Bernoulli **18**(1), 2012, 24–45
DOI: 10.3150/10-BEJ327

Stochastic delay equations with non-negativity constraints driven by fractional Brownian motion

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In this note we prove an existence and uniqueness result for the solution of multidimensional stochastic delay differential equations with normal reflection. The equations are driven by a fractional Brownian motion with Hurst parameter $H > 1/2$. The stochastic integral with respect to the fractional Brownian motion is a pathwise Riemann–Stieltjes integral.

Keywords: fractional Brownian motion; normal reflection; Riemann–Stieltjes integral; stochastic delay equation

1. Introduction

Consider a stochastic delay differential equation with positivity constraints. More precisely, we deal with a stochastic delay differential equation with normal reflection on $\mathbb{R}^d$ of the form

$$\begin{aligned} X(t) &= \eta(0) + \int_0^t b(s, X)\,\mathrm{d}s + \int_0^t \sigma\bigl(s, X(s-r)\bigr)\,\mathrm{d}W_s^H + Y(t), \qquad t \in (0, T], \\ X(t) &= \eta(t), \qquad t \in [-r, 0]. \end{aligned} \tag{1.1}$$

Here, r denotes a strictly positive time delay, $W^H = \{W^{H,j}, j = 1, \ldots, m\}$ are independent fractional Brownian motions with Hurst parameter $H > \frac{1}{2}$ defined in a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, the hereditary term $b(s, X)$ depends on the path $\{X(u), -r \le u \le s\}$, while $\eta : [-r, 0] \to \mathbb{R}_+^d$ is a non-negative smooth function with $\mathbb{R}_+^d = \{u \in \mathbb{R}^d; u_i \ge 0 \text{ for } i = 1, \ldots, d\}$ and Y is a vector-valued non-decreasing process which ensures that the non-negativity constraints on X are enforced.

Set

$$Z(t) = \eta(0) + \int_0^t b(s, X)\,\mathrm{d}s + \int_0^t \sigma\bigl(s, X(s-r)\bigr)\,\mathrm{d}W_s^H, \qquad t \in [0, T]. \tag{1.2}$$

It is known that we have an explicit formula for the regulator term Y in terms of Z: for each $i = 1, \ldots, d$,

$$Y^i(t) = \max_{s\in[0,t]} (Z^i(s))^-, \qquad t \in [0, T].$$

The solution of (1.1) then satisfies

$$X(t) = \begin{cases} Z(t) + Y(t), & t \in [0, T], \\ \eta(t), & t \in [-r, 0]. \end{cases}$$

We call (1.1) a delay differential equation with reflection with hereditary drift driven by a fractional Brownian motion. To the best of our knowledge, this problem has not been considered before in the extensive literature on stochastic differential equations.

There are many references to stochastic systems with delay (see, as a basic reference, [9]), but the literature concerning stochastic differential equations with delay driven by a fractional Brownian motion is scarce. The existence and uniqueness of solutions [2,7,10,12], existence and regularity of the density [7] and convergence when the delay goes to zero [3] have all been studied.

On the other hand, there has been little work on stochastic differential equations with delay and non-negativity constraints. We can only refer the reader to the book by Kushner [5] dedicated to the study of numerical methods for this class of equations, and the paper of Kinnally and Williams [6], where the authors obtain sufficient conditions for existence and uniqueness of stationary solutions for stochastic differential equations with delay and non-negativity constraints driven by standard Brownian motion.

As described in the paper of Kinnally and Williams [6], there are some models affected by some types of noise where the dynamics are related to propagation delay and some of them are naturally non-negative quantities, for instance, applications such as rates and prices in Internet models, and concentrations of ions or proportions of a population that are infected (see the references in [6]). Therefore, it is natural to consider stochastic differential equations with delay and non-negativity constraints. In this paper, we initiate the study when the noise is not a standard Brownian motion but a fractional Brownian motion.

The main novelty of this paper is the use of non-negative constraints dealing with fractional Brownian motion. We have used Skorokhod's mapping.

Set

$$\mathcal{C}_+(\mathbb{R}_+, \mathbb{R}^d) := \{x \in \mathcal{C}(\mathbb{R}_+, \mathbb{R}^d) : x(0) \in \mathbb{R}^d_+\}.$$

We now recall the Skorokhod problem.

Definition 1.1. *Given a path $z \in \mathcal{C}_+(\mathbb{R}_+, \mathbb{R}^d)$, we say that a pair (x, y) of functions in $\mathcal{C}_+(\mathbb{R}_+, \mathbb{R}^d)$ solves the Skorokhod problem for z with reflection if:*

1. $x(t) = z(t) + y(t)$ *for all* $t \geq 0$ *and* $x(t) \in \mathbb{R}^d_+$ *for each* $t \geq 0$;
2. *for each* $i = 1, \dots, d$, $y^i(0) = 0$ *and* y^i *is non-decreasing*;
3. *for each* $i = 1, \dots, d$, $\int_0^t x^i(s)\, \mathrm{d}y^i(s) = 0$ *for all* $t \geq 0$, *so* y^i *can increase only when* x^i *is at zero.*

It is known that we have an explicit formula for y in terms of z: for each $i = 1, \dots, d$,

$$y^i(t) = \max_{s \in [0,t]} \left(z^i(s)\right)^-.$$

The path z is called the *reflector* of x and the path y is called the *regulator* of x. We use the Skorokhod mapping for constraining a continuous real-valued function to be non-negative by means of reflection at the origin. We will apply it to each path of z defined by (1.2). Note that because we are dealing with a multidimensional case, the mapping will be applied to each component.

We must also explain how to understand the stochastic integral appearing in (1.1). Since $H > \frac{1}{2}$, the stochastic integral in (1.1) is defined using a pathwise approach. Indeed, if we have a stochastic process $\{u(t), t \geq 0\}$ whose trajectories are λ-Hölder continuous with $\lambda > 1 - H$, then the Riemann–Stieltjes integral $\int_0^T u(s)\, \mathrm{d}W_s^H$ exists for each trajectory (see Young [13]). Using the techniques introduced by Young [13] and the p-variation norm, Lyons [8] began the study of integral equations driven by functions with bounded p-variation, where $p \in [1, 2)$. Zähle [14] introduced a generalized Stieltjes integral using the techniques of fractional calculus. The integral is expressed in terms of fractional derivative operators and coincides with the Riemann–Stieltjes integral $\int_0^T f\, \mathrm{d}g$ when the functions f and g are Hölder continuous of orders λ and β, respectively, with $\lambda + \beta > 1$. Using this Riemann–Stieltjes integral, Nualart and Rascanu [11] obtained the existence and uniqueness of a solution for a class of multidimensional integral equations.

In this paper, also using the Riemann–Stieltjes integral, we will prove the existence and uniqueness of a solution to equation (1.1). Our results are inspired by those in Nualart and Rascanu [11] and Ferrante and Rovira [3]. Using some estimates presented in those papers, we will first prove our results for deterministic equations and will then easily apply them pathwise to fractional Brownian motion.

Since our definition of the stochastic integral holds for $H > \frac{1}{2}$, we cannot extend our approach to the case $H \in (\frac{1}{3}, \frac{1}{2})$. However, in a forthcoming paper, we will use the method used by Hu and Nualart in [4] to consider the case $H < \frac{1}{2}$.

The structure of the paper is as follows. In the next section, we give our hypothesis and we state the main results of our paper. In Section 3 we give some useful estimates for Lebesgue and Riemann–Stieltjes integrals inspired by the results in [11] and [3]. Section 4 is devoted to proving our main result: the existence, uniqueness and boundedness of solutions to deterministic equations. In Section 5 we recall how to apply the deterministic results to the stochastic case, while the Appendix is devoted to giving some technical results such as a fixed point theorem and some properties related to the Skorokhod problem.

2. Main results

Let $\alpha \in (0, \frac{1}{2})$ and $r > 0$. Let $(s, t) \subseteq [-r, T]$ and denote by $W_0^{\alpha,\infty}(s, t; \mathbb{R}^d)$ the space of measurable functions $f : [s, t] \to \mathbb{R}^d$ such that

$$\|f\|_{\alpha,\infty(s,t)} := \sup_{u\in[s,t]} \left(|f(u)| + \int_s^u \frac{|f(u) - f(v)|}{(u - v)^{\alpha+1}}\, \mathrm{d}v \right) < \infty.$$

For any $0 < \lambda \le 1$, denote by $C^\lambda(s,t;\mathbb{R}^d)$ the space of λ-Hölder continuous functions $f:[s,t]\to\mathbb{R}^d$ such that

$$\|f\|_{\lambda(s,t)} := \|f\|_{\infty(s,t)} + \sup_{s\le u<v\le t} \frac{|f(v)-f(u)|}{(v-u)^\lambda} < \infty,$$

where

$$\|f\|_{\infty(s,t)} := \sup_{u\in[s,t]} |f(u)|.$$

Since we will use the spaces $W_0^{\alpha,\infty}(-r,T;\mathbb{R}^d)$ and $C^\lambda(-r,T;\mathbb{R}^d)$ extensively, we will use the notation $\|f\|_{\alpha,\infty(r)} := \|f\|_{\alpha,\infty(-r,T)}$, $\|f\|_{\alpha,\lambda(r)} := \|f\|_{\alpha,\lambda(-r,T)}$, $\|f\|_{\lambda(r)} := \|f\|_{\lambda(-r,T)}$ and $\|f\|_{\infty(r)} := \|f\|_{\infty(-r,T)}$.

Consider the following hypothesis.

(H1) $\sigma:[0,T]\times\mathbb{R}^d\to\mathbb{R}^d\times\mathbb{R}^m$ is a measurable function such that there exist some constants $\beta>0$ and $M_0>0$ such that the following properties hold:

1. $|\sigma(t,x)-\sigma(t,y)| \le M_0|x-y|\ \forall x,y\in\mathbb{R}^d, \forall t\in[0,T]$;
2. $|\sigma(t,x)-\sigma(s,x)| \le M_0|t-s|^\beta\ \forall x\in\mathbb{R}^d, \forall t,s\in[0,T]$.

(H2) $b:[0,T]\times C(-r,T;\mathbb{R}^d)\to\mathbb{R}^d$ is a measurable function such that for every $t>0$ and $f\in C(-r,T;\mathbb{R}^d)$, $b(t,f)$ depends only on $\{f(s); -r\le s\le t\}$. Moreover, there exists some $b_0\in L^\rho(0,t;\mathbb{R}^d)$ with $\rho\ge 2$ and $\forall N\ge 0$ there exists some $L_N>0$ such that:

1. $|b(t,x)-b(t,y)| \le L_N \sup_{-r\le s\le t}|x(s)-y(s)|$ $\forall x,y$ such that $\|x\|_{\infty(r)}\le N$, $\|y\|_{\infty(r)}\le N\ \forall t\in[0,T]$;
2. $|b(t,x)| \le L_0\sup_{-r\le s\le t}|x(s)| + b_0(t)\ \forall t\in[0,T]$.

(H3) There exist some $\gamma\in[0,1]$ and $K_0>0$ such that

$$|\sigma(t,x)| \le K_0(1+|x|^\gamma) \qquad \forall x\in\mathbb{R}^d,\ \forall t\in[0,T].$$

Under these assumptions, we are able to prove that our problem admits a unique solution. Our main result reads as follows.

Theorem 2.1. *Assume that $\eta\in W_0^{\alpha,\infty}(-r,0;\mathbb{R}_+^d)$, and that b and σ satisfy hypotheses* (H1) *and* (H2), *respectively, with $\beta>1-H$. Set $\alpha_0 := \min\{\frac{1}{2},\beta\}$. If $\alpha\in(1-H,\alpha_0)$ and $\rho\le\frac{1}{\alpha}$, then the equation* (1.1) *has a unique solution,*

$$X\in L^0(\Omega,\mathcal{F},\mathbb{P}; W_0^{\alpha,\infty}(-r,T;\mathbb{R}^d)),$$

and for P-almost all $\omega\in\Omega$, $X(\omega,\cdot)\in C^{1-\alpha}(0,T;\mathbb{R}^d)$.

Moreover, if $\alpha\in(1-H,\alpha_0\vee(2-\gamma)/4)$ and (H3) *holds, then $E(\|X\|^p_{\alpha,\infty(r)})<\infty\ \forall p\ge 1$.*

Examples. Note that the following equations satisfy our hypothesis:

(a) (linear example) for any $a, b \in \mathbb{R}$,

$$X(t) = r + \int_0^t X(s-r)\,\mathrm{d}s + \int_0^t \big(aX(s-r)+b\big)\,\mathrm{d}W_s^H + Y(t), \qquad t \in (0,T],$$
$$X(t) = t + r, \qquad t \in [-r, 0];$$

(b) (non-linear example)

$$X(t) = \int_0^t \cos(X(s))\,\mathrm{d}s + \int_0^t \sin\big(s + X(s-r)\big)\,\mathrm{d}W_s^H + Y(t), \qquad t \in (0,T],$$
$$X(t) = t^2, \qquad t \in [-r, 0].$$

3. Preliminaries

In this section, we give some useful estimates for Lebesgue and Riemann–Stieltjes integrals. These types of estimates were presented in the work of Nualart and Rascanu [11] and adapted to the delay case by Ferrante and Rovira [3]. Since our results are directly inspired by these works, we do not give the proofs, but instead direct the reader to these references.

We will need to introduce a new norm in the space $W_0^{\alpha,\infty}(s,t;\mathbb{R}^d)$: for any $\lambda \ge 1$,

$$\|f\|_{\alpha,\lambda(s,t)} := \sup_{u\in[s,t]} \exp(-\lambda u)\left(|f(u)| + \int_s^u \frac{|f(u)-f(v)|}{(u-v)^{\alpha+1}}\,\mathrm{d}v\right).$$

It is easy to check that for any $\lambda \ge 1$, this norm is equivalent to $\|f\|_{\alpha,\infty(s,t)}$.

3.1. Lebesgue integral

We first consider the ordinary Lebesgue integral. Given a measurable function $f : [-r, T] \to \mathbb{R}^d$, we define

$$F^{(b)}(f)(t) = \int_0^t b(u, f)\,\mathrm{d}u.$$

We first recall some estimates that constitute an obvious adaptation of [3], Proposition 2.2.

Proposition 3.1. *Assume that b satisfies* (H2) *with $\rho = \frac{1}{\alpha}$ and $[s,t] \subseteq [0,T]$. If $f \in W_0^{\alpha,\infty}(-r, t;\mathbb{R}^d)$, then $F^{(b)}(f)(\cdot) = \int_0^{\cdot} b(u,f)\,\mathrm{d}u \in C^{1-\alpha}(s,t;\mathbb{R}^d)$ and*

1. $\|F^{(b)}(f)\|_{1-\alpha(s,t)} \le d^{(1)}\big(1 + \|f\|_{\infty(-r,t)}\big)$,
2. $\|F^{(b)}(f)\|_{\alpha,\lambda(s,t)} \le d^{(2)}\left(\frac{1}{\lambda^{1-2\alpha}} + \frac{\|f\|_{\alpha,\lambda(-r,t)}}{\lambda^{1-\alpha}}\right)$

for all $\lambda \ge 1$, where $d^{(i)}$, $i \in \{1,2\}$, are positive constants depending only on α, t, L_0 and $B_{0,\alpha} = \|b_0\|_{L^{1/\alpha}}$.

3.2. Riemann–Stieltjes integral

Let us now consider the Riemann–Stieltjes integral introduced by Zähle, which is based on fractional integrals and derivatives. We refer the reader to the paper of Zähle [14] and the references therein for a detailed presentation of this generalized Stieltjes integral and the associated fractional calculus. Here, we will just recall some basic facts.

Fix a parameter $0 < \alpha < \frac{1}{2}$. Denote by $W_T^{1-\alpha,\infty}(0,T;\mathbb{R})$ the space of measurable functions $g:[0,T] \to \mathbb{R}$ such that

$$\|g\|_{1-\alpha,\infty,T} := \sup_{0<s<t<T} \left(\frac{|g(t)-g(s)|}{(t-s)^{1-\alpha}} + \int_s^t \frac{|g(y)-g(s)|}{(y-s)^{2-\alpha}}\,\mathrm{d}y \right) < \infty.$$

Moreover, if g belongs to $W_T^{1-\alpha,\infty}(0,T;\mathbb{R})$, then we define

$$\begin{aligned}\Lambda_\alpha(g) &:= \frac{1}{\Gamma(1-\alpha)} \sup_{0<s<t<T} |(D_{t-}^{1-\alpha} g_{t-})(s)| \\ &\leq \frac{1}{\Gamma(1-\alpha)\Gamma(\alpha)} \|g\|_{1-\alpha,\infty,T} < \infty,\end{aligned}$$

where $D_{t-}^{1-\alpha} g_{t-}$ denotes a fractional Weyl derivative,

$$g_{t-}(s) = \big(g(s) - g(t-)\big)\mathbf{1}_{(0,t)}(s),$$

where $g(t-) = \lim_{\varepsilon \searrow 0} g(t-\varepsilon)$ and Γ is the Euler function. We also denote by $W_0^{\alpha,1}(0,T;\mathbb{R})$ the space of measurable functions f on $[0,T]$ such that

$$\|f\|_{\alpha,1} := \int_0^T \frac{|f(s)|}{s^\alpha}\,\mathrm{d}s + \int_0^T \int_0^s \frac{|f(s)-f(y)|}{(s-y)^{\alpha+1}}\,\mathrm{d}u\,\mathrm{d}s < \infty.$$

Note that if f is a function in the space $W_0^{\alpha,1}(0,T;\mathbb{R})$ and $g \in W_T^{1-\alpha,\infty}(0,T;\mathbb{R})$, then the integral $\int_0^t f\,\mathrm{d}g$ exists for all $t \in [0,T]$ and we can define

$$G(f)(t) := \int_0^t f(s)\,\mathrm{d}g_s = \int_0^T f(s)\mathbf{1}_{(0,t)}(s)\,\mathrm{d}g_s.$$

Furthermore, the following estimate holds:

$$\left| \int_0^t f\,\mathrm{d}g \right| \leq \Lambda_\alpha(g)\|f\|_{\alpha,1}. \tag{3.1}$$

Moreover, if $f \in W_0^{\alpha,\infty}(0,T)$, it is proved in [11], Proposition 4.1, that for each $s < t$,

$$\left| \int_s^t f\,\mathrm{d}g \right| \leq \Lambda_\alpha(g) C_{\alpha,T}(t-s)^{1-\alpha}\|f\|_{\alpha,\infty}. \tag{3.2}$$

Let us consider the term

$$G_r^{(\sigma)}(f)(t) = \int_0^t \sigma\big(s, f(s-r)\big)\,\mathrm{d}g_s.$$

For the Riemann–Stieltjes integral, we will also give a version of [3], Proposition 2.4.

Proposition 3.2. *Let $g \in W_T^{1-\alpha,\infty}(0,T)$. Assume that σ satisfies* (H1) *and $[s,t] \subseteq [0,T]$. If $f \in W_0^{\alpha,\infty}(-r,T;\mathbb{R}^d)$, then*

$$G_r^{(\sigma)}(f) \in C^{1-\alpha}(s,t;\mathbb{R}^d) \subset W_0^{\alpha,\infty}(s,t;\mathbb{R}^d)$$

and

1. $\|G_r^{(\sigma)}(f)\|_{1-\alpha(s,t)} \leq \Lambda_\alpha(g) d^{(3)}\big(1 + \|f\|_{\alpha,\infty(-r,t-r)}\big)$,
2. $\|G_r^{(\sigma)}(f)\|_{\alpha,\lambda(s,t)} \leq \frac{\Lambda_\alpha(g) d^{(4)}}{\lambda^{1-2\alpha}}\big(1 + \|f\|_{\alpha,\lambda(-r,t-r)}\big)$

for all $\lambda \geq 1$, where $d^{(i)}$, $i \in \{3,4\}$, are positive constants independent of λ, f and g.

Finally, we recall [3], Proposition 2.6. Consider $\varphi(\gamma,\alpha)$ defined such that $\varphi(\gamma,\alpha) = 2\alpha$ if $\gamma = 1$, $\varphi(\gamma,\alpha) > 1 + \frac{2\alpha-1}{\gamma}$ if $\frac{1-2\alpha}{1-\alpha} \leq \gamma < 1$ and $\varphi(\gamma,\alpha) = \alpha$ if $0 \leq \gamma < \frac{1-2\alpha}{1-\alpha}$. Note that $\varphi(\gamma,\alpha) \in [\alpha, 2\alpha]$.

Proposition 3.3. *Let $g \in W_T^{1-\alpha,\infty}(0,T)$. Assume that σ satisfies* (H1) *and* (H3). *If $f \in W_0^{\alpha,\infty}(-r,T;\mathbb{R}^d)$, then*

$$\|G_r^{(\sigma)}(f)\|_{\alpha,\lambda} \leq \Lambda_\alpha(g) d^{(5)}\left(1 + \frac{\|f\|_{\alpha,\lambda(r)}}{\lambda^{1-\varphi(\gamma,\alpha)}}\right)$$

for all $\lambda \geq 1$, where $d^{(5)}$ is a positive constant depending only on α, β, T, d, m and $B_{0,\alpha} = \|b_0\|_{L^{1/\alpha}}$.

4. Deterministic integral equations

In this section, we give all the deterministic results.

For simplicity, let us assume that $T = Mr$. Set $0 < \alpha < \frac{1}{2}$, $g \in W_T^{1-\alpha,\infty}(0,T;\mathbb{R}^d)$ and $\eta \in W_0^{\alpha,\infty}(-r,0;\mathbb{R}_+^d)$. Consider the deterministic stochastic differential equation on $\mathbb{R}^d$

$$\begin{aligned} x(t) &= \eta(0) + \int_0^t b(s,x)\,\mathrm{d}s + \int_0^t \sigma\big(s, x(s-r)\big)\,\mathrm{d}g_s + y(t), \qquad t \in (0,T], \\ x(t) &= \eta(t), \qquad t \in [-r,0], \end{aligned} \tag{4.1}$$

where, for each $i = 1,\dots,d$,

$$y^i(t) = \max_{s\in[0,t]} \big(z^i(s)\big)^-, \qquad t \in [0,T],$$

and

$$z(t) = \eta(0) + \int_0^t b(s, x)\, \mathrm{d}s + \int_0^t \sigma\bigl(s, x(s-r)\bigr)\, \mathrm{d}g_s, \qquad t \in [0, T].$$

The existence and uniqueness result reads as follows.

Theorem 4.1. *Assume that b and σ satisfy hypotheses* (H1) *and* (H2), *respectively, with* $\rho = 1/\alpha$ *and* $0 < \alpha < \min\{\frac{1}{2}, \beta\}$. *The equation* (4.1) *then has a unique solution* $x \in W_0^{\alpha,\infty}(-r, T; \mathbb{R}_+^d)$.

Proof. To prove that equation (4.1) admits a unique solution on $[-r, T]$, we shall use an induction argument. We will prove that if equation (4.1) admits a unique solution on $[-r, nr]$, then we can further prove that there is a unique solution on the interval $[-r, (n+1)r]$.

Our induction hypothesis, for $k \le M$, is the following:

($\mathrm{H_k}$) *The equation*

$$x^k(t) = \eta(0) + \int_0^t b(s, x^k)\, \mathrm{d}s + \int_0^t \sigma\bigl(s, x^k(s-r)\bigr)\, \mathrm{d}g_s + y_k(t), \qquad t \in [0, kr],$$
$$x^k(t) = \eta(t), \qquad t \in [-r, 0],$$

where, for each $i = 1, \ldots, d$,

$$y_k^i(t) = \max_{s \in [0,t]} (z_k^i(s))^-, \qquad t \in [0, kr],$$

with

$$z_k(t) = \eta(0) + \int_0^t b(s, x^k)\, \mathrm{d}s + \int_0^t \sigma\bigl(s, x^k(s-r)\bigr)\, \mathrm{d}g_s, \qquad t \in [0, kr],$$

has a unique solution $x^k \in W_0^{\alpha,\infty}(-r, kr; \mathbb{R}_+^d)$.

The initial case can be easily checked. Assume now that ($\mathrm{H_i}$) is true for all $i \le n$, where $n < M$. We wish to check ($\mathrm{H_{n+1}}$).

Clearly, for $t \in [-r, nr]$, $x^{n+1}(t)$ will coincide with $x^n(t)$, the solution of the equation of ($\mathrm{H_n}$). Moreover, for $t \in [-r, nr]$, $y_{n+1}(t)$ will coincide with $y_n(t)$. We can therefore write the equation of ($\mathrm{H_{n+1}}$) as

$$\begin{aligned} x^{n+1}(t) &= \eta(0) + \int_0^t b(s, x^{n+1})\, \mathrm{d}s + \int_0^t \sigma\bigl(s, x^n(s-r)\bigr)\, \mathrm{d}g_s \\ &\quad + y_{n+1}(t), \qquad t \in [0, (n+1)r], \\ x^{n+1}(t) &= \eta(t), \qquad t \in [-r, 0]. \end{aligned} \tag{4.2}$$

Moreover, using the notation introduced in the previous section, we have

$$x^{n+1}(t) = \eta(0) + F^{(b)}(x^{n+1}) + G^{(\sigma)}(x^n) + y_{n+1}(t), \qquad t \in [0, (n+1)r],$$
$$x^{n+1}(t) = \eta(t), \qquad t \in [-r, 0].$$

The proof will be divided into three steps:

1. if x^{n+1} is a solution of $(\mathrm{H_{n+1}})$ in the space $\mathcal{C}(-r,(n+1)r;\mathbb{R}_+^d)$, then $x^{n+1}\in W_0^{\alpha,\infty}(-r,(n+1)r;\mathbb{R}_+^d)$;
2. the solution is unique in the space $\mathcal{C}(-r,(n+1)r;\mathbb{R}_+^d)$;
3. there exists a solution in the space $\mathcal{C}(-r,(n+1)r;\mathbb{R}_+^d)$.

Step 1: *If* x^{n+1} *is a solution of* $(\mathrm{H_{n+1}})$ *in the space* $\mathcal{C}(-r,(n+1)r;\mathbb{R}_+^d)$, *then* $x^{n+1}\in W_0^{\alpha,\infty}(-r,(n+1)r;\mathbb{R}_+^d)$. We can write

$$\begin{aligned}
&\|x^{n+1}\|_{\alpha,\infty(-r,(n+1)r)}\\
&=\sup_{t\in[-r,(n+1)r]}\left(|x^{n+1}(t)|+\int_{-r}^{t}\frac{|x^{n+1}(t)-x^{n+1}(s)|}{(t-s)^{\alpha+1}}\,\mathrm{d}s\right)\\
&\le\sup_{t\in[-r,nr]}\left(|x^{n}(t)|+\int_{-r}^{t}\frac{|x^{n}(t)-x^{n}(s)|}{(t-s)^{\alpha+1}}\,\mathrm{d}s\right)\\
&\quad+\sup_{t\in[nr,(n+1)r]}\left(|x^{n+1}(t)|+\int_{-r}^{nr}\frac{|x^{n+1}(t)-x^{n}(s)|}{(t-s)^{\alpha+1}}\,\mathrm{d}s\right.\\
&\qquad\left.+\int_{nr}^{t}\frac{|x^{n+1}(t)-x^{n+1}(s)|}{(t-s)^{\alpha+1}}\,\mathrm{d}s\right)\\
&\le\|x^n\|_{\alpha,\infty(-r,nr)}+\|x^{n+1}\|_{\alpha,\infty(nr,(n+1)r)}\\
&\quad+\sup_{t\in[nr,(n+1)r]}\int_{-r}^{nr}\frac{|x^{n+1}(t)-x^{n}(nr)+x^{n}(nr)-x^{n}(s)|}{(t-s)^{\alpha+1}}\,\mathrm{d}s\\
&\le 2\|x^n\|_{\alpha,\infty(-r,nr)}+\|x^{n+1}\|_{\alpha,\infty(nr,(n+1)r)}+\sup_{t\in[nr,(n+1)r]}\frac{|x^{n+1}(t)-x^{n}(nr)|}{\alpha(t-nr)^{\alpha}}\\
&=2\|x^n\|_{\alpha,\infty(-r,nr)}+A_1+A_2,
\end{aligned}\tag{4.3}$$

where

$$\begin{aligned}
A_1&=\|x^{n+1}\|_{\alpha,\infty(nr,(n+1)r)},\\
A_2&=\sup_{t\in[nr,(n+1)r]}\frac{|x^{n+1}(t)-x^{n}(nr)|}{\alpha(t-nr)^{\alpha}}.
\end{aligned}$$

From our hypothesis, it is clear that $\|x^n\|_{\alpha,\infty(-r,nr)}<\infty$. So, to complete this step of the proof, it suffices to check that $A_1<\infty$ and $A_2<\infty$.

We begin with the study of A_1. Clearly,

$$\begin{aligned}
A_1\le|\eta(0)|&+\left\|F^{(b)}(x^{n+1})\right\|_{\alpha,\infty(nr,(n+1)r)}\\
&+\left\|G_r^{(\sigma)}(x^{n})\right\|_{\alpha,\infty(nr,(n+1)r)}+\|y_{n+1}(\cdot)\|_{\alpha,\infty(nr,(n+1)r)}.
\end{aligned}\tag{4.4}$$

One of the keys to our proof is the study of the behavior of y. We note that from its definition, it is clear that if y^i is increasing at t (i.e., $y^i(t) > y^i(t-\varepsilon)$ for $\varepsilon \le \varepsilon_0$ small enough), then $y^i(t) = -z^i(t) \ge 0$. Moreover, by construction, $y^i(s) \ge -z^i(s)$ for any s. So, if y^i is increasing at t, then for all $s < t$,

$$|y^i(t) - y^i(s)| = y^i(t) - y^i(s) \le -z^i(t) + z^i(s) \le |z^i(t) - z^i(s)|.$$

For $t \in (nr, (n+1)r)$ and $i \in \{1, \dots, d\}$, set

$$t_0^i = \inf\{u;\, y^i(u) = y^i(t)\} \vee nr.$$

Since y^i is increasing, we note that $y^i(s) = y^i(t_0^i)$ for all $s \in [t_0^i, t]$. Then,

$$\begin{aligned}\int_{nr}^{t} \frac{|y^i_{n+1}(t) - y^i_{n+1}(s)|}{(t-s)^{\alpha+1}}\,\mathrm{d}s &= \int_{nr}^{t_0^i} \frac{|y^i_{n+1}(t_0^i) - y^i_{n+1}(s)|}{(t-s)^{\alpha+1}}\,\mathrm{d}s \\ &\le \int_{nr}^{t_0^i} \frac{|y^i_{n+1}(t_0^i) - y^i_{n+1}(s)|}{(t_0^i-s)^{\alpha+1}}\,\mathrm{d}s \\ &\le \int_{nr}^{t_0^i} \frac{|z^i_{n+1}(t_0^i) - z^i_{n+1}(s)|}{(t_0^i-s)^{\alpha+1}}\,\mathrm{d}s.\end{aligned} \tag{4.5}$$

On the other hand, we have

$$|y^i_{n+1}(t)| = |y^i_{n+1}(t_0^i)| \le \sup_{0\le s\le t_0^i} |z^i_{n+1}(s)|. \tag{4.6}$$

So, combining (4.5) and (4.6), we have that

$$\|y_{n+1}\|_{\alpha,\infty(nr,(n+1)r)} \le d\big(\|z_{n+1}\|_{\alpha,\infty(nr,(n+1)r)} + \|z_n\|_{\infty(0,nr)}\big), \tag{4.7}$$

where we can use the bound

$$\begin{aligned}&\|z_{n+1}\|_{\alpha,\infty(nr,(n+1)r)} \\ &\quad\le |\eta(0)| + \big\|F^{(b)}(x^{n+1})\big\|_{\alpha,\infty(nr,(n+1)r)} + \big\|G_r^{(\sigma)}(x^n)\big\|_{\alpha,\infty(nr,(n+1)r)}.\end{aligned} \tag{4.8}$$

Now combining (4.4), (4.7) and (4.8), we obtain that

$$\begin{aligned}A_1 \le (d+1)\big(&|\eta(0)| + \big\|F^{(b)}(x^{n+1})\big\|_{\alpha,\infty(nr,(n+1)r)} \\ &+ \big\|G_r^{(\sigma)}(x^n)\big\|_{\alpha,\infty(nr,(n+1)r)}\big) + d\|z_n\|_{\infty(0,nr)}.\end{aligned} \tag{4.9}$$

From our hypothesis and Propositions 3.1 and 3.2, it is easy to obtain that $\|z_n\|_{\infty(0,nr)} < \infty$. It therefore only remains to check the norms of the Lebesgue and Riemann–Stieltjes integrals.

On the one hand,

$$
\begin{aligned}
&\big\|F^{(b)}(x^{n+1})\big\|_{\alpha,\infty(nr,(n+1)r)}\\
&\quad\le \sup_{t\in[nr,(n+1)r]}\bigg(\int_0^t |b(s,x^{n+1})|\,\mathrm{d}s+\int_{nr}^t \frac{\int_s^t |b(u,x^{n+1})|\,\mathrm{d}u}{(t-s)^{\alpha+1}}\,\mathrm{d}s\bigg)\\
&\quad\le \sup_{t\in[nr,(n+1)r]}\bigg(\int_0^t\Big(L_0\sup_{-r\le u\le s}|x^{n+1}(u)|+b_0(s)\Big)\,\mathrm{d}s\\
&\qquad\qquad+\int_{nr}^t \frac{L_0\int_s^t(\sup_{-r\le v\le u}|x^{n+1}(v)|+b_0(u))\,\mathrm{d}u}{(t-s)^{\alpha+1}}\,\mathrm{d}s\bigg)\\
&\quad\le L_0\Big(T+\frac{r^{1-\alpha}}{1-\alpha}\Big)\|x^{n+1}\|_{\infty(-r,(n+1)r)}+\Big(T^{1-\alpha}+\frac{r^{1-2\alpha}}{1-2\alpha}\Big)\|b_0\|_{L^{1/\alpha}}.
\end{aligned}
\tag{4.10}
$$

On the other hand, to study the Young integral, we will use Proposition 3.2 and the fact that $x^n\in W_0^{\alpha,\infty}(-r,nr;\mathbb{R}_+^d)$. For any $\lambda\ge 1$,

$$
\begin{aligned}
\big\|G_r^{(\sigma)}(x^n)\big\|_{\alpha,\infty(nr,(n+1)r)}&\le \mathrm{e}^{\lambda(n+1)r}\big\|G_r^{(\sigma)}(x^n)\big\|_{\alpha,\lambda(nr,(n+1)r)}\\
&\le \frac{\Lambda_\alpha(g)d^{(4)}}{\lambda^{1-2\alpha}}\mathrm{e}^{\lambda(n+1)r}\big(1+\|x^n\|_{\alpha,\lambda(-r,nr)}\big).
\end{aligned}
\tag{4.11}
$$

So, combining (4.9), (4.10) and (4.11), and using the facts that $\|x^n\|_{\alpha,\lambda(-r,nr)}<\infty$ and $\|x^{n+1}\|_{\infty(-r,(n+1)r)}<\infty$ for $\lambda\ge 1$, we get that $A_1<\infty$.

We now deal with the term A_2. We can write the decomposition

$$
\begin{aligned}
\frac{|x^{n+1}(t)-x^n(nr)|}{(t-nr)^\alpha}&\le \frac{|\int_{nr}^t b(s,x^{n+1})\,\mathrm{d}s|}{(t-nr)^\alpha}+\frac{|\int_{nr}^t \sigma(s,x^n(s-r))\,\mathrm{d}g_s|}{(t-nr)^\alpha}\\
&\quad+\frac{|y_{n+1}(t)-y_{n+1}(nr)|}{(t-nr)^\alpha}.
\end{aligned}
\tag{4.12}
$$

Using the same arguments as in (4.5), we get

$$
\begin{aligned}
&\sup_{t\in[nr,(n+1)r]}\frac{|y_{n+1}(t)-y_{n+1}(nr)|}{(t-nr)^\alpha}\\
&\quad\le d\sup_{t\in[nr,(n+1)r]}\frac{|z_{n+1}(t)-z_{n+1}(nr)|}{(t-nr)^\alpha}\\
&\quad\le d\bigg(\sup_{t\in[nr,(n+1)r]}\frac{|\int_{nr}^t b(s,x^{n+1})\,\mathrm{d}s|}{(t-nr)^\alpha}+\sup_{t\in[nr,(n+1)r]}\frac{|\int_{nr}^t \sigma(s,x^n(s-r))\,\mathrm{d}g_s|}{(t-nr)^\alpha}\bigg).
\end{aligned}
\tag{4.13}
$$

Moreover, from Propositions 3.1 and 3.2, we obtain the estimates

$$\sup_{t\in[nr,(n+1)r]} \frac{|\int_{nr}^t b(s,x^{n+1})\,\mathrm{d}s|}{(t-nr)^\alpha} \le \big\| F^{(b)}(x^{n+1})\big\|_{1-\alpha(nr,(n+1)r)} r^{1-2\alpha} \le \frac{d^{(1)}}{r^{2\alpha-1}}\big(1+\|x^{n+1}\|_{\infty(-r,(n+1)r)}\big), \tag{4.14}$$

$$\sup_{t\in[nr,(n+1)r]} \frac{|\int_{nr}^t \sigma(s,x^n(s-r))\,\mathrm{d}g_s|}{(t-nr)^\alpha} \le \big\| G^{(\sigma)}(x^{n})\big\|_{1-\alpha(nr,(n+1)r)} r^{1-2\alpha} \le \frac{d^{(3)}\Lambda_\alpha(g)}{r^{2\alpha-1}}\big(1+\|x^{n}\|_{\alpha,\infty(-r,nr)}\big). \tag{4.15}$$

So, using the facts that $\|x^n\|_{\alpha,\infty(-r,nr)} < \infty$ and $\|x^{n+1}\|_{\infty(-r,(n+1)r)} < \infty$, and combining (4.12), (4.13), (4.14) and (4.15), we obtain that $A_2 < \infty$.

The proof of the first step is now complete.

Step 2: *Uniqueness of the solution in the space* $\mathcal{C}(-r,(n+1)r;\mathbb{R}^d_+)$.

Let x and x' be two solutions of (4.2) in the space $\mathcal{C}(-r,(n+1)r;\mathbb{R}^d_+)$ and choose N large enough so that $\|x\|_{\infty(-r,(n+1)r)} \le N$ and $\|x'\|_{\infty(-r,(n+1)r)} \le N$.

For any $t \in [0,(n+1)r]$,

$$\sup_{s\in[0,t]} |x(s)-x'(s)| \le \sup_{s\in[0,t]} |z(s)-z'(s)| + \sup_{s\in[0,t]} |y(s)-y'(s)|.$$

Moreover, using Lemma A.2, we have

$$\sup_{s\in[0,t]} |y(s)-y'(s)| \le K_l \sup_{s\in[0,t]} |z(t)-z'(t)|.$$

So, combining the last two inequalities, we get that

$$\begin{aligned}\sup_{s\in[0,t]} |x(s)-x'(s)| &\le (1+K_l) \sup_{s\in[0,t]} |z(s)-z'(s)| \\ &\le (1+K_l) \sup_{s\in[0,t]} \left|\int_0^s \big(b(u,x)-b(u,x')\big)\,\mathrm{d}u\right| \\ &\le (1+K_l)L_N \sup_{s\in[0,t]} \left|\int_0^s \sup_{0\le v\le u} |x(v)-x'(v)|\,\mathrm{d}u\right| \\ &\le L_N(1+K_l)\int_0^t \sup_{v\in[0,u]} |x(v)-x'(v)|\,\mathrm{d}u.\end{aligned}$$

Now applying Gronwall's inequality, we have that for all $t\in[0,(n+1)r]$,

$$\sup_{s\in[0,t]} |x(s)-x'(s)| = 0.$$

So,

$$\|x - x'\|_{\infty(-r,(n+1)r)} = 0$$

and the uniqueness has been proven.

Step 3: *Existence of a solution in* $\mathcal{C}(-r, (n+1)r; \mathbb{R}^d_+)$.

In the space $\mathcal{C}(-r, (n+1)r; \mathbb{R}^d_+)$, we can deal with the reflection term using the Skorokhod mapping. Nevertheless, since the coefficient b is only locally Lipschitz, we will need to use a fixed point argument in $\mathcal{C}(-r, (n+1)r; \mathbb{R}^d_+)$ based on Lemma A.1.

Let us consider the operator

$$\mathcal{L} : \mathcal{C}\big(-r, (n+1)r; \mathbb{R}^d_+\big) \to \mathcal{C}\big(-r, (n+1)r; \mathbb{R}^d_+\big)$$

which is such that

$$\mathcal{L}(u)(t) = \eta(0) + \int_0^t b(s,u)\,\mathrm{d}s + \int_0^t \sigma\big(s, x^n(s-r)\big)\,\mathrm{d}g_s + y_{n+1,u}(t), \qquad t \in [0,(n+1)r],$$
$$\mathcal{L}(u)(t) = \eta(t), \qquad t \in [-r, 0],$$

where x^n is the solution obtained in $(\mathrm{H_n})$ and if

$$z_{n+1,u}(t) = \eta(0) + \int_0^t b(s,u)\,\mathrm{d}s + \int_0^t \sigma\big(s, x^n(s-r)\big)\,\mathrm{d}g_s,$$

then $y^i_{n+1,u}(t) = \max_{s\in[0,t]}(z^i_{n+1,u}(s))^-$ for all $i = 1,\dots,d$.

Note that $\mathcal{L}$ is well defined. Moreover, if $u = \mathcal{L}(u)$, then $u(t) = x^n(t)$ for any $t \in (-r, nr)$. We will use the notation $u^* = \mathcal{L}(u)$.

We next introduce a new norm in the space $\mathcal{C}(-r, (n+1)r; \mathbb{R}^d_+)$: for any $\lambda \geq 1$,

$$\|f\|_{\infty,\lambda(-r,(n+1)r)} := \sup_{t\in[-r,(n+1)r]} \mathrm{e}^{-\lambda t}|f(t)|.$$

These norms are equivalent to $\|f\|_{\infty(-r,(n+1)r)}$.

We now check that we can apply Lemma A.1. Note first that

$$\begin{aligned}
&\|u^*\|_{\infty,\lambda(-r,(n+1)r)} \\
&\quad \leq \|\eta\|_{\infty,\lambda(-r,0)} + |\eta(0)| + \sup_{t\in[0,(n+1)r]} \mathrm{e}^{-\lambda t}\left|\int_0^t b(s,u)\,\mathrm{d}s\right| \\
&\qquad + \sup_{t\in[0,(n+1)r]} \mathrm{e}^{-\lambda t}\left|\int_0^t \sigma\big(s, x^n(s-r)\big)\,\mathrm{d}g_s\right| + d \sup_{t\in[0,(n+1)r]} \mathrm{e}^{-\lambda t}|z_{n+1,u}(t)| \qquad (4.16) \\
&\quad \leq \|\eta\|_{\infty,\lambda(-r,0)} + (d+1)|\eta(0)| + (d+1) \sup_{t\in[0,(n+1)r]} \mathrm{e}^{-\lambda t}\left|\int_0^t b(s,u)\,\mathrm{d}s\right| \\
&\qquad + (d+1) \sup_{t\in[0,(n+1)r]} \mathrm{e}^{-\lambda t}\left|\int_0^t \sigma\big(s, x^n(s-r)\big)\,\mathrm{d}g_s\right|,
\end{aligned}$$

where we have used computations similar to those in (4.6). Indeed, for fixed t, let $t_1 := \inf\{u;\, y^i(u) = y^i(t)\}$. Then,

$$\mathrm{e}^{-\lambda t}|y^i_{n+1,u}(t)| \le \mathrm{e}^{-\lambda t_1}|y^i_{n+1,u}(t_1)| \le \mathrm{e}^{-\lambda t_1}|z^i_{n+1,u}(t_1)|,$$

and taking suprema, we have

$$\sup_{t\in[0,(n+1)r]} \mathrm{e}^{-\lambda t}|y_{n+1,u}(t)| \le d \sup_{t\in[0,(n+1)r]} \mathrm{e}^{-\lambda t}|z_{n+1,u}(t)|. \tag{4.17}$$

Moreover, we have

$$\begin{aligned} &\sup_{t\in[0,(n+1)r]} \mathrm{e}^{-\lambda t}\left|\int_0^t b(s,u)\,\mathrm{d}s\right| \\ &\quad\le L_0 \sup_{t\in[0,(n+1)r]} \mathrm{e}^{-\lambda t}\int_0^t \sup_{-r\le v\le s}|u(v)|\,\mathrm{d}s + \sup_{t\in[0,(n+1)r]} \mathrm{e}^{-\lambda t}\left|\int_0^t b_0(s)\,\mathrm{d}s\right| \qquad (4.18)\\ &\quad\le \frac{L_0}{\lambda}\|u\|_{\infty,\lambda(-r,(n+1)r)} + \frac{C_\alpha}{\lambda^{1-\alpha}}\|b_0\|_{L^{1/\alpha}} \end{aligned}$$

and, from Proposition 3.2,

$$\sup_{t\in[0,(n+1)r]} \mathrm{e}^{-\lambda t}\left|\int_0^t \sigma\big(s, x^n(s-r)\big)\,\mathrm{d}g_s\right| \le \frac{\Lambda_\alpha(g)d^{(4)}}{\lambda^{1-2\alpha}}\big(1+\|x^n\|_{\alpha,\lambda(-r,nr)}\big). \tag{4.19}$$

So, combining (4.16), (4.18) and (4.19), we have

$$\|u^*\|_{\infty,\lambda(-r,(n+1)r)} \le M_1(\lambda) + M_2(\lambda)\|u\|_{\infty,\lambda(-r,(n+1)r)},$$

where

$$\begin{aligned} M_1(\lambda) &= \|\eta\|_{\infty,\lambda(-r,0))} + (d+1)|\eta(0)| + \frac{(d+1)C_\alpha}{\lambda^{1-\alpha}}\|b_0\|_{L^{1/\alpha}} \\ &\qquad + (d+1)\frac{\Lambda_\alpha(g)d^{(4)}}{\lambda^{1-2\alpha}}\big(1+\|x^n\|_{\alpha,\lambda(-r,nr)}\big), \\ M_2(\lambda) &= (d+1)L_0\frac{1}{\lambda}. \end{aligned}$$

Choose $\lambda = \lambda_0$ large enough so that $M_2(\lambda_0) \le \frac{1}{2}$. Then, if $\|u\|_{\infty,\lambda_0(-r,(n+1)r)} \le 2M_1(\lambda_0)$, we have

$$\|u^*\|_{\infty,\lambda_0(-r,(n+1)r)} \le 2M_1(\lambda_0)$$

and so $\mathcal{L}(B_0) \subseteq B_0$, where

$$B_0 = \big\{u \in \mathcal{C}\big(-r,(n+1)r;\mathbb{R}^d_+\big);\, \|u\|_{\infty,\lambda_0(-r,(n+1)r)} \le 2M_1(\lambda_0)\big\}.$$

The first hypothesis in Lemma A.1 is thus satisfied with the metric ρ_0 associated with the norm $\|\cdot\|_{\infty,\lambda_0(-r,(n+1)r)}$.

To complete the proof, we only need to find a metric ρ_1 satisfying the second hypothesis of Lemma A.1.

Note first that if $u \in B_0$, then $\|u\|_{\infty(-r,(n+1)r)} \leq 2e^{\lambda_0(n+1)r} M_1(\lambda_0) := N_0$. Consider $u, u' \in B_0$ and $\lambda \geq 1$. We then have

$$\|\mathcal{L}(u) - \mathcal{L}(u')\|_{\infty,\lambda(-r,(n+1)r)} \leq \sup_{t\in[0,(n+1)r]} e^{-\lambda t}|z_{n+1,u}(t) - z'_{n+1,u}(t)| + \sup_{t\in[0,(n+1)r]} e^{-\lambda t}|y_{n+1,u}(t) - y'_{n+1,u}(t)|.$$

From Lemma A.2, note that given $t \in [0, (n+1)r]$, there exists some $t_2 \leq t$ such that

$$|y_{n+1,u}(t) - y'_{n+1,u}(t)| \leq K_l |z_{n+1,u}(t_2) - z'_{n+1,u}(t_2)|.$$

So,

$$e^{-\lambda t}|y_{n+1,u}(t) - y'_{n+1,u}(t)| \leq K_l e^{-\lambda t_2}|z_{n+1,u}(t_2) - z'_{n+1,u}(t_2)|$$

and it easily follows that

$$\sup_{t\in[0,(n+1)r]} e^{-\lambda t}|y_{n+1,u}(t) - y'_{n+1,u}(t)| \leq K_l \sup_{t\in[0,(n+1)r]} e^{-\lambda t}|z_{n+1,u}(t) - z'_{n+1,u}(t)|.$$

Then,

$$\begin{aligned}
&\|\mathcal{L}(u) - \mathcal{L}(u')\|_{\infty,\lambda(-r,(n+1)r)} \\
&\leq (1 + K_l) \sup_{t\in[0,(n+1)r]} e^{-\lambda t} \int_0^t |b(s,u) - b(s,u')|\, ds \\
&\leq L_{N_0}(1 + K_l) \sup_{t\in[0,(n+1)r]} e^{-\lambda t} \int_0^t \sup_{0\leq v\leq s} |u(v) - u'(v)|\, ds \\
&\leq L_{N_0}(1 + K_l) \sup_{t\in[0,(n+1)r]} \int_0^t e^{-\lambda(t-s)} e^{-\lambda s} \sup_{-r\leq v\leq s} |u(v) - u'(v)|\, ds \\
&\leq L_{N_0}(1 + K_l) \frac{1}{\lambda} \|u - u'\|_{\infty,\lambda(-r,(n+1)r)}.
\end{aligned}$$

So, if we choose $\lambda = \lambda_1$ such that $\frac{L_{N_0}(1+K_l)}{\lambda} \leq \frac{1}{2}$, then the second hypothesis is satisfied for the metric ρ_1 associated with the norm $\|\cdot\|_{\infty,\lambda_1(-r,(n+1)r)}$ and $a = \frac{L_{N_0}(1+K_l)}{\lambda_1}$. □

We now check that the solution is $(1-\alpha)$-Hölder continuous.

Proposition 4.2. *Assume that b and σ satisfy hypothesis* (H1) *and* (H2), *respectively, with $\rho = 1/\alpha$ and $0 < \alpha < \min\{\frac{1}{2}, \beta\}$. The solution x of equation* (4.1) *then belongs to $C^{1-\alpha}(0, T; \mathbb{R}^d)$*

with

$$\|x\|_{1-\alpha(0,T)} \le d^{(6)}\big(1+\Delta_\alpha(g)\big)\big(1+\|x\|_{\alpha,\infty(-r,T)}\big),$$

where $d^{(6)}$ is a positive constant independent of f and g.

Proof. Note that

$$\|x\|_{1-\alpha(0,T)} \le \|z\|_{1-\alpha(0,T)} + \|y\|_{1-\alpha(0,T)}.$$

For fixed $t \in [0,T]$, set $t_* = \inf\{u \le t; y^i(u) = y^i(t)\}$. Then, y^i is increasing in t_* and it is easy to check that $|y^i(t_*) - y^i(s)| \le |z^i(t_*) - z^i(s)|$ for all $s \in (0,t_*)$. For all $s \in (0,t_*)$, it thus holds that

$$\frac{|y^i(t)-y^i(s)|}{(t-s)^{1-\alpha}} \le \frac{|z^i(t_*)-z^i(s)|}{(t_*-s)^{1-\alpha}}$$

and it then follows easily that $\|y\|_{1-\alpha(0,T)} \le d\|z\|_{1-\alpha(0,T)}$. So,

$$\begin{aligned}\|x\|_{1-\alpha(0,T)} &\le (d+1)\|z\|_{1-\alpha(0,T)}\\ &\le (d+1)\big(|\eta(0)| + \big\|F^{(b)}(x)\big\|_{1-\alpha(0,T)} + \big\|G_r^{(\sigma)}(x)\big\|_{1-\alpha(0,T)}\big).\end{aligned}$$

Using Propositions 3.1 and 3.2, we easily complete the proof. □

We will now give an upper bound for the norm of the solution. Recall the definition of $\varphi(\gamma,\alpha)$:

$$\varphi(\gamma,\alpha) = \begin{cases} 2\alpha, & \gamma = 1,\\ > 1 + \dfrac{2\alpha-1}{\gamma}, & \dfrac{1-2\alpha}{1-\alpha} \le \gamma < 1,\\ \alpha, & 0 \le \gamma < \dfrac{1-2\alpha}{1-\alpha}.\end{cases}$$

Lemma 4.3. *Assume* (H1), (H2) *and* (H3). *The unique solution of equation* (4.1) *then satisfies*

$$\|x\|_{\alpha,\infty(r)} \le d_\alpha^{(3)}\big(\|\eta\|_{\alpha,\infty(-r,0)} + \Lambda_\alpha(g) + 1\big)\exp\big(T\big(d_\alpha^{(1)} + d_\alpha^{(2)}\Lambda_\alpha(g)^{1/(1-\varphi(\gamma,\alpha))}\big)\big).$$

Proof. First, we need to obtain an upper bound for $\|x\|_{\alpha,\lambda(r)}$. We begin with the estimates

$$\begin{aligned}\|x\|_{\alpha,\lambda(r)} &\le \|\eta\|_{\alpha,\lambda(-r,0)} + \sup_{t\in[0,T]} \mathrm{e}^{-\lambda t}\left(|x(t)| + \int_{-r}^{t} \frac{|x(t)-x(s)|}{(t-s)^{\alpha+1}}\,\mathrm{d}s\right)\\ &\le \|\eta\|_{\alpha,\lambda(-r,0)} + \sup_{t\in[0,T]} \mathrm{e}^{-\lambda t}\left(|x(t)| + \int_{-r}^{0} \frac{|x(t)-\eta(s)|}{(t-s)^{\alpha+1}}\,\mathrm{d}s\right.\\ &\qquad\left. + \int_{0}^{t} \frac{|x(t)-x(s)|}{(t-s)^{\alpha+1}}\,\mathrm{d}s\right)\\ &\le \|\eta\|_{\alpha,\lambda(-r,0)} + \|x\|_{\alpha,\lambda(0,T)} + \sup_{t\in[0,T]} \mathrm{e}^{-\lambda t}\int_{-r}^{0} \frac{|x(t)-\eta(s)|}{(t-s)^{\alpha+1}}\,\mathrm{d}s.\end{aligned} \tag{4.20}$$

Moreover,

$$\|x\|_{\alpha,\lambda(0,T)} \le \|z\|_{\alpha,\lambda(0,T)} + \|y\|_{\alpha,\lambda(0,T)} \tag{4.21}$$

and using the same arguments as in (4.5) and (4.6), we have,

$$\|y\|_{\alpha,\lambda(0,T)} \le d\|z\|_{\alpha,\lambda(0,T)}. \tag{4.22}$$

We also know that

$$\|z\|_{\alpha,\lambda(0,T)} \le |\eta(0)| + \left\|F^{(b)}(x)\right\|_{\alpha,\lambda(0,T)} + \left\|G_r^{(\sigma)}(x)\right\|_{\alpha,\lambda(0,T)}. \tag{4.23}$$

So, combining (4.20)–(4.23) and applying Propositions 3.1 and 3.3, we get that

$$\begin{aligned}\|x\|_{\alpha,\lambda(r)} \le{}& \|\eta\|_{\alpha,\lambda(-r,0)} + (d+1)|\eta(0)| + (d+1)d^{(2)}\left(\frac{1}{\lambda^{1-2\alpha}} + \frac{\|x\|_{\alpha,\lambda(r)}}{\lambda^{1-\alpha}}\right)\\ &+ \Lambda_\alpha(g)(d+1)d^{(5)}\left(1 + \frac{\|x\|_{\alpha,\lambda(r)}}{\lambda^{1-\varphi(\gamma,\alpha)}}\right) + B\end{aligned} \tag{4.24}$$

with

$$B := \sup_{t\in[0,T]} \mathrm{e}^{-\lambda t} \int_{-r}^{0} \frac{|x(t)-\eta(s)|}{(t-s)^{\alpha+1}}\,\mathrm{d}s.$$

It remains to investigate B. We can decompose this term as follows:

$$\begin{aligned}B &\le \sup_{t\in[0,T]} \mathrm{e}^{-\lambda t} \int_{-r}^{0} \frac{|x(t)-\eta(0)|}{(t-s)^{\alpha+1}}\,\mathrm{d}s + \sup_{t\in[0,T]} \mathrm{e}^{-\lambda t} \int_{-r}^{0} \frac{|\eta(0)-\eta(s)|}{(-s)^{\alpha+1}}\,\mathrm{d}s\\ &\le \frac{1}{\alpha} \sup_{t\in[0,T]} \frac{\mathrm{e}^{-\lambda t}}{t^\alpha}|x(t)-\eta(0)| + \|\eta\|_{\alpha,\lambda(-r,0)}\\ &\le \frac{1}{\alpha}(B_1+B_2+B_3) + \|\eta\|_{\alpha,\lambda(-r,0)},\end{aligned} \tag{4.25}$$

where

$$B_1 = \sup_{t\in[0,T]} \frac{\mathrm{e}^{-\lambda t}}{t^\alpha} \int_0^t |b(s,x)|\,\mathrm{d}s,$$

$$B_2 = \sup_{t\in[0,T]} \left|\frac{\mathrm{e}^{-\lambda t}}{t^\alpha} \int_0^t \sigma\big(s, x(s-r)\big)\,\mathrm{d}g_s\right|,$$

$$B_3 = \sup_{t\in[0,T]} \frac{\mathrm{e}^{-\lambda t}}{t^\alpha}|y(t)|.$$

Using the same arguments as in (4.17), we get that

$$B_3 = \sup_{t\in[0,T]} \frac{\mathrm{e}^{-\lambda t}}{t^\alpha}|y(t)| \le d \sup_{t\in[0,T]} \frac{\mathrm{e}^{-\lambda t}}{t^\alpha}|z(t)| \le d(B_1+B_2).$$

We now consider B_1 and B_2. For B_1, we can write

$$\begin{aligned}
B_1 &\le \sup_{t\in[0,T]} \frac{e^{-\lambda t}}{t^\alpha} \int_0^t \Big(L_0 \sup_{-r\le u\le s} |x(u)| + b_0(s) \Big)\, ds \\
&\le L_0 \Big(\sup_{s\in[-r,T]} e^{-\lambda s}|x(s)| \Big) \sup_{t\in[0,T]} \int_0^t \frac{e^{-\lambda(t-s)}}{(t-s)^\alpha}\, ds + \sup_{t\in[0,T]} \frac{e^{-\lambda t}}{t^{2\alpha-1}} \|b_0\|_{L^{1/\alpha}} \\
&\le L_0 \lambda^{\alpha-1}\Gamma(1-\alpha)\|x\|_{\alpha,\lambda(r)} + C_\alpha \lambda^{2\alpha-1}\|b_0\|_{L^{1/\alpha}}.
\end{aligned}$$

Next, we obtain a bound for B_2. We will use the hypothesis (H3).

$$\begin{aligned}
B_2 &\le \sup_{t\in[0,T]} \frac{e^{-\lambda t}}{t^\alpha} \Lambda_\alpha(g) \Bigg(\int_0^t \frac{|\sigma(s,x(s-r))|}{s^\alpha}\, ds \\
&\qquad + \alpha \int_0^t \int_0^s \frac{|\sigma(s,x(s-r)) - \sigma(y,x(y-r))|}{(s-y)^{\alpha+1}}\, dy\, ds \Bigg) \\
&\le \sup_{t\in[0,T]} \frac{e^{-\lambda t}}{t^\alpha} \Lambda_\alpha(g) \Bigg(K_0 \int_0^t \frac{1+|x(s-r)|^\gamma}{s^\alpha}\, ds \\
&\qquad + \alpha M_0 \int_0^t \int_0^s \left(\frac{|x(s-r)-x(y-r)|}{(s-y)^{\alpha+1}} + \frac{1}{(s-y)^{\alpha+1-\beta}} \right) dy\, ds \Bigg) \\
&\le \sup_{t\in[0,T]} \Lambda_\alpha(g) \Bigg(K_0 \frac{t^{1-2\alpha}}{1-\alpha} e^{-\lambda t} + K_0 \frac{e^{-\lambda t}}{t^\alpha} \int_{-r}^{t-r} \frac{|x(s)|^\gamma}{(s+r)^\alpha}\, ds \\
&\qquad + \alpha M_0 \|x\|_{\alpha,\lambda(r)} \int_{-r}^{t-r} \frac{e^{-\lambda(t-s)}}{(t-s)^\alpha}\, ds + \frac{\alpha M_0 t^{\beta-2\alpha+1} e^{-\lambda t}}{(\beta-\alpha)(\beta-\alpha+1)} \Bigg).
\end{aligned}$$

Now, using the inequalities

$$\sup_{t\in[0,T]} t^\mu e^{-\lambda t} \le \left(\frac{\mu}{\lambda}\right)^\mu e^{-\mu},$$

$$\begin{aligned}
\int_{-r}^{t-r} \frac{e^{-\lambda(t-s)}}{(t-s)^\alpha}\, ds &= e^{-\lambda r} \int_0^t \frac{e^{-\lambda(t-u)}}{(t-u+r)^\alpha}\, du \\
&\le e^{-\lambda r} \int_0^t \frac{e^{-\lambda(t-u)}}{(t-u)^\alpha}\, du \le e^{-\lambda r}\lambda^{\alpha-1}\Gamma(1-\alpha),
\end{aligned}$$

the Hölder inequality and the fact that $|f(s)|^\gamma \le |f(s)| + 1$, we get

$$\begin{aligned}
\frac{e^{-\lambda t}}{t^\alpha} \int_{-r}^{t-r} \frac{|x(s)|^\gamma}{(s+r)^\alpha}\, ds &\le e^{-\lambda t} t^{\varphi(\gamma,\alpha)\gamma - 2\alpha + 1 - \gamma} \left(\int_{-r}^{t-r} \frac{|x(s)|}{(s+r)^{\varphi(\gamma,\alpha)}}\, ds \right)^\gamma \\
&\le C_{\alpha,\gamma,T} e^{-\lambda t} \left(1 + \int_{-r}^{t-r} \frac{|x(s)|}{(s+r)^{\varphi(\gamma,\alpha)}}\, ds \right)
\end{aligned}$$

$$\begin{aligned}
&\le C_{\alpha,\gamma,T}\left(1+\|x\|_{\alpha,\lambda(r)}\mathrm{e}^{-\lambda r}\int_0^t \frac{\mathrm{e}^{-\lambda(t-u)}}{u^{\varphi(\gamma,\alpha)}}\,\mathrm{d}u\right)\\
&\le C_{\alpha,\gamma,T}\big(1+\|x\|_{\alpha,\lambda(r)}\mathrm{e}^{-\lambda r}\lambda^{\varphi(\gamma,\alpha)-1}\big),
\end{aligned}$$

where we have used the fact that $\varphi(\gamma,\alpha)\gamma-2\alpha+1-\gamma\ge 0$. We thus obtain

$$\begin{aligned}
B_2 &\le \Lambda_\alpha(g)\bigg(\frac{K_0}{1-\alpha}\left(\frac{1-2\alpha}{\mathrm{e}}\right)^{1-2\alpha}\lambda^{2\alpha-1}+\frac{\alpha M_0((\beta-2\alpha+1)\mathrm{e})^{\beta-2\alpha+1}}{(\beta-\alpha)(\beta-\alpha+1)}\lambda^{2\alpha-1-\beta}\\
&\qquad + C_{\alpha,\gamma,T}+\|x\|_{\alpha,\lambda(r)}\mathrm{e}^{-\lambda r}\big(\alpha M_0\Gamma(1-\alpha)\lambda^{\alpha-1}+K_0C_{\alpha,\gamma,T}\lambda^{\varphi(\gamma,\alpha)-1}\big)\bigg)\\
&\le \Lambda_\alpha(g)C_{\alpha,\beta,\gamma}\big(1+\lambda^{2\alpha-1}+\mathrm{e}^{-\lambda r}\lambda^{\varphi(\gamma,\alpha)-1}\|x\|_{\alpha,\lambda(r)}\big).
\end{aligned}$$

Finally, combining (4.24), (4.25) and the estimates for B, B_1 and B_2, we have

$$\|x\|_{\alpha,\lambda(r)}\le M_1(\lambda)+M_2(\lambda)\|x\|_{\alpha,\lambda(r)}$$

with

$$\begin{aligned}
M_1(\lambda)&=2\|\eta\|_{\alpha,\lambda(-r,0)}+(d+1)\bigg(|\eta(0)|+\Lambda_\alpha(g)d^{(5)}+C_{\alpha,\beta,\gamma}\\
&\qquad+\frac{C_{\alpha,\beta}}{\lambda^{1-2\alpha}}\big(d^{(2)}+\|b_0\|_{L^{1/\alpha}}+\Lambda_\alpha(g)\big)\bigg),\\
M_2(\lambda)&=\frac{(d+1)C_\alpha}{\lambda^{1-\varphi(\gamma,\alpha)}}\big(\Lambda_\alpha(g)\big(d^{(5)}+C_{\alpha,\beta,\gamma}\big)+L_0\Gamma(1-\alpha)+d^{(2)}\big).
\end{aligned}$$

Choosing $\lambda=\lambda_0$ large enough so that $M_2(\lambda_0)=\frac{1}{2}$, we then have

$$\|x\|_{\alpha,\lambda_0(r)}\le 2M_1(\lambda_0).$$

Set

$$\begin{aligned}
\lambda_0&=\big[2C_{\alpha,d}\big(d^{(2)}+\Lambda_\alpha(g)\big(d^{(5)}+1\big)+L_0\big)\big]^{1/(1-\varphi(\gamma,\alpha))}\\
&\le d_\alpha\big(2C_{\alpha,d}\big(d^{(2)}+L_0\big)\big)^{1/(1-\varphi(\gamma,\alpha))}+\Lambda_\alpha(g)^{1/(1-\varphi(\gamma,\alpha))}d_\alpha\big(2C_{\alpha,d}\big(1+d^{(5)}\big)\big)^{1/(1-\varphi(\gamma,\alpha))}\\
&\le d_\alpha^{(1)}+d_\alpha^{(2)}\Lambda_\alpha(g)^{1/(1-\varphi(\gamma,\alpha))}
\end{aligned}$$

with

$$\begin{aligned}
d_\alpha^{(1)}&=d_\alpha\big(2C_{\alpha,d}\big(d^{(2)}+L_0\big)\big)^{1/(1-\varphi(\gamma,\alpha))},\\
d_\alpha^{(2)}&=d_\alpha\big(2C_{\alpha,d}\big(1+d^{(5)}\big)\big)^{1/(1-\varphi(\gamma,\alpha))}.
\end{aligned}$$

This implies that

$$\|x\|_{\alpha,\infty(r)}\le \exp\Big(T\big(d_\alpha^{(1)}+d_\alpha^{(2)}\Lambda_\alpha(g)^{1/(1-\varphi(\gamma,\alpha))}\big)\Big)2M_1(\lambda_0)$$

and the proof is then easily completed. Note that we can choose $d_\alpha^{(1)}, d_\alpha^{(2)}$ which do not depend on β or γ. □

5. Stochastic integral equations

In this section, we apply the deterministic results in order to prove the main theorem of this paper.

The stochastic integral appearing throughout this paper, $\int_0^T u(s)\,\mathrm{d}W_s$, is a pathwise Riemann–Stieltjes integral and it is well know that this integral exists if the process $u(s)$ has Hölder continuous trajectories of order larger than $1-H$.

Set $\alpha \in (1-H, \frac{1}{2})$. For any $\delta \in (0,2)$, by Fernique's theorem (see [1], Theorem 1.3.2), we have

$$E(\exp(\Lambda_\alpha(W)^\delta)) < \infty.$$

Then, if $u = \{u_t, t \in [0,T]\}$ is a stochastic process whose trajectories belong to the space $W_T^{\alpha,1}(0,T)$, it follows that the Riemann–Stieltjes integral $\int_0^T u(s)\,\mathrm{d}W_s$ exists and we have that

$$\left|\int_0^T u(s)\,\mathrm{d}W_s\right| \le G\|u\|_{\alpha,1},$$

where G is a random variable with moments of all orders (see [11], Lemma 7.5). Moreover, if the trajectories of u belong to $W_0^{\alpha,\infty}(0,T)$, then the indefinite integral $\int_0^T u(s)\,\mathrm{d}W_s$ is Hölder continuous of order $1-\alpha$ and with trajectories in $W_0^{\alpha,\infty}(0,T)$.

The existence and uniqueness of a solution then follows from Theorem 4.1. In order to get the existence of a moment of any order, we need only note that if $\alpha < (2-\gamma)/4$, then $1/(1-\varphi(\gamma,\alpha)) < 2$ and $E(\exp(C\Lambda_\alpha(W)^{1/(1-\varphi(\gamma,\alpha))})) < \infty$.

Appendix

In this appendix, we just give a fixed point theorem, well posed to our problem, and recall a result with some properties of the solution of the Skorokhod problem.

Lemma A.1. *Let (X,ρ) be a complete metric space, and ρ_0 and ρ_1 two metrics on X which are equivalent to ρ. If $\mathcal{L}: X \to X$ satisfies*:

1. *there exists some $r_0 > 0$, $x_0 \in X$ such that if $B_0 = \{x \in X; \rho_0(x_0,x) \le r_0\}$ then $\mathcal{L}(B_0) \subseteq B_0$*;
2. *there exists some $a \in (0,1)$ such that $\rho_1(\mathcal{L}(x),\mathcal{L}(y)) \le a\rho_1(x,y)$ for all $x, y \in B_0$*,

then there exists some $x^* \in \mathcal{L}(B_0) \subseteq X$ such that $x^* = \mathcal{L}(x^*)$.

Proof. For all $n \in \mathbb{N}$, set

$$x_{n+1} = \mathcal{L}(x_n).$$

Clearly, $x_n \in \mathcal{L}(B_0)$ for all $n \in \mathbb{N}$. Moreover,

$$\rho_1(x_{n+1}, x_n) = \rho_1(\mathcal{L}(x_n), \mathcal{L}(x_{n-1})) \leq a\rho_1(x_n, x_{n-1}) \leq \cdots \leq a^n \rho_1(x_1, x_0)$$

and

$$\begin{aligned}\rho_1(x_{n+p}, x_n) &\leq \rho_1(x_{n+p}, x_{n+p-1}) + \cdots + \rho_1(x_{n+1}, x_n) \\ &\leq a^n(a^{p-1} + \cdots + a + 1)\rho_1(x_1, x_0) \leq \frac{a^n}{1-a}\rho_1(x_1, x_0) \to 0\end{aligned}$$

as $n \to \infty$.

Since (X, ρ) is a complete metric space and B_0 is closed in X, there exists some $x^* \in B_0$ such that $x_n \to x^*$. Furthermore, from the second hypothesis of the lemma, we get that

$$\rho_1(\mathcal{L}(x_n), \mathcal{L}(x^*)) \leq a\rho_1(x_n, x^*).$$

Since $\rho_1(x_n, x^*) \to 0$, $\mathcal{L}(x_n) \to \mathcal{L}(x^*)$ and it follows that $x^* = \mathcal{L}(x^*)$. □

Lemma A.2. *For each path $z \in \mathcal{C}(\mathbb{R}_+, \mathbb{R}^d)$, there exists a unique solution (x, y) to the Skorokhod problem for z. Thus, there exists a pair of functions $(\phi, \varphi) : \mathcal{C}_+(\mathbb{R}_+, \mathbb{R}^d) \to \mathcal{C}_+(\mathbb{R}_+, \mathbb{R}^{2d})$ defined by $(\phi(z), \varphi(z)) = (x, y)$ such that the following holds: There exists a constant $K_l > 0$ such that for any $z_1, z_2 \in \mathcal{C}_+(\mathbb{R}_+, \mathbb{R}^d)$, we have, for each $t \geq 0$,*

$$\begin{aligned}\|\phi(z_1) - \phi(z_2)\|_{\infty,[0,t]} &\leq K_l\|z_1 - z_2\|_{\infty,[0,t]}, \\ \|\varphi(z_1) - \varphi(z_2)\|_{\infty,[0,t]} &\leq K_l\|z_1 - z_2\|_{\infty,[0,t]}.\end{aligned}$$

Proof. See [6], Proposition A.0.1. □

Acknowledgement

This work was partially supported by DGES Grant MTM09-07203 (both authors).

References

[1] Fernique, X. (1975). Regularité des trajectoires des fonctions aléatoires gaussienes. In *École d'Été de Probabilités de Saint Flour, IV-1974. Lecture Notes in Math.* **480** 1–96. Berlin: Springer. MR0413238

[2] Ferrante, M. and Rovira, C. (2006). Stochastic delay differential equations driven by fractional Brownian motion with Hurst parameter $H > 1/2$. *Bernoulli* **12** 85–100. MR2202322

[3] Ferrante, M. and Rovira, C. (2010). Convergence of delay differential equations driven by fractional Brownian motion. *J. Evol. Equ.* **10** 761–783.

[4] Hu, Y. and Nualart, D. (2009). Rough path analysis via fractional calculus. *Trans. Amer. Math. Soc.* **361** 2689–2718. MR2471936

[5] Kusher, H.J. (2008). *Numerical Methods for Controlled Stochastic Delay Systems*. Boston: Birkhäuser. MR2441170

[6] Kinnally, M.S. and Williams, J. (2010). On existence and uniqueness of stationary distributions for stochastic delay differential equations with non-negativity constraints. *Electron. J. Probab.* **15** 409–451. MR2639731

[7] León, J. and Tindel, S. Malliavin calculus for fractional delay equations. Available at arXiv:1001.3344.

[8] Lyons, T. (1994). Differential equations driven by rough signals (I): An extension of an inequality of L. C. Young. *Math. Res. Lett.* **1** 451–464. MR1302388

[9] Mohammed, S.-E.A. (1998). Stochastic differential systems with memory: theory, examples and applications. In *Stochastic Analysis and Related Topics VI* (L. Decreusefond, J. Gjerde, B. Oksendal and A.S. Üstünel, eds.) 1–77. Boston: Birkhäuser. MR1652338

[10] Neuenkirch, A., Nourdin, I. and Tindel, S. (2008). Delay equations driven by rough paths. *Electron. J. Probab.* **13** 2031–2068. MR2453555

[11] Nualart, D. and Rascanu, A. (2002). Differential equations driven by fractional Brownian motion. *Collect. Math.* **53** 55–81. MR1893308

[12] Tindel, S. and Torrecilla, I. Some differential systems driven by a fBm with Hurst parameter greater than $1/4$. Available at arXiv:0901.2010.

[13] Young, L.C. (1936). An inequality of the Hölder type connected with Stieltjes integration. *Acta Math.* **67** 251–282. MR1555421

[14] Zähle, M. (1998). Integration with respect to fractal functions and stochastic calculus. I. *Probab. Theory Related Fields* **111** 333–374. MR1640795

Received March 2010 and revised August 2010