Bernoulli **17**(3), 2011, 916–941
DOI: 10.3150/10-BEJ311

Quasi Ornstein–Uhlenbeck processes

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The question of existence and properties of stationary solutions to Langevin equations driven by noise processes with stationary increments is discussed, with particular focus on noise processes of pseudo-moving-average type. On account of the Wold–Karhunen decomposition theorem, such solutions are, in principle, representable as a moving average (plus a drift-like term) but the kernel in the moving average is generally not available in explicit form. A class of cases is determined where an explicit expression of the kernel can be given, and this is used to obtain information on the asymptotic behavior of the associated autocorrelation functions, both for small and large lags. Applications to Gaussian- and Lévy-driven fractional Ornstein–Uhlenbeck processes are presented. A Fubini theorem for Lévy bases is established as an element in the derivations.

Keywords: fractional Ornstein–Uhlenbeck processes; Fubini theorem for Lévy bases; Langevin equations; stationary processes

1. Introduction

This paper studies the existence and properties of stationary solutions to Langevin equations driven by a noise process N with, in general, stationary dependent increments. We shall refer to such solutions as quasi Ornstein–Uhlenbeck (QOU) processes. Of particular interest are the cases where the noise process is of the pseudo-moving-average (PMA) type. In wide generality, the stationary solutions can, in principle, be written in the form of a Wold–Karhunen-type representation, but it is relatively rare that an explicit expression for the kernel of such a representation can be given. When this is possible it often provides a more direct and simpler access to the character and properties of the process; for instance, concerning the autocovariance function.

This will be demonstrated in applications to the case where the noise process N is of the pseudo-moving-average kind, including fractional Brownian motion and, more generally, fractional Lévy motions. Of some particular interest for turbulence theory is the large and small lags limit behavior of the autocovariance function of the Ornstein–Uhlenbeck-type process driven by fractional Brownian motion, which has been proposed as a representation of homogeneous Eulerian turbulent velocities; see Shao [37].

The fractional Brownian and Lévy motions are not of the semimartingale type. Another non-semimartingale case covered is $N_t = \int_{\mathcal{X}} B_t^{(x)} m(\mathrm{d}x)$, where the processes $B_\cdot^{(x)}$ are Brownian motions in different filtrations and m is a measure on some space $\mathcal{X}$.

In recent applications of stochastics, particularly in finance and in turbulence, modifications of classic noise processes by time change or by volatility modifications are of central importance; see Barndorff-Nielsen and Shephard [4] and Barndorff-Nielsen and Shiryaev [5] and references given therein. Prominent examples of such processes are $\mathrm{d}N_t = \sigma_t \,\mathrm{d}B_t$, where B is Brownian motion and $\sigma > 0$ is a predictable stationary process – for instance, the square root of a superposition of inverse Gaussian Ornstein–Uhlenbeck processes (cf. Barndorff-Nielsen and Shephard [3] and Barndorff-Nielsen and Stelzer [6]) – and $N_t = L_{T_t}$, where L is a Lévy process and T is a time change process with stationary increments (cf. Carr *et al.* [13]). The theory discussed in the present paper applies also to processes of this type.

The structure of the paper is as follows. Section 2 defines the concept of QOU processes and provides conditions for existence and uniqueness of stationary solutions to the Langevin equation. The form of the autocovariance function of the solutions is given and its asymptotic behavior for $t \to \infty$ is discussed. As an intermediate step, a Fubini theorem for Lévy bases is established in Section 3. In Section 4 explicit forms of Wold–Karhunen representations are derived and used to analyze the asymptotics, under more specialized assumptions, of the autocovariance functions, both for $t \to \infty$ and for $t \to 0$. The Appendix establishes an auxiliary continuity result of a technical nature.

2. Langevin equations and QOU processes

Let $N = (N_t)_{t\in\mathbb{R}}$ be a measurable process with stationary increments and let $\lambda > 0$ be a positive number. By a QOU process X driven by N and with parameter λ, we mean a stationary solution to the Langevin equation $\mathrm{d}X_t = -\lambda X_t\,\mathrm{d}t + \mathrm{d}N_t$, that is, $X = (X_t)_{t\in\mathbb{R}}$ is a stationary process that satisfies

$$X_t = X_0 - \lambda \int_0^t X_s\,\mathrm{d}s + N_t, \qquad t \in \mathbb{R}, \tag{2.1}$$

where the integral is a pathwise Lebesgue integral. For all $a < b$ we use the notation $\int_b^a := -\int_a^b$. Recall that a process $Z = (Z_t)_{t\in\mathbb{R}}$ is measurable if $(t,\omega) \mapsto Z_t(\omega)$ is $(\mathcal{B}(\mathbb{R}) \otimes \mathcal{F}, \mathcal{B}(\mathbb{R}))$-measurable, and that Z has stationary increments if, for all $s \in \mathbb{R}$, $(Z_t - Z_0)_{t\in\mathbb{R}}$ has the same finite distributions as $(Z_{t+s} - Z_s)_{t\in\mathbb{R}}$. For $p > 0$ we will say that a process Z has finite p moments if $\mathrm{E}[|Z_t|^p] < \infty$ for all $t \in \mathbb{R}$. Moreover, for $t \to 0$ or ∞, we will write $f(t) \sim g(t)$, $f(t) = \mathrm{o}(g(t))$ or $f(t) = \mathrm{O}(g(t))$, provided that $f(t)/g(t) \to 1$, $f(t)/g(t) \to 0$ or $\limsup_t |f(t)/g(t)| < \infty$, respectively. For each process Z with finite second moments, let $\mathrm{V}_Z(t) = \mathrm{Var}(Z_t)$ denote its variance function. When Z, in addition, is stationary, let $\mathrm{R}_Z(t) = \mathrm{Cov}(Z_t, Z_0)$ denote its autocovariance function and $\bar{\mathrm{R}}_X(t) = \mathrm{R}_X(0) - \mathrm{R}_X(t) = \frac{1}{2}\mathrm{E}[(X_t - X_0)^2]$ its complementary autocovariance function.

Before discussing the general setting further, we recall some well-known cases. The stationary solution X to (2.1) when $N_t = \mu t + \sigma B_t$ (with B the Brownian motion) is the Gaussian Ornstein–Uhlenbeck process, μ/λ is the mean level, λ is the speed of reversion and σ is the volatility. When N is a Lévy process, the corresponding QOU process, X, exists if and only if $\mathrm{E}[\log^+ |N_1|] < \infty$ or, equivalently, if and only if $\int_{\{|x|>1\}} \log|x|\,\nu(\mathrm{d}x) < \infty$, where ν is the Lévy measure of N; see Sato and Yamazato [36] or Wolfe [39]. In this case X is called an Ornstein–Uhlenbeck-type process; for applications of such processes in financial economics, see Barndorff-Nielsen and Shephard [3,4].

2.1. Existence and uniqueness of QOU processes

The first result below shows the existence and uniqueness for the stationary solution X to the Langevin equation $\mathrm{d}X_t = -\lambda X_t \,\mathrm{d}t + \mathrm{d}N_t$ in the case where the noise N is integrable. That is, we show existence and uniqueness of QOU processes X. Moreover, we provide an explicit form of the solution that is used to calculate the mean and variance of X.

Theorem 2.1. *Let N be a measurable process with stationary increments and finite first moments, and let $\lambda > 0$ be a positive real number. Then, $X = (X_t)_{t\in\mathbb{R}}$, given by*

$$X_t = N_t - \lambda \mathrm{e}^{-\lambda t} \int_{-\infty}^{t} \mathrm{e}^{\lambda s} N_s \,\mathrm{d}s, \qquad t \in \mathbb{R}, \tag{2.2}$$

is a QOU process driven by N with parameter λ (the integral is a pathwise Lebesgue integral). Furthermore, any other QOU process driven by N and with parameter λ equals X in law. Finally, if N has finite p moments, $p \geq 1$, then X also has finite p moments and is continuous in L^p.

Remark 2.2. It is an open problem to relax the integrability of N in Theorem 2.1; that is, is it enough that N has finite log moments? Recall that when N is a Lévy process, finite log moments is a necessary and sufficient condition for the existence of the corresponding Ornstein–Uhlenbeck-type process.

Proof of Theorem 2.1. Existence: Let $p \geq 1$ and assume that N has finite p moments. Choose $\alpha, \beta > 0$, according to Corollary A.3, such that $\|N_t\|_p \leq \alpha + \beta|t|$ for all $t \in \mathbb{R}$. By Jensen's inequality,

$$\begin{aligned}
\mathrm{E}\left[\left(\int_{-\infty}^{t} \mathrm{e}^{\lambda s} |N_s| \,\mathrm{d}s\right)^p\right] &\leq (\mathrm{e}^{\lambda t}/\lambda)^{p-1} \int_{-\infty}^{t} \mathrm{e}^{\lambda s} \mathrm{E}[|N_s|^p] \,\mathrm{d}s \\
&\leq (\mathrm{e}^{\lambda t}/\lambda)^{p-1} \int_{-\infty}^{t} \mathrm{e}^{\lambda s} (\alpha + \beta|s|)^p \,\mathrm{d}s < \infty,
\end{aligned}$$

which shows that the integral in (2.2) exists almost surely as a Lebesgue integral and that X_t, given by (2.2), is p-integrable. Using substitution we obtain from (2.2),

$$X_t = \lambda \int_{-\infty}^{0} \mathrm{e}^{\lambda u} (N_t - N_{t+u}) \,\mathrm{d}u, \qquad t \in \mathbb{R}. \tag{2.3}$$

By Corollary A.3, N is L^p-continuous and, therefore, it follows that the right-hand side of (2.3) exists as a limit of Riemann sums in L^p. Hence the stationarity of the increments of N implies that X is stationary. Moreover, using integration by parts on $t \mapsto \int_{-\infty}^{t} \mathrm{e}^{\lambda s} N_s(\omega) \,\mathrm{d}s$, we get

$$\int_0^t X_s \,\mathrm{d}s = \mathrm{e}^{-\lambda t} \int_{-\infty}^{t} \mathrm{e}^{\lambda s} N_s \,\mathrm{d}s - \int_{-\infty}^{0} \mathrm{e}^{\lambda s} N_s \,\mathrm{d}s,$$

which shows that X satisfies (2.1), and hence X is a QOU process driven by N with parameter λ.

Since X is a measurable process with stationary increments and finite p moments, Proposition A.3 shows that it is continuous in L^p.

To show uniqueness in law, let Y be a QOU process driven by N with parameter $\lambda > 0$, that is, Y is a stationary process that satisfies (2.1). For all $t_0 \in \mathbb{R}$ we have, with $Z_t = N_t - N_{t_0} + Y_{t_0}$, that

$$Y_t = Z_t - \lambda \int_{t_0}^{t} Y_s \, \mathrm{d}s, \qquad t \geq t_0. \tag{2.4}$$

Solving (2.4) pathwise, it follows that for all $t \geq t_0$,

$$\begin{aligned} Y_t &= Z_t - \lambda \mathrm{e}^{-\lambda t} \int_{t_0}^{t} \mathrm{e}^{\lambda s} Z_s \, \mathrm{d}s \\ &= N_t - \lambda \mathrm{e}^{-\lambda t} \int_{t_0}^{t} \mathrm{e}^{\lambda s} N_s \, \mathrm{d}s + (Y_{t_0} - N_{t_0}) \mathrm{e}^{-\lambda (t - t_0)}. \end{aligned}$$

Note that $\lim_{t\to\infty}(Y_{t_0} - N_{t_0})\mathrm{e}^{-\lambda(t-t_0)} = 0$ a.s., thus for all $n \geq 1$ and $t_0 < t_1 < \cdots < t_n$, the stationarity of Y implies that for $k \to \infty$, $(Y_{t_i+k})_{i=1}^n \Rightarrow (Y_{t_i})_{i=1}^n$ (for all random vectors, $\Rightarrow$ denotes convergence in distribution). Therefore, as $k \to \infty$,

$$\left(N_{t_i+k} - \lambda \mathrm{e}^{-\lambda(t_i+k)} \int_{t_0}^{t_i+k} \mathrm{e}^{\lambda s} N_s \, \mathrm{d}s \right)_{i=1}^{n} \Rightarrow (Y_{t_i})_{i=1}^n.$$

This shows that the distribution of Y only depends on N and λ, and completes the proof. □

Proposition 2.1 in Surgailis *et al.* [38] and Proposition 2.1 in Maejima and Yamamoto [23] provide existence results for stationary solutions to Langevin equations. However, these results do not cover Theorem 2.1. The first result considers only Bochner-type integrals and the second result requires, in particular, that the sample paths of N are Riemann integrable.

Let $B = (B_t)_{t\in\mathbb{R}}$ denote an $\mathcal{F}$-Brownian motion indexed by $\mathbb{R}$ and $\sigma = (\sigma_t)_{t\in\mathbb{R}}$ be a predictable process; that is, σ is measurable with respect to

$$\mathcal{P} = \sigma\big((s,t] \times A\colon\ s,t \in \mathbb{R}, s < t, A \in \mathcal{F}_s\big).$$

Assume that, for all $u \in \mathbb{R}$, $(\sigma_t, B_t)_{t\in\mathbb{R}}$ has the same finite-dimensional distributions as $(\sigma_{t+u}, B_{t+u} - B_u)_{t\in\mathbb{R}}$ and that $\sigma_0 \in L^2$. Then N, given by

$$N_t = \int_0^t \sigma_s \, \mathrm{d}B_s, \qquad t \in \mathbb{R}, \tag{2.5}$$

is a well-defined continuous process with stationary increments and finite second moments. (Recall that for $t < 0$, $\int_0^t := -\int_t^0$.)

Corollary 2.3. *Let N be given by (2.5). Then, there exists a unique-in-law QOU process X driven by N with parameter $\lambda > 0$, and X is given by*

$$X_t = \int_{-\infty}^{t} \mathrm{e}^{-\lambda(t-s)} \sigma_s \,\mathrm{d}B_s, \qquad t \in \mathbb{R}. \tag{2.6}$$

Proof. Since N is a measurable process with finite second moments, it follows by Theorem 2.1 that there exists a unique-in-law QOU process X, and it is given by

$$\begin{aligned} X_t &= N_t - \lambda \mathrm{e}^{-\lambda t} \int_{-\infty}^{t} \mathrm{e}^{\lambda s} N_s \,\mathrm{d}s = \lambda \int_{-\infty}^{0} \mathrm{e}^{\lambda s} (N_t - N_{t+s}) \,\mathrm{d}s \\ &= \lambda \int_{-\infty}^{0} \left(\int_{\mathbb{R}} 1_{(t+s,t]}(u) \mathrm{e}^{\lambda s} \sigma_u \,\mathrm{d}B_u \right) \mathrm{d}s. \end{aligned} \tag{2.7}$$

By an extension of Protter [29], Chapter IV, Theorem 65, from finite intervals to infinite intervals we may switch the order of integration in (2.7) and hence we obtain (2.6). □

Let us conclude this section with formulas for the mean and variance of a QOU process X. In the rest of this section, let N be a measurable process with stationary increments and finite first moments and let X be a QOU process driven by N with parameter $\lambda > 0$ (which exists by Theorem 2.1). Since X is unique in law, it makes sense to consider the mean and variance function of X. Let us assume for simplicity that $N_0 = 0$ a.s. The following proposition gives the mean and variance of X.

Proposition 2.4. *Let N and X be given as above. Then,*

$$\mathrm{E}[X_0] = \frac{\mathrm{E}[N_1]}{\lambda} \quad \textit{and} \quad \mathrm{Var}(X_0) = \frac{\lambda}{2} \int_0^\infty \mathrm{e}^{-\lambda s} \mathrm{V}_N(s) \,\mathrm{d}s.$$

In the part concerning the variance of X_0, we assume that N has finite second moments.

Note that Proposition 2.4 shows that the variance of X_0 is $\lambda/2$ times the Laplace transform of V_N. In particular, if $N_t = \mu t + \sigma B_t^H$, where B^H is a fractional Brownian motion (fBm) of index $H \in (0,1)$ (see [11] or [27] for properties of the fBm), then $\mathrm{E}[N_1] = \mu$ and $\mathrm{V}_N(s) = \sigma^2 |s|^{2H}$ and hence, by Proposition 2.4, we have that

$$\mathrm{E}[X_0] = \frac{\mu}{\lambda} \quad \text{and} \quad \mathrm{Var}(X_0) = \frac{\sigma^2 \Gamma(1+2H)}{2\lambda^{2H}}. \tag{2.8}$$

For $H = 1/2$, (2.8) is well known, and in this case $\mathrm{Var}(X_0) = \sigma^2/(2\lambda)$.

Before proving Proposition 2.4, let us note that $\mathrm{E}[N_t] = \mathrm{E}[N_1]t$ for all $t \in \mathbb{R}$. Indeed, this follows by the continuity of $t \mapsto \mathrm{E}[N_t]$ (see Corollary A.3) and the stationarity of the increments of N.

Proof of Proposition 2.4. Recall that, by Corollary A.3, we have that $\mathrm{E}[|N_t|] \le \alpha + \beta|t|$ for some $\alpha, \beta > 0$. Hence, by (2.2) and Fubini's theorem, we have that

$$\begin{aligned}\mathrm{E}[X_0] &= \mathrm{E}\left[-\lambda \int_{-\infty}^{0} \mathrm{e}^{\lambda s} N_s \,\mathrm{d}s\right] = -\lambda \int_{-\infty}^{0} \mathrm{e}^{\lambda s} \mathrm{E}[N_s] \,\mathrm{d}s \\ &= -\lambda \mathrm{E}[N_1] \int_{-\infty}^{0} \mathrm{e}^{\lambda s} s \,\mathrm{d}s = \mathrm{E}[N_1]/\lambda.\end{aligned}$$

This shows the part concerning the mean of X_0.

To show the last part, assume that N has finite second moments. By using $\mathrm{E}[X_0] = \mathrm{E}[N_1]/\lambda$, (2.2) shows that, with $\tilde{N}_t := N_t - \mathrm{E}[N_1]t$, we have

$$\mathrm{Var}(X_0) = \mathrm{E}\big[(X_0 - \mathrm{E}[X_0])^2\big] = \mathrm{E}\left[\left(\lambda \int_{-\infty}^{0} \mathrm{e}^{\lambda s} \tilde{N}_s \,\mathrm{d}s\right)^2\right].$$

Since $\|\tilde{N}_t\|_2 \le \alpha + \beta|t|$ for some $\alpha, \beta > 0$ by Corollary A.3, Fubini's theorem shows

$$\mathrm{Var}(X_0) = \lambda^2 \int_{-\infty}^{0} \int_{-\infty}^{0} (\mathrm{e}^{\lambda s} \mathrm{e}^{\lambda u} \mathrm{E}[\tilde{N}_s \tilde{N}_u]) \,\mathrm{d}s \,\mathrm{d}u,$$

and since $\mathrm{E}[\tilde{N}_s \tilde{N}_u] = \frac{1}{2}[\mathrm{V}_N(s) + \mathrm{V}_N(u) - \mathrm{V}_N(s-u)]$, we have

$$\begin{aligned}\mathrm{Var}(X_0) &= \frac{\lambda^2}{2} \int_{-\infty}^{0} \int_{-\infty}^{0} \big(\mathrm{e}^{\lambda s} \mathrm{e}^{\lambda u} \big(\mathrm{V}_N(s) + \mathrm{V}_N(u) - \mathrm{V}_N(s-u)\big)\big) \,\mathrm{d}s \,\mathrm{d}u \\ &= \lambda \int_{-\infty}^{0} \mathrm{e}^{\lambda s} \mathrm{V}_N(s) \,\mathrm{d}s - \frac{\lambda^2}{2} \int_{-\infty}^{0} \mathrm{e}^{\lambda u} \left(\int_{-\infty}^{-u} \mathrm{e}^{\lambda(s+u)} \mathrm{V}_N(s) \,\mathrm{d}s\right) \mathrm{d}u.\end{aligned} \tag{2.9}$$

Moreover,

$$\begin{aligned}&\frac{\lambda^2}{2} \int_{-\infty}^{0} \mathrm{e}^{\lambda u} \left(\int_{-\infty}^{-u} \mathrm{e}^{\lambda(s+u)} \mathrm{V}_N(s) \,\mathrm{d}s\right) \mathrm{d}u \\ &\quad = \frac{\lambda^2}{2} \int_{\mathbb{R}} \mathrm{V}_N(s) \mathrm{e}^{\lambda s} \left(\int_{-\infty}^{(-s)\wedge 0} \mathrm{e}^{2\lambda u} \,\mathrm{d}u\right) \mathrm{d}s \\ &\quad = \frac{\lambda^2}{2} \left(\int_{-\infty}^{0} \mathrm{V}_N(s) \mathrm{e}^{\lambda s} \left(\int_{-\infty}^{0} \mathrm{e}^{2\lambda u} \,\mathrm{d}u\right) \mathrm{d}s + \int_{0}^{\infty} \mathrm{V}_N(s) \mathrm{e}^{\lambda s} \left(\int_{-\infty}^{-s} \mathrm{e}^{2\lambda u} \,\mathrm{d}u\right) \mathrm{d}s\right) \\ &\quad = \frac{\lambda}{4} \left(\int_{-\infty}^{0} \mathrm{V}_N(s) \mathrm{e}^{\lambda s} \,\mathrm{d}s + \int_{0}^{\infty} \mathrm{V}_N(s) \mathrm{e}^{\lambda s} (\mathrm{e}^{-2\lambda s}) \,\mathrm{d}s\right) \\ &\quad = \frac{\lambda}{2} \int_{0}^{\infty} \mathrm{e}^{-\lambda s} \mathrm{V}_N(s) \,\mathrm{d}s,\end{aligned}$$

which, by (2.9), gives the expression for the variance of X_0. □

2.2. Asymptotic behavior of the autocovariance function

The next result shows that the autocovariance function of a QOU process X driven by N with parameter λ has the same asymptotic behavior at infinity as the second derivative of the variance function of N divided by $2\lambda^2$.

Proposition 2.5. *Let N be a measurable process with stationary increments, $N_0 = 0$ a.s., and finite second moments. Let X be a QOU process driven by N with parameter $\lambda > 0$.*

(i) *Assume that V_N is three times continuous differentiable in a neighborhood of ∞, and for $t \to \infty$ we have that $\mathrm{V}_N''(t) = \mathrm{O}(\mathrm{e}^{(\lambda/2)t})$, $\mathrm{e}^{-\lambda t} = \mathrm{o}(\mathrm{V}_N''(t))$ and $\mathrm{V}_N'''(t) = \mathrm{o}(\mathrm{V}_N''(t))$. Then, for $t \to \infty$, we have $\mathrm{R}_X(t) \sim (\frac{1}{2\lambda^2})\mathrm{V}_N''(t)$.*

(ii) *Assume for $t \to 0$ that $t^2 = \mathrm{o}(\mathrm{V}_N(t))$, then, for $t \to 0$, we have $\bar{\mathrm{R}}_X(t) \sim \frac{1}{2}\mathrm{V}_N(t)$. More generally, let $p \geq 1$ and assume that N has finite p moments and $t = \mathrm{o}(\|N_t\|_p)$ as $t \to 0$. Then, for $t \to 0$, we have $\|X_t - X_0\|_p \sim \|N_t\|_p$.*

Note that by Proposition 2.5(ii) the short-term asymptotic behavior of $\bar{\mathrm{R}}_X$ is not influenced by λ.

Proof of Proposition 2.5. (i) Let $\beta > 0$ and assume that V_N is three times continuous differentiable on (β, ∞); that is, $\mathrm{V}_N \in C^3((\beta, \infty); \mathbb{R})$. Let $t_0 = \beta + 1$, and let us show that for $t \geq t_0$ and $t \to \infty$,

$$\mathrm{R}_X(t) = \frac{\mathrm{e}^{-\lambda t}}{4\lambda} \int_{t_0}^{t} \mathrm{e}^{\lambda u} \mathrm{V}_N''(u)\,\mathrm{d}u + \frac{\mathrm{e}^{\lambda t}}{4\lambda} \int_{t}^{\infty} \mathrm{e}^{-\lambda u} \mathrm{V}_N''(u)\,\mathrm{d}u + \mathrm{O}(\mathrm{e}^{-\lambda t}). \tag{2.10}$$

If we have shown (2.10), then, by using that $\mathrm{e}^{-\lambda t} = \mathrm{o}(\mathrm{V}_N''(t))$, $\mathrm{V}_N'''(t) = \mathrm{o}(\mathrm{V}_N''(t))$ and l'Hôpital's rule, (i) follows.

Similar to the proof of Proposition 2.4, let $\tilde{N}_t = N_t - \mathrm{E}[N_1]t$. To show (2.10), recall that by Corollary A.3 we have $\|\tilde{N}_t\|_2 \leq \alpha + \beta|t|$ for some $\alpha, \beta > 0$. Hence, by (2.2) and Fubini's theorem, we find that

$$\mathrm{R}_X(t) = \mathrm{E}\big[(X_t - \mathrm{E}[X_t])(X_0 - \mathrm{E}[X_0])\big] = g(t) - \lambda \mathrm{e}^{-\lambda t} \int_{-\infty}^{t} \mathrm{e}^{\lambda s} g(s)\,\mathrm{d}s, \tag{2.11}$$

where

$$g(t) = -\lambda \int_{-\infty}^{0} \mathrm{e}^{\lambda s} \mathrm{E}[\tilde{N}_s \tilde{N}_t]\,\mathrm{d}s, \qquad t \in \mathbb{R}.$$

Since $\mathrm{E}[\tilde{N}_s \tilde{N}_t] = \frac{1}{2}[\mathrm{V}_N(t) + \mathrm{V}_N(s) - \mathrm{V}_N(s-t)]$, we have that

$$\begin{aligned} g(t) &= -\frac{\lambda}{2} \int_{-\infty}^{0} \mathrm{e}^{\lambda s}[\mathrm{V}_N(t) + \mathrm{V}_N(s) - \mathrm{V}_N(t-s)]\,\mathrm{d}s \\ &= -\frac{1}{2}\left(\mathrm{V}_N(t) - \lambda \mathrm{e}^{\lambda t} \int_{t}^{\infty} \mathrm{e}^{-\lambda s} \mathrm{V}_N(s)\,\mathrm{d}s\right) - \frac{\lambda}{2} \int_{-\infty}^{0} \mathrm{e}^{\lambda s} \mathrm{V}_N(s)\,\mathrm{d}s. \end{aligned} \tag{2.12}$$

From (2.12) it follows that $g \in C^1((\beta, \infty); \mathbb{R})$ and hence, using partial integration on (2.11), we have for $t \geq t_0$,

$$\mathrm{R}_X(t) = \mathrm{e}^{-\lambda t} \int_{t_0}^{t} \mathrm{e}^{\lambda s} g'(s)\,\mathrm{d}s + \mathrm{e}^{-\lambda t}\left(\mathrm{e}^{\lambda t_0} g(t_0) - \lambda \int_{-\infty}^{t_0} \mathrm{e}^{\lambda s} g(s)\,\mathrm{d}s\right). \tag{2.13}$$

Moreover, from (2.12) and for $t \geq t_0$, we find

$$g'(t) = -\frac{1}{2}\left(\mathrm{V}'_N(t) - \lambda^2 \mathrm{e}^{\lambda t} \int_t^{\infty} \mathrm{e}^{-\lambda s} \mathrm{V}_N(s)\,\mathrm{d}s + \lambda \mathrm{V}_N(t)\right). \tag{2.14}$$

For $t \to \infty$ we have, by assumption, that $\mathrm{V}''_N(t) = \mathrm{O}(\mathrm{e}^{(\lambda/2)t})$, and hence $\mathrm{V}'_N(t) = \mathrm{O}(\mathrm{e}^{(\lambda/2)t})$. Thus, from (2.14) and a double use of partial integration, we obtain that

$$g'(t) = \frac{\mathrm{e}^{\lambda t}}{2} \int_t^{\infty} \mathrm{e}^{-\lambda s} \mathrm{V}''_N(s)\,\mathrm{d}s, \qquad t \geq t_0. \tag{2.15}$$

Using (2.15), Fubini's theorem and that $\mathrm{V}''_N(t) = \mathrm{O}(\mathrm{e}^{(\lambda/2)t})$, we have for $t \geq t_0$,

$$\begin{aligned}
&\mathrm{e}^{-\lambda t} \int_{t_0}^{t} \mathrm{e}^{\lambda s} g'(s)\,\mathrm{d}s \\
&\quad = \mathrm{e}^{-\lambda t} \int_{t_0}^{t} \mathrm{e}^{\lambda s} \left(\frac{\mathrm{e}^{\lambda s}}{2} \int_s^{\infty} \mathrm{e}^{-\lambda u} \mathrm{V}''_N(u)\,\mathrm{d}u\right) \mathrm{d}s \\
&\quad = \mathrm{e}^{-\lambda t} \int_{t_0}^{\infty} \mathrm{e}^{-\lambda u} \mathrm{V}''_N(u) \left(\int_{t_0}^{t \wedge u} \frac{1}{2} \mathrm{e}^{2\lambda s}\,\mathrm{d}s\right) \mathrm{d}u \\
&\quad = \mathrm{e}^{-\lambda t} \int_{t_0}^{\infty} \mathrm{e}^{-\lambda u} \mathrm{V}''_N(u) \left(\frac{1}{4\lambda}\left(\mathrm{e}^{2\lambda(t \wedge u)} - \mathrm{e}^{2\lambda t_0}\right)\right) \mathrm{d}u \\
&\quad = \frac{\mathrm{e}^{-\lambda t}}{4\lambda} \int_{t_0}^{t} \mathrm{e}^{\lambda u} \mathrm{V}''_N(u)\,\mathrm{d}u + \frac{\mathrm{e}^{\lambda t}}{4\lambda} \int_t^{\infty} \mathrm{e}^{-\lambda u} \mathrm{V}''_N(u)\,\mathrm{d}u - \mathrm{e}^{-\lambda t}\left(\frac{\mathrm{e}^{2\lambda t_0}}{4\lambda} \int_{t_0}^{\infty} \mathrm{e}^{-\lambda u} \mathrm{V}''_N(u)\,\mathrm{d}u\right).
\end{aligned}$$

Combining this with (2.13) we obtain (2.10), and the proof of (i) is complete.

(ii) Using (2.1) we have for all $t > 0$ that

$$\|X_t - X_0\|_p \leq \|N_t\|_p + \lambda \int_0^t \|X_s\|_p\,\mathrm{d}s = \|N_t\|_p + \lambda t \|X_0\|_p.$$

On the other hand,

$$\|X_t - X_0\|_p \geq \|N_t\|_p - \lambda \int_0^t \|X_s\|_p\,\mathrm{d}s = \|N_t\|_p - \lambda t \|X_0\|_p,$$

which shows that

$$1 - \lambda \|X_0\|_p \frac{t}{\|N_t\|_p} \leq \frac{\|X_t - X_0\|_p}{\|N_t\|_p} \leq 1 + \lambda \|X_0\|_p \frac{t}{\|N_t\|_p}.$$

A similar inequality is available when $t < 0$, and hence for $t \to 0$ we have that $\|X_t - X_0\|_p \sim \|N_t\|_p$ if $\lim_{t\to 0}(t/\|N_t\|_p) = 0$. □

When N is an fBm of index $H \in (0, 1)$, then $\mathrm{V}_N(t) = |t|^{2H}$, and hence

$$\mathrm{V}_N''(t) = 2H(2H-1)t^{2H-2}, \qquad t > 0.$$

The conditions in Proposition 2.5 are clearly fulfilled and thus we have the following corollary.

Corollary 2.6. *Let N be an fBm of index $H \in (0, 1)$, and let X be a QOU process driven by N with parameter $\lambda > 0$. For $H \in (0, 1) \setminus \{\frac{1}{2}\}$ and $t \to \infty$, we have $\mathrm{R}_X(t) \sim (H(2H-1)/\lambda^2)t^{2H-2}$. For $H \in (0, 1)$ and $t \to 0$, we have $\bar{\mathrm{R}}_X(t) \sim \frac{1}{2}|t|^{2H}$.*

The above result concerning the behavior of R_X for $t \to \infty$ when N is an fBm has been obtained previously via a different approach by Cheridito *et al.* [14], Theorem 2.3.

A square-integrable stationary process $Y = (Y_t)_{t\in\mathbb{R}}$ is said to have *long-range dependence* of order $\alpha \in (0, 1)$ if R_Y is regularly varying at ∞ of index $-\alpha$. Recall that a function $f : \mathbb{R} \to \mathbb{R}$ is regularly varying at ∞ of index $\beta \in \mathbb{R}$ if, for $t \to \infty$, $f(t) \sim t^\beta l(t)$, where l is slowly varying, which means that for all $a > 0$, $\lim_{t\to\infty} l(at)/l(t) = 1$. Many empirical observations have shown evidence for long-range dependence in various fields, such as finance, telecommunication and hydrology; see Doukhan *et al.* [18]. Let X be a QOU process driven by N; then Proposition 2.5(i) shows that X has long-range dependence of order $\alpha \in (0, 1)$ if and only if V_N'' is regularly varying at ∞ of order $-\alpha$. Furthermore, Proposition 4.9(i) below shows how to construct QOUs with long-range dependence. More precisely, if X is a QOU driven by N, where N is given by (4.9), and for some $\alpha \in (0, 1)$ and $t \to \infty$, $f'(t) \sim ct^{(\alpha-1)/2}$, then X has long-range dependence of order α. The example $f(t) = (\delta \vee t)^{H-1/2}$, with $\delta \geq 0$ and $H \in (\frac{1}{2}, 1)$ is considered in Corollary 4.10 and it follows that the QOU process X has long-range dependence of order $2 - 2H$. Here X is a fractional Ornstein–Uhlenbeck process if $\delta = 0$, and a semimartingale if and only if $\delta > 0$. A quite different type of semimartingale with long-range dependence is obtained for $N = \sigma \bullet B$ with σ and B independent and σ^2 being a supOU process with long-range dependence, cf. Barndorff-Nielsen [1], Barndorff-Nielsen and Stelzer [6] and Barndorff-Nielsen and Shephard [4]. Hence, by considering more general processes than the fractional type, we can easily construct stationary processes with long-range dependence within the semimartingale framework.

3. A Fubini theorem for Lévy bases

Let $\Lambda = \{\Lambda(A)\colon A \in \mathcal{S}\}$ denote a centered Lévy basis on a non-empty space S equipped with a δ-ring $\mathcal{S}$, see Rajput and Rosiński [30]. (A Lévy basis is an infinitely divisible, independently scattered random measure. Recall also that a δ-ring on S is a family of subsets of S that is closed under union, countable intersection and set difference). As usual, we assume that $\mathcal{S}$ is σ-finite, meaning that there exists $(S_n)_{n\geq 1} \subseteq \mathcal{S}$ such that $\bigcup_{n\geq 1} S_n = S$. All integrals $\int_S f(s)\Lambda(\mathrm{d}s)$ will be defined in the sense of Rajput and Rosiński [30]. We can now find a measurable parametrization of Lévy measures $\nu(\mathrm{d}u, s)$ on $\mathbb{R}$, a σ-finite measure m on S and a positive measurable function $\sigma^2 : S \to \mathbb{R}_+$, such that for all $A \in \mathcal{S}$,

$$\mathrm{E}\big[\mathrm{e}^{\mathrm{i}y\Lambda(A)}\big] = \exp\biggl(\int_A \biggl[-\sigma^2(s)y^2/2 + \int_{\mathbb{R}} (\mathrm{e}^{\mathrm{i}yu} - 1 - \mathrm{i}yu)\nu(\mathrm{d}u, s)\biggr] m(\mathrm{d}s)\biggr), \qquad y \in \mathbb{R}, \quad (3.1)$$

see [30]. Let $\phi : \mathbb{R} \times S \mapsto \mathbb{R}$ be given by

$$\phi(y, s) = y^2\sigma^2(s) + \int_{\mathbb{R}} \big[(uy)^2 1_{\{|uy| \le 1\}} + (2|uy| - 1)1_{\{|uy|>1\}}\big]\nu(\mathrm{d}u, s),$$

and for all measurable functions $g : S \to \mathbb{R}$ define

$$\|g\|_\phi = \inf\biggl\{c > 0\colon \int_S \phi(c^{-1}g(s), s)m(\mathrm{d}s) \le 1\biggr\} \in [0, \infty].$$

Moreover, let $L^\phi = L^\phi(S, \sigma(\mathcal{S}), m)$ denote the *Musielak–Orlicz space* of measurable functions g with

$$\int_S \biggl[g(s)^2\sigma^2(s) + \int_{\mathbb{R}} \big(|ug(s)|^2 \wedge |ug(s)|\big)\nu(\mathrm{d}u, s)\biggr] m(\mathrm{d}s) < \infty,$$

equipped with the Luxemburg norm $\|g\|_\phi$. Note that $g \in L^\phi$ if and only if $\|g\|_\phi < \infty$, since $\phi(2x, s) \le C\phi(x, s)$ for some $C > 0$ and all $s \in S, x \in \mathbb{R}$. We refer to Musielak [26] for the basic properties of Musielak–Orlicz spaces. When $\sigma^2 \equiv 0$ and $g \in L^\phi$, Theorem 2.1 in Marcus and Rosiński [24] shows that $\int_S g(s)\Lambda(\mathrm{d}s)$ is well defined, integrable and centered and

$$c_1\|g\|_\phi \le \mathrm{E}\biggl[\biggl|\int_S g(s)\Lambda(\mathrm{d}s)\biggr|\biggr] \le c_2\|g\|_\phi,$$

and we may choose $c_1 = 1/8$ and $c_2 = 17/8$. Hence for general σ^2 it is easily seen that for all $g \in L^\phi$, $\int_S g(s)\Lambda(\mathrm{d}s)$ is well defined, integrable and centered and

$$\mathrm{E}\biggl[\biggl|\int_S g(s)\Lambda(\mathrm{d}s)\biggr|\biggr] \le 2c_2\|g\|_\phi. \quad (3.2)$$

Let T denote a complete separable metric space, and $Y = (Y_t)_{t\in T}$ be given by

$$Y_t = \int_S f(t, s)\Lambda(\mathrm{d}s), \qquad t \in T,$$

for some measurable function $f(\cdot,\cdot)$ for which the integrals are well defined. Then we can choose a measurable modification of Y. Indeed, the existence of a measurable modification of Y is equivalent to measurability of $(t \in T) \mapsto (Y_t \in L^0)$ according to Theorem 3 and the remark in Cohn [15]. Hence, since f is measurable, the maps $(t \in T) \mapsto (\|f(t,\cdot) - g(\cdot)\|_\phi \in \mathbb{R})$ for all $g \in L^\phi$ are measurable. This shows that $(t \in \mathbb{R}) \mapsto (f(t,\cdot) \in L^\phi)$ is measurable since L^ϕ is a separable Banach space. Hence by continuity of $(f(t,\cdot) \in L^\phi) \mapsto (Y_t \in L^0)$, see Rajput and Rosiński [30], it follows that $(t \in T) \mapsto (Y_t \in L^0)$ is measurable.

Assume that μ is a σ-finite measure on a complete and separable metric space T. Then we have the following stochastic Fubini result extending Rosiński [33], Lemma 7.1; Pérez-Abreu and Rocha-Arteaga [28], Lemma 5; and Basse and Pedersen [9], Lemma 4.9. Stochastic Fubini-type results for semimartingales can be founded in Protter [29] and Ikeda and Watanabe [19]; however, the assumptions in these results are too strong for our purpose.

Theorem 3.1 (Fubini). *Let $f : T \times S \mapsto \mathbb{R}$ be an $\mathcal{B}(T) \otimes \sigma(\mathcal{S})$-measurable function such that*

$$f_x = f(x, \cdot) \in L^\phi \qquad \text{for } x \in T \text{ and } \int_E \|f_x\|_\phi \mu(\mathrm{d}x) < \infty. \tag{3.3}$$

Then $f(\cdot, s) \in L^1(\mu)$ for m-a.a. $s \in S$ and $s \mapsto \int_T f(x, s)\mu(\mathrm{d}x)$ belongs to L^ϕ, all of the below integrals exist and

$$\int_T \left(\int_S f(x, s)\Lambda(\mathrm{d}s) \right) \mu(\mathrm{d}x) = \int_S \left(\int_T f(x, s)\mu(\mathrm{d}x) \right) \Lambda(\mathrm{d}s) \qquad a.s. \tag{3.4}$$

Remark 3.2. If μ is a finite measure, then the last condition in (3.3) is equivalent to

$$\int_T \left[\int_S f(x, s)^2 \sigma^2(s) + \int_{\mathbb{R}} \left(|uf(x, s)|^2 \wedge |uf(x, s)| \right) \nu(\mathrm{d}u, s) \right] m(\mathrm{d}s)\mu(\mathrm{d}x) < \infty.$$

We will need Theorem 3.1 to be able to prove Proposition 4.2. That proposition yields, in particular, examples for which the conditions of Theorem 3.1 are fulfilled. But before proving Theorem 3.1, we will need the following observation.

Lemma 3.3. *For all measurable functions $f : T \times S \to \mathbb{R}$ we have*

$$\left\| \int_T |f(x, \cdot)| \mu(\mathrm{d}x) \right\|_\phi \le \int_T \|f(x, \cdot)\|_\phi \mu(\mathrm{d}x). \tag{3.5}$$

Moreover, if $f : T \times S \to \mathbb{R}$ is a measurable function such that $\int_T \|f(x, \cdot)\|_\phi \mu(\mathrm{d}x) < \infty$, then for m-a.a. $s \in S$, $f(\cdot, s) \in L^1(\mu)$ and $s \mapsto \int_T f(x, s)\mu(\mathrm{d}x)$ is a well-defined function that belongs to L^ϕ.

Proof. Let us sketch the proof of (3.5). For f of the form

$$f(x, s) = \sum_{i=1}^k g_i(s) 1_{A_i}(x),$$

where $k \ge 1$, $g_1, \dots, g_k \in L^\phi$ and $A_1, \dots, A_k$ are disjoint measurable subsets of T of finite μ-measure, (3.5) easily follows. Hence, by a monotone class lemma argument, it is possible to show (3.5) for all measurable f. The second statement is a consequence of (3.5). □

Recall that if $(F, \|\cdot\|)$ is a separable Banach space, μ is a measure on T and $f : T \to F$ is a measurable map such that $\int_T \|f(x)\|\mu(\mathrm{d}x) < \infty$, then the Bochner integral $\mathrm{B}\int_T f(x)\mu(\mathrm{d}x)$ exists in F and $\|\mathrm{B}\int_T f(x)\mu(\mathrm{d}x)\| \le \int_T \|f(x)\|\mu(\mathrm{d}x)$. Even though $(L^\phi, \|\cdot\|_\phi)$ is a Banach space, this result does not cover Lemma 3.3.

Proof of Theorem 3.1. For f of the form

$$f(x,s) = \sum_{i=1}^{n} \alpha_i 1_{A_i}(x) 1_{B_i}(s), \qquad x \in T, s \in S, \tag{3.6}$$

where $n \ge 1$, $A_1, \dots, A_n$ are measurable subsets of T of finite μ-measure, $B_1, \dots, B_n \in \mathcal{S}$ and $\alpha_1, \dots, \alpha_n \in \mathbb{R}$, the theorem is trivially true. Thus, for a general f as in the theorem, choose f_n for $n \ge 1$ of the form (3.6) such that $\int_T \|f_n(x,\cdot) - f(x,\cdot)\|_\phi \mu(\mathrm{d}x) \to 0$. Indeed, the existence of such a sequence follows by an application of the monotone class lemma. Let

$$X_n = \int_E \left(\int_S f_n(x,s)\Lambda(\mathrm{d}s)\right)\mu(\mathrm{d}x), \qquad X = \int_E \left(\int_S f(x,s)\Lambda(\mathrm{d}s)\right)\mu(\mathrm{d}x),$$

and let us show that X is well defined and $X_n \to X$ in L^1. This follows since

$$\mathrm{E}\left[\int_E \left|\int_S f(x,s)\Lambda(\mathrm{d}s)\right|\mu(\mathrm{d}x)\right] \le 2c_2 \int_E \|f(x,\cdot)\|_\phi \mu(\mathrm{d}x) < \infty$$

and

$$\mathrm{E}[|X_n - X|] \le 2c_2 \int_E \|f_n(x,\cdot) - f(x,\cdot)\|_\phi \mu(\mathrm{d}x).$$

Similarly, let

$$Y_n = \int_S \left(\int_E f_n(x,s)\mu(\mathrm{d}x)\right)\Lambda(\mathrm{d}s), \qquad Y = \int_S \left(\int_E f(x,s)\mu(\mathrm{d}x)\right)\Lambda(\mathrm{d}s)$$

and let us show that Y is well defined and $Y_n \to Y$ in L^1. By Remark 3.3, $s \mapsto \int_E f(x,s)\mu(\mathrm{d}x)$ is a well-defined function that belongs to L^ϕ, which shows that Y is well defined. By (3.2) and (3.5) we have

$$\mathrm{E}[|Y_n - Y|] \le 2c_2 \int_E \|f_n(x,\cdot) - f(x,\cdot)\|_\phi \mu(\mathrm{d}x),$$

which shows that $Y_n \to Y$ in L^1. We have, therefore, proved (3.4), since $Y_n = X_n$ a.s., $X_n \to X$ and $Y_n \to Y$ in L^1. $\square$

Let $Z = (Z_t)_{t\in\mathbb{R}}$ denote an integrable and centered Lévy process with Lévy measure ν and Gaussian component σ^2. Then Z induces a Lévy basis Λ on $S = \mathbb{R}$ and $\mathcal{S} = \mathcal{B}_b(\mathbb{R})$, the bounded

Borel sets, which is uniquely determined by $\Lambda((a,b]) = Z_b - Z_a$ for all $a, b \in \mathbb{R}$ with $a < b$. In this case m is the Lebesgue measure on $\mathbb{R}$ and

$$\phi(y,s) = \phi(y) = \sigma^2 + \int_{\mathbb{R}} \bigl(|uy|^2 1_{\{|uy|\le 1\}} + (2|uy| - 1) 1_{\{|uy|>1\}}\bigr)\nu(\mathrm{d}u).$$

We will write $\int f(s)\mathrm{d}Z_s$ instead of $\int f(s)\Lambda(\mathrm{d}s)$. Note that, $\int_{\mathbb{R}} f(s)\,\mathrm{d}Z_s$ exists and is integrable if and only if $f \in L^\phi$, that is,

$$\int_{\mathbb{R}} \biggl(f(s)^2\sigma^2 + \int_{\mathbb{R}} \bigl(|uf(s)|^2 \wedge |uf(s)|\bigr)\nu(\mathrm{d}x)\biggr)\mathrm{d}s < \infty. \tag{3.7}$$

Moreover, if Z is a symmetric α-stable Lévy process, $\alpha \in (0,2]$, then $L^\phi = L^\alpha(\mathbb{R},\lambda)$, where $L^\alpha(\mathbb{R},\lambda)$ is the space of α-integrable functions with respect to the Lebesgue measure λ.

4. Moving average representations

In wide generality, if X is a continuous-time stationary process, then it is representable, in principle, as a moving average (MA), that is,

$$X_t = \int_{-\infty}^{t} \psi(t-s)\,\mathrm{d}\Xi_s,$$

where ψ is a deterministic function and Ξ has stationary and orthogonal increments, at least in the second-order sense. (For a precise statement, see the beginning of Section 4.1.) However, an explicit expression for ϕ is seldom available.

We show in Section 4.2 that an expression can be found in cases where the process X is the stationary solution to a Langevin equation for which the driving noise process N is a PMA, that is,

$$N_t = \int_{\mathbb{R}} \bigl(f(t-s) - f(-s)\bigr)\,\mathrm{d}Z_s, \qquad t \in \mathbb{R}, \tag{4.1}$$

where $Z = (Z_t)_{t\in\mathbb{R}}$ is a suitable process specified later on and $f : \mathbb{R} \to \mathbb{R}$ is a deterministic function for which the integrals exist.

In Section 4.3, continuing the discussion from Section 2.2, we use the MA representation to study the asymptotic behavior of the associated autocovariance functions. Section 4.4 comments on a notable cancellation effect. But first, in Section 4.1 we summarize known results concerning Wold–Karhunen-type representations of stationary continuous-time processes.

4.1. Wold–Karhunen-type decompositions

Let $X = (X_t)_{t\in\mathbb{R}}$ be a second-order stationary process of mean zero and continuous in quadratic mean. Let F_X denote the spectral measure of X, that is, F_X is a finite and symmetric measure on

$\mathbb{R}$ satisfying

$$\mathrm{E}[X_t X_u] = \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}(t-u)x} F_X(\mathrm{d}x), \qquad t, u \in \mathbb{R},$$

and let F'_X denote the density of the absolutely continuous part of F_X. For each $t \in \mathbb{R}$ let $\mathcal{X}_t = \overline{\operatorname{span}}\{X_s\colon s \le t\}$, $\mathcal{X}_{-\infty} = \bigcap_{t\in\mathbb{R}} \mathcal{X}_t$ and $\mathcal{X}_\infty = \overline{\operatorname{span}}\{X_s\colon s \in \mathbb{R}\}$ ($\overline{\operatorname{span}}$ denotes the L^2-closure of the linear span). Then X is called deterministic if $\mathcal{X}_{-\infty} = \mathcal{X}_\infty$ and purely non-deterministic if $\mathcal{X}_{-\infty} = \{0\}$. The following result, which is due to Satz 5–6 in Karhunen [20] (cf. also Doob [17], Chapter XII, Theorem 5.3), provides a decomposition of stationary processes as a sum of a deterministic process and a purely non-deterministic process.

Theorem 4.1 (Karhunen). *Let X and F_X be given as above. If*

$$\int_{\mathbb{R}} \frac{|\log F'_X(x)|}{1+x^2}\,\mathrm{d}x < \infty, \tag{4.2}$$

then there exists a unique decomposition of X as

$$X_t = \int_{-\infty}^{t} \psi(t-s)\,\mathrm{d}\Xi_s + V_t, \qquad t \in \mathbb{R}, \tag{4.3}$$

where $\psi:\mathbb{R}\to\mathbb{R}$ is a Lebesgue square-integrable deterministic function and Ξ is a process with second-order stationary and orthogonal increments, $\mathrm{E}[|\Xi_u - \Xi_s|^2] = |u-s|$. Furthermore, for all $t \in \mathbb{R}$, $\mathcal{X}_t = \overline{\operatorname{span}}\{\Xi_s - \Xi_u\colon -\infty < u < s \le t\}$, and V is a deterministic second-order stationary process.

Moreover, if F_X is absolutely continuous and (4.2) is satisfied, then $V \equiv 0$ and hence X is a backward MA. Finally, the integral in (4.2) is infinite if and only if X is deterministic.

The results in Karhunen [20] are formulated for complex-valued processes; however, if X is real-valued (as it is in our case), then one can show that all the above processes and functions are real-valued as well. Note also that if X is Gaussian, then the process Ξ in (4.3) is a standard Brownian motion. If σ is a stationary process with $\mathrm{E}[\sigma_0^2] = 1$ and B is a Brownian motion, then $\mathrm{d}\Xi_s = \sigma_s\,\mathrm{d}B_s$ is of the above type.

A generalization of the classical Wold–Karhunen result to a broad range of non-Gaussian, infinitely divisible processes was given in Rosiński [34].

4.2. Explicit MA solutions of Langevin equations

Assume initially that Z is an integrable and centered Lévy process, and recall that L^ϕ is the space of all measurable functions $f:\mathbb{R}\to\mathbb{R}$ satisfying (3.7). Let $f:\mathbb{R}\to\mathbb{R}$ be a measurable function such that $f(t-\cdot) - f(-\cdot) \in L^\phi$ for all $t \in \mathbb{R}$, and let N be given by

$$N_t = \int_{\mathbb{R}} \bigl(f(t-s) - f(-s)\bigr)\,\mathrm{d}Z_s, \qquad t \in \mathbb{R}.$$

Proposition 4.2. *Let N be given as above. Then there exists a unique-in-law QOU process X driven by N with parameter $\lambda > 0$, and X is an MA of the form*

$$X_t = \int_{\mathbb{R}} \psi_f(t-s)\,\mathrm{d}Z_s, \qquad t \in \mathbb{R}, \tag{4.4}$$

where $\psi_f : \mathbb{R} \to \mathbb{R}$ belongs to L^ϕ and is given by

$$\psi_f(t) = \left(f(t) - \lambda \mathrm{e}^{-\lambda t} \int_{-\infty}^{t} \mathrm{e}^{\lambda s} f(s)\,\mathrm{d}s \right), \qquad t \in \mathbb{R}. \tag{4.5}$$

Proof. Since $(t,s) \mapsto f(t-s) - f(-s)$ is measurable we may choose a measurable modification of N – see Section 3 – and hence, by Theorem 2.1, there exists a unique-in-law QOU process X driven by N with parameter λ. For fixed $t \in \mathbb{R}$, we have by (2.2) and with $h_u(s) = f(t-s) - f(t+u-s)$ for all $u, s \in \mathbb{R}$ and $\mu(\mathrm{d}u) = 1_{\{u \le 0\}} \mathrm{e}^{\lambda u}\,\mathrm{d}u$ that

$$X_t = \lambda \int_{-\infty}^{0} \mathrm{e}^{\lambda u} (N_t - N_{t+u})\,\mathrm{d}u = \int_{-\infty}^{0} \left(\int_{\mathbb{R}} h_u(s)\,\mathrm{d}Z_s \right) \mu(\mathrm{d}u).$$

By Theorem A.1 there exist $\alpha, \beta > 0$ such that $\|h_u\|_\phi \le \alpha + \beta|t|$ for all $u \in \mathbb{R}$, implying that $\int_{\mathbb{R}} \|h_u\|_\phi \mu(\mathrm{d}u) < \infty$. By Theorem 3.1, $(u \mapsto h_u(s)) \in L^1(\mu)$ for Lebesgue almost all $s \in \mathbb{R}$, which implies that $\int_{-\infty}^{t} |f(u)| \mathrm{e}^{\lambda u}\,\mathrm{d}u < \infty$ for all $t > 0$, and hence ψ_f, defined in (4.5), is a well-defined function. Moreover, by Theorem 3.1, $\psi_f \in L^\phi(\mathbb{R}, \lambda)$ and

$$X_t = \int_{\mathbb{R}} \left(\int_{-\infty}^{0} h(u,s) \mu(\mathrm{d}u) \right) \mathrm{d}Z_s = \int_{\mathbb{R}} \psi_f(t-s)\,\mathrm{d}Z_s, \qquad t \in \mathbb{R},$$

which completes the proof. □

Note that for $f = 1_{\mathbb{R}_+}$, we have $N_t = Z_t$ and $\psi_f(t) = \mathrm{e}^{-\lambda t} 1_{\mathbb{R}_+}(t)$. Thus, in this case we recover the well-known result that the QOU process X driven by Z with parameter $\lambda > 0$ is an MA of the form $X_t = \int_{-\infty}^{t} \mathrm{e}^{-\lambda(t-s)}\,\mathrm{d}Z_s$.

Let us use the notation $x_+ := x 1_{\{x \ge 0\}}$, and let c_H be given by

$$c_H = \frac{\sqrt{2H \sin(\pi H) \Gamma(2H)}}{\Gamma(H + 1/2)}.$$

A PMA N of the form (4.1), where Z is an α-stable Lévy process with $\alpha \in (0,2]$ and f is given by $t \mapsto c_H t_+^{H-1/\alpha}$, is called a *linear fractional α-stable motion* of index $H \in (0,1)$; see Samorodnitsky and Taqqu [35]. Moreover, PMAs with $f(t) = t^\alpha$ for $\alpha \in (0, \frac{1}{2})$ and where Z is a square-integrable and centered Lévy process are called *fractional Lévy processes* in Marquardt [25]; these processes provide examples of f and Z for which Proposition 4.2 applies. Moreover, [25], Theorems 6.2 and 6.3, studies MAs driven by fractional Lévy processes, which in some cases also have a representation of the form (4.4).

Corollary 4.3. *Let $\alpha \in (1,2]$ and N be a linear fractional α-stable motion of index $H \in (0,1)$. Then there exists a unique-in-law QOU process X driven by N with parameter $\lambda > 0$, and X is an MA of the form*

$$X_t = \int_{-\infty}^{t} \psi_{\alpha,H}(t-s)\,\mathrm{d}Z_s, \qquad t \in \mathbb{R},$$

where $\psi_{\alpha,H} : \mathbb{R}_+ \to \mathbb{R}$ is given by

$$\psi_{\alpha,H}(t) = c_H\biggl(t^{H-1/\alpha} - \lambda \mathrm{e}^{-\lambda t}\int_0^t \mathrm{e}^{\lambda u} u^{H-1/\alpha}\,\mathrm{d}u\biggr), \qquad t \geq 0.$$

For $t \to \infty$, we have $\psi_{\alpha,H}(t) \sim (c_H(H-1/\alpha)/\lambda)t^{H-1/\alpha-1}$, and for $t \to 0$, $\psi_{\alpha,H}(t) \sim c_H t^{H-1/\alpha}$.

Remark 4.4. A QOU process driven by a linear fractional α-stable motion is called a fractional Ornstein–Uhlenbeck process. In Maejima and Yamamoto [23], the existence of the fractional Ornstein–Uhlenbeck process is shown in the case where $\alpha > 1$ and $1/\alpha < H < 1$. (The case $H = 1/\alpha$ is trivial since $X = N$.) The existence in the case $H \in (0, 1/\alpha)$ (see Corollary 4.3) is somewhat unexpected due to the fact that the sample paths of the linear fractional α-stable motion are unbounded on each compact interval; cf. page 4 in Maejima and Yamamoto [23], where non-existence is surmised. In the case $\alpha = 2$ (i.e., N is a fractional Brownian motion), Cheridito *et al.* [14] show the existence of the fractional Ornstein–Uhlenbeck process.

In the next lemma we will show a special property of ψ_f, given by (4.5); namely that $\int_0^\infty \psi_f(s)\,\mathrm{d}s = 0$ whenever this integral is well defined and f tends to zero at ∞. This property has a great impact on the behavior of the autocovariance function of QOU processes. We will return to this point in Section 4.4.

Lemma 4.5. *Let $f : \mathbb{R} \to \mathbb{R}$ be a locally integrable function that is zero on $(-\infty, 0)$ and $\lim_{t\to\infty} f(t) = 0$. Then, $\lim_{t\to\infty} \int_0^t \psi_f(s)\,\mathrm{d}s = 0$.*

Proof. For $t > 0$,

$$\begin{aligned}\int_0^t \biggl(\lambda \mathrm{e}^{-\lambda s}\int_0^s \mathrm{e}^{\lambda u} f(u)\,\mathrm{d}u\biggr)\mathrm{d}s &= \int_0^t \biggl(\biggl(\int_u^t \lambda \mathrm{e}^{-\lambda s}\,\mathrm{d}s\biggr)\mathrm{e}^{\lambda u} f(u)\,\mathrm{d}u \\ &= \int_0^t f(u)\,\mathrm{d}u - \mathrm{e}^{-\lambda t}\int_0^t \mathrm{e}^{\lambda u} f(u)\,\mathrm{d}u,\end{aligned}$$

and hence, by using that $\lim_{t\to\infty} f(t) = 0$, we obtain that

$$\lim_{t\to\infty} \int_0^t \psi_f(s)\,\mathrm{d}s = \lim_{t\to\infty}\biggl(\mathrm{e}^{-\lambda t}\int_0^t \mathrm{e}^{\lambda u} f(u)\,\mathrm{d}u\biggr) = 0. \qquad \square$$

Proposition 4.2 carries over to a much more general setting. For example, if N is of the form

$$N_t = \int_{\mathbb{R}\times V} [f(t-s,x) - f(-s,x)]\Lambda(\mathrm{d}s,\mathrm{d}x), \qquad t\in\mathbb{R},$$

where Λ is a centered Lévy basis on $\mathbb{R}\times V$ (V is a non-empty space) with control measure $m(\mathrm{d}s,\mathrm{d}x) = \mathrm{d}s n(\mathrm{d}x)$; $a(s,x), \sigma^2(s,x)$ and $\nu(\mathrm{d}u,(s,x))$, from (3.1), do not depend on $s\in\mathbb{R}$; and $f(t-\cdot,\cdot) - f(-\cdot,\cdot)\in L^\phi$ for all $t\in\mathbb{R}$, then, using Theorems A.1, 2.1 and 3.1, the arguments from Proposition 4.2 show that there exists a unique-in-law QOU process X driven by N with parameter $\lambda>0$, and X is given by

$$X_t = \int_{\mathbb{R}\times V} \psi_f(t-s,x)\Lambda(\mathrm{d}s,\mathrm{d}x), \qquad t\in\mathbb{R},$$

where

$$\psi_f(s,x) = f(s,x) - \lambda \mathrm{e}^{-\lambda s}\int_{-\infty}^{s} f(u,x)\mathrm{e}^{\lambda u}\,\mathrm{d}u, \qquad s\in\mathbb{R}, x\in V.$$

We recover Proposition 4.2 when $V=\{0\}$ and $n=\delta_0$ is the Dirac delta measure at 0.

4.3. Asymptotic behavior of the autocovariance function

The representation, from the previous section, of QOU processes as MAs enables us to handle the autocovariance function analytically. In Section 4.3.1 we discuss how the tail behavior of the kernel ψ of a general MA process determines that of the covariance function. By use of those results, Section 4.3.2 relates the asymptotic behavior of the kernel of the noise N to the asymptotic behavior of the autocovariance function of the QOU process X driven by N, both for $t\to 0$ and $t\to\infty$.

4.3.1. *Autocovariance function of general MAs*

Let ψ be a Lebesgue square-integrable function and Z be a centered process with stationary and orthogonal increments. Assume for simplicity that $Z_0=0$ a.s. and $\mathrm{V}_Z(t)=t$. Let $X=\psi * Z = (\int_{-\infty}^t \psi(t-s)\,\mathrm{d}Z_s)_{t\in\mathbb{R}}$ be a backward MA; R_X be its autocovariance function, that is

$$\mathrm{R}_X(t) = \mathrm{E}[X_t X_0] = \int_0^\infty \psi(t+s)\psi(s)\,\mathrm{d}s, \qquad t\in\mathbb{R};$$

and $\bar{\mathrm{R}}_X(t) = \mathrm{R}_X(0) - \mathrm{R}_X(t) = \frac{1}{2}\mathrm{E}[(X_t - X_0)^2]$. The behavior of R_X at 0 or ∞ corresponds in large extent to the behavior of the kernel ψ at 0 or ∞, respectively.

Indeed, we have the following result, in which k_α and j_α are constants given by

$$k_\alpha = \Gamma(1+\alpha)\Gamma(-1-2\alpha)\Gamma(-\alpha)^{-1}, \qquad \alpha\in(-1,-1/2),$$
$$j_\alpha = (2\alpha+1)\sin\bigl(\pi(\alpha+1/2)\bigr)\Gamma(2\alpha+1)\Gamma(\alpha+1)^{-2}, \qquad \alpha\in(-1/2,1/2).$$

Proposition 4.6. *Let the setting be as described above.*

(i) *For $t \to \infty$ and $\alpha \in (-1, -\frac{1}{2})$, $\psi(t) \sim ct^\alpha$ implies $\mathrm{R}_X(t) \sim (c^2 k_\alpha) t^{2\alpha+1}$, provided $|\psi(t)| \le c_1 t^\alpha$ for all $t > 0$ and some $c_1 > 0$.*

(ii) *For $t \to \infty$ and $\alpha \in (-\infty, -1)$, $\psi(t) \sim ct^\alpha$ implies $\mathrm{R}_X(t)/t^\alpha \to c\int_0^\infty \psi(s)\,\mathrm{d}s$, and hence $\mathrm{R}_X(t) \sim (c\int_0^\infty \psi(s)\,\mathrm{d}s) t^\alpha$, provided $\int_0^\infty \psi(s)\,\mathrm{d}s \ne 0$.*

(iii) *For $t \to 0$ and $\alpha \in (-\frac{1}{2}, \frac{1}{2})$, $\psi(t) \sim ct^\alpha$ implies $\bar{\mathrm{R}}_X(t) \sim (c^2 j_\alpha/2)|t|^{2\alpha+1}$, provided ψ is absolutely continuous on $(0,\infty)$ with density ψ' satisfying $|\psi'(t)| \le c_2 t^{\alpha-1}$ for all $t > 0$ and some $c_2 > 0$.*

Proof. (i) Let $\alpha \in (-1, -\frac{1}{2})$ and assume that $\psi(t) \sim ct^\alpha$ as $t \to \infty$ and $|\psi(t)| \le c_1 t^\alpha$ for $t > 0$. Then

$$
\begin{aligned}
\mathrm{R}_X(t) &= \int_0^\infty \psi(t+s)\psi(s)\,\mathrm{d}s \\
&= t\int_0^\infty \psi\bigl(t(s+1)\bigr)\psi(ts)\,\mathrm{d}s \\
&= t^{2\alpha+1}\int_0^\infty \frac{\psi(t(1+s))\psi(ts)}{(t(1+s))^\alpha (ts)^\alpha}(1+s)^\alpha s^\alpha\,\mathrm{d}s \\
&\sim t^{2\alpha+1}c^2\int_0^\infty (1+s)^\alpha s^\alpha\,\mathrm{d}s \qquad \text{as } t \to \infty.
\end{aligned}
\tag{4.6}
$$

Since

$$
\int_0^\infty (1+s)^\alpha s^\alpha\,\mathrm{d}s = \frac{\Gamma(1+\alpha)\Gamma(-1-2\alpha)}{\Gamma(-\alpha)} = k_\alpha,
$$

(4.6) shows that $\mathrm{R}_X(t) \sim (c^2 k_\alpha) t^{2\alpha+1}$ for $t \to \infty$.

(ii) Let $\alpha \in (-\infty, -1)$ and assume that $\psi(t) \sim ct^\alpha$ for $t \to \infty$. Note that $\psi \in L^1(\mathbb{R}_+, \lambda)$ and for some $K > 0$ we have for all $t \ge K$ and $s > 0$ that $|\psi(t+s)|/t^\alpha \le 2|c|(t+s)^\alpha/t^\alpha \le 2|c|$. Hence, by applying Lebesgue's dominated convergence theorem, we obtain

$$
\mathrm{R}_X(t) = t^\alpha \int_0^\infty \left(\frac{\psi(t+s)}{t^\alpha}\psi(s)\right)\mathrm{d}s \sim t^\alpha c\int_0^\infty \psi(s)\,\mathrm{d}s \qquad \text{for } t \to \infty.
$$

(iii) By letting

$$
f_t(s) := \frac{\psi(t(s+1)) - \psi(ts)}{t^\alpha}, \qquad t > 0, s \in \mathbb{R},
$$

we have

$$
\mathrm{E}[(X_t - X_0)^2] = t\int \bigl[\psi\bigl(t(s+1)\bigr) - \psi(ts)\bigr]^2\,\mathrm{d}s = t^{2\alpha+1}\int |f_t(s)|^2\,\mathrm{d}s.
\tag{4.7}
$$

As $t \to 0$, we find

$$f_t(s) = \frac{\psi(t(s+1))}{(t(s+1))^\alpha}(s+1)^\alpha - \frac{\psi(ts)}{(ts)^\alpha}s^\alpha \to c\big((s+1)_+^\alpha - s_+^\alpha\big).$$

Choose $\delta > 0$ such that $|\psi(x)| \le 2x^\alpha$ for $x \in (0,\delta)$. By our assumptions we have for all $s \ge \delta$ that

$$\begin{aligned}|f_t(s)| &= t^{-\alpha}\left|\int_{ts}^{t(1+s)} \psi'(u)\,\mathrm{d}u\right| \le t^{-\alpha+1} \sup_{u\in[st,t(s+1)]} |\psi'(u)| \\ &\le c_2 t^{-\alpha+1} \sup_{u\in[st,t(s+1)]} |u|^{\alpha-1} = c_2 t^{-\alpha+1}|ts|^{\alpha-1} = c_2 s^{\alpha-1},\end{aligned}$$

and for $s \in [-1,\delta)$, $|f_t(s)| \le 2c[(1+s)^\alpha + s_+^\alpha]$. This shows that there exists a function $g \in L^2(\mathbb{R}_+,\lambda)$ such that $|f_t| \le g$ for all $t > 0$, and thus, by Lebesgue's dominated convergence theorem, we have

$$\int |f_t(s)|^2\,\mathrm{d}s \underset{t\to 0}{\longrightarrow} c^2 \int \big((s+1)_+^\alpha - s_+^\alpha\big)^2\,\mathrm{d}s = c^2 j_\alpha. \tag{4.8}$$

Together with (4.7), (4.8) shows that $\bar{\mathrm{R}}_X(t) \sim (c^2 j_\alpha/2)t^{2\alpha+1}$ for $t \to 0$. □

Remark 4.7. It would be of interest to obtain a general result covering Proposition 4.6(ii) in the case $\int_0^\infty \psi(s)\,\mathrm{d}s = 0$. Recall that ψ_f, given by (4.5), often satisfies that $\int_0^\infty \psi_f(s)\,\mathrm{d}s = 0$, according to Lemma 4.5.

Example 4.8. Consider the case where $\psi(t) = t^\alpha \mathrm{e}^{-\lambda t}$ for $\alpha \in (-\frac{1}{2},\infty)$ and $\lambda > 0$. For $t \to 0$, $\psi(t) \sim t^\alpha$, and hence $\bar{\mathrm{R}}_X(t) \sim (j_\alpha/2)t^{2\alpha+1}$ for $t \to 0$ and $\alpha \in (-\frac{1}{2},\frac{1}{2})$, by Proposition 4.6(iii) (compare with Barndorff-Nielsen *et al.* [2]).

Note that if $X = \psi * Z$ is a moving average, as above, then by Proposition 4.6(i) and for $t \to \infty$, $\mathrm{R}_X(t) \sim c_1 t^{-\alpha}$ with $\alpha \in (0,1)$, provided that $\psi(t) \sim c_2 t^{-(\alpha+1)/2}$ and $|\psi(t)| \le c_3 t^{-(\alpha+1)/2}$. This shows that X has long-range dependence of order α.

Let us conclude this section with a short discussion of when an MA $X = \psi * Z$ is a semimartingale. It is often very important that the process of interest is a semimartingale, especially in finance, where the semimartingale property of the asset price is equivalent to the property that the capital process depends continuously on the chosen strategy; see Section 8.1.1 in Cont and Tankov [16]. In the case where Z is a Brownian motion, Theorem 6.5, in Knight [22] shows that X is an $\mathcal{F}^Z$ semimartingale if and only if ψ is absolutely continuous on $[0,\infty)$ with a square-integrable density. (Here $\mathcal{F}_t^Z := \sigma(Z_s\colon s \in (-\infty,t])$.) For a further study of the semimartingale property of PMA and more general processes, see [7,8,10] in the Gaussian case, and Basse and Pedersen [9] for the infinitely divisible case.

4.3.2. *QOU processes with PMA noise*

Let us return to the case of a QOU process driven by a PMA. Let Z be a centered Lévy process, $f:\mathbb{R}\to\mathbb{R}$ be a measurable function that is 0 on $(-\infty,0)$ and satisfies $f(t-\cdot)-f(-\cdot)\in L^{\phi}$ for all $t\in\mathbb{R}$ and N be given by

$$N_t=\int_{\mathbb{R}}[f(t-s)-f(-s)]\,\mathrm{d}Z_s,\qquad t\in\mathbb{R}. \tag{4.9}$$

First, we will consider the relationship between the behavior of the kernel of the noise N and that of the kernel ψ_f of the corresponding moving average X.

Proposition 4.9. *Let N be given by* (4.9), *and X be a QOU process driven by N with parameter* $\lambda>0$.

(i) *Let $\alpha\in(-1,-\frac{1}{2})$ and assume that, for some $c\neq 0$, f is continuous differentiable in a neighborhood of ∞ with $f'(t)\sim ct^{\alpha}$ for $t\to\infty$. Then, for $t\to\infty$, we have $\mathrm{R}_X(t)\sim(\frac{c^2k_\alpha}{\lambda^2})t^{2\alpha+1}$, provided $|f(t)|\le rt^{\alpha}$ for all $t>0$ and some $r>0$.*

(ii) *Let $\alpha\in(-\frac{1}{2},\frac{1}{2})$ and $f(t)\sim ct^{\alpha}$ for $t\to 0$. Then, for $t\to 0$, we have $\bar{\mathrm{R}}_X(t)\sim(c^2j_\alpha/2)|t|^{2\alpha+1}$, provided f is two times continuous differentiable in a neighborhood of ∞ with $f''(t)=\mathrm{O}(t^{\alpha-1})$ for $t\to\infty$, and that f is absolutely continuous on $(0,\infty)$ with a density f' satisfying $\sup_{t\in(0,t_o)}|f'(t)|t^{1-\alpha}<\infty$ for all $t_0>0$.*

Proof. (i) Choose $\beta>0$ such that f is continuous differentiable on $[\beta,\infty)$. By partial integration, we have for $t\ge\beta$,

$$\psi_f(t)=\mathrm{e}^{-\lambda t}\left(\mathrm{e}^{\lambda a}f(a)-\lambda\int_{-\infty}^{a}\mathrm{e}^{\lambda s}f(s)\,\mathrm{d}s\right)+\mathrm{e}^{-\lambda t}\int_a^t\mathrm{e}^{\lambda s}f'(s)\,\mathrm{d}s, \tag{4.10}$$

showing that $\psi_f(t)\sim(\frac{c}{\lambda})t^{\alpha}$ for $t\to\infty$. Choose $k>0$ such that $|\psi_f(t)|\le(2c/\lambda)t^{\alpha}$ for all $t\ge k$. By (4.5) we have that $\sup_{t\in[0,k]}|\psi_f(t)t^{-\alpha}|<\infty$, since $\sup_{t\in[0,k]}|f(t)t^{-\alpha}|<\infty$, and hence there exists a constant $c_1>0$ such that $|\psi_f(t)|\le c_1t^{\alpha}$ for all $t>0$. Therefore, (i) follows by Proposition 4.6(i).

(ii) Choose $\beta>0$ such that f is two times continuous differentiable on $[\beta,\infty)$. By (4.10) and partial integration we have for $t>\beta$ and $t\to\infty$,

$$\begin{aligned}\psi_f'(t)&=f'(t)-\lambda\psi_f(t)=f'(t)-\lambda\mathrm{e}^{-\lambda t}\int_\beta^t\mathrm{e}^{\lambda s}f'(s)\,\mathrm{d}s+\mathrm{O}(\mathrm{e}^{-\lambda t})\\&=\mathrm{e}^{-\lambda t}\int_\beta^t\mathrm{e}^{\lambda s}f''(s)\,\mathrm{d}s+\mathrm{O}(\mathrm{e}^{-\lambda t})=\mathrm{O}(t^{\alpha-1}),\end{aligned}$$

where we in the last equality have used that $f''(t)=\mathrm{O}(t^{\alpha-1})$ for $t\to\infty$. Using that $|\psi_f'(t)|\le|f'(t)|+\lambda|\psi_f(t)|$ and $\sup_{t\in(0,t_0)}|f'(t)t^{1-\alpha}|<\infty$ for all $t_0>0$, it follows that there exists a $c_2>0$ such that $|\psi_f'(t)|\le c_1t^{\alpha-1}$ for all $t>0$. Moreover, for $t\to 0$, we have that $\psi_f(t)\sim ct^{\alpha}$. Hence, (ii) follows by Proposition 4.6(iii). $\square$

Now consider the following set-up: Let $Z = (Z_t)_{t\in\mathbb{R}}$ be a centered and square-integrable Lévy process, and for $H \in (0,1)$, $r_0 \neq 0$, $\delta \geq 0$, let

$$f(t) = r_0(\delta \vee t)^{H-1/2} \quad \text{and} \quad N_t^{H,\delta} = \int_{\mathbb{R}} [f(t-s) - f(-s)]\,\mathrm{d}Z_s. \tag{4.11}$$

Note that when $\delta = 0$ and Z is a Brownian motion, $N^{H,\delta}$ is a constant times the fBm of index H, and when $\delta > 0$, $N^{H,\delta}$ is a semimartingale. We have the following corollary to Proposition 4.9.

Corollary 4.10. *Let $N^{H,\delta}$ be given by (4.11), and let $X^{H,\delta}$ be a QOU process driven by $N^{H,\delta}$ with parameter $\lambda > 0$. Then, for $H \in (\frac{1}{2}, 1)$ and $t \to \infty$,*

$$\mathrm{R}_{X^{H,\delta}}(t) \sim \bigl(r_0^2 k_{H-3/2}(H-1/2)/\lambda^2\bigr)t^{2H-2}, \qquad \delta \geq 0,$$

and for $H \in (0,1)$ and $t \to 0$,

$$\bar{\mathrm{R}}_{X^{H,\delta}}(t) \sim \begin{cases} (r_0^2\delta^{2H-1}/2)|t|, & \delta > 0, \\ (r_0^2 j_{H-1/2}/2)|t|^{2H}, & \delta = 0. \end{cases} \tag{4.12}$$

Proof. For $H \in (\frac{1}{2}, 1)$, let $\beta = \delta$. Then $f \in C^1((\beta,\infty);\mathbb{R})$ and, for $t > \beta$, $f'(t) = ct^\alpha$, where $\alpha = H - 3/2 \in (-1, -\frac{1}{2})$ and $c = r(H-1/2)$. Moreover, $|f(t)| \leq r\delta t^\alpha$. Thus, Proposition 4.9(i) shows that $\mathrm{R}_{X^{H,\delta}}(t) \sim (c^2 k_\alpha/\lambda^2)t^{2\alpha+1} = (r^2(H-1/2)^2 k_{H-3/2}/\lambda^2)t^{2H-2}$. To show (4.12) assume that $H \in (0,1)$. For $t \to 0$, we have $f(t) \sim ct^\alpha$, where $c = r_0$ and $\alpha = H - 1/2 \in (-\frac{1}{2}, \frac{1}{2})$ when $\delta = 0$, and $c = r_0\delta^{H-1/2}$ and $\alpha = 0$ when $\delta > 0$. For $\beta = \delta$, $f \in C^2((\beta,\infty);\mathbb{R})$ with $f''(t) = r_0(H-1/2)(H-3/2)t^{H-5/2}$, showing that $f''(t) = \mathrm{O}(t^{\alpha-1})$ for $t \to \infty$ (both for $\delta > 0$ and $\delta = 0$). Moreover, f is absolutely continuous on $(0,\infty)$ with density $f'(t) = r_0(H-1/2)t^{H-3/2}1_{[\delta,\infty)}(t)$. This shows that $\sup_{t\in(0,t_0)}|f'(t)t^{1-\alpha}| < \infty$ for all $t_0 > 0$ (both for $\delta > 0$ and $\delta = 0$). Hence (4.12) follows by Proposition 4.9(ii). □

4.4. Stability of the autocovariance function

Let N be a PMA of the form (4.1), where Z is a centered square-integrable Lévy process, and $f(t) = c_H t_+^{H-1/2}$, where $H \in (0,1)$. (Recall that if Z is a Brownian motion, then N is an fBm of index H.) Let X be a QOU process driven by N with parameter $\lambda > 0$, and recall that by Proposition 4.2, X is an MA of the form

$$X_t = \int_{-\infty}^t \psi_H(t-s)\,\mathrm{d}Z_s, \qquad t \in \mathbb{R},$$

where

$$\psi_H(t) = c_H\left(t^{H-2/2} - \lambda\mathrm{e}^{-\lambda t}\int_0^t \mathrm{e}^{\lambda u}u^{H-1/2}\,\mathrm{d}u\right), \qquad t \geq 0.$$

Below we will discus some stability properties for the autocovariance function under minor modification of the kernel function.

For all bounded measurable functions $f:\mathbb{R}_+ \to \mathbb{R}$ with compact support, let $X_t^f = \int_{-\infty}^t (\psi_H(t-s) - f(t-s))\,\mathrm{d}Z_s$. We will think of X^f as an MA where we have made a minor change of X's kernel. Note that if we let $Y_t^f = X_t - X_t^f = \int_{-\infty}^t f(t-s)\,\mathrm{d}Z_s$, then the autocovariance function $\mathrm{R}_{Y^f}(t)$, of Y^f, is zero whenever t is large enough due to the fact that f has compact support.

Corollary 4.11. *We have the following two situations in which $c_1, c_2, c_3 \neq 0$ are non-zero constants.*

(i) *For $H \in (0, \frac{1}{2})$ and $\int_0^\infty f(s)\,\mathrm{d}s \neq 0$, we have for $t \to \infty$,*

$$\mathrm{R}_{X^f}(t) \sim c_2 \mathrm{R}_X(t) t^{1/2-H} \sim c_1 t^{H-3/2}.$$

(ii) *For $H \in (\frac{1}{2}, 1)$, we have for $t \to \infty$,*

$$\mathrm{R}_{X^f}(t) \sim \mathrm{R}_X(t) \sim c_3 t^{2H-2}.$$

Thus for $H \in (0, \frac{1}{2})$, the above shows that the behavior of the autocovariance function at infinity is changed dramatically by making a minor change of the kernel. In particular, if f is a positive function, not the zero function, then $\mathrm{R}_{X^f}(t)$ behaves as $t^{1/2-H}\mathrm{R}_X(t)$ at infinity. On the other hand, when $H \in (\frac{1}{2}, 1)$, the behavior of the autocovariance function at infinity does not change if we make a minor change to the kernel. That is, in this case the autocovariance function has a stability property, contrary to the case where $H \in (0, \frac{1}{2})$.

Remark 4.12. Note that the dramatic effect appearing from Corollary 4.11(i) is associated with the fact that $\int_0^\infty \psi_H(s)\,\mathrm{d}s = 0$, as shown in Lemma 4.5.

Proof of Corollary 4.11. By Corollary 4.3 we have for $t \to \infty$ that $\psi_H(t) \sim ct^\alpha$, where $c = c_H(H-1/2)/\lambda$ and $\alpha = H - 3/2$. To show (i), assume that $H \in (0, \frac{1}{2})$ and hence $\alpha \in (-\infty, -1)$. According to Lemma 4.5, we have that $\int_0^\infty \psi_H(s)\,\mathrm{d}s = 0$ and hence $\int_0^\infty [\psi_H(s) - f(s)]\,\mathrm{d}s \neq 0$, since $\int_0^\infty f(s)\,\mathrm{d}s \neq 0$ by assumption. From Proposition 4.6(ii) and for $t \to \infty$, we have that $\mathrm{R}_{X^f}(t)(t) \sim c_1 t^{2\alpha+1} = c_1 t^{H-3/2}$, where $c_1 = c\int_0^\infty [\psi_H(s) - f(s)]\,\mathrm{d}s$. On the other hand, by Corollary 2.6 we have that $\mathrm{R}_X(t) \sim (H(H-1/2)/\lambda^2)t^{2H-2}$ for $t \to \infty$, and hence we have shown (i) with $c_2 = c_1\lambda^2/(H(H-1/2)$. For $H \in (\frac{1}{2}, 1)$ we have that $\alpha \in (-1, -\frac{1}{2})$, and hence (ii) follows by Proposition 4.6(i). □

Appendix

In this Appendix we will show an auxiliary continuity result used several times in the paper. The main result in this Appendix is Theorem A.1; Corollary A.3 is used in Theorem 2.1, while the general modular setting is needed to prove Proposition 4.2. For the basic definitions and properties of linear metric spaces, modulars and F-norms, we refer to Rolewicz [32].

Let $(E, \mathcal{E}, \mu)$ be a σ-finite measure space, and $\phi : \mathbb{R} \to \mathbb{R}_+$ an even and continuous function that is non-decreasing on $\mathbb{R}_+$, with $\phi(0) = 0$. Assume there exists a constant $C > 0$ such that $\phi(2x) \leq C\phi(x)$ for all $x \in \mathbb{R}$ (that is, ϕ satisfies the Δ_2 condition). Let $L^0 = L^0(E, \mathcal{E}, \mu)$ denote the space of all measurable functions from E into $\mathbb{R}$; Φ denote the modular on L^0 given by

$$\Phi(g) = \int_E \phi(g)\,\mathrm{d}\mu, \qquad g \in L^0;$$

and $L^\phi = \{g \in L^0 : \Phi(g) < \infty\}$ denote the corresponding modular space. Furthermore, for $g \in L^0$, define

$$\rho(g) = \inf\{c > 0\colon\ \Phi(g/c) \leq c\} \quad \text{and} \quad \|g\|_\phi = \inf\{c > 0\colon\ \Phi(g/c) \leq 1\}.$$

Then ρ is an F-norm on L^ϕ and, in particular, $d_\phi(f, g) = \rho(f - g)$ is an invariant metric on L^ϕ. Moreover, when ϕ is convex, the *Luxemburg norm* $\|\cdot\|_\phi$ is a norm on L^ϕ; see Khamsi [21].

Theorem A.1. *Let $f : \mathbb{R} \times E \to \mathbb{R}$ denote a measurable function satisfying $f_t = f(t, \cdot) \in L^\phi$ for all $t \in \mathbb{R}$, and*

$$d_\phi(f_{t+u}, f_{v+u}) = d_\phi(f_t, f_v) \qquad \text{for all } t, u, v \in \mathbb{R}. \tag{A.1}$$

Then, $(t \in \mathbb{R}) \mapsto (f_t \in L^\phi)$ is continuous. Moreover, if ϕ is convex, then there exist $\alpha, \beta > 0$ such that $\|f_t\|_\phi \leq \alpha + \beta|t|$ for all $t \in \mathbb{R}$.

To prove Theorem A.1, we shall need the following lemma.

Lemma A.2. *Let $f : \mathbb{R} \times E \to \mathbb{R}$ denote a measurable function, such that $f_t \in L^\phi$ for all $t \in \mathbb{R}$. Then, $(t \in \mathbb{R}) \mapsto (f_t \in L^\phi)$ is Borel measurable and has a separable range.*

Recall that $f : E \to F$ has a separable range, if $f(E)$ is a separable subset of F.

Proof of Lemma A.2. We will use a monotone class lemma argument to prove this result, so let $\mathcal{M}_2$ be the set of all functions f for which Lemma A.2 holds and $\mathcal{M}_1$ be the set of all functions f of the form

$$f_t(s) = \sum_{i=1}^n \alpha_i 1_{A_i}(t) 1_{B_i}(s), \qquad t \in \mathbb{R}, s \in E,$$

where, for $n \geq 1$, $A_1, \ldots, A_n$ are measurable subsets of $\mathbb{R}$, $B_1, \ldots, B_n$ are measurable subsets of E of finite μ measure and $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$. Let us show that $\Psi_f : (t \in \mathbb{R}) \mapsto (f_t \in L^\phi)$ is measurable. Since, for all $g \in L^\phi$, $t \mapsto d_\phi(f_t, g)$ is measurable, we get that for all $g \in L^\phi$ and $r > 0$, $\Psi_f^{-1}(B(g, r))$ is measurable (we use the notation, $B(g, r) = \{h \in L^\phi\colon\ d_\phi(g, h) < r\}$). Therefore, since Ψ_f has separable range, it follows that Ψ_f is measurable (recall that the Borel σ-field in a separable metric space is generated by the open balls). This shows that $\mathcal{M}_1 \subseteq \mathcal{M}_2$.

Note that the set $\mathrm{b}\mathcal{M}_2$ of bounded elements from $\mathcal{M}_2$ is a vector space with $1 \in \mathrm{b}\mathcal{M}_2$, and that $(f_n)_{n\geq 1} \subseteq \mathrm{b}\mathcal{M}_2$ with $0 \leq f_n \uparrow f \leq K$ implies that $f \in \mathrm{b}\mathcal{M}_2$. Moreover, since $\mathcal{M}_1$ is stable under pointwise multiplication, the monotone class lemma (see [31], Chapter II, Theorem 3.2) shows that

$$\mathrm{bM}\big(\mathcal{B}(\mathbb{R}) \times \mathcal{F}\big) = \mathrm{bM}(\sigma(\mathcal{M}_1)) \subseteq \mathrm{b}\mathcal{M}_2.$$

(For a family of functions $\mathcal{M}$, $\sigma(\mathcal{M})$ denotes the least σ-algebra for which all the functions are measurable, and for each σ-algebra $\mathcal{E}$, $\mathrm{bM}(\mathcal{E})$ denotes the space of all bounded $\mathcal{E}$-measurable functions.) For a general function f, define $f^{(n)}$ by $f_t^{(n)} = f_t 1_{\{|f_t|\leq n\}}$. For all $n \geq 1$, $f^{(n)}$ is a bounded measurable function and hence $\Psi_{f^{(n)}}$ is a measurable map with a separable range. Moreover, $\lim_n \Psi_{f^{(n)}} = \Psi_f$ pointwise in L^ϕ, showing that Ψ_f is measurable and has a separable range. □

Proof of Theorem A.1. Let Ψ_f denote the map $(t \in \mathbb{R}) \mapsto (f_t \in L^\phi)$, and for fixed $\epsilon > 0$ and arbitrary $t \in \mathbb{R}$, consider the ball $B_t = \{s \in \mathbb{R}\colon d_\phi(f_t, f_s) < \epsilon\}$. By Lemma A.2, Ψ_f is measurable, and hence B_t is a measurable subset of $\mathbb{R}$ for all $t \in \mathbb{R}$. According to Lemma A.2, Ψ_f has a separable range and, therefore, there exists a countable set $(t_n)_{n\geq 1} \subseteq \mathbb{R}$ such that the range of Ψ_f is included in $\bigcup_{n\geq 1} B(f_{t_n}, \epsilon)$, implying that $\mathbb{R} = \bigcup_{n\geq 1} B_{t_n}$. In particular, there exists an $n \geq 1$ such that B_{t_n} has a strictly positive Lebesgue measure. By the Steinhaus lemma, see [12], Theorem 1.1.1, there exists a $\delta > 0$ such that $(-\delta, \delta) \subseteq B_{t_n} - B_{t_n}$. Note that by (A.1) it is enough to show continuity of Ψ_f at $t = 0$. For $|t| < \delta$ there exists, by definition, $s_1, s_2 \in \mathbb{R}$ such that $d_\phi(f_{t_n}, f_{s_i}) < \epsilon$ for $i = 1, 2$, showing that

$$d_\phi(f_t, f_0) \leq d_\phi(f_t, f_{s_1}) + d_\phi(f_t, f_{s_2}) < 2\epsilon,$$

which completes the proof of the continuity part.

To show the last part of the theorem, assume that ϕ is convex. For each $t > 0$ choose $n = 0, 1, 2, \ldots$ such that $n \leq t < n + 1$. Then,

$$\begin{aligned} \|f_t - f_0\|_\phi &\leq \sum_{i=1}^{n} \|f_i - f_{i-1}\|_\phi + \|f_t - f_n\|_\phi \\ &\leq n\|f_1 - f_0\|_\phi + \|f_{t-n} - f_0\|_\phi \leq t\beta + a, \end{aligned} \tag{A.2}$$

where $\beta = \|f_1 - f_0\|_\phi$ and $a = \sup_{s\in[0,1]} \|f_s - f_0\|_\phi$. We have already shown that $t \mapsto f_t$ is continuous, and hence $a < \infty$. Since $\|f_{-t} - f_0\|_\phi = \|f_t - f_0\|_\phi$ for all $t \in \mathbb{R}$, (A.2) shows that $\|f_t - f_0\|_\phi \leq a + \beta|t|$ for all $t \in \mathbb{R}$, implying that $\|f_t\|_\phi \leq \alpha + \beta|t|$, where $\alpha = a + \|f_0\|_\phi$. □

For $(E, \mathcal{E}, \mu) = (\Omega, \mathcal{F}, P)$ and $\phi(t) = |t|^p$ for $p > 0$ or $\phi(t) = |t| \wedge 1$ for $p = 0$, we have the following corollary to Theorem A.1.

Corollary A.3. *Let $p \geq 0$ and $X = (X_t)_{t\in\mathbb{R}}$ be a measurable process with stationary increments and finite p moments. Then X is continuous in L^p. Moreover, if $p \geq 1$, then there exist $\alpha, \beta > 0$ such that $\|X_t\|_p \leq \alpha + \beta|t|$ for all $t \in \mathbb{R}$.*

Note that in Corollary A.3 the reversed implication is also true; in fact, all stochastic processes $X = (X_t)_{t\in\mathbb{R}}$ that are continuous in L^0 have a measurable modification according to Theorem 2 in Cohn [15]. The idea of using the Steinhaus lemma to prove Theorem A.1 is borrowed from Surgailis *et al*. [38], where Corollary A.3 is shown for $p = 0$. Furthermore, when μ is a probability measure and $\phi(t) = |t| \wedge 1$, Lemma A.2 is known from Cohn [15].

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Received December 2009 and revised June 2010