

# Some stochastic inequalities for weighted sums

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We compare weighted sums of i.i.d. positive random variables according to the usual stochastic order. The main inequalities are derived using majorization techniques under certain log-concavity assumptions. Specifically, let  $Y_i$  be i.i.d. random variables on  $\mathbf{R}_+$ . Assuming that  $\log Y_i$  has a log-concave density, we show that  $\sum a_i Y_i$  is stochastically smaller than  $\sum b_i Y_i$ , if  $(\log a_1, \dots, \log a_n)$  is majorized by  $(\log b_1, \dots, \log b_n)$ . On the other hand, assuming that  $Y_i^p$  has a log-concave density for some  $p > 1$ , we show that  $\sum a_i Y_i$  is stochastically larger than  $\sum b_i Y_i$ , if  $(a_1^q, \dots, a_n^q)$  is majorized by  $(b_1^q, \dots, b_n^q)$ , where  $p^{-1} + q^{-1} = 1$ . These unify several stochastic ordering results for specific distributions. In particular, a conjecture of Hitzenko [Sankhyā A 60 (1998) 171–175] on Weibull variables is proved. Potential applications in reliability and wireless communications are mentioned.

*Keywords:* gamma distribution; log-concavity; majorization; Prékopa–Leindler inequality; Rayleigh distribution; tail probability; usual stochastic order; Weibull distribution; weighted sum

## 1. Main results and examples

This paper aims to unify and generalize certain stochastic comparison results concerning weighted sums. Let  $Y_1, \dots, Y_n$  be i.i.d. random variables on  $\mathbf{R}_+$ . We are interested in comparing two weighted sums,  $\sum_{i=1}^n a_i Y_i$  and  $\sum_{i=1}^n b_i Y_i$ ,  $a_i, b_i \in \mathbf{R}_+$ , with respect to the usual stochastic order. A random variable  $X$  is said to be no larger than  $Y$  in the usual stochastic order, written as  $X \leq_{\text{st}} Y$ , if  $\Pr(X > t) \leq \Pr(Y > t)$  for all  $t \in \mathbf{R}$ . For an introduction to various stochastic orders, see [19]. Ordering in terms of  $\leq_{\text{st}}$  may be used to bound the tail probability of  $\sum a_i Y_i$ , for example, in terms of the tail probability of  $\sum Y_i$ . For specific distributions, such comparisons have been explored in several contexts, including reliability [2,3].

We shall use the notion of *majorization* [15]. A real vector  $b = (b_1, \dots, b_n)$  is said to majorize  $a = (a_1, \dots, a_n)$ , written as  $a < b$ , if (i)  $\sum_{i=1}^n a_i = \sum_{i=1}^n b_i$ , and (ii)  $\sum_{i=k}^n a_i \leq \sum_{i=k}^n b_i$ ,  $k = 2, \dots, n$ , where  $a_{(1)} \leq \dots \leq a_{(n)}$  and  $b_{(1)} \leq \dots \leq b_{(n)}$  are  $(a_1, \dots, a_n)$  and  $(b_1, \dots, b_n)$  arranged in increasing order, respectively. A function  $\phi(a)$  symmetric in the coordinates of  $a = (a_1, \dots, a_n)$  is said to be *Schur-concave* if  $a < b$  implies  $\phi(a) \geq \phi(b)$ . A function  $\phi(a)$  is *Schur-convex* if  $-\phi(a)$  is Schur-concave.

A non-negative function  $f(x)$ ,  $x \in \mathbf{R}^n$ , is *log-concave* if  $\text{supp}(f)$  is convex and  $\log f(x)$  is concave on  $\text{supp}(f)$ . Log-concavity plays a critical role in deriving our main results. For other stochastic comparison results involving log-concavity, see, for example, [10,22,24,25].

In this section, after stating our main results (Theorems 1 and 2), we illustrate with several examples and mention potential applications. The main results are proved in Section 2. Some technical details in the proof of Theorem 2 are collected in the [Appendix](#).

**Theorem 1.** Let  $Y_1, \dots, Y_n$  be i.i.d. random variables with density  $f(y)$  on  $\mathbf{R}_+$  such that  $f(e^x)$  is log-concave in  $x \in \mathbf{R}$ . Then, for each  $t > 0$ ,  $\Pr(\sum a_i Y_i \leq t)$  is a Schur-concave function of

$$\log a \equiv (\log a_1, \dots, \log a_n).$$

Equivalently, if  $a, b \in \mathbf{R}_+^n$ , then

$$\log a \prec \log b \implies \sum a_i Y_i \leq_{\text{st}} \sum b_i Y_i. \tag{1.1}$$

**Theorem 2.** Let  $p > 1$ , and let  $Y_1, \dots, Y_n$  be i.i.d. random variables with density  $f(y)$  on  $\mathbf{R}_+$  such that the function

$$\min\{0, 2/p - 1\} \log x + \log f(x^{1/p}) \tag{1.2}$$

is concave in  $x \in \mathbf{R}_+$ . Then, for each  $t > 0$ ,  $\Pr(\sum a_i Y_i \leq t)$  is a Schur-convex function of

$$a^q \equiv (a_1^q, \dots, a_n^q) \in \mathbf{R}_+^n,$$

where  $p^{-1} + q^{-1} = 1$ . Equivalently, if  $a, b \in \mathbf{R}_+^n$ , then

$$a^q \prec b^q \implies \sum b_i Y_i \leq_{\text{st}} \sum a_i Y_i. \tag{1.3}$$

**Remark.** In Theorem 1, the condition that  $f(e^x)$  is log-concave is equivalent to  $\log Y_i$  having a log-concave density (see, e.g., [18]). In Theorem 2, a sufficient condition for (1.2) is that  $x^{1/p-1} f(x^{1/p})$  is log-concave, or, equivalently,  $Y_i^p$  has a log-concave density (this special case is mentioned in the abstract). Theorems 1 and 2 are quite applicable, since log-concavity is associated with many well-known densities (see Corollaries 1 and 2).

Theorem 1 is reminiscent of the following result of [17], originally stated in terms of the peakedness order.

**Theorem 3.** Let  $Y_i, i = 1, \dots, n$ , be i.i.d. random variables on  $\mathbf{R}$  with a log-concave density that is symmetric about zero. Then for each  $t > 0$ ,  $\Pr(\sum a_i Y_i \leq t)$  is a Schur-concave function of  $a \in \mathbf{R}_+^n$ .

Theorem 2 is closely related to Theorem 4, which is a version (with a stronger assumption) of Theorem 24 of [11].

**Theorem 4.** Let  $0 < p < 1$ , and let  $Y_1, \dots, Y_n$  be i.i.d. random variables on  $\mathbf{R}_+$  such that  $Y_i^p$  has a log-concave density. Then, for each  $t > 0$ ,  $\Pr(\sum a_i Y_i \leq t)$  is a Schur-concave function of  $(a_1^q, \dots, a_n^q) \in \mathbf{R}_+^n$ , where  $p^{-1} + q^{-1} = 1$ .

Karlin and Rinott [11] gave an elegant proof of Theorem 4 using the Prékopa–Leindler inequality. Our proofs of Theorems 1 and 2 (Section 2) borrow ideas from both [17] and [11]. See [20] for more related inequalities.

Bounds on the distribution function of  $\sum a_i Y_i$  are readily obtained in terms of the distribution function of  $\sum Y_i$ . In Theorem 1, for example, (1.1) gives

$$\Pr\left(\sum b_i Y_i \leq t\right) \leq \Pr\left(b_* \sum Y_i \leq t\right), \quad b_* = \left(\prod b_i\right)^{1/n}, t > 0. \tag{1.4}$$

In Theorem 2, (1.3) gives

$$\Pr\left(\sum b_i Y_i \leq t\right) \geq \Pr\left(b^* \sum Y_i \leq t\right), \quad b^* = \left(n^{-1} \sum b_i^q\right)^{1/q}, t > 0. \tag{1.5}$$

More generally, we obtain inequalities for the expectations of monotone functions, since  $X \leq_{st} Y$  implies  $Eg(X) \leq Eg(Y)$  for every increasing function  $g$  such that the expectations exist.

Let us mention some specific distributions to which Theorems 1 and 2 can be applied. Corollary 1 follows from Theorem 1. The log-concavity condition is easily verified in each case (for more distributions that satisfy this condition, see [8], Example 1). Related results on sums of uniform variables can be found in [13]. The gamma case has recently been discussed by Khaledi and Kochar [12], Yu [23] and Zhao and Balakrishnan [26].

**Corollary 1.** For  $a, b \in \mathbf{R}_+^n$ , (1.1) holds when  $Y_i$  are i.i.d. having one of the following distributions:

- (1) uniform on the interval  $(0, s)$ ,  $s > 0$ ;
- (2)  $\text{gamma}(\alpha, \beta)$ ,  $\alpha, \beta > 0$ ;
- (3) any log-normal distribution;
- (4) the Weibull distribution with parameter  $p > 0$ , whose density is

$$f(y) = py^{p-1}e^{-y^p}, \quad y > 0;$$

- (5) the generalized Rayleigh distribution with parameter  $\nu > 0$ , whose density is

$$f(y) \propto y^{\nu-1}e^{-y^2/2}, \quad y > 0.$$

The inequality (1.4) holds for each of these distributions. The gamma case is interesting in that the upper bound in (1.4) is in terms of a single gamma variable,  $\sum Y_i$ . The gamma case with  $\alpha = 1/2$  dates back to [16]. See also [1,21] for related inequalities.

**Corollary 2.** Let  $p > 1$ , and define  $q$  by  $p^{-1} + q^{-1} = 1$ . Then, for  $a, b \in \mathbf{R}_+^n$ , (1.3) holds in the following cases:

- (1)  $Y_i$  are i.i.d. Weibull variables with parameter  $p$ ;
- (2)  $p = q = 2$ , and  $Y_i$  are i.i.d. generalized Rayleigh variables with parameter  $\nu \geq 1$ .

Corollary 2 follows from Theorem 2. The condition (1.2) is easily verified. For example, in case 1,  $Y_i^p$  has a log-concave density, which implies (1.2). Case 1 confirms a conjecture of [5]. Case 2 recovers some results of [6,7].

The Weibull case and the generalized Rayleigh case are interesting in that Corollary 1 is also applicable, and we obtain a double bound through (1.4) and (1.5). For example, if  $Y_i$  are i.i.d. generalized Rayleigh variables with parameter  $\nu \geq 1$ , then

$$\Pr\left(a^* \sum Y_i \leq t\right) \leq \Pr\left(\sum a_i Y_i \leq t\right) \leq \Pr\left(a_* \sum Y_i \leq t\right), \quad a_i > 0, t > 0,$$

where  $a^* = (n^{-1} \sum a_i^2)^{1/2}$  and  $a_* = (\prod a_i)^{1/n}$ . Manesh and Khaledi [14] present related inequalities.

We briefly mention some applications:

- Weighted sums of independent  $\chi^2$  variables arise naturally in multivariate statistics as quadratic forms in normal variables. Stochastic comparisons between such weighted sums are therefore statistically interesting, and can lead to bounds on the distribution functions.
- Suppose the component lifetimes of a redundant standby system (without repairing) are modeled by a scale family of distributions. Then the total lifetime is of the form  $\sum_i a_i Y_i$ . When  $Y_i$  are i.i.d. exponential variables, Bon and Paltanea [3] obtain comparisons of the total lifetime with respect to several stochastic orders. Our Corollary 1 shows that, for the usual stochastic order, (1.1) actually holds for a broad class of distributions, including the commonly used gamma, Weibull, and log-normal distributions.
- When  $Y_i$  are i.i.d. exponential variables and  $a_i \in \mathbf{R}_+$ , the quantity  $E \log(1 + \sum a_i Y_i)$  appears in certain wireless communications problems [9]. By the monotonicity of  $\log(1 + x)$ , we have

$$\sum a_i Y_i \leq_{st} \sum b_i Y_i \implies E \log\left(1 + \sum a_i Y_i\right) \leq E \log\left(1 + \sum b_i Y_i\right).$$

Corollary 1 therefore leads to qualitative comparisons for this expected value. Other weighted sums (e.g., of Rayleigh variables) also appear in the context of communications.

It would be interesting to see whether results similar to Theorems 1, 2 and 4 can be obtained for the hazard rate order, or the likelihood ratio order. For sums of independent gamma variables, such results have been obtained by Boland, El-Newehi and Proschan [2], Bon and Paltanea [3], Korwar [13], Khaledi and Kochar [12], Yu [23] and Zhao and Balakrishnan [26].

## 2. Proofs

Two proofs are presented for Theorem 1. The first one uses the Prékopa–Leindler inequality (Lemma 1) and is inspired by Karlin and Rinott [11].

**Lemma 1.** *If  $g(x, y)$  is log-concave in  $(x, y) \in \mathbf{R}^m \times \mathbf{R}^n$ , then  $\int_{\mathbf{R}^m} g(x, y) dx$  is log-concave in  $y \in \mathbf{R}^n$ .*

We also use a basic criterion for Schur-concavity.

**Proposition 1.** *If  $h(\alpha), \alpha \in \mathbf{R}^n$ , is log-concave and permutation invariant in  $\alpha$ , then it is Schur-concave.*

**First proof of Theorem 1.** For  $t > 0$ , define

$$g(x, \alpha) \equiv 1_K \prod_{i=1}^n e^{x_i} f(e^{x_i}), \quad K \equiv \left\{ (x, \alpha) \in \mathbf{R}^{2n} : \sum_{i=1}^n e^{x_i + \alpha_i} \leq t \right\},$$

where  $x = (x_1, \dots, x_n), \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbf{R}^n$ . Note that  $K$  is a convex set ( $1_K$  denotes the indicator function). Since  $f(e^{x_i})$  is log-concave, we know that  $g(x, \alpha)$  is log-concave in  $(x, \alpha)$ . By Lemma 1,

$$h(\alpha) \equiv \Pr\left(\sum e^{\alpha_i} Y_i \leq t\right) = \int_{\mathbf{R}^n} g(x, \alpha) dx$$

is log-concave in  $\alpha \in \mathbf{R}^n$ . Since  $h(\alpha)$  is permutation invariant, it is Schur-concave in  $\alpha$  by Proposition 1, and the claim is proved.  $\square$

The second proof is inspired by Proschan [17], and serves as an introduction to the proof of Theorem 2. Properties of majorization imply that it suffices to prove (1.1) for  $a < b$  such that  $a$  and  $b$  differ only in two components. Since  $\leq_{st}$  is closed under convolution [19], we only need to prove (1.1) for  $n = 2$ .

We shall use log-concavity in the following form. If  $g(x), x \in \mathbf{R}$ , is log-concave, and  $(x_1, x_2) < (y_1, y_2)$ , then

$$g(x_1)g(x_2) - g(y_1)g(y_2) \geq 0.$$

**Second proof of Theorem 1.** Fix  $t > 0$ , and let  $F$  denote the distribution function of  $Y_1$ . It suffices to show that

$$h(\beta) \equiv \Pr(\beta^{-1}Y_1 + \beta Y_2 \leq t) = \int_0^\infty F(t\beta - \beta^2 y) f(y) dy$$

increases in  $\beta \in (0, 1]$ . We may assume that  $\text{supp}(f) \subset [\varepsilon, \infty)$  for some  $\varepsilon > 0$ . The general case follows by a standard limiting argument. We can then justify differentiation under the integral sign and obtain

$$\begin{aligned} h'(\beta) &= \int_0^\infty (t - 2\beta y) f(t\beta - \beta^2 y) f(y) dy \\ &= \int_0^{t/(2\beta)} (t - 2\beta y) f(t\beta - \beta^2 y) f(y) dy + \int_{t/(2\beta)}^{t/\beta} (t - 2\beta y) f(t\beta - \beta^2 y) f(y) dy. \end{aligned} \tag{2.1}$$

By a change of variables  $y \rightarrow t/\beta - y$  in the second integral in (2.1), we get

$$h'(\beta) = \int_0^{t/(2\beta)} (t - 2\beta y) [f(t\beta - \beta^2 y) f(y) - f(\beta^2 y) f(t/\beta - y)] dy.$$

If  $0 < y < t/(2\beta)$  and  $0 < \beta \leq 1$ , then  $\beta^2 y \leq \min\{y, t\beta - \beta^2 y\}$ . That is,

$$(\log(t\beta - \beta^2 y), \log y) < (\log(\beta^2 y), \log(t/\beta - y)).$$

Since  $f(e^x)$  is log-concave, we have

$$f(t\beta - \beta^2 y)f(y) - f(\beta^2 y)f(t/\beta - y) \geq 0, \quad 0 < y < t/(2\beta),$$

which leads to  $h'(\beta) \geq 0$ , as required. □

Our proof of Theorem 2 is similar to (but more involved than) the second proof of Theorem 1. Under the stronger assumption that  $Y_i^p$  has a log-concave density, we actually obtain a simpler proof of Theorem 2 following the first proof of Theorem 1 (see [11]). It seems difficult, however, to extend this argument assuming only that (1.2) is concave.

**Proof of Theorem 2.** We may assume  $n = 2$  as in the second proof of Theorem 1. Fix  $t > 0$ . Effectively, we need to show that

$$h(\beta) \equiv \Pr(\beta^{1/q} Y_1 + (1 - \beta)^{1/q} Y_2 \leq t) = \int_0^\infty F(t\beta^{-1/q} - (\beta^{-1} - 1)^{1/q} y) f(y) dy$$

increases in  $\beta \in [1/2, 1)$  ( $F$  denotes the distribution function of  $Y_1$ ). We have

$$q(1 - \beta)^{1/p} \beta^{1/q+1} h'(\beta) = \int_0^\infty g(y) dy = \int_0^{y_0} g(y) dy + \int_{y_0}^{y_1} g(y) dy,$$

where

$$y_0 = t(1 - \beta)^{1/p}, \quad y_1 = t(1 - \beta)^{-1/q} \tag{2.2}$$

and

$$g(y) = (y - y_0) f(x(y)) f(y), \quad x(y) = (\beta^{-1} - 1)^{1/q} (y_1 - y). \tag{2.3}$$

Differentiation under the integral sign is permitted because

$$|g(y)| \leq (y_1 - y_0) M f(y), \quad 0 < y < y_1,$$

where  $M = \sup_{y>0} f(y)$ . We know  $M < \infty$  because (1.2) implies that  $f(x^{1/p})$  is log-concave in  $x \in \mathbf{R}_+$ .

In the [Appendix](#), we prove:

**Claim 1.** For each  $y \in (0, y_0)$ , there exists a unique  $\tilde{y} \in (y_0, y_1)$  such that

$$y^p + x^p(y) = \tilde{y}^p + x^p(\tilde{y}), \tag{2.4}$$

where  $x(y)$  is given by (2.3).

Henceforth let  $y$  and  $\tilde{y}$  be related by (2.4). Direct calculation using the implicit function theorem gives

$$\frac{d\tilde{y}}{dy} = \frac{(\beta y)^{p/q} - ((1 - \beta)(y_1 - y))^{p/q}}{(\beta \tilde{y})^{p/q} - ((1 - \beta)(y_1 - \tilde{y}))^{p/q}}.$$

A change of variables  $y \rightarrow \tilde{y}$  in  $\int_0^{y_0} g(y) dy$  yields

$$q(1 - \beta)^{1/p} \beta^{1/q+1} h'(\beta) = \int_A g(y) \left| \frac{dy}{d\tilde{y}} \right| d\tilde{y} + \int_{y_0}^{y_1} g(z) dz,$$

where  $A \subset (y_0, y_1)$  is the image of the interval  $(0, y_0)$  under the mapping  $y \rightarrow \tilde{y}$ . Note that  $g(z) \geq 0$  for  $y_0 < z < y_1$ . Hence

$$\begin{aligned} q(1 - \beta)^{1/p} \beta^{1/q+1} h'(\beta) &\geq \int_A \left( g(\tilde{y}) + g(y) \left| \frac{dy}{d\tilde{y}} \right| \right) d\tilde{y} \\ &\geq \int_A (\tilde{y} - y_0) \left[ f(x(\tilde{y}))f(\tilde{y}) - \left( \frac{x(y)y}{x(\tilde{y})\tilde{y}} \right)^\delta f(x(y))f(y) \right] d\tilde{y}, \end{aligned} \tag{2.5}$$

where  $\delta = \min\{0, 2 - p\}$ . The inequality (2.5) is deduced from Claim 2, which we prove in the Appendix.

**Claim 2.** *We have*

$$\left| \frac{d\tilde{y}}{dy} \right| \geq \left( \frac{x(\tilde{y})\tilde{y}}{x(y)y} \right)^\delta \left( \frac{y_0 - y}{\tilde{y} - y_0} \right), \quad 0 < y < y_0. \tag{2.6}$$

In the Appendix we also show:

**Claim 3.** *For  $0 < y < y_0$ , we have*

$$\beta \tilde{y} \geq (1 - \beta)(y_1 - y); \tag{2.7}$$

$$\beta y \leq (1 - \beta)(y_1 - \tilde{y}). \tag{2.8}$$

For  $0 < y < y_0$ , (2.8) yields  $y \leq \min\{\tilde{y}, y_1 - \tilde{y}\}$ , that is,

$$(\tilde{y}, y_1 - \tilde{y}) < (y, y_1 - y).$$

Thus,  $y(y_1 - y) \leq \tilde{y}(y_1 - \tilde{y})$ , or, equivalently,  $y^p x^p(y) \leq \tilde{y}^p x^p(\tilde{y})$ . By (2.4), this implies the relation

$$(\tilde{y}^p, x^p(\tilde{y})) < (y^p, x^p(y)), \quad 0 < y < y_0.$$

Assumption (1.2) then yields ( $\delta = \min\{0, 2 - p\}$ )

$$(x(\tilde{y})\tilde{y})^\delta f(x(\tilde{y}))f(\tilde{y}) - (x(y)y)^\delta f(x(y))f(y) \geq 0, \quad \tilde{y} \in A.$$

It follows that the integrand in (2.5) is non-negative, and  $h'(\beta) \geq 0, \beta \in [1/2, 1)$ , as required.  $\square$

**Remark 1.** The main complication in the proof of Theorem 2 is that the mapping  $y \rightarrow \tilde{y}$  is not in closed form. In the special case  $p = q = 2$ , where  $\tilde{y}$  is explicitly available, the proof can be simpler.

### Appendix: Proofs of Claims 1–3

It is convenient to prove Claims 1, 3 and 2 in that order. We emphasize that no circular argument is involved.

**Proof of Claim 1.** Define

$$L(y) = \beta^{p/q} y^p + (1 - \beta)^{p/q} (y_1 - y)^p, \quad 0 \leq y \leq y_1, \tag{A.1}$$

where  $y_1$  is given by (2.2). We have

$$L'(y) = p\beta^{p/q} y^{p-1} - p(1 - \beta)^{p/q} (y_1 - y)^{p-1},$$

and the unique solution of  $L'(y) = 0$  is  $y_0 = t(1 - \beta)^{1/p}$ . Moreover,

$$L''(y) = p(p - 1)[\beta^{p/q} y^{p-2} + (1 - \beta)^{p/q} (y_1 - y)^{p-2}] > 0.$$

Hence  $L(y)$  strictly decreases on the interval  $(0, y_0)$  and strictly increases on  $(y_0, y_1)$ . We have  $L(0) \leq L(y_1)$  because  $\beta \in [1/2, 1)$ . By continuity, for any  $0 < y < y_0$  there exists a unique  $\tilde{y} \in (y_0, y_1)$  that satisfies

$$L(y) = L(\tilde{y}),$$

which reduces to (2.4) after routine algebra. □

**Proof of Claim 3.** We only prove (2.7); the proof of (2.8) is similar. For  $0 < y < y_0$ , define

$$D(y) = L((\beta^{-1} - 1)(y_1 - y)) - L(y).$$

Direct calculation using (A.1) gives

$$\begin{aligned} D(y) &= (1 - \beta)^{p/q} [(y_1 - (\beta^{-1} - 1)(y_1 - y))^p \\ &\quad - (\beta^{-1} - 1)((1 - \beta)^{-1}\beta y)^p - (2 - \beta^{-1})(y_1 - y)^p] \\ &\leq 0, \end{aligned}$$

where the inequality follows from Jensen’s inequality

$$(\alpha u + (1 - \alpha)v)^p \leq \alpha u^p + (1 - \alpha)v^p, \quad p > 1,$$

with

$$\alpha = \beta^{-1} - 1, \quad u = \frac{\beta y}{1 - \beta}, \quad v = y_1 - y.$$

That is,

$$L((\beta^{-1} - 1)(y_1 - y)) \leq L(y) = L(\tilde{y}), \quad 0 < y < y_0. \tag{A.2}$$

By the strict monotonicity of  $L(\cdot)$  on the interval  $(y_0, y_1)$ , if  $(\beta^{-1} - 1)(y_1 - y) > \tilde{y}$ , then  $L((\beta^{-1} - 1)(y_1 - y)) > L(\tilde{y})$ , which contradicts (A.2). Hence

$$(\beta^{-1} - 1)(y_1 - y) \leq \tilde{y},$$

as required.  $\square$

To prove Claim 2, we use Proposition 2. Define

$$Q_\alpha(u, v) = \begin{cases} \frac{u^\alpha - v^\alpha}{u - v}, & u, v > 0, u \neq v, \\ \alpha u^{\alpha-1}, & u = v > 0. \end{cases}$$

**Proposition 2.** *If  $0 < \alpha \leq 1$ , then  $Q_\alpha(u, v)$  decreases in each of  $u, v > 0$ ; if  $\alpha > 1$ , then  $Q_\alpha(u, v)$  increases in each of  $u, v > 0$ .*

Proposition 2 follows from basic properties of the generalized logarithmic mean ([4], pages 386–387).

**Proof of Claim 2.** For  $0 < y < y_0$ , define

$$u = \beta y, \quad v = (1 - \beta)(y_1 - y), \quad \tilde{u} = \beta \tilde{y}, \quad \tilde{v} = (1 - \beta)(y_1 - \tilde{y}).$$

Claim 3 says that  $v \leq \tilde{u}$  and  $u \leq \tilde{v}$ . Applying Proposition 2, we obtain

$$Q_{p/q}(u, v) \geq Q_{p/q}(\tilde{v}, \tilde{u}), \quad 1 < p \leq 2, \quad (\text{A.3})$$

and

$$Q_{p/q}(u^{-1}, v^{-1}) \geq Q_{p/q}(\tilde{v}^{-1}, \tilde{u}^{-1}), \quad p > 2. \quad (\text{A.4})$$

After routine algebra, (A.3) (for  $1 < p \leq 2$ ) and (A.4) (for  $p > 2$ ) reduce to (2.6).  $\square$

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