-1} N_t = \xi_t^i M_t^{-1} N_t,$$

where for $i = 0, \dots, n$,

$$\xi_t^i = \mathbf{E}\{M_T \xi^i | \mathcal{F}_t\} = \mathbf{E}(M_T \xi^i) + \int_0^t \bar{\eta}_s^i \, \mathrm{d}W_s. \tag{3.11}$$

Recalling (3.18), $M_T \in L^p(\mathcal{F}_T)$ and $\nabla X \in L^p(\mathbf{F})$ for all $p \geq 2$.

Therefore for each $p \geq 1$, (3.2) leads to

$$\mathbf{E}\left\{\sum_{j=1}^n |M_T \xi^j|\right\}^p \leq C_p \mathbf{E}\Big\{|M_T|^p \sup_{0\leq t\leq T} |\nabla X_t|^p\Big\}. \tag{3.12}$$

In particular, for each j, $M_T\xi^j \in L^2(\mathcal{F}_T)$, thus (3.11) makes sense. Finally the result follows by (3.10). $\square$

Proof of Theorem 3.1. Let us recall the partition π_0 defined above, and consider $|\pi_0|$, its mesh, defined by

$$|\pi_0| = \max_{0\leq i\leq m} |s_i - s_{i-1}|.$$

Applying Proposition 2.2, we get

$$\mathbf{E}(|Y_t - Y_{s_{i-1}}|^2) \leq C|\pi_0|, \qquad t \in [s_{i-1}, s_i), i = 1, \dots, m,$$

which, together with Burkölder–Davis–Gundy inequality, implies

$$\mathbf{E}\Big[\max_{1\leq i\leq m} \sup_{s_{i-1}\leq t\leq s_i} |Y_t - Y_{s_{i-1}}|^2\Big] \leq C|\pi_0|. \tag{3.13}$$

The estimate for Z is a little involved. This part will be divide in two steps.

Step 1. First we assume that $b, \sigma, f, \ g \in C_b^1$. It follows from Lemma 3.1, together with Proposition 2.3, that

$$\lim_{|\pi_0|\to 0} \mathbf{E}\left\{\sup_{0\leq t\leq T} |Y_t^\pi - Y_t|^2 + \int_0^T |Z_t^\pi - Z_t|^2 \, \mathrm{d}t\right\} = 0. \tag{3.14}$$

On the other hand, according to (3.4), $\tilde{Z}_{s_{i-1}}^{\pi_0} \in L^2(\Omega, \mathcal{F}_{s_{i-1}})$. Then since $Z_{s_{i-1}}^\pi \in L^2(\Omega, \mathcal{F}_{s_{i-1}})$, it follows from Lemma 3.4.2 of Zang [19], page 71, that

$$\begin{aligned}
\mathbf{E}\left[\sum_{i=1}^m \int_{s_{i-1}}^{s_i} |Z_s - \tilde{Z}_{s_{i-1}}^{\pi_0}|^2 \, \mathrm{d}s\right] &\leq \mathbf{E}\left[\sum_{i=1}^m \int_{s_{i-1}}^{s_i} |Z_s - Z_{s_{i-1}}^\pi|^2 \, \mathrm{d}s\right] \\
&\leq 2\mathbf{E}\left[\sum_{i=1}^m \int_{s_{i-1}}^{s_i} (|Z_s - Z_s^\pi|^2 + |Z_s^\pi - Z_{s_{i-1}}^\pi|^2) \, \mathrm{d}s\right] \qquad (3.15) \\
&\leq C|\pi_0| + \mathbf{E}\left[\sum_{i=1}^m \int_{s_{i-1}}^{s_i} |Z_s^\pi - Z_{s_{i-1}}^\pi|^2 \, \mathrm{d}s\right].
\end{aligned}$$

By (3.14) and (3.15), it remains to prove that

$$\sum_{i=1}^{m} \mathbf{E}\left[\int_{s_{i-1}}^{s_i} |Z_s^\pi - Z_{s_{i-1}}^\pi|^2 \,\mathrm{d}s\right] \le C|\pi_0|, \tag{3.16}$$

where C is independent of π or π_0. Now we fix i_0. For $t \in [s_{i_0-1}, s_{i_0})$, it follows from Proposition 2.3 of Pardoux and Peng [18], together with Lemma 3.2,

$$\begin{aligned} Z_t^\pi &= \left[\left(\xi_t^0 + \sum_{j\ge i} \xi_t^j\right) M_t^{-1} - \int_0^t f_x(\Theta_r^\pi)\nabla X_r N_r^{-1}\,\mathrm{d}r - \int_0^t g_x(\Xi_r^\pi)\nabla X_r N_r^{-1}\,\overleftarrow{\mathrm{d}B_r}\right] \\ &\quad \times N_t[\nabla X_t]^{-1}\sigma(X_t). \end{aligned}$$

Therefore,

$$|Z_t^\pi - Z_{s_{i_0-1}}^\pi| \le I_t^1 + I_t^2 + I_t^3 + I_t^4, \tag{3.17}$$

where (recalling that $s_{i_0-1} = t_{l_{i_0-1}}$)

$$\begin{aligned} I_t^1 &= \left|\left[\xi_t^0 + \sum_{j\ge i}\xi_t^j\right] - \left[\xi_{s_{i_0-1}}^0 + \sum_{j\ge l_{i_0-1}+1} \xi_{s_{i_0-1}}^j\right]\right| \times |M_{s_{i_0-1}}^{-1} N_{s_{i_0-1}}[\nabla X_{s_{i_0-1}}]^{-1}\sigma(X_{s_{i_0-1}})|, \\ I_t^2 &= \left|\xi_t^0 + \sum_{j\ge i}\xi_t^j\right| |M_t^{-1}N_t[\nabla X_t]^{-1}\sigma(X_t) - M_{s_{i_0-1}}^{-1} N_{s_{i_0-1}}[\nabla X_{s_{i_0-1}}]^{-1}\sigma(X_{s_{i_0-1}})|, \\ I_t^3 &= |A_t^1|, \\ I_t^4 &= |A_t^2| \end{aligned}$$

with

$$\begin{aligned} A_t^1 &= \left(\int_0^t f_x(\Theta_r^\pi)\nabla X_r N_r^{-1}\,\mathrm{d}r\right) N_t[\nabla X_t]^{-1}\sigma(X_t) \\ &\quad - \left(\int_0^{s_{i_0-1}} f_x(\Theta_r^\pi)\nabla X_r N_r^{-1}\,\mathrm{d}r\right) N_{s_{i_0-1}}[\nabla X_{s_{i_0-1}}]^{-1}\sigma(X_{s_{i_0-1}}) \end{aligned}$$

and

$$\begin{aligned} A_t^2 &= \left(\int_0^t g_x(\Xi_r^\pi)\nabla X_r N_r^{-1}\,\overleftarrow{\mathrm{d}B_r}\right) N_t[\nabla X_t]^{-1}\sigma(X_t) \\ &\quad - \left(\int_0^{t_{s_0-1}} g_x(\Xi_r^\pi)\nabla X_r N_r^{-1}\,\overleftarrow{\mathrm{d}B_r}\right) N_{s_{i_0-1}}[\nabla X_{s_{i_0-1}}]^{-1}\sigma(X_{s_{i_0-1}}). \end{aligned}$$

Recalling (3.8), and noting that f_y, f_z and g_y are uniformly bounded, one can deduce that, for all $p \geq 1$, there exists a constant C_p depending only on T, K and p, such that

$$\begin{aligned}
&\mathbf{E}\Big(\sup_{0\leq t\leq T} |N_t|^p + |N_t^{-1}|^p\Big) \leq C_p; \qquad \mathbf{E}\Big(\sup_{0\leq t\leq T} [|M_t|^p + |M_t^{-1}|^p]\Big) \leq C_p;\\
&\mathbf{E}(|N_t - N_s|^p + |N_t^{-1} - N_s^{-1}|^p) \leq C_p|t-s|^{p/2};\\
&\mathbf{E}(|M_t - M_s|^p + |M_t^{-1} - M_s^{-1}|^p) \leq C_p|t-s|^{p/2}.
\end{aligned} \tag{3.18}$$

Thus, applying Proposition 2.1 and 2.2, one can show that

$$\mathbf{E}(|I_t^3|^2) \leq C|\pi_0|, \tag{3.19}$$

and

$$\mathbf{E}(|I_t^4|^2) \leq C|\pi_0|. \tag{3.20}$$

Recalling (3.11) and (3.2) we have

$$\Big|\xi_t^0 + \sum_{j\geq i} \xi_t^j\Big| \leq C\mathbf{E}\Big\{\sup_{0\leq t\leq T} \nabla X_t \big| \mathcal{F}_t\Big\}.$$

Thus by using again Propositions 2.1 and 2.2 together with (3.18), we get

$$\mathbf{E}(|I_t^2|^2) \leq C|\pi_0|. \tag{3.21}$$

As proved in Zhang [20] (see the proof of Theorem 3.1), we have

$$\mathbf{E}(|I_t^1|^2) \leq C|\pi_0|. \tag{3.22}$$

Combining (3.19), (3.20), (3.21) and (3.22), we deduce from (3.17) that (3.16) holds, which ends the proof for the smooth case.

Step 2. Let us consider the general case, that is, b, σ, f, g are only Lipschitz. For $\varphi = b, \sigma, f, g$, it not difficult to construct via a convolution method, for any $\varepsilon > 0$, the function $\varphi^\varepsilon \in C_b^1$ be the smooth mollifiers of φ such that the derivatives of φ^ε are uniformly bounded by K and $\lim_{\varepsilon\to 0} \varphi^\varepsilon = \varphi$. Let $(X^\varepsilon, Y^\varepsilon, Z^\varepsilon)$ and $(X^\varepsilon, Y^{\pi,\varepsilon}, Z^{\pi,\varepsilon})$ denote the solution to corresponding FBDSDE replaced φ by φ^ε, and set

$$\begin{aligned}
N_t^\varepsilon &= \exp\Big(\int_0^t f_y^\varepsilon(\Theta_r^{\pi,\varepsilon})\,\mathrm{d}r + \int_0^t g_y^\varepsilon(\Xi_r^{\pi,\varepsilon})\,\overleftarrow{\mathrm{d}B}_r - \frac{1}{2}\int_0^t |g_y^\varepsilon(\Xi_r^{\pi,\varepsilon})|^2\,\mathrm{d}r\Big),\\
M_t^\varepsilon &= \exp\Big\{\int_0^t f_z^\varepsilon(\Theta_r^{\pi,\varepsilon})\,\mathrm{d}W_r - \frac{1}{2}\int_0^t |f_z^\varepsilon(\Theta_r^{\pi,\varepsilon})|^2\,\mathrm{d}r\Big\}.
\end{aligned}$$

Then one can derive, since the function f_y^ε, f_z^ε and g_y^ε are uniformly bounded by K, with the standard calculus about BSDEs, that, for all $p \geq 1$, there exists a constant C_p independent on ε

(depending only on T, K and p), such that

$$\mathbf{E}\Big(\sup_{0\le t\le T}|N_t^\varepsilon|^p+|(N^\varepsilon)_t^{-1}|^p\Big)\le C_p;\qquad \mathbf{E}\Big(\sup_{0\le t\le T}[|M_t^\varepsilon|^p+|(M^\varepsilon)_t^{-1}|^p]\Big)\le C_p;$$

$$\mathbf{E}\big(|N_t^\varepsilon-N_s^\varepsilon|^p+|(N^\varepsilon)_t^{-1}-(N^\varepsilon)_s^{-1}|^p\big)\le C_p|t-s|^{p/2};$$

$$\mathbf{E}\big(|M_t^\varepsilon-M_s^\varepsilon|^p+|(M^\varepsilon)_t^{-1}-(M^\varepsilon)_s^{-1}|^p\big)\le C_p|t-s|^{p/2}.$$

Next, define

$$\tilde{Z}_{s_{i-1}}^{\varepsilon,\pi_0}=\frac{1}{s_i-s_{i-1}}\mathbf{E}\bigg[\int_{s_{i-1}}^{s_i}Z_s^\varepsilon\,\mathrm{d}s\Big|\mathcal{F}_{s_{i-1}}\bigg].$$

Then it follows from Step 1 that

$$\sum_{i=1}^m\int_{s_{i-1}}^{s_i}|Z_s^\varepsilon-\tilde{Z}_{t_{i-1}}^{\pi_0,\varepsilon}|^2\,\mathrm{d}s\le C|\pi_0|.$$

Therefore using again Lemma 3.4.2 of Zang [19], page 71, we obtain

$$\begin{aligned}\sum_{i=1}^m\mathbf{E}\bigg[\int_{s_{i-1}}^{s_i}|Z_s^\pi-\tilde{Z}_{s_{i-1}}^{\pi_0}|^2\,\mathrm{d}s\bigg]&\le\sum_{i=1}^m\mathbf{E}\bigg[\int_{s_{i-1}}^{s_i}|Z_s^\pi-\tilde{Z}_{s_{i-1}}^{\pi_0,\varepsilon}|^2\,\mathrm{d}s\bigg]\\&\le\sum_{i=1}^m\mathbf{E}\bigg[\int_{s_{i-1}}^{s_i}[|Z_s-Z_s^\varepsilon|^2+|Z_s^\varepsilon-\tilde{Z}_{t_{i-1}}^{\pi_0,\varepsilon}|^2]\,\mathrm{d}s\bigg]\qquad(3.23)\\&\le\mathbf{E}\bigg[\int_0^T|Z_s-Z_s^\varepsilon|^2\,\mathrm{d}s\bigg]+C|\pi_0|.\end{aligned}$$

Applying Proposition 2.3, we have

$$\lim_{\varepsilon\to0}\mathbf{E}\bigg[\int_0^T|Z_s-Z_s^\varepsilon|^2\,\mathrm{d}s\bigg]=0,$$

which, combined with (3.23), proves the theorem. □

4. Numerical scheme and rate of convergence

In this section, we consider the BDSDE (2.2) in the special case $\Phi(X)=h(X_T)$, where $h\in W^{1,\infty}(\mathbf{R}^d)$, such that $h(0)$ is bounded by K. The goal of this section is to construct an approximation of the solution (X,Y,Z) by using the "step processes." Let us recall $\pi\colon t_0<t_1<\cdots<t_n=T$ the partition of $[0,T]$ and $|\pi|=\max_{1\le i\le n}|\Delta_i^\pi|$, with $\Delta_i^\pi=t_i-t_{i-1}$. We set also $\Delta^\pi W_i=W_{t_i}-W_{t_{i-1}}$, $\Delta^\pi B_i=B_{t_i}-B_{t_{i-1}}$ and for all $0\le i\le n$ define

$$\mathcal{F}_i^\pi=\sigma(X_{t_j},j\le i)\vee\mathcal{F}_{t_i}^B,$$

the discrete-time filtration. Let us briefly review the Euler scheme for the forward diffusion X. Define $\pi(t) = t_{i-1}$, for $t \in [t_{i-1}, t_i)$. Let X^π be the solution of the following SDE:

$$X_t^\pi = x + \int_0^t b\big(\pi(s), X_{\pi(s)}^\pi\big)\,\mathrm{d}s + \int_0^t \sigma\big(\pi(s), X_{\pi(s)}^\pi\big)\,\mathrm{d}W_s, \tag{4.1}$$

and we define a "step process" $\hat{X}^\pi$ as follows:

$$\hat{X}_t^\pi = X_{\pi(t)}^\pi, \qquad t \in [0, T]. \tag{4.2}$$

The following estimate is well known (see, e.g., Kloeden and Platen [12]).

Proposition 4.1. *Assume b and σ satisfy the assumptions* (H1) *and* (H3). *Then there exists a constant C depending only on T and K, such that*

$$\max_{1 \le i \le n} \mathbf{E}\Big[\sup_{0 \le t \le T} |X_t^\pi - X_t|^2 + \sup_{t_{i-1} \le t \le t_i} |X_t - X_{t_{i-1}}|^2\Big] \le C|\pi|.$$

Moreover, we get the following estimate involving the step process $\hat{X}^\pi$ due to Zhang [20].

Proposition 4.2. *Assume b and σ satisfy the assumptions* (H1) *and* (H3). *Then there exists a constant C depending only on T and K, such that*

$$\sup_{0 \le t \le T} \mathbf{E}[|\hat{X}_t^\pi - X_t|^2] \le C|\pi|;$$

$$\mathbf{E}\Big[\sup_{0 \le t \le T} |\hat{X}_t^\pi - X_t|^2\Big] \le C|\pi| \log\Big(\frac{1}{|\pi|}\Big).$$

The backward component (Y, Z) will be approximated by the following numerical scheme:

$$Y_{t_n}^\pi = h(X_T^\pi), \qquad Z_{t_n}^\pi = 0,$$

$$Z_{t_{i-1}}^\pi = \frac{1}{\Delta_i^\pi} \mathbf{E}_{i-1}^\pi[\tilde{Y}_{t_i}^\pi \Delta^\pi W_i], \tag{4.3}$$

$$Y_{t_{i-1}}^\pi = \mathbf{E}_{i-1}^\pi[\tilde{Y}_{t_i}^\pi] + f(t_{i-1}, X_{t_{i-1}}^\pi, Y_{t_{i-1}}^\pi, Z_{t_{i-1}}^\pi)\Delta_i^\pi, \tag{4.4}$$

where $\mathbf{E}_i^\pi[\cdot] = \mathbf{E}[\cdot|\mathcal{F}_i^\pi]$ and $\tilde{Y}_{t_i}^\pi = Y_{t_i}^\pi + g(t_i, X_{t_i}^\pi, Y_{t_i}^\pi)\Delta^\pi B_i$.

Remark 4.1.

(i) Our approximation scheme differs from the one appearing in Bouchard and Touzi [5]. Indeed, actually we use the conditional expectation with respect the enlarged filtration $\sigma(X_j, j \le i) \vee \mathcal{F}_{t_i}^B$, which is necessary to extend Itô's representation theorem for backward doubly SDE (see Pardoux and Peng [18]).

(ii) The backward component and the associated control (Y, Z), which solves the backward doubly SDE, can be expressed as a function of X and B, that is, $(Y_t, Z_t) = (u(t, B_t, X_t), v(t, B_t, X_t))$, for some deterministic functions u and v. Then, the conditional expectations, involved in the above discretization scheme, reduce to the regression of $\tilde{Y}^\pi_{t_i}$ and $\tilde{Y}^\pi_{t_i}(W_{t_i} - W_{t_{i-1}})$ on the random variable $(X^\pi_{t_{i-1}}, B_{t_{i-1}})$.

Next, for all $0 \le i \le n$, on can show that $\tilde{Y}^\pi_{t_i}$ belongs to $L^2(\Omega, \mathcal{F}_{t_i})$, thus an obvious extension of Itô martingale representation theorem yields the existence of the $(\mathcal{F}_s)_{s\in[t_{i-1},t_i)}$-jointly measurable and square integrable process $\bar{Z}^\pi$ satisfying

$$\tilde{Y}^\pi_{t_i} = \mathbf{E}[\tilde{Y}^\pi_{t_i} | \mathcal{F}^\pi_{i-1}] + \int_{t_{i-1}}^{t_i} \bar{Z}^\pi_s \, \mathrm{d}W_s. \tag{4.5}$$

Therefore we define the following continuous version:

$$\begin{aligned} Y^\pi_t = {} & Y^\pi_{t_{i-1}} - (t - t_{i-1}) f(t_{i-1}, X^\pi_{t_{i-1}}, Y^\pi_{t_{i-1}}, Z^\pi_{t_{i-1}}) - g(t_i, X^\pi_{t_i}, Y^\pi_{t_i})(B_t - B_{t_{i-1}}) \\ & + \int_{t_{i-1}}^{t} \bar{Z}^\pi_s \, \mathrm{d}W_s, \qquad t_{i-1} < t \le t_i. \end{aligned} \tag{4.6}$$

Note that the process $\bar{Z}^\pi$ is given by the representation theorem; thus it is useful and even necessary to find a relationship with Z^π, defined by (4.3). We have:

Lemma 4.1. *Assume b, σ, f and g satisfy the assumptions* (H1), (H2) *and* (H3) *and let $h \in W^{1,\infty}(\mathbf{R}^d)$ such that $h(0)$ is bounded by K. Then for all $1 \le i \le n$, we have*

$$Z^\pi_{t_{i-1}} = \frac{1}{\Delta^\pi_i} \mathbf{E}^\pi_{i-1}\left[\int_{t_{i-1}}^{t_i} \bar{Z}^\pi_s \, \mathrm{d}s\right].$$

Proof. Let us recall

$$\Delta^\pi_i Z^\pi_{t_{i-1}} = \frac{1}{\Delta^\pi_i} \mathbf{E}^\pi_{i-1}\big[\big(Y^\pi_{t_i} + g(t_i, X^\pi_{t_i}, Y^\pi_{t_i}) \Delta^\pi B_i\big) \Delta^\pi W_i\big].$$

Then it follows from (4.5) that

$$Z^\pi_{t_{i-1}} = \frac{1}{\Delta^\pi_i} \mathbf{E}^\pi_{i-1}\left[\Delta^\pi W_i \int_{t_{i-1}}^{t_i} \bar{Z}^\pi_s \, \mathrm{d}W_s\right].$$

The result follows by Itô's isometry. □

We also need the following, which is the particular case of Theorem 3.1.

Lemma 4.2. *Assume b, σ, f and g satisfy the assumptions* (H1), (H2) *and* (H3), *and let $h \in W^{1,\infty}(\mathbf{R}^d)$ such that $h(0)$ is bounded by K. Let define, for each $1 \le i \le n$,*

$$\tilde{Z}^{\pi}_{t_{i-1}} = \frac{1}{\Delta^{\pi}_i} \mathbf{E}^{\pi}_{i-1}\left[\int_{t_{i-1}}^{t_i} Z_s \,\mathrm{d}s\right].$$

Then there exists a constant C depending only on T and K, such that

$$\mathbf{E}\left[\max_{1\le i\le n} \sup_{t_{i-1}\le t\le t_i} |Y_t - Y_{t_{i-1}}|^2 + \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} |Z_s - \tilde{Z}^{\pi}_{t_{i-1}}|^2 \,\mathrm{d}s\right] \le C|\pi|. \tag{4.7}$$

We are now ready to state our main result of this section, which provides the rate of convergence of the numerical scheme (4.3)–(4.4).

Theorem 4.1. *Assume b, σ, f and g satisfy assumptions* (H1), (H2) *and* (H3), *and let $h \in W^{1,\infty}(\mathbf{R}^d)$ such that $h(0)$ is bounded by K. Then there exists a constant C depending only on T and K, such that*

$$\sup_{0\le t\le T} \mathbf{E}|Y_t - Y^{\pi}_t|^2 + \mathbf{E}\left[\int_0^T |Z_s - \hat{Z}^{\pi}_s|^2 \,\mathrm{d}s\right] \le C|\pi|.$$

Proof. The proof follows the steps of the proof of Theorem 3.1 in Bouchard and Touzi [5], so we will only outline. In the sequel, $C > 0$ will denote the generic constant independent of i and n, and may vary line to line. For $i \in \{0, \dots, n-1\}$, we set

$$\delta^{\pi} Y_t = Y_t - Y^{\pi}_t, \qquad \delta^{\pi} Z_t = Z_t - \bar{Z}^{\pi}_t, \qquad \delta^{\pi} f(t) = f(t, X_t, Y_t, Z_t) - f(t_i, X^{\pi}_{t_i}, Y^{\pi}_{t_i}, Z^{\pi}_{t_i})$$

and

$$\delta^{\pi} g(t) = g(t, X_t, Y_t,) - g(t_{i+1}, X^{\pi}_{t_{i+1}}, Y^{\pi}_{t_{i+1}}), \qquad t \in [t_i, t_{i+1}).$$

By Itô's formula, it follows from a Lipschitz condition on f, g and h, together with the inequality $ab \le \beta a^2 + b^2/\beta$, that

$$\begin{aligned} V_t &= \mathbf{E}|\delta^{\pi} Y_t|^2 + \mathbf{E}\int_t^{t_{i+1}} |\delta^{\pi} Z_s|^2 \,\mathrm{d}s - |\delta^{\pi} Y_{t_{i+1}}|^2 \\ &= 2\mathbf{E}\int_t^{t_{i+1}} \langle \delta^{\pi} Y_s, \delta^{\pi} f(s)\rangle \,\mathrm{d}s + \int_t^{t_{i+1}} |\delta^{\pi} g(s)|^2 \,\mathrm{d}s \\ &\le \frac{C}{\beta}\int_t^{t_{i+1}} \mathbf{E}\{|\pi|^2 + |X_s - X^{\pi}_{t_i}|^2 + |Y_s - Y^{\pi}_{t_i}|^2 + |Z_s - Z^{\pi}_{t_i}|^2\} \,\mathrm{d}s \qquad (4.8) \\ &\quad + \int_t^{t_{i+1}} C\mathbf{E}\{|\pi|^2 + |X_s - X^{\pi}_{t_{i+1}}|^2 + |Y_s - Y^{\pi}_{t_{i+1}}|^2\} \,\mathrm{d}s \\ &\quad + \beta\int_t^{t_{i+1}} \mathbf{E}|\delta^{\pi} Y_s|^2 \,\mathrm{d}s, \qquad t \in [t_i, t_{i+1}). \end{aligned}$$

Proposition 2.1, Proposition 2.2 and Lemma 4.1 yield that

$$
\begin{aligned}
\mathbf{E}|X_s - X^\pi_{t_i}|^2 + \mathbf{E}|X_s - X^\pi_{t_{i+1}}|^2 &\le C|\pi|, \\
\mathbf{E}|Y_s - Y^\pi_{t_i}|^2 &\le 2(\mathbf{E}|Y_s - Y_{t_i}|^2 + \mathbf{E}|\delta^\pi Y_{t_i}|^2) \le C(|\pi| + \mathbf{E}|\delta^\pi Y_{t_i}|^2), \\
\mathbf{E}|Y_s - Y^\pi_{t_{i+1}}|^2 &\le 2(\mathbf{E}|Y_s - Y_{t_{i+1}}|^2 + \mathbf{E}|\delta^\pi Y_{t_{i+1}}|^2) \\
&\le C(|\pi| + \mathbf{E}|\delta^\pi Y_{t_{i+1}}|^2), \\
\mathbf{E}|Z_s - Z^\pi_{t_i}|^2 &\le 2\left(\mathbf{E}|Z_s - \tilde{Z}^\pi_{t_i}|^2 + \frac{1}{\Delta^\pi_{i+1}} \int_{t_i}^{t_{i+1}} \mathbf{E}|\delta^\pi Z_r|^2 \,\mathrm{d}r\right).
\end{aligned} \tag{4.9}
$$

Plugging (4.9) into (4.8), we get

$$
\begin{aligned}
V_t \le{}& \frac{C}{\beta} \int_t^{t_{i+1}} \mathbf{E}\{|\pi| + |\delta^\pi Y_{t_i}|^2 + |Z_s - \tilde{Z}^\pi_{t_i}|^2\} \,\mathrm{d}s + C \int_t^{t_{i+1}} \mathbf{E}\{|\pi| + |\delta^\pi Y_{t_{i+1}}|^2\} \,\mathrm{d}s \\
&+ \frac{C}{\beta} \int_t^{t_{i+1}} \mathbf{E}|\delta^\pi Z_s|^2 \,\mathrm{d}s + \beta \int_t^{t_{i+1}} \mathbf{E}|\delta^\pi Y_s|^2 \,\mathrm{d}s,
\end{aligned}
$$

which, along with the definitions of V_t, provides, for $t_i \le t \le t_{i+1}$,

$$
\mathbf{E}|\delta^\pi Y_t|^2 + \int_t^{t_{i+1}} \mathbf{E}|\delta^\pi Z_s|^2 \,\mathrm{d}s \le \beta \int_t^{t_{i+1}} \mathbf{E}|\delta^\pi Y_s|^2 \,\mathrm{d}s + A_i, \tag{4.10}
$$

where

$$
\begin{aligned}
A_i ={}& (1 + C\pi)\mathbf{E}|\delta^\pi Y_{t_{i+1}}|^2 + \frac{C}{\beta}\left[|\pi|^2 + |\pi|\mathbf{E}|Y^\pi_{t_i}| + \int_{t_i}^{t_{i+1}} \mathbf{E}|Z_s - \tilde{Z}^\pi_{t_i}|^2 \,\mathrm{d}s\right] \\
&+ \frac{C}{\beta} \int_{t_i}^{t_{i+1}} \mathbf{E}|\delta^\pi Z_s|^2 \,\mathrm{d}s.
\end{aligned}
$$

Next, by some calculus used in Gronwall's lemma, we have

$$
\mathbf{E}|\delta^\pi Y_t|^2 + \int_t^{t_{i+1}} \mathbf{E}|\delta^\pi Z_s|^2 \,\mathrm{d}s \le (1 + C\beta|\pi|)A_i; \tag{4.11}
$$

hence for $t = t_i$ and β sufficiently larger than C, such that $\frac{C}{\beta} < 1$, we obtain

$$
\begin{aligned}
&\mathbf{E}|\delta^\pi Y_{t_i}|^2 + \left(1 - \frac{C}{\beta}\right) \int_{t_i}^{t_{i+1}} \mathbf{E}|\delta^\pi Z_s|^2 \,\mathrm{d}s \\
&\quad \le (1 + C|\pi|)\left\{\mathbf{E}|\delta^\pi Y_{t_{i+1}}|^2 + |\pi|^2 + \int_{t_i}^{t_{i+1}} \mathbf{E}[|Z_s - \tilde{Z}^\pi_{t_i}|^2] \,\mathrm{d}s\right\}
\end{aligned}
$$

for small $|\pi|$.

Iterating the last inequality, we get

$$\begin{aligned}&\mathbf{E}|\delta^\pi Y_{t_i}|^2+\left(1-\frac{C}{\beta}\right)\int_{t_i}^{t_{i+1}}\mathbf{E}|\delta^\pi Z_s|^2\,\mathrm{d}s\\&\quad\le(1+C|\pi|)^{T/|\pi|}\left\{\mathbf{E}|\delta^\pi Y_T|^2+|\pi|+\sum_{i=1}^n\int_{t_{i-1}}^{t_i}\mathbf{E}[|Z_s-\tilde{Z}^\pi_{t_{i-1}}|^2]\,\mathrm{d}s\right\}.\end{aligned}$$

Moreover, it follows from Lemma 4.2, the Lipschitz condition on g and Proposition 4.1 that

$$\begin{aligned}&\mathbf{E}|\delta^\pi Y_{t_i}|^2+\left(1-\frac{C}{\beta}\right)\int_{t_i}^{t_{i+1}}\mathbf{E}|\delta^\pi Z_s|^2\,\mathrm{d}s\\&\quad\le(1+C|\pi|)^{T/\pi}\{\mathbf{E}|\delta^\pi Y_T|^2+|\pi|+C|\pi|\}\le C|\pi|\end{aligned}\tag{4.12}$$

for small $|\pi|$.

On the other hand, summing up inequality (4.11) with $t=t_i$, we get

$$\begin{aligned}&\left[1-\frac{C}{\beta}(1+C\beta|\pi|)\right]\int_0^T\mathbf{E}|\delta^\pi Z_s|^2\,\mathrm{d}s\\&\quad\le(1+C\beta|\pi|)\frac{C}{\beta}|\pi|+(1+C\beta|\pi|)(1+C|\pi|)\mathbf{E}|\delta^\pi Y_T|^2\\&\qquad+\left[(1+C\beta|\pi|)\frac{C}{\beta}|\pi|-1\right]\mathbf{E}|\delta^\pi Y_0|^2\\&\qquad+\left[(1+C\beta|\pi|)\left((1+C|\pi|)+\frac{C}{\beta}|\pi|\right)-1\right]\sum_{i=1}^{n-1}\mathbf{E}|\delta^\pi Y_{t_i}|^2\\&\qquad+(1+C\beta|\pi|)\frac{C}{\beta}\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}}\mathbf{E}|Z_s-\tilde{Z}^\pi_{t_i}|^2\,\mathrm{d}s.\end{aligned}$$

Therefore, by inequality (4.12) and again Lemma 4.2, one derives that

$$\int_0^T\mathbf{E}|\delta^\pi Z_s|^2\,\mathrm{d}s\le C|\pi|$$

and then

$$\sup_{0\le t\le T}|\delta^\pi Y_t|^2\le C|\pi|.$$ □

To end this section, let us give the following bound on $Y^\pi_{t_i}$'s, which will be used in the approximating of the discrete conditional expectation $\mathbf{E}^\pi_i$, for all $0\le i\le n-1$.

Lemma 4.3. *Assume b,σ,f and g satisfy the assumptions* (H1), (H2) *and* (H3), *and let $h\in W^{1,\infty}(\mathbf{R}^d)$ such that $h(0)$ is bounded by K. For all $0\le i\le n-1$, define the sequences of* *random variables by backward induction*:

$$
\begin{aligned}
\alpha_n^\pi &= 2C, \qquad \beta_n = C,\\
\alpha_i^\pi &= (1 - C|\pi|)^{-1}(1 + C^2|\pi|)^{1/2}\\
&\quad \times \{(1 + 2C|\pi|)[(1 + C|\Delta^\pi B_{i+1}|)\beta_{i+1}^\pi + C|\Delta^\pi B_{i+1}|] + C|\pi|\},\\
\beta_i^\pi &= (1 - C|\pi|)^{-1}(1 + C^2|\pi|)^{1/2}\\
&\quad \times \{(1 + C|\Delta^\pi B_{i+1}|)\alpha_{i+1}^\pi + 6C^2|\pi|(1 + 2C|\Delta^\pi B_{i+1}|) + 3C|\pi|\}.
\end{aligned}
$$

Then, for all $0 \le i \le n$,

$$
|Y_{t_i}^\pi| \le \alpha_i^\pi + \beta_i^\pi |X_{t_i}^\pi|^2, \tag{4.13}
$$

$$
\begin{aligned}
&\mathbf{E}_{i-1}^\pi |Y_{t_i}^\pi + g(t_i, X_{t_i}^\pi, Y_{t_i}^\pi)\Delta^\pi B_i|\\
&\quad \le \big(\mathbf{E}_{i-1}^\pi |Y_{t_i}^\pi + g(t_i, X_{t_i}^\pi, Y_{t_i}^\pi)\Delta^\pi B_i|^2\big)^{1/2}\\
&\quad \le (1 + 2C|\pi|)\{(1 + C|\Delta^\pi B_{i+1}|)\beta_{i+1}^\pi + C|\Delta^\pi B_{i+1}|\}|X_{t_i}^\pi|^2\\
&\qquad + (1 + C|\Delta^\pi B_{i+1}|)\alpha_{i+1}^\pi + 6C^2|\pi|(1 + 2C|\Delta^\pi B_{i+1}|),
\end{aligned} \tag{4.14}
$$

$$
\begin{aligned}
&\mathbf{E}_{i-1}^\pi \big|\big(Y_{t_i}^\pi + g(t_i, X_{t_i}^\pi, Y_{t_i}^\pi)\Delta^\pi B_i\big)\Delta^\pi W_{i+1}\big|\\
&\quad \le \sqrt{|\pi|}(1 + 2C|\pi|)\{(1 + C|\Delta^\pi B_{i+1}|)\beta_{i+1}^\pi + C|\Delta^\pi B_{i+1}|\}|X_{t_i}^\pi|^2\\
&\qquad + \sqrt{|\pi|}(1 + C|\Delta^\pi B_{i+1}|)\alpha_{i+1}^\pi + 6C^2|\pi|(1 + 2C|\Delta^\pi B_{i+1}|).
\end{aligned} \tag{4.15}
$$

Moreover,

$$
\limsup_{|\pi|\to 0} \max_{0 \le i \le n} (\alpha_i^\pi + \beta_i^\pi) < \infty \qquad a.s.
$$

Proof. First, since h is K-Lipschitz, $h(0)$ is bounded by K,

$$
|Y_{t_n}| = |h(X_T^\pi)| \le C(|X_T^\pi| + 1) \le 2C + C|X_T^\pi|^2 = \alpha_n + \beta_n |X_T^\pi|^2.
$$

Next, we assume that

$$
|Y_{t_{i+1}}| \le \alpha_{i+1}^\pi + \beta_{i+1}^\pi |X_{t_{i+1}}^\pi|^2, \tag{4.16}
$$

for some fixed $0 \le i \le n - 1$. Then by the definition of Y^π in (4.4), there exists a $\mathcal{F}_{t_i}$-measurable random variable ζ_i such that

$$
\begin{aligned}
(1 - C|\pi|)|Y_{t_i}^\pi| &\le \mathbf{E}_i^\pi\big[\big(Y_{t_{i+1}}^\pi + g(t_{i+1}, X_{t_{i+1}}^\pi, Y_{t_{i+1}}^\pi)\Delta^\pi B_{i+1}\big)(1 + \zeta_i \Delta^\pi W_{i+1})\big]\\
&\quad + C|\pi|(2 + |X_{t_i}^\pi|)\\
&\le \big(\mathbf{E}_i^\pi |Y_{t_{i+1}}^\pi + g(t_{i+1}, X_{t_{i+1}}^\pi, Y_{t_{i+1}}^\pi)\Delta^\pi B_{i+1}|^2\big)^{1/2}(\mathbf{E}_i^\pi | + \zeta_i \Delta^\pi W_{i+1}|^2)^{1/2}\\
&\quad + C|\pi|(3 + |X_{t_i}^\pi|^2).
\end{aligned} \tag{4.17}
$$

It show in Bouchard and Touzi [5] that

$$\mathbf{E}_i^\pi|1+\zeta_i\Delta^\pi W_{i+1}|^2 \le 1 + C^2|\pi|.$$

This provides, from (4.17),

$$\begin{aligned}(1-C|\pi|)|Y_{t_i}^\pi| \le (1+C^2|\pi|)^{1/2}\big(\mathbf{E}_i^\pi|Y_{t_{i+1}}^\pi + g(t_{i+1}, X_{t_{i+1}}^\pi, Y_{t_{i+1}}^\pi)\Delta^\pi B_{i+1}|^2\big)^{1/2}\\ + C|\pi|(3+|X_{t_i}^\pi|^2).\end{aligned} \tag{4.18}$$

But it follows from the Lipschitz property of g that

$$\begin{aligned}&\mathbf{E}_i^\pi\big(|Y_{t_{i+1}}^\pi + g(t_{i+1}, X_{t_{i+1}}^\pi, Y_{t_{i+1}}^\pi)\Delta^\pi B_{i+1}|^2\big)^{1/2}\\
&\le (1+C|\Delta^\pi B_{i+1}|)(\mathbf{E}_i^\pi|Y_{t_{i+1}}^\pi|^2)^{1/2} + C|\Delta^\pi B_{i+1}|(\mathbf{E}_i^\pi|X_{t_{i+1}}^\pi|^2)^{1/2}\\
&\le (1+C|\Delta^\pi B_{i+1}|)\{\alpha_{i+1}^\pi + \beta_{i+1}^\pi[(1+2C|\pi|)|X_{t_i}^\pi|^2 + 6K^2|\pi|]\}\\
&\quad + C|\Delta^\pi B_{i+1}|[(1+2C|\pi|)|X_{t_i}^\pi|^2 + 6K^2|\pi|]\\
&= (1+2C|\pi|)\{(1+C|\Delta^\pi B_{i+1}|)\beta_{i+1}^\pi + C|\Delta^\pi B_{i+1}|\}|X_{t_i}^\pi|^2\\
&\quad + (1+C|\Delta^\pi B_{i+1}|)\alpha_{i+1}^\pi + 6C^2|\pi|(1+2C|\Delta^\pi B_{i+1}|).\end{aligned}$$

Finally (4.18) becomes

$$\begin{aligned}|Y_{t_i}^\pi| &\le (1-C|\pi|)^{-1}(1+C^2|\pi|)^{1/2}\\
&\quad \times\{(1+2C|\pi|)[(1+C|\Delta^\pi B_{i+1}|)\beta_{i+1}^\pi + C|\Delta^\pi B_{i+1}|] + C|\pi|\}|X_{t_i}^\pi|^2\\
&\quad + (1-C|\pi|)^{-1}(1+C^2|\pi|)^{1/2}\\
&\quad \times\{(1+C|\Delta^\pi B_{i+1}|)\alpha_{i+1}^\pi + 6C^2|\pi|(1+2C|\Delta^\pi B_{i+1}|) + 3C|\pi|\}\\
&= \alpha_i^\pi + \beta_i^\pi|X_{t_i}^\pi|^2.\end{aligned}$$

□

5. Rate of convergence of the regression approximation

In this section, we try to give some ideas for a method of simulating a numerical scheme derived in the above section. It well known that the process X^π defined by (4.1) is simulated by the classical Monte Carlo method. We are reduced to simulate the process (Y^π, Z^π) defined in (4.3) and (4.4). In practice, the main tool used to define an approximation of Y^π, and then of Z^π, is to replace the conditional expectation $\mathbf{E}_i^\pi$ by its estimator $\widehat{E}_i^\pi$ in the backward scheme (4.3) and (4.4). We first establish the following bound on the $Y_{t_i}^\pi$s which help us to derive this simulation.

For the regression approximation, we consider $\{\mathcal{P}_i^\pi\}_{0\le i\le n}$, $\{\mathcal{R}_i^\pi\}_{0\le i\le n}$ and $\{\mathcal{J}_i^\pi\}_{0\le i\le n}$ defined by

$$\begin{aligned}
\mathcal{P}_i^\pi &= \alpha_i^\pi + \beta_i^\pi |X_{t_i}^\pi|^2,\\
\mathcal{R}_i^\pi &= (1+2C|\pi|)\{(1+C|\Delta^\pi B_{i+1}|)\beta_{i+1}^\pi + C|\Delta^\pi B_{i+1}|\}|X_{t_i}^\pi|^2\\
&\quad + (1+C|\Delta^\pi B_{i+1}|)\alpha_{i+1}^\pi + 6C^2|\pi|(1+2C|\Delta^\pi B_{i+1}|),\\
\mathcal{J}_i^\pi &= \sqrt{|\pi|}(1+2C|\pi|)\{(1+C|\Delta^\pi B_{i+1}|)\beta_{i+1}^\pi + C|\Delta^\pi B_{i+1}|\}|X_{t_i}^\pi|^2\\
&\quad + \sqrt{|\pi|}(1+C|\Delta^\pi B_{i+1}|)\alpha_{i+1}^\pi + 6C^2|\pi|(1+2C|\Delta^\pi B_{i+1}|).
\end{aligned}$$

Therefore, thanks to Lemma 4.3, we have

$$-\mathcal{P}_i^\pi(X_{t_i}^\pi, \Delta^\pi B_{i+1}) \le Y_{t_i}^\pi \le \mathcal{P}_i^\pi(X_{t_i}^\pi, \Delta^\pi B_{i+1}), \tag{5.1}$$

$$-\mathcal{R}_i^\pi(X_{t_i}^\pi, \Delta^\pi B_{i+1}) \le \mathbf{E}_i^\pi[\tilde{Y}_{t_{i+1}}^\pi] \le \mathcal{R}_i^\pi(X_{t_i}^\pi, \Delta^\pi B_{i+1}), \tag{5.2}$$

$$-\mathcal{J}_i^\pi(X_{t_i}^\pi, \Delta^\pi B_{i+1}) \le \mathbf{E}[\tilde{Y}_{t_{i+1}}^\pi \Delta^\pi W_{i+1}] \le \mathcal{J}_i^\pi(X_{t_i}^\pi, \Delta^\pi B_{i+1}). \tag{5.3}$$

Next, for an $\mathbf{R}$-valued random variable ξ, we define

$$\begin{aligned}
\mathbf{T}_i^{\mathcal{P}^\pi}(\xi) &= -\mathcal{P}_i^\pi(X_{t_i}^\pi, \Delta^\pi B_{i+1}) \vee \xi \wedge \mathcal{P}_i^\pi(X_{t_i}^\pi, \Delta^\pi B_{i+1}),\\
\mathbf{T}_i^{\mathcal{R}^\pi}(\xi) &= -\mathcal{R}_i^\pi(X_{t_i}^\pi, \Delta^\pi B_{i+1}) \vee \xi \wedge \mathcal{R}^\pi{}_i(X_{t_i}^\pi, \Delta^\pi B_{i+1}),\\
\mathbf{T}_i^{\mathcal{J}^\pi}(\xi) &= -\mathcal{J}_i^\pi(X_{t_i}^\pi, \Delta^\pi B_{i+1}) \vee \xi \wedge \mathcal{J}_i^\pi(X_{t_i}^\pi, \Delta^\pi B_{i+1}).
\end{aligned}$$

Given an approximation $\widehat{\mathbf{E}}_i^\pi$ of $\mathbf{E}_i^\pi$, we are ready to get the process $(\hat{Y}^\pi, \hat{Z}^\pi)$ defined by following backward induction scheme,

$$\hat{Y}_{t_n}^\pi = g(X_T^\pi), \tag{5.4}$$

$$\hat{Z}_{t_{i-1}}^\pi = \frac{1}{\Delta_i^\pi}\widehat{\mathbf{E}}_{i-1}^\pi\big[\big(\hat{Y}_{t_i}^\pi + g(t_i, X_{t_i}^\pi, \hat{Y}_{t_i}^\pi)\Delta^\pi B_i\big)\Delta^\pi W_i\big] \tag{5.5}$$

$$Y_{t_{i-1}}^\pi = \widehat{\mathbf{E}}_{i-1}^\pi[\hat{Y}_{t_i}^\pi + g(t_i, X_{t_i}^\pi, \hat{Y}_{t_i}^\pi)\Delta^\pi B_i] + f(t_{i-1}, X_{t_{i-1}}^\pi, Y_{t_{i-1}}^\pi, \hat{Z}_{t_{i-1}}^\pi)\Delta_i^\pi \tag{5.6}$$

$$\hat{Y}_{t_{i-1}}^\pi = \mathbf{T}_{t_{i-1}}^{\mathcal{P}^\pi}(Y_{t_{i-1}}^\pi), \tag{5.7}$$

for all $1 \le i \le n$.

Remark 5.1. Using the above notation and replacing X^π by $U^\pi = (X^\pi, B^\pi)$, one can state analogously to Examples 4.1 and 4.2, appearing in Bouchard and Touzi [5].

To end this section, we derive the following L^p estimate of the error $\hat{Y}^\pi - Y^\pi$ in terms of the regression errors $\widehat{\mathbf{E}}_i^\pi - \mathbf{E}_i^\pi$.

Theorem 5.1. *Let $p \geq 1$ be given and $\mathcal{P}^\pi$ be a sequence defined above. Then there is a constant C depending only on T, K and p such that*

$$\begin{aligned}
&\|\hat{Y}^\pi_{t_i} - Y^\pi_{t_i}\|_{L^p} \\
&\quad \leq \frac{C}{\pi} \max_{1\leq j\leq n-1} \big\{ \|(\widehat{\mathbf{E}}_j - \mathbf{E}_j)[\hat{Y}^\pi_{t_{j+1}} + g(t_{j+1}, X^\pi_{t_{j+1}}, \hat{Y}^\pi_{t_{j+1}})\Delta^\pi B_{j+1}]\|_{L^p} \\
&\qquad + \big\|\big(\widehat{\mathbf{E}}_j - \mathbf{E}_j\big)\big[\big(\hat{Y}^\pi_{t_{j+1}} + g(t_{j+1}, X^\pi_{t_{j+1}}, \hat{Y}^\pi_{t_{j+1}})\Delta^\pi B_{j+1}\big)\Delta^\pi W_{j+1}\big]\big\|_{L^p}\big\}.
\end{aligned}$$

Proof. For $0 \leq i \leq n-1$ to be fixed, with calculus similar to that used in the proof of Theorem 4.1 in Bouchard and Touzi [5], we have

$$\begin{aligned}
(1 - C\pi)|Y^\pi_{t_i} - \hat{Y}^\pi_{t_i}| &\leq |\varepsilon_i| + (1 + C|\Delta^\pi B_{i+1}|)(\mathbf{E}|Y^\pi_{t_i} - \hat{Y}^\pi_{t_i}|^p)^{1/p} \\
&\quad \times (\mathbf{E}^\pi_i|1 + \zeta_i\Delta^\pi W_{i+1}|^{2k})^{1/2k},
\end{aligned} \tag{5.8}$$

where k is an arbitrary integer greater than the conjugate of p and

$$\begin{aligned}
\varepsilon_i &= (\widehat{\mathbf{E}}_i - \mathbf{E}_i)[\hat{Y}^\pi_{t_{i+1}} + g(t_{i+1}, X^\pi_{t_{i+1}}, \hat{Y}^\pi_{t_{i+1}})\Delta^\pi B_{i+1}] \\
&\quad + \Delta^\pi_{i+1}\big\{f\big(t_i, X^\pi_{t_i}, Y^\pi_{t_i}, (\Delta^\pi_{i+1})^{-1}\mathbf{E}^\pi_i\big[\big(\hat{Y}^\pi_{t_{i+1}} + g(t_{i+1}, X^\pi_{t_{i+1}}, \hat{Y}^\pi_{t_{i+1}})\Delta^\pi B_{i+1}\big)\Delta^\pi W_{i+1}\big]\big) \\
&\qquad - f\big(t_i, X^\pi_{t_i}, Y^\pi_{t_i}, (\Delta^\pi_{i+1})^{-1} \\
&\qquad\quad \times \widehat{\mathbf{E}}^\pi_i\big[\big(\hat{Y}^\pi_{t_{i+1}} + g(t_{i+1}, X^\pi_{t_{i+1}}, \hat{Y}^\pi_{t_{i+1}})\Delta^\pi B_{i+1}\big)\Delta^\pi W_{i+1}\big]\big)\big\}.
\end{aligned}$$

Since

$$\begin{aligned}
\|\varepsilon_i\|_{L^p} \leq \eta_i &= C\big(\|(\widehat{\mathbf{E}}_i - \mathbf{E}_i)[\hat{Y}^\pi_{t_{i+1}} + g(t_{i+1}, X^\pi_{t_{i+1}}, \hat{Y}^\pi_{t_{i+1}})\Delta^\pi B_{i+1}]\|_{L^p} \\
&\quad + \big\|(\widehat{\mathbf{E}}_i - \mathbf{E}_i)\big[\big(\hat{Y}^\pi_{t_{i+1}} + g(t_{i+1}, X^\pi_{t_{i+1}}, \hat{Y}^\pi_{t_{i+1}})\Delta^\pi B_{i+1}\big)\Delta^\pi W_{i+1}\big]\big\|_{L^p},
\end{aligned}$$

and

$$(\mathbf{E}^\pi_i|1 + \zeta_i\Delta^\pi W_{i+1}|^{2k})^{1/2k} \leq (1 + C|\pi|),$$

we get from (5.8)

$$(1 - C\pi)\|Y^\pi_{t_i} - \hat{Y}^\pi_{t_i}\|_{L^p} \leq \eta_i + (1 + C|\pi|)^{1/2k}\|Y^\pi_{t_{i+}} - \hat{Y}^\pi_{t_{i+1}}\|_{L^p}, \tag{5.9}$$

and thus the result follows as in Bouchard and Touzi [5]. □

Remark 5.2. With the foregoing results, it is possible, with some not very difficult adjustments, to obtain similar results to those obtained by Bouchard and Touzi (see Sections 5 and 6 of Bouchard and Touzi [5]).

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References

[1] Aman, A. (2010). Reflected generalized backward doubly SDEs driven by Lévy processes and applications. *J. Theoret. Probab.* DOI:10.1007/s10959-010-0328-1.

[2] Aman, A., N'zi, M. and Owo, J.M. (2010). A note on homeomorphism for backward doubly SDEs and applications. *Stoch. Dyn.* **10** 1–12.

[3] Bahlali, S. and Gherbal, B. (2010). Optimality conditions of controlled backward doubly stochastic differential equations. *Random Oper. Stoch. Equ.* **18** 247–265. MR2718124

[4] Bally, V. (1997). Approximation scheme for solutions of BSDE. In *Backward Stochastic Differential Equations* (*Paris*, 1995–1996). *Pitman Res. Notes Math. Ser.* **364** 177–191. Harlow: Longman. MR1752682

[5] Bouchard, B. and Touzi, N. (2004). Discrete-time approximation and Monte-Carlo simulation of backward stochastic differential equations. *Stochastic Process. Appl.* **111** 175–206. MR2056536

[6] Briand, P., Delyon, B. and Mémin, J. (2001). Donsker-type theorem for BSDEs. *Electron. Commun. Probab.* **6** 1–14 (electronic). MR1817885

[7] Buckdahn, R. and Ma, J. (2001). Stochastic viscosity solutions for nonlinear stochastic partial differential equations. I. *Stochastic Process. Appl.* **93** 181–204. MR1828772

[8] Buckdahn, R. and Ma, J. (2001). Stochastic viscosity solutions for nonlinear stochastic partial differential equations. II. *Stochastic Process. Appl.* **93** 205–228. MR1831830

[9] Chevance, D. (1997). Numerical methods for backward stochastic differential equations. In *Numerical Methods in Finance* (L.C.G. Rogers and D. Talay, eds.). *Publ. Newton Inst.* 232–244. Cambridge: Cambridge Univ. Press. MR1470517

[10] Douglas, J. Jr., Ma, J. and Protter, P. (1996). Numerical methods for forward–backward stochastic differential equations. *Ann. Appl. Probab.* **6** 940–968. MR1410123

[11] Karatzas, I. and Shreve, S.E. (1998). *Brownian Motion and Stochastic Calculus. Graduate Texts in Mathematics* **113**. New York: Springer.

[12] Kloeden, P.E. and Platen, E. (1992). *Numerical Solution of Stochastic Differential Equations. Applications of Mathematics* (*New York*) **23**. Berlin: Springer. MR1214374

[13] Lemor, J.P., Gobet, E. and Warin, X. (2006). Rate of convergence of an empirical regression method for solving generalized backward stochastic differential equations. *Bernoulli* **12** 889–916. MR2265667

[14] Ma, J., Protter, P., San Martín, J. and Torres, S. (2002). Numerical method for backward stochastic differential equations. *Ann. Appl. Probab.* **12** 302–316. MR1890066

[15] Ma, J., Protter, P. and Yong, J.M. (1994). Solving forward–backward stochastic differential equations explicitly—A four step scheme. *Probab. Theory Related Fields* **98** 339–359. MR1262970

[16] Ma, J. and Zhang, J. (2002). Path regularity for solutions of backward SDE's. *Probab. Theory Related Fields* **122** 163–190.

[17] Milstein, G.N. and Tretyakov, M.V. (2006). Numerical algorithms for forward–backward stochastic differential equations. *SIAM J. Sci. Comput.* **28** 561–582 (electronic). MR2231721

[18] Pardoux, É. and Peng, S.G. (1994). Backward doubly stochastic differential equations and systems of quasilinear SPDEs. *Probab. Theory Related Fields* **98** 209–227. MR1258986

[19] Zhang, J. (2001). Some fine properties of backward stochastic differential equations. Ph.D. thesis, Purdue Univ. MR2703162

[20] Zhang, J. (2004). A numerical scheme for BSDEs. *Ann. Appl. Probab.* **14** 459–488. MR2023027

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