Distributions of exponential integrals of independent increment processes related to generalized gamma convolutions

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It is known that in many cases distributions of exponential integrals of Lévy processes are infinitely divisible and in some cases they are also selfdecomposable. In this paper, we give some sufficient conditions under which distributions of exponential integrals are not only selfdecomposable but furthermore are generalized gamma convolution. We also study exponential integrals of more general independent increment processes. Several examples are given for illustration.

Keywords: exponential integral; generalized gamma convolutions; Lévy process; selfdecomposable distribution

1. Introduction

Let $(\xi, \eta) = \{(\xi_t, \eta_t), t \geq 0\}$ be a bivariate càdlàg independent increment process. In most cases, (ξ, η) is assumed as a bivariate Lévy process, but we also treat more general cases where ξ or η is a compound sum process, which is not necessarily a Lévy process but is another typical independent increment process. Our concern in this paper is to examine distributional properties of the exponential integral

$$V := \int_{(0,\infty)} e^{-\xi_{t-}} \, d\eta_t, \tag{1.1}$$

provided that this integral converges almost surely. More precisely, we are interested in when $\mathcal{L}(V)$, the law of V, is selfdecomposable and moreover is a generalized gamma convolution.

We say that a probability distribution μ on $\mathbb{R}$ (resp. an $\mathbb{R}$-valued random variable X) is selfdecomposable, if for any $b > 1$ there exists a probability distribution μ_b (resp. a random variable Y_b independent of X) such that

$$\mu = D_{b^{-1}}(\mu) * \mu_b \qquad (\text{resp. } X \overset{d}{=} b^{-1}X + Y_b),$$

where $D_a(\mu)$ means the distribution induced by $D_a(\mu)(aB) := \mu(B)$ for $B \in \mathfrak{B}(\mathbb{R})$, $*$ is the convolution operator and $\overset{d}{=}$ denotes equality in distribution. Every selfdecomposable distribution is infinitely divisible. Some well-known distributional properties of nontrivial selfdecomposable distributions are absolute continuity and unimodality (see Sato [15], pages 181 and 404).

First, we review existing results on $\mathcal{L}(V)$. Bertoin *et al.* [3] (in the case when $\eta = \{\eta_t\}$ is a one-dimensional Lévy process) and Kondo *et al.* [9] (in the case when η is a multi-dimensional Lévy process) showed that if $\xi = \{\xi_t\}$ is a spectrally negative Lévy process satisfying $\lim_{t \to \infty} \xi_t = +\infty$ a.s. and if the integral (1.1) converges a.s., or equivalently, if $\int_{\mathbb{R}} \log^+ |y| \nu_\eta(dy) < \infty$ for the Lévy measure ν_η of η_1, then $\mathcal{L}(V)$ is selfdecomposable.

On the other hand, there is an example of noninfinitely divisible $\mathcal{L}(V)$, which is due to Samorodnitsky (see Klüppelberg *et al.* [8]). In fact if $(\xi_t, \eta_t) = (S_t + at, t)$, where $\{S_t\}$ is a subordinator and $a > 0$ some constant, then the support of $\mathcal{L}(V)$ is bounded so that $\mathcal{L}(V)$ is not infinitely divisible.

Recently, Lindner and Sato [11] considered the exponential integral

$$\int_{(0,\infty)} \exp(-(\log c) N_{t-}) \, dY_t = \int_{(0,\infty)} c^{-N_{t-}} \, dY_t, \qquad c > 0,$$

where $\{(N_t, Y_t)\}$ is a bivariate compound Poisson process whose Lévy measure is concentrated on $(1, 0)$, $(0, 1)$ and $(1, 1)$, and showed a necessary and sufficient condition for the infinite divisibility of $\mathcal{L}(V)$. They also pointed out that $\mathcal{L}(V)$ is always c^{-1}-decomposable, namely there exists a probability distribution ρ such that $\mu = D_{c^{-1}}(\mu) * \rho$. Note that a c^{-1}-decomposable distribution is not necessarily infinitely divisible, unless ρ is infinitely divisible. In their second paper (Lindner and Sato [12]), they also gave a condition under which $\mathcal{L}(V)$, generated by a bivariate compound Poisson process $\{(N_t, Y_t)\}$ whose Lévy measure is concentrated on $(1, 0)$, $(0, 1)$ and $(1, c^{-1})$, is infinitely divisible.

For other distributional properties of exponential integrals, like the tail behavior, see, e.g., Maulik and Zwart [13], Rivero [14] and Behme [2].

In this paper, we focus on "Generalized Gamma Convolutions" (GGCs, for short) to get more explicit distributional informations of V than selfdecomposability.

Throughout this paper, we say that for $r > 0$ and $\lambda > 0$ a random variable $\gamma_{r,\lambda}$ has a gamma(r, λ) distribution if its probability density function f on $(0, \infty)$ is

$$f(x) = \frac{\lambda^r}{\Gamma(r)} x^{r-1} e^{-\lambda x}.$$

A gamma$(1, \lambda)$ distribution is an exponential distribution with parameter $\lambda > 0$. When we do not have to emphasize the parameters (r, λ), we just write γ for a gamma random variable.

The class of GGCs is defined to be the smallest class of distributions on the positive half line that contains all gamma distributions and is closed under convolution and weak convergence. By including gamma distributions on the negative real axis, we obtain the class of distributions on $\mathbb{R}$ which will be called "Extended Generalized Gamma Convolutions" (EGGCs, for short). We refer to Bondesson [4] and Steutel and van Harn [16] for many properties of GGCs and EGGCs with relations among other subclasses of infinitely divisible distributions.

One well-known concrete example of exponential integrals is the following. When $(\xi_t, \eta_t) = (B_t + at, t)$ with a standard Brownian motion $\{B_t\}$ and a drift $a > 0$, the law of (1.1) equals $\mathcal{L}(1/(2\gamma))$ which is GGC (and thus is selfdecomposable).

When choosing ξ to be deterministic, that is, $(\xi_t, \eta_t) = (t, \eta_t)$, the exponential integral (1.1) is defined and is an EGGC if and only if η admits a finite log-moment (needed for the convergence) and $\mathcal{L}(\eta_1)$ is included in the Goldie–Steutel–Bondesson Class, a superclass of EGGC as defined for example, in Barndorff-Nielsen *et al.* [1]. This fact follows directly from [1], equation (2.28). In the same paper, the authors characterized the class of GGCs by using stochastic integrals with respect to Lévy processes as follows. Let $e(x) = \int_x^\infty u^{-1} e^{-u} \, \mathrm{d}u$, $x > 0$, and let $e^*(\cdot)$ be its inverse function. Then $\mathcal{L}(\int_{(0,\infty)} e^*(s) \, \mathrm{d}\eta_s)$ is GGC if η is a Lévy process and $\mathcal{L}(\eta_1)$ has a finite log-moment.

In this paper, via concrete examples, we investigate distributional properties of exponential integrals connected with GGCs.

The paper is organized as follows. In Section 2, we give some preliminaries. In Section 3, we consider exponential integrals for two independent Lévy processes ξ and η such that ξ or η is a compound Poisson process, and construct concrete examples related to our question. In the special case that both ξ and η are compound Poisson processes, we also allow dependence between the two components of (ξ, η). In Section 4, we consider exponential integrals for independent increment processes such that ξ and η are independent and one is a compound sum process (which is not necessarily a Lévy process) while the other is a Lévy process.

2. Preliminaries

The class of all infinitely divisible distributions on $\mathbb{R}$ (resp. $\mathbb{R}_+$) is denoted by $I(\mathbb{R})$ (resp. $I(\mathbb{R}_+)$). We denote the class of selfdecomposable distributions on $\mathbb{R}$ (resp. $\mathbb{R}_+$) by $L(\mathbb{R})$ (resp. $L(\mathbb{R}_+)$). The class of EGGCs on $\mathbb{R}$ (resp. GGCs on $\mathbb{R}_+$) is denoted by $T(\mathbb{R})$ (resp. $T(\mathbb{R}_+)$). The moment generating function of a random variable X and of a distribution μ are written as $\mathbb{L}_X$ and $\mathbb{L}_\mu$, respectively. If X is positive and μ has support in $\mathbb{R}_+$, $\mathbb{L}_X$ and $\mathbb{L}_\mu$ coincide with the Laplace transforms.

We are especially interested in distributions on $\mathbb{R}_+$. The class $T(\mathbb{R}_+)$ is characterized by the Laplace transform as follows: A probability distribution μ is GGC if and only if there exist $a \geq 0$ and a measure U satisfying

$$\int_{(0,1)} |\log x^{-1}| U(\mathrm{d}x) < \infty \quad \text{and} \quad \int_{(1,\infty)} x^{-1} U(\mathrm{d}x) < \infty,$$

such that the Laplace transform $\mathbb{L}_\mu(z)$ can be uniquely represented as

$$\mathbb{L}_\mu(u) = \int_{[0,\infty)} e^{-ux} \mu(\mathrm{d}x) = \exp\left\{ -au + \int_{(0,\infty)} \log\left(\frac{x}{x+u}\right) U(\mathrm{d}x) \right\}. \tag{2.1}$$

Another class of distributions which we are interested in is the class of distributions on $\mathbb{R}_+$ whose densities are hyperbolically completely monotone (HCM, for short). Here we say that a function $f(x)$ on $(0, \infty)$ with values in $\mathbb{R}_+$ is HCM if for every $u > 0$, the mapping $f(uv) \cdot$

$f(u/v)$ is completely monotone with respect to the variable $w = v + v^{-1}$, $v > 0$. Examples of HCM functions are x^β ($\beta \in \mathbb{R}$), e^{-cx} ($c > 0$) and $(1 + cx)^{-\alpha}$ ($c > 0, \alpha > 0$). The class of all distributions on $\mathbb{R}_+$ whose probability densities are HCM is denoted by $H(\mathbb{R}_+)$. Note that $H(\mathbb{R}_+) \subset T(\mathbb{R}_+) \subset L(\mathbb{R}_+) \subset I(\mathbb{R}_+)$.

For illustration, we give some examples. Log-normal distributions are in $H(\mathbb{R}_+)$ [4], Example 5.2.1. So these are also GGCs. Positive strictly stable distributions with Laplace transform $\mathbb{L}(u) = \exp(-u^\alpha)$ for $\alpha \in \{1/2, 1/3, \ldots\}$ are in $H(\mathbb{R}_+)$ [4], Example 5.6.2, while they are GGCs for all $\alpha \in (0, 1]$ [16], Proposition 5.7. Let $Y = \exp(\gamma_{r,\lambda}) - 1$. If $r \geq 1$, then $\mathcal{L}(Y)$ is in $H(\mathbb{R}_+)$, but if $r < 1$, $\mathcal{L}(Y)$ is not in $H(\mathbb{R}_+)$ [4], page 88. But $\mathcal{L}(Y)$ or equivalently $\mathcal{L}(\exp(\gamma_{r,\lambda}))$ is always in $T(\mathbb{R}_+)$, independent of the value of r [4], Theorem 6.2.3. Remark that by treating $H(\mathbb{R}_+)$ we cannot replace $\exp(\gamma_{r,\lambda}) - 1$ by $\exp(\gamma_{r,\lambda})$. Namely, set $r = 1$ and observe that the probability density function $\lambda(x + 1)^{-\lambda-1} 1_{[0,\infty)}(x)$ is HCM, but the probability density function $\lambda x^{-\lambda-1} 1_{[1,\infty)}(x)$ is not HCM. It follows from this that $\mathcal{L}(\exp(\gamma_{1,\lambda}) - 1)$ is in $H(\mathbb{R}_+)$ but $\mathcal{L}(\exp(\gamma_{1,\lambda}))$ is not in $H(\mathbb{R}_+)$.

In addition, we also investigate the modified HCM class denoted by $\widetilde{H}(\mathbb{R})$, which gives some interesting examples of $\mathcal{L}(V)$ on $\mathbb{R}$. The class $\widetilde{H}(\mathbb{R})$ is characterized to be the class of distributions of random variables $\sqrt{X}Z$, where $\mathcal{L}(X) \in H(\mathbb{R}_+)$ and Z is a standard normal random variable independent of X (see Bondesson [4], page 115). By the definition, any distribution in $\widetilde{H}(\mathbb{R})$ is a type G distribution, which is the distribution of the variance mixture of a standard normal random variable. Note that $\widetilde{H}(\mathbb{R}) \subset T(\mathbb{R})$. As will be seen in Proposition 2.1, there are nice relations between $\widetilde{H}(\mathbb{R})$ and $T(\mathbb{R})$ in common with those of $H(\mathbb{R}_+)$ and $T(\mathbb{R}_+)$.

Here we state some known facts that we will use later.

Proposition 2.1 (Bondesson [4] and Steutel and van Harn [16]).

(1) *A continuous function $\mathbb{L}(u)$, $u > 0$, with $\mathbb{L}(0+) = 1$ is HCM if and only if it is the Laplace transform of a GGC.*

(2) *If $\mathcal{L}(X) \in H(\mathbb{R}_+)$, $\mathcal{L}(Y) \in T(\mathbb{R}_+)$ and X and Y are independent, then $\mathcal{L}(XY) \in T(\mathbb{R}_+)$.*

(3) *Suppose that $\mathcal{L}(X) \in H(\mathbb{R}_+)$, $\mathcal{L}(Y) \in T(\mathbb{R})$ and that X and Y are independent. If $\mathcal{L}(Y)$ is symmetric, then $\mathcal{L}(\sqrt{X}Y) \in T(\mathbb{R})$.*

(4) *Suppose that $\mathcal{L}(X) \in \widetilde{H}(\mathbb{R})$ and $\mathcal{L}(Y) \in T(\mathbb{R})$ and that X and Y are independent. If $\mathcal{L}(Y)$ is symmetric, then $\mathcal{L}(XY) \in T(\mathbb{R})$.*

(5) *If $\mathcal{L}(X) \in \widetilde{H}(\mathbb{R})$, then $\mathcal{L}(|X|^q) \in H(\mathbb{R}_+)$ for all $|q| \geq 2, q \in \mathbb{R}$. Furthermore, $\mathcal{L}(|X|^q \times \text{sign}(X)) \in \widetilde{H}(\mathbb{R})$ for all $q \in \mathbb{N}, q \neq 2$, but not always for $q = 2$.*

Remark 2.2. Notice that the distribution of a sum of independent random variables with distributions in $H(\mathbb{R}_+)$ does not necessarily belong to $H(\mathbb{R}_+)$. See Bondesson [4], page 101.

Some distributional properties of GGCs are stated in the following proposition [4], Theorems 4.1.1. and 4.1.3.

Proposition 2.3.

(1) *The probability density function of a GGC without Gaussian part satisfying $0 < \int_{(0,\infty)} U(du) = \beta < \infty$ with the measure U as in (2.1) admits the representation $x^{\beta-1}h(x)$, where $h(x)$ is some completely monotone function.*

(2) *Let f be the probability density of a GGC distribution without Gaussian part satisfying $1 < \int_{(0,\infty)} U(du) = \beta \leq \infty$. Let k be a nonnegative integer such that $k < \beta - 1$. Then the density f is continuously differentiable any times on $(0, \infty)$, and at 0 at least k times differentiable with $f^{(j)}(0) = 0$ for $j \leq k$.*

Examples of GGCs and the explicit calculation of their Lévy measure are found in Bondesson [4] and James *et al.* [7].

Necessary and sufficient conditions for the convergence of (1.1) for bivariate Lévy processes were given by Erickson and Maller [5]. More precisely, in their Theorem 2.1, they showed that V converges a.s. if and only if for some $\varepsilon > 0$ such that $A_\xi(x) > 0$ for all $x > \varepsilon$ it holds

$$\lim_{t \to \infty} \xi_t = \infty \qquad \text{a.s.} \quad \text{and}$$

$$\int_{(e^\varepsilon, \infty)} \left(\frac{\log y}{A_\xi(\log y)} \right) |\nu_\eta((dy, \infty)) + \nu_\eta((-\infty, -dy))| < \infty. \qquad (2.2)$$

Here $A_\xi(x) = a_\xi + \nu_\xi((1, \infty)) + \int_{(1,x]} \nu_\xi((y, \infty)) \, dy$ while (Σ_X, ν_X, a_X) denotes the Lévy–Khintchine triplet of a Lévy process X.

3. Exponential integrals for compound Poisson processes

In this section, we study exponential integrals of the form (1.1), where either ξ or η is a compound Poisson process and the other is an arbitrary Lévy process. First, we assume the two processes to be independent, later we also investigate the case that (ξ, η) is a bivariate compound Poisson process.

3.1. Independent component case

We start with a general lemma which gives a sufficient condition for distributions of perpetuities to be GGCs.

Lemma 3.1. *Suppose A and B are two independent random variables such that $\mathcal{L}(A) \in H(\mathbb{R}_+)$ and $\mathcal{L}(B) \in T(\mathbb{R}_+)$. Let (A_j, B_j), $j = 0, 1, 2, \ldots$, be i.i.d. copies of (A, B). Then, given its a.s. convergence, the distribution of the perpetuity $Z := \sum_{k=0}^{\infty} (\prod_{i=0}^{k-1} A_i) B_k$ belongs to $T(\mathbb{R}_+)$. Furthermore, if $\mathcal{L}(A) \in \widetilde{H}(\mathbb{R})$, $\mathcal{L}(B) \in T(\mathbb{R})$ and $\mathcal{L}(B)$ is symmetric, then $\mathcal{L}(Z) \in T(\mathbb{R})$.*

Proof. If we put

$$Z_n := \sum_{k=0}^{n} \left(\prod_{i=0}^{k-1} A_i \right) B_k,$$

then we can rewrite

$$Z_n = B_0 + A_0 \big(B_1 + A_1 \big(B_2 + \cdots + A_{n-2}(B_{n-1} + A_{n-1} B_n) \cdots \big) \big).$$

Since A_{n-1} and B_n are independent, $\mathcal{L}(A_{n-1}) \in H(\mathbb{R}_+)$ and $\mathcal{L}(B_n) \in T(\mathbb{R}_+)$, we get $\mathcal{L}(A_{n-1}B_n) \in T(\mathbb{R}_+)$ by Proposition 2.1. Further B_{n-1} and $A_{n-1}B_n$ are independent and both distributions belong to $T(\mathbb{R}_+)$ and hence $\mathcal{L}(B_{n-1} + A_{n-1}B_n) \in T(\mathbb{R}_+)$. By induction, we can conclude that $\mathcal{L}(Z_n) \in T(\mathbb{R}_+)$. Since the class $T(\mathbb{R}_+)$ is closed under weak convergence, the first part follows immediately. A similar argument with (4) in Proposition 2.1 gives the second part. □

Case 1: *The process ξ is a compound Poisson process*

Proposition 3.2. *Suppose that the processes ξ and η are independent Lévy processes and further that $\xi_t = \sum_{i=1}^{N_t} X_i$ is a compound Poisson process with i.i.d. jump heights $X_i, i = 1, 2, \ldots,$ such that $0 < E[X_1] < \infty$, $\mathcal{L}(e^{-X_1}) \in H(\mathbb{R}_+)$ and η has finite log-moment $E \log^+ |\eta_1|$ and it holds $\mathcal{L}(\eta_\tau) \in T(\mathbb{R}_+)$ for an exponential random variable τ independent of η having the same distribution as the waiting times of N. Then the integral (1.1) converges a.s. and*

$$\mathcal{L}\left(\int_{(0,\infty)} e^{-\xi_{t-}} \, d\eta_t\right) \in T(\mathbb{R}_+).$$

Furthermore, if $\mathcal{L}(e^{-X_1}) \in \widetilde{H}(\mathbb{R})$, $\mathcal{L}(\eta_\tau) \in T(\mathbb{R})$ and $\mathcal{L}(\eta_\tau)$ is symmetric, then $\mathcal{L}(V) \in T(\mathbb{R})$.

Proof. Convergence of the integral follows from (2.2).

Set $T_0 = 0$ and let $T_j, j = 1, 2, \ldots,$ be the time of the jth jump of $\{N_t, t \geq 0\}$. Then we can write

$$\int_{(0,\infty)} e^{-\xi_{t-}} \, d\eta_t = \sum_{j=0}^{\infty} \int_{(T_j, T_{j+1}]} e^{-\sum_{i=1}^{j} X_i} \, d\eta_t = \sum_{j=0}^{\infty} e^{-\sum_{i=1}^{j} X_i} \int_{(T_j, T_{j+1}]} d\eta_t$$

$$=: \sum_{j=0}^{\infty} \left(\prod_{i=1}^{j} A_i\right) B_j,$$

where $\sum_{i=1}^{0} = 0$, $\prod_{i=1}^{0} = 1$, $A_i = e^{-X_i}$ and $B_j = \int_{(T_j, T_{j+1}]} d\eta_t \overset{d}{=} \eta_{T_{j+1}-T_j}$. Now Lemma 3.1 yields the conclusion. □

In the following, we first give some examples for possible choices of ξ fulfilling the assumptions of Proposition 3.2 and then continue with examples for η. Hence, any combination of them yields an exponential integral which is a GGC.

Example 3.3 (The case when X_1 is a normal random variable with positive mean). We see that $\mathcal{L}(e^{-X_1})$ is log-normal and hence is in $H(\mathbb{R}_+)$.

Example 3.4 (The case when X_1 is the logarithm of the power of a gamma random variable). Let $Y = \gamma_{r,\lambda}$ and $X_1 = c \log Y$ for $c \in \mathbb{R}$. Recall that $\mathcal{L}(\log Y) \in T(\mathbb{R})$ and so $\mathcal{L}(cX_1) = \mathcal{L}(\log Y^c) \in T(\mathbb{R})$ for $c \in \mathbb{R}$. Note that $E[X_1] = c\psi(\lambda)$, where $\psi(x)$ the derivative of $\log \Gamma(x)$. If we take $c \in \mathbb{R}$ such that $c\psi(\lambda) > 0$, we conclude that $\mathcal{L}(e^{-X_1}) = \mathcal{L}(\gamma_{r,\lambda}^{-c}) \in H(\mathbb{R}_+)$.

Example 3.5 (The case when X_1 is logarithm of some positive strictly stable random variable).
Let $X_1 = \log Y$ be a random variable, where Y is a positive stable random variable with parameter $0 < \alpha < 1$. Then X_1 is in the class of EGGCs when $\alpha = 1/n, n = 2, 3, \ldots$ (see Bondesson [4], Example 7.2.5) and

$$E[e^{uX_1}] = E[Y^u] = \frac{\Gamma(1 - u/\alpha)}{\Gamma(1 - u)}.$$

It follows that $E[X_1] = -\frac{1}{\alpha}\psi(1) + \psi(1) = (1 - 1/\alpha)\psi(1) > 0$ and $\mathcal{L}(e^{-X_1}) = \mathcal{L}(Y^{-1}) \in H(\mathbb{R}_+)$ by [4], Example 5.6.2.

Example 3.6 (The case when X_1 is the logarithm of the ratio of two exponential random variables).
Let $X_1 = \log(Y_1/Y_2)$, where $Y_j, j = 1, 2$, are independent exponential random variables with parameters $\lambda_j > 0, j = 1, 2$. The density function of X_1 is given by [4], Example 7.2.4,

$$f(x) = \frac{1}{B(\lambda_1, \lambda_2)} \frac{e^{-\lambda_1 x}}{(1 + e^{-x})^{\lambda_1 + \lambda_2}}, \qquad x \in \mathbb{R},$$

where $B(\cdot, \cdot)$ denotes the Beta-function. Now if $E[X_1] > 0$ we can set X_1 to be a jump distribution of ξ. It is easy to see that $\mathcal{L}(e^{-X_1}) = \mathcal{L}(Y_2/Y_1) \in H(\mathbb{R}_+)$ since $\mathcal{L}(Y_j) \in H(\mathbb{R}_+)$.

Example 3.7 (The case when η is nonrandom). When $\eta_t = t$, it holds that $\mathcal{L}(\eta_\tau) = \mathcal{L}(\tau) \in T(\mathbb{R}_+)$.

Example 3.8 (The case when η is a stable subordinator). The Laplace transform of $B := \eta_\tau$ with $\tau = \gamma_{1,\lambda}$ is given by (see, e.g., Steutel and van Harn [16], page 10)

$$\mathbb{L}_B(u) = \frac{\lambda}{\lambda - \log \mathbb{L}_\eta(u)}.$$

Now consider η to be a stable subordinator without drift. Then the Laplace transform of η_1 is given by $\mathbb{L}_{\eta_1}(u) = \exp\{-u^\alpha\}$. Therefore, we have

$$\mathbb{L}_B(u) = \frac{\lambda}{\lambda + u^\alpha}.$$

This function is HCM, since $\frac{\lambda}{\lambda+u}$ is HCM by the definition and due to the fact that the composition of an HCM function and $x^\alpha, |\alpha| \leq 1$, is also HCM (see [4], page 68). Thus, the Laplace transform of B is HCM by Proposition 2.1 and we conclude that $\mathcal{L}(\eta_\tau)$ is GGC.

Remark that if η admits an additional drift term, the distribution $\mathcal{L}(B)$ is not GGC. This result was pointed out by Kozubowski [10].

Example 3.9 (The case when η is an inverse Gaussian Lévy process). We suppose η to be an inverse Gaussian subordinator with parameters $\beta > 0$ and $\delta > 0$. The Laplace transform of η_t is

$$\mathbb{L}_{\eta_t}(u) = \exp\left(-\delta t\left(\sqrt{\beta^2 + 2u} - \beta\right)\right).$$

Now by choosing the parameters satisfying $\lambda \geq \delta\beta$, we have, for $B = \eta_\tau$,

$$\mathbb{L}_B(u) = \frac{\lambda}{\lambda - \delta\beta + \delta\sqrt{\beta^2 + 2u}}.$$

This function is HCM by argumentation as in Example 3.8 with $\alpha = 1/2$ and using Property (xi) in [4], page 68.

Remark 3.10. Although the Lévy measure of η_τ is known explicitly, it is an open question whether the parameter of the exponentially distributed random variable τ has an influence on the GGC-property of η_τ, or not. Examples 3.7 and 3.8 lead to the conjecture that there is no influence. So far, no counterexamples to this conjecture are known to the authors.

Case 2: *The process η is a compound Poisson process*

In the following, we assume the integrator η to be a compound Poisson process, while ξ is an arbitrary Lévy process, independent of η. We can argue similarly as above to obtain the following result.

Proposition 3.11. *Let ξ and η be independent and assume $\eta_t = \sum_{i=1}^{N_t} Y_i$ to be a compound Poisson process with i.i.d. jump heights Y_i, $i = 1, 2, \ldots$. Suppose that $E[\xi_1] > 0$, $E\log^+ |\eta_1| < \infty$, $\mathcal{L}(Y_1) \in T(\mathbb{R}_+)$ and $\mathcal{L}(e^{-\xi_\tau}) \in H(\mathbb{R}_+)$ for an exponentially distributed random variable τ independent of ξ having the same distribution as the waiting times of N. Then the integral (1.1) converges a.s. and it holds that*

$$\mathcal{L}\left(\int_{(0,\infty)} e^{-\xi_{t-}}\,\mathrm{d}\eta_t\right) \in T(\mathbb{R}_+).$$

Furthermore, if $\mathcal{L}(e^{-\xi_\tau}) \in \widetilde{H}(\mathbb{R})$, $\mathcal{L}(Y_1) \in T(\mathbb{R})$ and $\mathcal{L}(Y_1)$ is symmetric, then $\mathcal{L}(V) \in T(\mathbb{R})$.

Proof. Convergence of the integral is guaranteed by (2.2). Now set $T_0 = 0$ and let T_j, $j = 1, 2, \ldots$, be the jump times of $\{N_t, t \geq 0\}$. Then we have

$$\int_{(0,\infty)} e^{-\xi_{t-}}\,\mathrm{d}\eta_t = \sum_{j=1}^{\infty} e^{-\xi_{T_j}} Y_j = \sum_{j=1}^{\infty} e^{-(\xi_{T_j} - \xi_{T_{j-1}})} \cdots e^{-(\xi_{T_1} - \xi_{T_0})} Y_j$$

$$= \sum_{j=1}^{\infty} \left(\prod_{i=1}^{j} e^{-(\xi_{T_i} - \xi_{T_{i-1}})}\right) Y_j =: \sum_{j=1}^{\infty} \left(\prod_{i=1}^{j} A_i\right) B_j,$$

where $A_i = e^{-(\xi_{T_i} - \xi_{T_{i-1}})} \overset{d}{=} e^{-\xi_{T_i - T_{i-1}}}$ and $B_j = Y_j$. Remark that the proof of Lemma 3.1 remains valid even if the summation starts from $j = 1$. Hence, the assertion follows from Lemma 3.1. □

3.2. Dependent component case

In this subsection, we generalize a model of Lindner and Sato [11] and study to which class $\mathcal{L}(V)$ belongs.

Let $0 < p < 1$. Suppose that (ξ, η) is a bivariate compound Poisson process with parameter $\lambda > 0$ and normalized Lévy measure

$$\nu(\mathrm{d}x, \mathrm{d}y) = p\delta_0(\mathrm{d}x)\rho_0(\mathrm{d}y) + (1 - p)\delta_1(\mathrm{d}x)\rho_1(\mathrm{d}y),$$

where ρ_0 and ρ_1 are probability measures on $(0, \infty)$ and $[0, \infty)$, respectively, such that

$$\int_{(1,\infty)} \log y \, \mathrm{d}\rho_0(y) < \infty \quad \text{and} \quad \int_{(1,\infty)} \log y \, \mathrm{d}\rho_1(y) < \infty.$$

For the bivariate compound Poisson process (ξ, η), we have the following representation (see Sato [15], page 18):

$$(\xi_t, \eta_t) = \sum_{k=0}^{N_t} S_k = \left(\sum_{k=0}^{N_t} S_k^{(1)}, \sum_{k=0}^{N_t} S_k^{(2)} \right),$$

where $S_0^{(1)} = S_0^{(2)} = 0$ and $\{S_k = (S_k^{(1)}, S_k^{(2)})\}_{k=1}^{\infty}$ is a sequence of two-dimensional i.i.d. random variables. It implies that the projections of the compound Poisson process on $\mathbb{R}^2$ are also compound Poisson processes. Precisely in the given model, since $\mathbb{P}(S_1^{(1)} = 0) = p$ and $\mathbb{P}(S_1^{(1)} = 1) = 1 - p$, the marginal process ξ is a Poisson process with parameter $(1 - p)\lambda > 0$. Note that $S_k^{(1)}$ and $S_k^{(2)}$ may be dependent for any $k \in \mathbb{N}$. In this case, $\rho_i(B)$ is equal to $\mathbb{P}(S_k^{(2)} \in B | S_k^{(1)} = i)$ for $i = 0, 1$ and $B \in \mathcal{B}(\mathbb{R})$.

Example 3.12. In Lindner and Sato [11], the authors considered the bivariate compound Poisson process with parameter $u + v + w$, $u, v, w \geq 0$ and normalized Lévy measure

$$\nu(\mathrm{d}x, \mathrm{d}y) = \frac{v}{u + v + w}\delta_0(\mathrm{d}x)\delta_1(\mathrm{d}y) + \frac{u + w}{u + v + w}\delta_1(\mathrm{d}x)\left(\frac{u}{u + w}\delta_0(\mathrm{d}y) + \frac{w}{u + w}\delta_1(\mathrm{d}y) \right).$$

So in their setting $p = \frac{v}{u+v+w}$, $\rho_0 = \delta_1$ and $\rho_1 = \frac{u}{u+w}\delta_0 + \frac{w}{u+w}\delta_1$.

In the following theorem, we give a sufficient condition for $\mathcal{L}(V)$, given by (1.1) with (ξ, η) as described above, to be GGC.

Theorem 3.13. *If the function $\frac{(1-p)\mathbb{L}_{\rho_1}(u)}{1-p\mathbb{L}_{\rho_0}(u)}$ is HCM, then $\mathcal{L}(V)$ is GGC.*

Proof. Define T_ξ and M to be the first jump time of the Poisson process ξ and the number of the jumps of the bivariate compound Poisson process (ξ, η) before T_ξ, respectively. Due to the

strong Markov property of the Lévy process (ξ, η), we have

$$\int_{(0,\infty)} \exp(-\xi_{s-}) \, d\eta_s = \int_{(0,T_\xi]} \exp(-\xi_{s-}) \, d\eta_s + \int_{(T_\xi,\infty]} \exp(-\xi_{s-}) \, d\eta_s$$

$$= \eta_{T_\xi} + \int_{(0,\infty)} \exp(-\xi_{T_\xi+s-}) \, d\eta_{s+T_\xi}$$

$$= \eta_{T_\xi} + \exp(-\xi_{T_\xi}) \int_{(0,\infty)} \exp\left(-(\xi_{T_\xi+s-} - \xi_{T_\xi})\right) d\left((\eta_{(s+T_\xi)} - \eta_{T_\xi}) + \eta_{T_\xi}\right)$$

$$\overset{d}{=} \eta_{T_\xi} + e^{-1} \int_{(0,\infty)} \exp(-\widetilde{\xi}_{s-}) \, d\widetilde{\eta}_s,$$

where the process $(\widetilde{\xi}, \widetilde{\eta})$ is independent of $\{(\xi_t, \eta_t), t \le T_\xi\}$ and has the same law as (ξ, η). Therefore, we have

$$\mathbb{L}_\mu(u) = \mathbb{L}_\mu(e^{-1}u)\mathbb{L}_\rho(u), \tag{3.1}$$

with $\mu = \mathcal{L}(V)$ and ρ denoting the distribution of η_{T_ξ}. Thus, μ is e^{-1}-decomposable and it follows that

$$\mathbb{L}_\mu(u) = \prod_{n=0}^{\infty} \mathbb{L}_\rho(e^{-n}u).$$

In the given setting, we have

$$\eta_{T_\xi} = \left(\sum_{k=0}^{M} S_k^{(2)}\right) + S_{M+1}^{(2)},$$

where M is geometrically distributed with parameter p, namely,

$$\mathbb{P}(M = k) = (1 - p)p^k \qquad \text{for any } k \in \mathbb{N}_0.$$

Hence, we obtain

$$\mathbb{L}_\rho(u) = \mathbb{E}[\exp(-u\eta_{T_\xi})]$$

$$= \mathbb{E}\left[\mathbb{E}\left[\exp\left(-u\left(\sum_{k=0}^{M} S_k^{(2)} + S_{M+1}^{(2)}\right)\right)\Big| M\right]\right] = \mathbb{E}[(\mathbb{L}_{\rho_0}(u))^M \mathbb{L}_{\rho_1}(u)]$$

$$= (1 - p)\mathbb{L}_{\rho_1}(u) \sum_{k=0}^{\infty} (p\mathbb{L}_{\rho_0}(u))^k = \frac{(1 - p)\mathbb{L}_{\rho_1}(u)}{1 - p\mathbb{L}_{\rho_0}(u)}.$$

The class of HCM functions is closed under scale transformation, multiplication and limit. Therefore, $\mathbb{L}_\mu(u)$ is HCM if $\mathbb{L}_\rho(u)$ is HCM and hence μ is GGC if ρ is GGC by Proposition 2.1(6). As a result, μ is GGC if $\frac{(1-p)\mathbb{L}_{\rho_1}(u)}{1-p\mathbb{L}_{\rho_0}(u)}$ is HCM. $\qquad \square$

A distribution with the Laplace transform $\frac{(1-p)}{1-p\mathbb{L}_{\rho_0}(u)}$ is called a compound geometric distribution. It is compound Poisson, because every geometric distribution is compound Poisson with Lévy measure given by

$$\nu_p(\{k\}) = -\frac{1}{\log(1-p)}\frac{1}{k+1}p^{k+1}, \qquad k = 1, 2, \dots$$

(see page 147 in Steutel and van Harn [16]). Since HCM functions are closed under multiplication, we have the following.

Corollary 3.14. *If ρ_1 and the compound geometric distribution of ρ_0 are GGCs, then so is $\mathcal{L}(V)$.*

In addition, we observe the following.

Corollary 3.15.

(1) *For any $c > 1$, the distribution $\mu_c = \mathcal{L}(\int_{(0,\infty)} c^{-\xi_{s-}} \, d\eta_s)$ is c^{-1}-selfdecomposable. Thus, in the nondegenerate case it is absolutely continuous or continuous singular (Wolfe [17]) and Theorem 3.13 holds true also for μ_c instead of μ.*

(2) *If ρ_1 is infinitely divisible, then μ_c is also infinitely divisible.*

(3) *Let $B(\mathbb{R}_+)$ be the Goldie–Steutel–Bondesson class, which is the smallest class that contains all mixtures of exponential distributions and is closed under convolution and weak convergence.*

If $\frac{(1-p)}{1-p\mathbb{L}_{\rho_0}(u)}$ is the Laplace transform of a distribution in $B(\mathbb{R}_+)$, then μ_c is in $B(\mathbb{R}_+)$. Moreover, μ_c will be a c^{-1}-semi-selfdecomposable distribution.

About the definition and basic properties of semi-selfdecomposable distributions, see [15]. The proof of (1) is obvious. For (2), remark that a distribution with Laplace transform $\frac{(1-p)}{1-p\mathbb{L}_{\rho_0}(u)}$ as compound Poisson distribution is always infinitely divisible. Hence, only ρ_1 has influence on that property. The proof of (3) follows from the characterization of the class $B(\mathbb{R}_+)$ in Chapter 9 of [4] and our proof of Theorem 3.13.

Example 3.16. Let ρ_1 be a GGC, that is, $\mathbb{L}_{\rho_1}(u)$ is HCM. Then if $\frac{(1-p)}{1-p\mathbb{L}_{\rho_0}(u)}$ is HCM, μ is found to be GGC. For example, if ρ_0 is an exponential random variable with density $f(x) = be^{-bx}$, $b > 0$, then $\frac{1}{1-p\mathbb{L}_{\rho_0}(u)}$ is HCM. To see this, for $u > 0$, $v > 0$, write

$$\frac{1}{1-p\mathbb{L}_{\rho_0}(uv)}\frac{1}{1-p\mathbb{L}_{\rho_0}(u/v)} = \frac{1+(u/b)(v+v^{-1})+u^2/b^2}{(1-p)^2+(u/b)(v+v^{-1})(1-p)+u^2/b^2}$$

$$= \frac{1}{1-p} + \frac{p+(1-1/(1-p))u^2/b^2}{(1-p)^2+(u/b)(v+v^{-1})(1-p)+u^2/b^2}.$$

This is nonnegative and completely monotone as a function of $v + v^{-1}$.

Example 3.17. In the case of Example 3.12, the Lévy measure of μ_c is

$$\nu_{\mu_c} = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} a_m \delta_{c^{-n}m},$$

where $a_m = \frac{1}{m}(q^m - (-r/p)^m)$. This Lévy measure is not absolutely continuous. Thus, μ_c is never GGC for any parameters u, v, w and c.

4. Exponential integrals for independent increment processes

We say that a process $X = \{X_t = \sum_{i=1}^{M_t} X_i, t \geq 0\}$ is a compound sum process, if the $\{X_i\}$ are i.i.d. random variables, $\{M_t, t \geq 0\}$ is a renewal process and they are independent. When $\{M_t\}$ is a Poisson process, X is nothing but a compound Poisson process and is a Lévy process. Unless $\{M_t\}$ is a Poisson process, X is no Lévy process. In this section, we consider the case when either ξ or η is a compound sum process and the other is an arbitrary Lévy process. Although (ξ, η) is not a Lévy process, the exponential integral (1.1) can be defined and its distribution can be infinitely divisible and/or GGC in many cases as we will show in the following.

Case 1: The process ξ is a compound sum process

First, we give a condition for the convergence of the exponential integral (1.1) when (ξ, η) is not a Lévy process.

Proposition 4.1. *Suppose that $(\xi_t, \eta_t)_{t \geq 0}$ is a stochastic process where ξ and η are independent, η is a Lévy process and $\xi_t = \sum_{i=1}^{M_t} X_i$ is a compound sum process with i.i.d. jump heights X_i, $i = 1, 2, \ldots$, and i.i.d. waiting times W_i. Then (1.1) converges in probability to a finite random variable if and only if*

$$\xi_t \to \infty \quad a.s. \quad and \quad \int_{(1,\infty)} \left(\frac{\log q}{A_\xi(\log q)} \right) P(|\eta_{W_1}| \in dq) < \infty \tag{4.1}$$

for $A_\xi(x) = \int_{(0,x)} P(X_1 > u)\,du$.

Proof. As argued in the proof of Proposition 3.2, we can rewrite the exponential integral as perpetuity

$$\int_{(0,\infty)} e^{-\xi_{t-}}\,d\eta_t = \sum_{j=0}^{\infty} \left(\prod_{i=1}^{j} A_i \right) B_j,$$

where $A_i = e^{-X_i}$ and $B_j \overset{d}{=} \eta_{W_j}$. By Theorem 2.1 of [6] the above converges a.s. to a finite random variable if and only if $\prod_{i=1}^{n} A_i \to 0$ a.s. and

$$\int_{(1,\infty)} \left(\frac{\log q}{A(\log q)} \right) P(|B_1| \in dq) < \infty$$

for $A(x) = \int_{(0,x)} P(-\log A_1 > u)\,\mathrm{d}u$. Using the given expressions for A_1 and B_1 in our setting, we observe that this is equivalent to (4.1). It remains to show that a.s. convergence of the perpetuity implies convergence in probability of (1.1). Therefore, remark that

$$\int_{(0,t]} \mathrm{e}^{-\xi_{s-}}\,\mathrm{d}\eta_s = \int_{(0,T_{M_t}]} \mathrm{e}^{-\xi_{s-}}\,\mathrm{d}\eta_s + \mathrm{e}^{-\xi_{T_{M_t}}}(\eta_t - \eta_{T_{M_t}}),$$

where the first term converges to a finite random variable while the second converges in probability to 0 since $\sup_{t \in [T_{M_t}, T_{M_t+1})} |\eta_t - \eta_{T_{M_t}}| \overset{d}{=} \sup_{t \in [0, W_1)} |\eta_t|$. $\square$

Now we can extend Proposition 3.2 in this new setting as follows.

Proposition 4.2. *Suppose that the processes ξ and η are independent and that $\xi_t = \sum_{i=1}^{M_t} X_i$ is a compound sum process with i.i.d. jump heights X_i, $i = 1, 2, \ldots$, and i.i.d. waiting times W_i, $i = 1, 2, \ldots$, such that (4.1) is fulfilled. Suppose that $\mathcal{L}(\mathrm{e}^{-X_1}) \in H(\mathbb{R}_+)$ and $\mathcal{L}(\eta_\tau) \in T(\mathbb{R}_+)$ for τ being a random variable with the same distribution as W_1 and independent of η. Then*

$$\mathcal{L}\left(\int_{(0,\infty)} \mathrm{e}^{-\xi_{t-}}\,\mathrm{d}\eta_t \right) \in T(\mathbb{R}_+).$$

Furthermore, if $\mathcal{L}(\mathrm{e}^{-X_1}) \in \widetilde{H}(\mathbb{R})$, $\mathcal{L}(\eta_\tau) \in T(\mathbb{R})$ and $\mathcal{L}(\eta_\tau)$ is symmetric, then $\mathcal{L}(V) \in T(\mathbb{R})$.

In the following, we give some examples fulfilling the assumptions of Proposition 4.2.

Example 4.3 (The case when η is nonrandom and $\mathcal{L}(W_1)$ is GGC). For the case $\eta_t = t$, $\mathcal{L}(\eta_\tau)$ belongs to $T(\mathbb{R}_+)$ if and only if $\mathcal{L}(\tau)$ does. Hence, for all waiting times which are GGCs and for a suitable jump heights of ξ, we have $\mathcal{L}(V) \in T(\mathbb{R}_+)$.

Example 4.4 (The case when η is a stable subordinator and $\mathcal{L}(W_1)$ is GGC). Consider η to be a stable subordinator having Laplace transform $\mathbb{L}_\eta(u) = \exp\{-u^\alpha\}$ with $0 < \alpha < 1$. Then the Laplace transform of $B := \eta_\tau$ is given by $\mathbb{L}_B(u) = \mathbb{L}_\tau(u^\alpha)$. This function is HCM if and only if τ is GGC, since by Proposition 2.1, $\mathbb{L}_\tau$ is HCM and hence also its composition with x^α. Thus, whenever $\mathcal{L}(\tau) = \mathcal{L}(W_1)$ is GGC, $\mathcal{L}(\eta_\tau)$ is GGC, too, fulfilling the assumption of Proposition 4.2.

Example 4.5 (The case when η is a standard Brownian motion and $\mathcal{L}(W_1)$ is GGC). Given that η is a standard Brownian motion, η_1 has characteristic function $E\mathrm{e}^{\mathrm{i}z\eta_1} = \exp(-z^2/2)$, which yields $\mathbb{L}_B(u) = \mathbb{L}_\tau(u^2/2)$. We can not see $\mathcal{L}(B) \in T(\mathbb{R}_+)$ from this, and in fact $\mathcal{L}(B)$ is in $T(\mathbb{R})$ but not in $T(\mathbb{R}_+)$ (see Bondesson [4], page 117). Then using that η is symmetric, we can apply (4) in Proposition 2.1 and conclude that $\mathcal{L}(V) \in T(\mathbb{R})$ for suitable jump heights of ξ.

Example 4.6 (The case when η is a Lévy subordinator and $\mathcal{L}(W_1)$ is a half normal distribution). The $1/2$-stable subordinator η and the standard half normal random variable τ have densities,

respectively, given by,

$$f_{\eta_t}(x) = \frac{t}{2\sqrt{\pi}}x^{-3/2}e^{-t^2/2x} \quad \text{and} \quad f_\tau(x) = \sqrt{\frac{2}{\pi}}e^{-x^2/2}, \qquad x > 0.$$

These yield the density function of η_τ as

$$f_{\eta_\tau}(x) = \int f_{\eta_y}(x)f_\tau(y)\,\mathrm{d}y = \frac{1}{\sqrt{2\pi}}\frac{x^{-1/2}}{1+x}.$$

Interestingly, this is an F distribution (see Sato [15], page 46) and since a random variable with an F distribution is constricted to be the quotient of two independent gamma random variables, we have that $\mathcal{L}(\eta_\tau) \in T(\mathbb{R}_+)$.

Case 2: The process η_t is a compound sum process

Again we start with a condition for the convergence of (1.1). It can be shown similar to Proposition 4.1.

Proposition 4.7. *Suppose ξ to be a Lévy process and that $\eta_t = \sum_{i=1}^{M_t} Y_i$ is a compound sum process with i.i.d. jump heights $Y_i, i = 1, 2, \ldots$, and i.i.d. waiting times $U_i, i = 1, 2, \ldots$. Then (1.1) converges a.s. to a finite random variable if and only if*

$$\xi_t \to \infty \qquad a.s. \quad and \quad \int_{(1,\infty)} \left(\frac{\log q}{A_\eta(\log q)}\right) P(|Y_1| \in \mathrm{d}q) < \infty \tag{4.2}$$

for $A_\eta(x) = \int_{(0,x)} P(\xi_{U_1} > u)\,\mathrm{d}u$.

In the same manner as before, we can now extend Proposition 3.11 to the new setting and obtain the following result.

Proposition 4.8. *Let ξ and η be independent and assume $\eta_t = \sum_{i=1}^{M_t} Y_i$ to be a compound renewal process with i.i.d. jump heights $Y_i, i = 1, 2, \ldots$, and i.i.d. waiting times U_i such that (4.2) holds. Suppose that $\mathcal{L}(Y_1) \in T(\mathbb{R}_+)$ and $\mathcal{L}(e^{-\xi_\tau}) \in H(\mathbb{R}_+)$ for a random variable τ having the same distribution as U_1 and being independent of ξ. Then*

$$\mathcal{L}\left(\int_{(0,\infty)} e^{-\xi_{t-}}\,\mathrm{d}\eta_t\right) \in T(\mathbb{R}_+).$$

Furthermore, if $\mathcal{L}(e^{-\xi_\tau}) \in \widetilde{H}(\mathbb{R})$, $\mathcal{L}(Y_1) \in T(\mathbb{R})$ and $\mathcal{L}(Y_1)$ is symmetric, then $\mathcal{L}(V) \in T(\mathbb{R})$.

The following is a very simple example fulfilling the assumptions in Proposition 4.8.

Example 4.9 (The case when η is a random walk and ξ is a standard Brownian motion with drift). Suppose $\xi_t = B_t + at$ is a standard Brownian motion with drift $a > 0$ and U_1 is degenerated at 1. Then $\mathcal{L}(\mathrm{e}^{-\xi_\tau}) = \mathcal{L}(\mathrm{e}^{-a}\mathrm{e}^{-B_1})$ is a scaled log-normal distribution and hence in $H(\mathbb{R}_+)$. So for all GGC jump heights $\mathcal{L}(Y_1)$, the exponential integral is GGC.

Acknowledgements

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