

Stability for random measures, point processes and discrete semigroups

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Discrete stability extends the classical notion of stability to random elements in discrete spaces by defining a scaling operation in a randomised way: an integer is transformed into the corresponding binomial distribution. Similarly defining the scaling operation as thinning of counting measures we characterise the corresponding discrete stability property of point processes. It is shown that these processes are exactly Cox (doubly stochastic Poisson) processes with strictly stable random intensity measures. We give spectral and LePage representations for general strictly stable random measures without assuming their independent scattering. As a consequence, spectral representations are obtained for the probability generating functional and void probabilities of discrete stable processes. An alternative cluster representation for such processes is also derived using the so-called Sibuya point processes, which constitute a new family of purely random point processes. The obtained results are then applied to explore stable random elements in discrete semigroups, where the scaling is defined by means of thinning of a point process on the basis of the semigroup. Particular examples include discrete stable vectors that generalise discrete stable random variables and the family of natural numbers with the multiplication operation, where the primes form the basis.

Keywords: cluster process; Cox process; discrete semigroup; discrete stability; random measure; Sibuya distribution; spectral measure; strict stability; thinning

1. Introduction

Stability for random variables was introduced by Paul Lévy and thereafter became one of the key concepts in probability. A random vector ξ (or its probability law) is called *strictly α -stable* (notation: $\text{St}\alpha\text{S}$) if for any positive numbers a and b the following identity is satisfied:

$$a^{1/\alpha}\xi' + b^{1/\alpha}\xi'' \stackrel{\mathcal{D}}{=} (a+b)^{1/\alpha}\xi,$$

where ξ' , ξ'' are independent vectors distributed as ξ and $\stackrel{\mathcal{D}}{=}$ denotes equality in distribution. Non-trivial distributions satisfying this exist only for $\alpha \in (0, 2]$. Using $t = a/(a+b)$ in the above definition, the stability property can be equivalently expressed as

$$t^{1/\alpha}\xi' + (1-t)^{1/\alpha}\xi'' \stackrel{\mathcal{D}}{=} \xi \tag{1}$$

for any $t \in [0, 1]$.

The recent work Davydov, Molchanov and Zuyev [8] explored the notion of strict stability for a much more general situation of random elements taking values in a commutative semigroup X . The stability property relies on two basic operations defined on X : the semigroup operation, addition, and the rescaling that plays a role of multiplication by numbers. Many properties of the classical stability still hold for this general situation, but there are many notable differences related to specific algebraic properties of X ; for instance, the validity of distributivity laws, relation between the zero and the neutral elements, compactness, etc. Most important, the stable random elements arise as a sum of points of a Poisson process whose intensity measure has a specific scalable form. In the classical case of d -dimensional vectors, this result turns into the *LePage representation* of strictly α -stable laws: any such law corresponds to the distribution of the sum of points of a Poisson process Π_α in \mathbb{R}^d with the density (intensity) function having a product form $\theta_\alpha(d\rho)\sigma(ds)$ in the polar coordinates (ρ, s) . Here σ is any finite measure on the unit sphere \mathbb{S}^{d-1} , called the *spectral measure*, and $\theta_\alpha((r, \infty)) = r^{-\alpha}$. The underlying reason is that the process Π_α is itself stable with respect to scaling and superposition, that is,

$$t^{1/\alpha}\Pi'_\alpha + (1-t)^{1/\alpha}\Pi''_\alpha \stackrel{D}{=} \Pi_\alpha; \tag{2}$$

this property being granted by the measure θ_α that scales in a ‘right’ way and the superposition property of Poisson processes. This fundamental stability property leads to a characterisation of strictly stable laws in very general Abelian semigroups forming a cone with respect to rescaling with a continuous argument $t \geq 0$; see [8].

The case of discrete spaces, however, cannot be treated the same way since the scaling by a continuous argument cannot be defined in such spaces. While infinite divisibility of random elements in general semigroups, at least in the commutative case, is well understood (see [26, 27]), a systematic exploration of stable laws on possibly discrete semigroups is not available. This prompts us to define rescaling as a direct transformation of probability distributions rather than being inherited from rescaling of the underlying phase space of the random elements.

Prior to the current work, the family \mathbb{Z}_+ of non-negative integers with addition was the only discrete semigroup for which stability was defined. The discrete stability concept for non-negative integer-valued random variables was introduced by [29], who defined the result $t \circ n$ of rescaling of $n \in \mathbb{Z}_+$ by $t \in [0, 1]$ to be the binomial probability measure $\text{Bin}(n, t)$ with the convention that $0 \circ n = 0$ for any n . Since $\text{Bin}(n, t)$ corresponds to the sum of n independent Bernoulli random variables $\text{Bin}(1, t)$ with parameter t , one can view $t \circ n$ as the total count of positive integers between 1 and n where each number is counted with probability t independently of others. Thus a \mathbb{Z}_+ -valued random variable ξ is mapped to a random variable $t \circ \xi$ having distribution of $\sum_{n=0}^\xi \beta_n$, where $\{\beta_n\}$ is a sequence of independent $\text{Bin}(1, t)$ random variables. Thus $t \circ \xi$ can also be viewed as a doubly stochastic random variable whose distribution depends on realisations of ξ . This multiplication operation, though in a different context, goes back to [24].

In terms of the probability generating function (p.g.f.), if $g_\xi(s) = \mathbb{E}s^\xi$ denotes the p.g.f. of ξ , then the p.g.f. of $t \circ \xi$ is given by the composition of g_ξ and the p.g.f. of the Bernoulli $\text{Bin}(1, t)$ law:

$$g_{t \circ \xi}(s) = g_\xi(1 - t(1 - s)). \tag{3}$$

Similarly to (1), a random variable ξ is called *discrete α -stable* if for all $0 \leq t \leq 1$ the following identity is satisfied

$$t^{1/\alpha} \circ \xi' + (1 - t)^{1/\alpha} \circ \xi'' \stackrel{\mathcal{D}}{=} \xi, \tag{4}$$

where ξ', ξ'' are independent copies of ξ . The full characterisation of discrete α -stable random variables (denoted by $D\alpha S$ in the sequel) is provided in [29] namely the laws satisfying (4) exist only for $\alpha \in (0, 1]$ and each such law has a p.g.f. of the form

$$\mathbf{E}s^\xi = \exp\{-c(1 - s)^\alpha\} \tag{5}$$

for some $c > 0$.

It has also been shown that the above multiplication, defined using the Binomial distribution, can be embedded in the family of multiplication operations that correspond to branching processes or, alternatively, to semigroups of probability generating functions; see [13]. A variant of this operation for integer-valued vectors with multiplication defined by operator-scaling of the probability generating function was studied by Van Harn and Steutel [14] in view of queueing applications. This setting has been extended to some functional equations for discrete random variables in [15,23].

In this paper we show that the discrete stability of non-negative integer random variables is a particular case of stability with respect to the thinning operation of point processes defined on a rather general phase space. If the phase space consists of one element, a point process may have multiple points – the total number of which is a non-negative integer random variable – and the thinning of the points is equivalent to the above-defined scaling operation \circ on discrete random variables. All known properties of discrete stable laws have their more general counterparts in point process settings. In particular, we show in Section 3 that α -stable point processes with respect to thinning are, in fact, Cox processes (see, e.g., Chapter 6.2 in [5]) with the (random) parametric measure being a positive α -stable measure with $\alpha \in (0, 1]$ on the phase space. Because of this, we first address general strictly stable random measures in Section 2. Note in this relation that, so far, only *independently scattered* stable measures have received much attention in the literature; see [28] and, more recently, [7,16] on the subject. It should be noted that Cox processes driven by various random measures are often used in spatial statistics (see [17,21,22]). A particularly novel feature of discrete stable processes is that the point counts have discrete α -stable distributions, which have infinite expectations unless for the degenerate case of $\alpha = 1$. Thus, these point processes open new possibilities for modelling point patterns with non-integrable point counts and non-existing moment measures of all orders.

It is known that discrete stable random variables can be represented as the sum of a Poisson number of Sibuya distributed integer random variables; see [10]. In Section 4 we show that an analogous cluster representation of finite discrete stable point processes also holds. The clusters are Sibuya point processes, which seem to be a new class of point processes not considered so far.

Some of our results for general point processes, especially defined on non-compact spaces, become trivial or do not have counterparts for discrete random variables. This concerns the results of Section 5, which draws an analogy with infinitely divisible point processes.

Another important model arising from the point process setting is the case of random elements in discrete semigroups that possess an at most countable basis. We show in Section 6 that the uniqueness of representation of each element as a linear finite combination with natural coefficients of the basis elements is a necessary condition to be able to define discrete stability. We give characterisation of D α S elements in these semigroups and establish a discrete analogue of their LePage representation.

The presentation of the theoretical material is complemented by examples from random measures, point processes and discrete semigroups all over the text. In particular, in Section 6 we introduce a new concept of multiplicatively stable natural numbers that is interesting on its own right and characterise their distributions.

2. Strictly stable random measures

Let X be a locally compact second countable space with the Borel σ -algebra \mathcal{B} . The family of Radon measures on \mathcal{B} is denoted by \mathbb{M} . These are measures that have finite values on the family \mathcal{B}_0 of relatively compact Borel sets, that is, the sets with compact topological closure. Note that all measures considered in this paper are assumed to be non-negative. The zero measure is denoted by 0 .

A *random measure* is a random element in the measurable space $[\mathbb{M}, \mathcal{M}]$, where the σ -algebra \mathcal{M} is generated by the following system of sets:

$$\{\mu \in \mathbb{M} : \mu(B_i) \leq t_i, i = 1, \dots, n\}, \quad B_i \in \mathcal{B}, t_i \geq 0.$$

The distribution of a random measure ζ can be characterised by its finite-dimensional distributions, that is, the distributions of $(\zeta(B_1), \dots, \zeta(B_n))$ for each finite collection of disjoint sets $B_1, \dots, B_n \in \mathcal{B}_0$.

It is well known (see, e.g., [6], Chapter 9.4) that the distribution of ζ is uniquely determined by its *Laplace functional*

$$L_\zeta[h] = \mathbf{E} \exp \left\{ - \int_X h(x) \zeta(dx) \right\} \quad (6)$$

defined on non-negative bounded measurable functions $h : X \mapsto \mathbb{R}_+$ with compact support (denoted by $h \in \text{BM}(X)$). From now on we adapt a shorter notation

$$\langle h, \mu \rangle = \int_X h(x) \mu(dx), \quad \mu \in \mathbb{M}.$$

The family \mathbb{M} can be endowed with the operation of addition and multiplication by non-negative numbers as

$$(\mu_1 + \mu_2)(\cdot) = \mu_1(\cdot) + \mu_2(\cdot); \quad (7)$$

$$(t\mu)(\cdot) = t\mu(\cdot), \quad t \geq 0. \quad (8)$$

If μ is a finite non-negative measure, it is possible to normalise it by dividing it by its total mass and so arriving at a probability measure. The normalisation procedure can be extended to all

locally finite measures as follows. Let B_1, B_2, \dots be a fixed countable base of the topology on X that consists of relatively compact sets. Append $B_0 = X$ to this base. For each $\mu \in \mathbb{M} \setminus \{0\}$ consider the sequence of its values $(\mu(B_0), \mu(B_1), \mu(B_2), \dots)$, possibly starting with infinity, but otherwise finite. Let $i(\mu)$ be the smallest non-negative integer i for which $0 < \mu(B_i) < \infty$; in particular, $i(\mu) = 0$ if μ is a finite measure. The set

$$\mathbb{S} = \{\mu \in \mathbb{M} : \mu(B_{i(\mu)}) = 1\} \tag{9}$$

is measurable, since

$$\mathbb{S} = \mathbb{M}_1 \cup \bigcup_{n=1}^{\infty} \{\mu \in \mathbb{M} : \mu(B_0) = \infty, \mu(B_1) = \dots = \mu(B_{n-1}) = 0, \mu(B_n) = 1\},$$

where \mathbb{M}_1 is the family of all probability measures on X . Note that $\mathbb{S} \cap \{\mu : \mu(X) < \infty\} = \mathbb{M}_1$. Furthermore, every $\mu \in \mathbb{M} \setminus \{0\}$ can be uniquely associated with the pair $(\hat{\mu}, \mu(B_{i(\mu)})) \in \mathbb{S} \times \mathbb{R}_+$, so that $\mu = \mu(B_{i(\mu)})\hat{\mu}$. It is straightforward to check that the mapping $\mu \mapsto (\mu(B_{i(\mu)}), \hat{\mu})$ is measurable. Hence we have the following *polar decomposition*: $\mathbb{M} = \mathbb{S} \times \mathbb{R}_+$.

Definition 1. A random measure ζ is called strictly α -stable (notation $\text{St}\alpha\text{S}$) if

$$t^{1/\alpha}\zeta' + (1-t)^{1/\alpha}\zeta'' \stackrel{D}{=} \zeta \tag{10}$$

for all $0 \leq t \leq 1$, where ζ', ζ'' are independent copies of the random measure ζ .

Definition 1 yields that for any $B_1, \dots, B_n \in \mathcal{B}_0$, the vector $(\zeta(B_1), \dots, \zeta(B_n))$ is a non-negative $\text{St}\alpha\text{S}$ n -dimensional random vector implying that $\alpha \in (0, 1]$. It is well known that one-sided – that is, concentrated on \mathbb{R}_+ – strictly stable laws corresponding to $\alpha = 1$ are degenerated so that $\text{St}\alpha\text{S}$ measures with $\alpha = 1$ are deterministic; see, for example, [28].

Theorem 2. A locally finite random measure ζ is $\text{St}\alpha\text{S}$ if and only if ζ is deterministic in the case $\alpha = 1$ and in the case $\alpha \in (0, 1)$ if and only if its Laplace functional is given by

$$L_\zeta[h] = \exp\left\{-\int_{\mathbb{M} \setminus \{0\}} (1 - e^{-\langle h, \mu \rangle}) \Lambda(d\mu)\right\}, \quad h \in \text{BM}(X), \tag{11}$$

where Λ is a Lévy measure, that is, a Radon measure on $\mathbb{M} \setminus \{0\}$ such that

$$\int_{\mathbb{M} \setminus \{0\}} (1 - e^{-\langle h, \mu \rangle}) \Lambda(d\mu) < \infty \tag{12}$$

for all $h \in \text{BM}(X)$, and Λ is homogeneous of order $-\alpha$, that is, $\Lambda(tA) = t^{-\alpha} \Lambda(A)$ for all measurable $A \subset \mathbb{M} \setminus \{0\}$ and $t > 0$.

Proof. *Sufficiency.* It is obvious that each deterministic measure is $\text{St}\alpha\text{S}$ with $\alpha = 1$. Consider now $h_1, \dots, h_k \in \text{BM}(X)$ and the image $\tilde{\Lambda}$ of Λ under the map $\mu \mapsto (\langle h_1, \mu \rangle, \dots, \langle h_k, \mu \rangle)$. It

is easy to see that $\tilde{\Lambda}$ is a homogeneous measure on \mathbb{R}_+^k . Then $L_\zeta[\sum_{i=1}^k t_i h_i]$ as a function of t_1, \dots, t_k is the Laplace transform of a totally skewed to the right (i.e., almost surely positive) strictly stable random vector $(\langle h_1, \zeta \rangle, \dots, \langle h_k, \zeta \rangle)$ with the Lévy measure $\tilde{\Lambda}$. If we show that (11) defines a Laplace functional of a random measure this would imply its stability.

We have $L_\zeta[0] = 1$ and $L_\zeta[h_n] \rightarrow L_\zeta[h]$ if $h_n \uparrow h$ pointwise. To see this, $H_n(\mu) = \langle h_n, \mu \rangle \uparrow \langle h, \mu \rangle = H(\mu)$ by the monotone convergence theorem. In its turn, $\langle H_n, \Lambda \rangle$ converges to $\langle H, \Lambda \rangle$ again by the monotone convergence. Now, by an analogue of [6], Theorem 9.4.II, for the Laplace functional, (11) indeed is the Laplace functional of a random measure.

Necessity. Strictly stable random measures can be treated by means of the general theory of strictly stable laws on semigroups developed in [8]. Consider the cone \mathbb{M} of locally finite measures with the addition and scaling operation defined by (7) and (8). In the terminology of [8], this set becomes a pointed cone with the origin and neutral element being the zero measure. The second distributivity law holds and the gauge function $\|\mu\| = \mu(B_{i(\mu)})$, $\mu \in \mathbb{M}$, is homogeneous with respect to multiplication of measures by a number. As in [8], Example 8.6, we argue that the characters on \mathbb{M}

$$\chi_h(\mu) = \exp\{-\langle h, \mu \rangle\}, \quad h \in \text{BM}(X),$$

have a strictly separating countable subfamily and are continuous, so that condition (C) of [8] holds. Furthermore, $\chi_h(s\mu) \rightarrow 1$ as $s \downarrow 0$ for all $\mu \in \mathbb{M}$, so that condition (E) of [8] is also satisfied. By Davydov, Molchanov and Zuyev [8], Theorem 5.16, the log-Laplace transform of a $\text{St}\alpha\text{S}$ measure satisfies $L_\zeta[s h] = s^\alpha L_\zeta[h]$ for $s > 0$, where $\alpha \in (0, 1]$ by Davydov, Molchanov and Zuyev [8], Theorem 5.20(ii). If $\alpha = 1$, this identity implies that the values of ζ are deterministic. By Davydov, Molchanov and Zuyev [8], Theorems 6.5(ii) and 6.7(i), the log-Laplace functional of a $\text{St}\alpha\text{S}$ random measure with $\alpha \in (0, 1)$ can be represented as the integral similar to (11) with the integration taken with respect to the Lévy measure of ζ over the second dual semigroup, that is, the family of all characters acting on functions from $\text{BM}(X)$.

In order to reduce the integration domain to $\mathbb{M} \setminus \{0\}$, it suffices to show that the convergence $\langle h, \mu_k \rangle \rightarrow g(h)$ for all $h \in \text{BM}(X)$ implies that $g(h) = \langle h, \mu \rangle$ for some $\mu \in \mathbb{M}$. Taking $h = \mathbb{1}_B$ here implies convergence and hence boundedness of the sequence $\{\mu_k(B), k \geq 1\}$ for each $B \in \mathcal{B}_0$. By Daley and Vere-Jones [5], Corollary A2.6.V, the sequence $\{\mu_k, k \geq 1\}$ is relatively compact in the vague topology. Given that $\langle h, \mu_k \rangle$ converges for each $h \in \text{BM}(X)$, in particular for all continuous $h \in \text{BM}(X)$, we obtain that all vaguely convergent subsequences of $\{\mu_k, k \geq 1\}$ share the same limit that can be denoted by μ and used to represent $g(h) = \langle h, \mu \rangle$ for continuous h . However, convergence $\mu_n \rightarrow \mu$ also happens in the strong local topology corresponding to test functions h from whole $\text{BM}(X)$. Indeed, if two subsequences $\mu_{i_1}, \mu_{i_2}, \dots$ and $\mu_{j_1}, \mu_{j_2}, \dots$ have different limits in this topology, say, μ' and μ'' , then there is $h \in \text{BM}(X)$ such that $\langle h, \mu' \rangle \neq \langle h, \mu'' \rangle$. Then for the subsequence $\mu_{i_1}, \mu_{j_1}, \mu_{i_2}, \mu_{j_2}, \dots$ (possibly after removing the repeating members) the limit of the integrals of h does not exist, contradicting the assumption. Now (11) follows from [8], Theorem 7.7. □

The properties of $\text{St}\alpha\text{S}$ measure ζ are determined by its Lévy measure Λ . For instance, if $X = \mathbb{R}^d$, then ζ is stationary if and only if Λ is invariant with respect to the map $\mu(\cdot) \mapsto \mu(\cdot + a)$

for all $a \in \mathbb{R}^d$ and all μ from the support of Λ . Note also that (12) is equivalent to

$$\int_{\mathbb{M} \setminus \{0\}} (1 - e^{-\mu(B)}) \Lambda(d\mu) < \infty \tag{13}$$

for all $B \in \mathcal{B}_0$.

Because of homogeneity, it is also useful to decompose Λ into the radial and directional components using the polar decomposition of \mathbb{M} described above. Note that the map $\mu \mapsto (\hat{\mu}, \mu(B_i(\mu)))$ is measurable. Introduce a measure $\hat{\sigma}$ such that

$$\hat{\sigma}(A) = \Lambda(\{t\mu : \mu \in A, t \geq 1\})$$

for all measurable $A \subset \mathbb{S}$. Then $\Lambda(A \times [a, b]) = \hat{\sigma}(A)(a^{-\alpha} - b^{-\alpha})$, so that Λ is represented as the product of $\hat{\sigma}$ and the radial component given by the measure θ_α defined as $\theta_\alpha([r, \infty)) = r^{-\alpha}$, $r > 0$. By the reason which will become apparent in the proof of Theorem 3 below, it is more convenient to scale $\hat{\sigma}$ by the value $\Gamma(1 - \alpha)$ of the gamma function. The measure $\sigma = \Gamma(1 - \alpha)\hat{\sigma}$ will be called the *spectral measure* of ζ in the sequel. Note that σ is not necessarily finite unless X is compact.

Condition (13) can be reformulated for the spectral measure σ as

$$\int_{\mathbb{S}} \mu(B)^\alpha \sigma(d\mu) < \infty \tag{14}$$

for all $B \in \mathcal{B}_0$. Note that σ can also be defined on any other reference sphere $\mathbb{S}' \subset \mathbb{M}$, provided each $\mu \in \mathbb{M} \setminus \{0\}$ can be uniquely represented as $t\mu'$ for some $\mu' \in \mathbb{S}'$.

Theorem 3. *Let σ be the spectral measure of a St α S random measure ζ with Lévy measure Λ . Then*

$$L_\zeta[h] = \exp\left\{-\int_{\mathbb{S}} \langle h, \mu \rangle^\alpha \sigma(d\mu)\right\}, \quad h \in \text{BM}(X). \tag{15}$$

Furthermore,

- (i) ζ is a.s. finite if and only if its Lévy measure Λ (resp., spectral measure σ) is supported by finite measures and $\sigma(\mathbb{S}) = \sigma(\mathbb{M}_1)$ is finite.
- (ii) The Laplace functional (11) defines a non-random measure if and only if $\alpha = 1$. In this case $\zeta = \bar{\mu}(\cdot) = \int_{\mathbb{S}} \mu(\cdot) \sigma(d\mu)$.

Proof. The representation (15) follows from (11), since

$$\begin{aligned} \int_{\mathbb{M} \setminus \{0\}} (1 - e^{-\langle h, \mu \rangle}) \Lambda(d\mu) &= \int_{\mathbb{S}} \int_0^\infty (1 - e^{-t\langle h, \mu \rangle}) \theta_\alpha(dt) \hat{\sigma}(d\mu) \\ &= \Gamma(1 - \alpha) \int_{\mathbb{S}} \langle h, \mu \rangle^\alpha \hat{\sigma}(d\mu), \end{aligned}$$

implying (15).

(i) Taking $h_n = \mathbb{1}_{X_n}$, $n \geq 1$, where $X_n \in \mathcal{B}_0$ form a nested sequence of relatively compact sets such that $X = \bigcup_n X_n$, we obtain that the Laplace transform of $\zeta(X)$ is

$$\lim_n L_\zeta[h_n] = \exp\left\{-\int_{\mathbb{M} \setminus \{0\}} (1 - e^{-\mu(X)}) \Lambda(d\mu)\right\},$$

where the integral is finite if and only if Λ is supported by finite measures. If this is the case, the spectral measure σ is defined on \mathbb{M}_1 and

$$\lim_n L_\zeta[h_n] = \exp\{-\sigma(\mathbb{M}_1)\}$$

defines a finite random variable if and only if $\sigma(\mathbb{M}_1)$ is finite.

(ii) If ζ is $\text{St}\alpha\text{S}$ with $\alpha = 1$, then its values on all relatively compact sets are deterministic, and so ζ is a deterministic measure. Furthermore, (10) clearly holds with $\alpha = 1$ for any deterministic ζ . Finally, (15) with $\alpha = 1$ yields

$$L_\zeta[h] = \exp\left\{-\int_{\mathbb{S}} \langle h, \mu \rangle \sigma(d\mu)\right\} = \exp\{-\langle h, \bar{\mu} \rangle\}. \quad \square$$

If the spectral measure is degenerate, the corresponding $\text{St}\alpha\text{S}$ measure has a particularly simple structure.

Theorem 4. Assume that the spectral measure σ of a $\text{St}\alpha\text{S}$ measure ζ with $0 < \alpha < 1$ is concentrated on a single measure so that $\sigma = c\delta_\mu$ for some $\mu \in \mathbb{S}$ and $c > 0$. Then ζ can be represented as $\zeta = c^{1/\alpha} \zeta_\alpha \mu$, where ζ_α is a positive $\text{St}\alpha\text{S}$ random variable with Laplace transform $\mathbb{E}e^{-z\zeta_\alpha} = \exp\{-z^\alpha\}$.

Proof. It suffices to verify that the Laplace transforms of both measures coincide and equal $L_\zeta[h] = \exp\{-c\langle h, \mu \rangle^\alpha\}$. □

Example 5. Let $X = \mathbb{R}^d$ and let σ be concentrated on the Lebesgue measure ℓ . By Theorem 4, the corresponding $\text{St}\alpha\text{S}$ measure ζ is stationary and proportional to $\zeta_\alpha \ell$.

If X is compact, then all measures $\mu \in \mathbb{M}$ are finite and condition (14) implies that $\sigma(\mathbb{M}_1)$ is finite, so that the spectral measure is defined on the family of probability measures on X .

Example 6 (Finite phase space). In the special case of a finite $X = \{x_1, \dots, x_d\}$ the family of probability measures becomes the unit simplex $\Delta_d = \{x \in \mathbb{R}_+^d : \sum x_i = 1\}$ and a $\text{St}\alpha\text{S}$ random measure ζ is nothing else but a d -dimensional totally skewed (or one-sided) strictly stable random vector $\zeta = (\zeta_1, \dots, \zeta_n)$. Its Laplace transform is given by

$$\mathbb{E}e^{-\langle h, \zeta \rangle} = \exp\left\{-\int_{\Delta_d} \langle h, x \rangle^\alpha \sigma(dx)\right\}, \quad h \in \mathbb{R}_+^d, \quad (16)$$

where σ is a finite measure on the unit simplex. Alternatively, the integration can be taken over the unit sphere. It is shown in [20] that this Laplace functional can be written as

$$\mathbf{E}e^{-\langle h, \zeta \rangle} = e^{-H_K(h^\alpha)}, \quad h \in \mathbb{R}_+^d, \tag{17}$$

where $h^\alpha = (h_1^\alpha, \dots, h_d^\alpha)$ and H_K is the support function of a certain convex set K that appears to be a generalisation of zonoids.

Example 7 (St α S random measures on \mathbb{R}^d). Fix a probability measure μ on $X = \mathbb{R}^d$ and let σ be the image under the map $x \mapsto \mu(\cdot - x)$ of a Radon measure ν . The corresponding St α S random measure ζ has the Laplace transform

$$L_\zeta[h] = \exp\left\{-\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} h(x+y)\mu(dx)\right)^\alpha \nu(dy)\right\}.$$

To ensure condition (14) it is necessary to assume that

$$\int_{\mathbb{R}^d} (\mu(B-y))^\alpha \nu(dy) < \infty \tag{18}$$

for all relatively compact B . In particular,

$$\mathbf{E}e^{-z\zeta(B)} = \exp\left\{-z^\alpha \int_{\mathbb{R}^d} (\mu(B-x))^\alpha \nu(dx)\right\}, \quad B \in \mathcal{B}_0. \tag{19}$$

A similar construction applies if μ is a general Radon measure. Since, in this case, it is difficult to ensure that measures $\mu(\cdot - x)$ belong to \mathbb{S} , the spectral measure σ is defined as the projection onto \mathbb{S} of the image of ν .

Example 8 (Stationary St α S random measures on \mathbb{R}^d). Consider the group T_y of shifts on \mathbb{M}_1 acting as $T_y\mu(\cdot) = \mu(\cdot - y)$. Call *centroid* any measurable map $C : \mathbb{M}_1 \mapsto \mathbb{R}^d$ such that $C(T_y\mu) = C(\mu) + y$ for every $y \in \mathbb{R}^d$ and $\mu \in \mathbb{M}_1$. For example, if μ^1, \dots, μ^d are the marginals of μ , the i th component of $C(\mu)$ is $C^i(\mu) = \inf\{t : \mu^i(-\infty, t] \geq 1/2\}$. Let \mathbb{M}_1^0 denote the set of probability measures with centroid $C(\mu)$ in the origin. A St α S random measure with the spectral measure supported by \mathbb{M}_1 is stationary if and only if the spectral measure σ is invariant with respect to T_y , that is, when it is decomposable into the product $\ell \times \sigma^0$ of the d -dimensional Lebesgue measure and a measure on \mathbb{M}_1^0 satisfying

$$\int_{\mathbb{M}_1^0} \mu(B)^\alpha \sigma^0(d\mu) < \infty \quad \text{for all } B \in \mathcal{B}_0.$$

Then the Laplace transform of a stationary St α S measure on \mathbb{R}^d has the following form:

$$L_\zeta[h] = \exp\left\{-\int_{\mathbb{M}_1^0} \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} h(x+y)\mu(dx)\right)^\alpha dy \sigma^0(d\mu)\right\}. \tag{20}$$

In particular, if σ^0 charges only one probability measure μ , we obtain ζ from Example 7 with ν being the Lebesgue measure.

More generally, take a homogeneous of order $-(\alpha + d)$ measure Λ_0 on $\mathbb{M} \setminus \{0\}$ and set $\Lambda(d\mu) = \int_{\mathbb{R}^d} \Lambda_0(d\mu - x) dx$. Provided (13) is satisfied, Λ is the Lévy measure of a stationary St α S measure.

Note that usually in the literature, and particularly in [28], Section 3.3, the term α -stable measure is reserved for an *independently scattered* measure, that is, a measure with independent α -stable values on disjoint sets. Our notion is more general as the following proposition shows.

Theorem 9. *A St α S random measure with $\alpha \in (0, 1)$ is independently scattered if and only if its spectral measure σ is supported by the set $\{\delta_x : x \in X\} \subseteq \mathbb{S}$ of Dirac measures.*

Proof. *Sufficiency.* The Laplace functional (15) of ζ becomes

$$L_\zeta[h] = \exp\left\{-\int_X h^\alpha(x)\tilde{\sigma}(dx)\right\},$$

where $\tilde{\sigma}$ is the image of σ under the map $\delta_x \mapsto x$. It is easy to see that $L_\zeta[\sum_{i=1}^n h_i] = \prod_{i=1}^n L_\zeta[h_i]$ for all functions h_1, \dots, h_n with disjoint supports, meaning, in particular, that for any disjoint Borel sets B_1, \dots, B_n the variables $\zeta(B_i)$ with Laplace transforms $L_\zeta[z_i \mathbb{1}_{B_i}]$, $i = 1, \dots, n$, are independent, so that ζ is independently scattered.

Necessity. Take two disjoint sets B_1, B_2 from an at most countable base of the topology on X . Since $\zeta(B_1)$ and $\zeta(B_2)$ are independent, we have that

$$L_\zeta[\mathbb{1}_{B_1} + \mathbb{1}_{B_2}] = L_\zeta[\mathbb{1}_{B_1}]L_\zeta[\mathbb{1}_{B_2}].$$

By (15),

$$\int_{\mathbb{S}} [\mu^\alpha(B_1) + \mu^\alpha(B_2) - (\mu(B_1) + \mu(B_2))^\alpha] \sigma(d\mu) = 0.$$

Since $\alpha \in (0, 1)$, the integrand is a non-negative expression whatever $\mu(B_1), \mu(B_2) \geq 0$ are. Since σ is a positive measure, the integral is zero only if the integrand vanishes on the support of σ , that is, for σ -almost all μ either $\mu(B_1) = 0$ or $\mu(B_2) = 0$. Since this is true for all disjoint B_1, B_2 from the base, μ is concentrated at a single point. \square

Along the same lines it is possible to prove the following result.

Theorem 10. *The values $\zeta(B_1), \dots, \zeta(B_n)$ of a St α S random measure ζ with $\alpha \in (0, 1)$ on disjoint sets B_1, \dots, B_n are independent if and only if the support of the spectral measure σ (or of the Lévy measure Λ) does not include any measure that has positive values on at least two sets from B_1, \dots, B_n .*

Example 11 (Self-similar random measures). Recently, Vere-Jones [31] has introduced a wide class of self-similar random measures on \mathbb{R}^d , that is, the measures satisfying

$$\zeta(B) \stackrel{\mathcal{D}}{=} a^{-H} \zeta(aB) \tag{21}$$

for all Borel B , all positive a and some H , which is then called the *similarity index*. These measures are generally not independently scattered although stationary independently scattered St α S measures are self-similar with index $H = 1/\alpha$.

Introduce operation D_a on measures $\mu \in \mathbb{M}$ by setting $(D_a\mu)(B) = \mu(aB)$, $B \in \mathcal{B}$, and the corresponding operation $(\tilde{D}_a\sigma)(M) = \sigma(D_aM)$ uplifted to σ on measurable $M \subset \mathbb{S}$. Assume that the spectral measure is supported by \mathbb{M}_1 . Since

$$L_{a^{-H} D_a \zeta}[h(x)] = L_\zeta[a^{-H} h(a^{-1}x)],$$

then writing property (21) for the Laplace transform (15), we obtain

$$\begin{aligned} \int_{\mathbb{S}} \langle h(x), \mu \rangle^\alpha \sigma(d\mu) &= \int_{\mathbb{S}} \langle a^{-H} h(a^{-1}x), \mu \rangle^\alpha \sigma(d\mu) \\ &= a^{-\alpha H} \int_{\mathbb{M}_1} \langle h, D_a\mu \rangle^\alpha \sigma(d\mu) = a^{-\alpha H} \int_{\mathbb{M}_1} \langle h, \mu \rangle^\alpha (\tilde{D}_{a^{-1}}\sigma)(d\mu), \end{aligned}$$

where in the last equality we used the fact that that $D_a\mathbb{M}_1 = \mathbb{M}_1$. Therefore a St α S random measure is self-similar with index H if and only if $\sigma = a^{-\alpha H} \tilde{D}_{a^{-1}}\sigma$.

As in the proof of Theorem 9 above, in the case of independently scattered St α S measures the last identity could be written as $\tilde{\sigma}(dx) = a^{-\alpha H} \tilde{D}_{a^{-1}}\tilde{\sigma}(dx) = a^{-\alpha H} \tilde{\sigma}(a dx)$ for the image $\tilde{\sigma}$ of σ under the map $\delta_x \mapsto x$. In particular, in the stationary case $\tilde{\sigma}$ is necessarily proportional to the Lebesgue measure and the corresponding random measure becomes self-similar if and only if $\alpha = 1/H$.

The following theorem provides a LePage representation of a St α S random measure.

Theorem 12. *A random measure ζ is St α S if and only if*

$$\zeta \stackrel{\mathcal{D}}{=} \sum_{\mu_i \in \Psi} \mu_i, \tag{22}$$

where Ψ is the Poisson process on $\mathbb{M} \setminus \{0\}$ driven by an intensity measure Λ satisfying (12) and such that $\Lambda(tA) = t^{-\alpha} \Lambda(A)$ for all $t > 0$ and any measurable A . In this case Λ is exactly the Lévy measure of ζ . Convergence of the series in (22) is in the sense of the vague convergence of measures.

If the spectral measure σ corresponding to Λ satisfies $c = \sigma(\mathbb{S}) < \infty$, then

$$\zeta \stackrel{\mathcal{D}}{=} b \sum_{k=1}^{\infty} \gamma_k^{-1/\alpha} \varepsilon_k, \quad b = \left(\frac{c}{\Gamma(1-\alpha)} \right)^{1/\alpha}, \tag{23}$$

where $\varepsilon_1, \varepsilon_2, \dots$ are i.i.d. random measures with distribution $c^{-1}\sigma$ and $\gamma_k = \xi_1 + \dots + \xi_k, k \geq 1$, for a sequence of independent exponentially distributed random variables ξ_k with mean one.

Proof. Consider a continuous function $h \in \text{BM}(X)$ and define a map $\mu \mapsto \langle h, \mu \rangle$ from \mathbb{M} to $\mathbb{R}_+ = [0, \infty)$. The image of Ψ under this map becomes a Poisson process $\{x_i, i \geq 1\}$ on \mathbb{R}_+ with intensity measure θ that satisfies

$$\theta([r, \infty)) = \Lambda(\{\mu : \langle h, \mu \rangle \geq r\}).$$

Since $1 - e^{-x} \geq a\mathbf{1}_{x \geq r}$ for any $r > 0$ and some constant $a > 0$, Condition (12) implies that $\theta([r, \infty)) < \infty$. Then the homogeneity property of Λ yields that $\theta([r, \infty)) = cr^{-\alpha}$ for all $r > 0$ and some constant $c > 0$.

It is well known (see [8,28]) that for such intensity measure θ the sum $\sum_i x_i$ converges almost surely. Thus, $\sum_{\mu_i \in \Psi} \langle h, \mu_i \rangle$ converges for each continuous $h \in \text{BM}(X)$, so that the series (22) converges in the vague topology to a random measure see [25], equation (3.14).

By Resnick [25], Proposition 3.19, a sequence of random measures converges weakly if and only if the values of their Laplace functionals on any continuous function converge. This is seen by noticing that the Laplace functional (11) of ζ coincides with the Laplace functional of the right-hand side of (22), where the latter can be computed as the probability generating functional (p.g.fl.) of a Poisson process, see (25).

Finally, (23) follows from the polar decomposition for Λ . □

3. Discrete stability for point processes

Point processes are counting random measures, that is, random elements with realisations in the set of locally finite counting measures. Each counting measure φ can be represented as the sum $\varphi = \sum \delta_{x_i}$ of unit masses, where we allow for the multiplicity of the support points $\{x_i, i \geq 1\}$. The distribution of a point process Φ can be characterised by its p.g.fl. defined on functions $u : X \mapsto (0, 1]$ such that $1 - u \in \text{BM}(X)$ by means of

$$G_\Phi[u] = \mathbf{E} \exp\{\langle \log u, \Phi \rangle\} = \mathbf{E} \prod_{x_i \in \text{supp } \Phi} u(x_i)^{\Phi(\{x_i\})} = L_\Phi[-\log u] \tag{24}$$

with the convention that the product is 1 if Φ is a null measure (i.e., the realisation of the process contains no points). The p.g.fl. can be extended to pointwise monotone limits of the functions from $\text{BM}(X)$ at the expense of allowing for infinite and zero values; see, for example, [6], Chapter 9.4.

A *Poisson process* Π_μ with intensity measure μ is characterised by the property that for any finite collection of disjoint sets $B_1, \dots, B_n \in \mathcal{B}_0$ the variables $\Pi_\mu(B_1), \dots, \Pi_\mu(B_n)$ are mutually independent Poisson distributed random variables with means $\mu(B_1), \dots, \mu(B_n)$. The p.g.fl. of a Poisson process Π_μ is given by

$$G_{\Pi_\mu}[u] = \exp\{-\langle 1 - u, \mu \rangle\}. \tag{25}$$

If the intensity measure itself is a random measure ζ , then the obtained (doubly stochastic) point process is called a *Cox process*. Its p.g.fl. is given by

$$G_{\Pi_\zeta}[u] = \mathbb{E} \exp\{-\langle 1 - u, \zeta \rangle\} = L_\zeta[1 - u]. \tag{26}$$

Note that the Cox process is stationary if and only if the random measure ζ is stationary.

Addition of counting measures is well defined and leads to the definition of the superposition operation for point processes. However, multiplication of the counting measure by positive numbers cannot be defined by multiplying its values – they no longer remain integers for arbitrary multiplication factors. In what follows we define a stochastic multiplication operation that corresponds to the thinning operation for point processes. Namely, each unit mass δ_{x_i} in the representation of the counting measure $\varphi = \sum_i \delta_{x_i}$ is removed with probability $1 - t$ and retained with probability t independently of other masses. The resulting counting measure $t \circ \varphi$ becomes random (even if φ is deterministic) and is known under the name of *independent thinning* in the point process literature; see, for example, [6], Chapter 11.3, or [19], Chapter 7, where relations to cluster and Cox processes were established.

To complement this pathwise description, it is possible to define the thinning operation on probability distributions of point processes. Namely, the thinned process $t \circ \Phi$ has the p.g.fl.

$$G_{t \circ \Phi}[u] = G_\Phi[tu + 1 - t] = G_\Phi[1 - t(1 - u)]; \tag{27}$$

see, for example, [6], page 155. From this, it is easy to establish the following properties of the thinning operation.

Theorem 13. *The thinning operation \circ is associative, commutative and distributive with respect to superposition of point processes, that is,*

$$t_1 \circ (t_2 \circ \Phi) \stackrel{\mathcal{D}}{=} (t_1 t_2) \circ \Phi \stackrel{\mathcal{D}}{=} t_2 \circ (t_1 \circ \Phi)$$

and

$$t \circ (\Phi + \Phi') \stackrel{\mathcal{D}}{=} (t \circ \Phi) + (t \circ \Phi')$$

for any $t, t_1, t_2 \in [0, 1]$ and any independent point processes Φ and Φ' . For disjoint Borel sets B_1 and B_2 , random variables $(t \circ \Phi)(B_1)$ and $(t \circ \Phi)(B_2)$ are conditionally independent given Φ and there exists a coupling of Φ and $t \circ \Phi$ (described above) such that $(t \circ \Phi)(B) \leq \Phi(B)$ almost surely for any $B \in \mathcal{B}$.

If the phase space X consists of one point, the point process Φ becomes a non-negative integer random variable $\Phi(X)$. Since (27) turns into (3), the thinning operation becomes the classical discrete multiplication operation acting on positive integer random variables. Keeping this in mind, we can define the notion of discrete stability for point processes.

Definition 14. *A point process Φ (or its probability distribution) is called discrete α -stable or α -stable with respect to thinning (notation $D\alpha S$), if for any $0 \leq t \leq 1$ one has*

$$t^{1/\alpha} \circ \Phi' + (1 - t)^{1/\alpha} \circ \Phi'' \stackrel{\mathcal{D}}{=} \Phi, \tag{28}$$

where Φ' and Φ'' are independent copies of Φ .

Definition 14 and (27) yield that Φ is D α S if and only if its p.g.fl. possesses the property:

$$G_\Phi[1 - t^{1/\alpha}(1 - u)]G_\Phi[1 - (1 - t)^{1/\alpha}(1 - u)] = G_\Phi[u] \tag{29}$$

for any positive function u such that $1 - u \in \text{BM}(X)$ and all $0 < t < 1$.

Relation (28) implies that $\Phi(B)$ is a discrete α -stable random variable for any $B \in \mathcal{B}$. Hence D α S point processes exist only for $\alpha \in (0, 1]$ due to the already mentioned result (5). Now we come to the main result of this section.

Theorem 15. *A point process Φ is D α S if and only if it is a Cox process Π_ζ with a St α S intensity measure ζ .*

Proof. *Sufficiency.* If ζ is a St α S random measure, then the corresponding Cox process Π_ζ has the p.g.fl.

$$G_{\Pi_\zeta}[u] = L_\zeta[1 - u] = \exp\left\{-\int_{\mathbb{M} \setminus \{0\}} (1 - e^{-(1-u, \mu)}) \Lambda(d\mu)\right\}. \tag{30}$$

Thus,

$$\begin{aligned} G_\Phi[1 - t(1 - u)] &= \exp\left\{-\int_{\mathbb{M} \setminus \{0\}} (1 - e^{-(1-u, t\mu)}) \Lambda(d\mu)\right\} \\ &= \exp\left\{-t^\alpha \int_{\mathbb{M} \setminus \{0\}} (1 - e^{-(1-u, \mu)}) \Lambda(d\mu)\right\} \end{aligned}$$

by the homogeneity of Λ . Then (29) clearly holds.

Necessity. By iterating (28) m times we arrive at

$$m^{-1/\alpha} \circ \Phi_1 + \dots + m^{-1/\alpha} \circ \Phi_m \stackrel{\mathcal{D}}{=} \Phi$$

for i.i.d. $\Phi, \Phi_1, \dots, \Phi_m$, implying that D α S point processes are necessarily infinitely divisible and

$$(G_\Phi[1 - m^{-1/\alpha}(1 - u)])^m = G_\Phi[u] \tag{31}$$

for all $m \geq 2$.

The crucial step of the proof aims to show that the functional

$$L[u] = G_\Phi[1 - u], \quad u \in \text{BM}(X), \tag{32}$$

is the Laplace functional of a St α S random measure ζ . While the functional L on the left-hand side should be defined on all (bounded) functions with compact supports, it is apparent that the p.g.fl. G_Φ on the right-hand side of (32) may not be well defined on $1 - u$. Indeed, in contrast

to (24), $1 - u$ does not necessarily take values from the unit interval. To overcome this difficulty, we employ (31) and define

$$L[u] = (G_\Phi[1 - m^{-1/\alpha}u])^m. \tag{33}$$

Since $u \in \text{BM}(X)$, for sufficiently large m the function $1 - m^{-1/\alpha}u$ does take values in $[0, 1]$ and equals 1 outside a compact set. Since (33) holds for all m , it is possible to pass to the limit as $m \rightarrow \infty$ to see that

$$L[u] = \exp\left\{-\lim_{m \rightarrow \infty} m(1 - G_\Phi[1 - m^{-1/\alpha}u])\right\}.$$

By the Schoenberg theorem (see, e.g., [3], Theorem 3.2.2) L is positive definite if $1 - G_\Phi[1 - m^{-1/\alpha}u]$ is negative definite, that is,

$$\sum_{i,j=1}^n c_i c_j (1 - G_\Phi[1 - m^{-1/\alpha}(u_i + u_j)]) \leq 0$$

for all $n \geq 2$, $u_1, \dots, u_n \in \text{BM}(X)$ and c_1, \dots, c_n with $\sum c_i = 0$. In view of the latter condition, we need to show the inequality

$$\sum_{i,j=1}^n c_i c_j G_\Phi[1 - m^{-1/\alpha}(u_i + u_j)] \geq 0.$$

Note that

$$\lim_{m \rightarrow \infty} \frac{1 - m^{-1/\alpha}u}{e^{-m^{-1/\alpha}u}} = 1.$$

Denote $v_i = e^{-m^{-1/\alpha}u_i}$, $i = 1, \dots, n$. Referring to the continuity of the p.g.fl. G_Φ , it suffices to check that

$$\sum_{i,j=1}^n c_i c_j G_\Phi[v_i v_j] \geq 0,$$

which is exactly the positive definiteness of G_Φ .

Thus, $L_\zeta[\sum_{i=1}^k t_i h_i]$ as a function of $t_1, \dots, t_k \geq 0$ is the Laplace transform of a random vector. Arguing as in the proof of sufficiency in Theorem 2, it is easy to check the continuity of L , so that L is indeed the Laplace functional of a random measure ζ . Condition (29) rewritten for L means that ζ is St α S. □

By Theorem 3, a St α S random measure ζ has the characteristic exponent $\alpha = 1$ if and only if ζ is deterministic. Respectively, a D α S processes with $\alpha = 1$ is a Poisson process driven by (non-random) intensity measure ζ .

Using decomposition $\Lambda = \hat{\sigma} \times \theta_\alpha$, the first identity in (30) and (15), we obtain the following result.

Corollary 16 (Spectral representation). *A point process Φ is $D\alpha S$ with $\alpha \in (0, 1]$ if and only if its p.g.fl. has the form*

$$G_\Phi[u] = \exp \left\{ - \int_{\mathbb{S}} \langle 1 - u, \mu \rangle^\alpha \sigma(d\mu) \right\}, \quad 1 - u \in \text{BM}(X), \tag{34}$$

for some locally finite spectral measure σ on \mathbb{S} that satisfies (14).

The number of points $\Phi(B)$ of a $D\alpha S$ process Φ in a relatively compact Borel set B has the p.g.f.

$$\mathbf{E}_s^{\Phi(B)} = \exp \left\{ -(1 - s)^\alpha \int_{\mathbb{S}} \mu(B)^\alpha \sigma(d\mu) \right\} \tag{35}$$

and so $\Phi(B)$ is either zero a.s. or has infinite expectation for $0 < \alpha < 1$, while $\mathbf{E}\Phi(B)$ is finite in the Poisson case $\alpha = 1$. Furthermore, the avoidance probability is given by

$$\mathbf{P}\{\Phi(B) = 0\} = \exp \left\{ - \int_{\mathbb{S}} \mu(B)^\alpha \sigma(d\mu) \right\},$$

so that (14) guarantees that the avoidance probabilities are positive. In other words, a $D\alpha S$ point process does not have fixed points.

Theorems 12 and 15 immediately imply the following result.

Corollary 17 (LePage representation of a $D\alpha S$ process). *A $D\alpha S$ process Φ with Lévy measure Λ can be represented as a Cox process:*

$$\Phi \stackrel{\mathcal{D}}{=} \sum_{\mu_i \in \Psi} \Pi_{\mu_i}, \tag{36}$$

where Ψ is a Poisson process on $\mathbb{M} \setminus \{0\}$ with intensity measure Λ . In particular, if its spectral measure σ is finite, then

$$\Phi \stackrel{\mathcal{D}}{=} \sum_{k=1}^{\infty} \Pi_{b\gamma_k^{-1/\alpha} \varepsilon_k}, \tag{37}$$

where ε_k , γ_k and b are defined in Theorem 12 and the sum in (37) almost surely contains only finitely many terms.

Proof. By Theorem 15, Φ is a Cox process Π_ζ , where ζ is representable as (22). Conditioned on a realisation of Ψ , we have that $\Pi_\zeta = \sum_i \Pi_{\mu_i}$ by the superposition property of a Poisson process. Now the statement easily follows.

The finiteness of the sum in (37) follows from the Borel–Cantelli lemma, since

$$\mathbf{P}\{\Pi_{b\gamma_k^{-1/\alpha} \varepsilon_k}(X) > 0 \mid \gamma_k\} = 1 - \exp\{-b\gamma_k^{-1/\alpha}\} = \mathcal{O}(k^{-1/\alpha})$$

for almost all realisations of $\{\gamma_k\}$. □

Corollary 18 (see [11]). *A random variable ξ with non-negative integer values is discrete α -stable if and only if ξ is a mixture of Poisson laws with parameter given by a positive strictly stable random variable. The p.g.f. of ξ is given by*

$$\mathbf{E}s^\xi = e^{-c(1-s)^\alpha}, \quad u \in (0, 1],$$

for some $c > 0$.

Example 19 (Discrete stable vectors, cf. Example 6). A point process Φ on a finite space X can be described by a random vector (ξ_1, \dots, ξ_d) of dimension d with non-negative integer components. If Φ is $D\alpha S$ then (ξ_1, \dots, ξ_d) are said to form a $D\alpha S$ random vector. For instance, if Φ is a $D\alpha S$ point process in any space X , then the point counts $\Phi(B_1), \dots, \Phi(B_n)$ form a $D\alpha S$ random vector for $B_1, \dots, B_n \in \mathcal{B}_0$.

Theorem 15 implies that $\xi = (\xi_1, \dots, \xi_d)$ is $D\alpha S$ if and only if its components are mixtures of Poisson random variables with parameters $\zeta = (\zeta_1, \dots, \zeta_d)$ and are conditionally independent given ζ , where ζ is a strictly stable non-negative random vector with the Laplace transform (16). Notice that, in general, the components of ξ are *dependent* $D\alpha S$ random variables unless the spectral measure σ of ζ is supported by the vertices of the simplex Δ_d only. In view of (17), the p.g.f. of ξ can be represented as

$$\mathbf{E} \prod_{i=1}^d s_i^{\xi_i} = \exp \left\{ - \int_{\Delta_d} \langle \mathbf{1} - s, \mu \rangle^\alpha \sigma(d\mu) \right\} = e^{-H_K((1-s)^\alpha)},$$

where $s = (s_1, \dots, s_d)$ and $\mathbf{1} = (1, \dots, 1)$. Thus, the distribution of ξ is determined by the values of the support function of H_K and its derivatives at $\mathbf{1}$.

Example 20 (Mixed Poisson process with stable intensity, cf. Example 5). Let $X = \mathbb{R}^d$ and let σ attach a mass c to the measure $a\ell$, where ℓ is the Lebesgue measure on \mathbb{R}^d and $a > 0$ is chosen so that $a\ell \in \mathbb{S}$. Then $G_\Phi[u] = \exp\{-ca^\alpha \langle 1 - u, \ell \rangle^\alpha\}$, which is the p.g.fl. of a stationary Cox process with a $S\alpha S$ density, that is, a stationary Poisson process in \mathbb{R}^d driven by the random intensity measure $ac^{1/\alpha} \zeta_\alpha \ell$, where ζ_α is a positive $S\alpha S$ random variable with Laplace transform $\mathbf{E}e^{-z\zeta_\alpha} = e^{-z^\alpha}$. This type of Cox process is also known as a *mixed Poisson process*. In particular, if $c = a^{-1/\alpha}$, then $\mathbf{E}s^{-\Phi(B)} = e^{-(1-s)^\alpha \ell(B)^\alpha}$, so that $\mathbf{P}\{\Phi(B) = 0\} = e^{-\ell(B)^\alpha}$ for all $B \in \mathcal{B}_0$. Note that this point process has an infinite intensity measure.

Discrete α -stable point processes appear naturally as limits for thinned superpositions of point processes. Let Ψ be a point process on X and let $S_n = \Psi_1 + \dots + \Psi_n$ be the sum of independent copies of Ψ . The p.g.fl. of the thinned point process $a_n \circ S_n$ is given by

$$G_{a_n \circ S_n}[1 - h] = (G_\Psi[1 - a_n h])^n,$$

where $a_n \rightarrow 0$ is a certain normalising sequence. On the other hand, the superposition can be normalised by scaling its values as $a_n S_n$, so that the result of scaling is a random measure, but no longer counting. The following basic result establishes a correspondence between the convergence of the thinned and the scaled superpositions.

Theorem 21. *Let $\{S_n, n \geq 1\}$, be a sequence of point processes. Then for some sequence $\{a_n\}$, $a_n S_n$ weakly converges to a non-trivial random measure that is necessarily $\text{St}\alpha\text{S}$ with a spectral measure σ if and only if $a_n \circ S_n$ weakly converges to a non-trivial point process that is necessarily $\text{D}\alpha\text{S}$ with the same spectral measure σ .*

If S_n is the sum of n independent copies of a process Ψ , the measure σ can be defined by its finite-dimensional distributions: if $\xi = (\Psi(B_1), \dots, \Psi(B_d))$ for $B_1, \dots, B_d \in \mathcal{B}_0$, then

$$\sigma(\{\mu : (\mu_1, \dots, \mu_d) \in A\}) = \Gamma(1 - \alpha) \lim_{n \rightarrow \infty} n \mathbf{P}\{\xi / \|\xi\|_1 \in A; \|\xi\|_1 > a_n\} \tag{38}$$

for all measurable A from the unit ℓ_1 -sphere $\{x \in \mathbb{R}_+^d : \|x\|_1 = 1\}$.

Proof. The equivalence of the convergence statements is established in [18], Theorem 8.4; see also [6], Theorem 11.3.III.

Finally, (38) is the standard condition for $a_n S_n$ to have a limit that is valid in a much more general setting than for random measures; see [8], Theorem 4.3, and [1]. The gamma factor stems from the particular normalisation of the spectral measure adopted in Section 2. \square

If a point process Ψ has a finite intensity measure, then the strong law of large numbers applied to its values on any relatively compact set implies that the limit of $n^{-1} S_n$ is a deterministic measure, so that the limiting $\text{D}\alpha\text{S}$ process is, in fact, Poisson. Therefore a $\text{D}\alpha\text{S}$ limit with $\alpha \in (0, 1)$ is only possible for a point process Ψ with infinite intensity. If X consists of one point, then we recover the result of [4] concerning a relationship between discrete stable random variables and conventional stable laws.

Remark 22. Note that not all point processes can give a non-trivial limit in the above schemes. Consider, for instance, a point process Ψ defined on $X = \{1, 2, \dots\}$ with the point multiplicities $\Psi(i)$ being independent discrete stable random variables with characteristic exponents $\alpha_i = 1/2 + 1/(2i)$, $i = 1, 2, \dots$. Then $n^{-1/\alpha} \circ S_n$ for $\alpha \leq 1/2$ gives null process as the limit, while for larger α the limiting process is infinite on all sufficiently large i .

4. Sibuya point process and cluster representation

Recall that a general cluster process is defined by means of a *centre point process* N_c in some phase space Y and a countable family of independent *daughter point processes* $N(\cdot|y)$ in a phase space X indexed by the points of Y . Their superposition in X defines a *cluster process*. The p.g.fl. $G[h]$ of a cluster process is then a composition $G_c[G_d[h]\cdot]$ of the centre and the daughter processes; see, for example, [5], Proposition 6.3.II.

Noting (25), the form of p.g.fl. (30) suggests that a $\text{D}\alpha\text{S}$ process can be regarded as a cluster process with centre processes being Poisson with intensity measure Λ in $Y = \mathbb{M} \setminus \{0\}$ and daughter processes being also Poisson parametrised by their intensity measure $\mu \in \text{supp } \Lambda$. We will embark on exploration of the general case in the next section, but here we concentrate on the case when measure Λ charges only finite measures and give an alternative explicit cluster characterisation of such $\text{D}\alpha\text{S}$ processes.

Definition 23. Let μ be a probability measure on X . A point process Υ on X defined by the p.g.fl.

$$G_\Upsilon[u] = G_{\Upsilon(\mu)}[u] = 1 - (1 - u, \mu)^\alpha \tag{39}$$

is called the Sibuya point process with exponent α and parameter measure μ . Its distribution is denoted by $\text{Sib}(\alpha, \mu)$.

If X consists of one point, then the point multiplicity is a random variable η with the p.g.f.

$$\mathbf{E}z^\eta = 1 - (1 - z)^\alpha, \quad z \in (0, 1].$$

In this case we say that η has the Sibuya distribution and denote it by $\text{Sib}(\alpha)$. The Sibuya distribution corresponds to the number of trials to get the first success in a series of Bernoulli trials with probability of success in the k th trial being α/k ; see also [10,11] for efficient algorithms of its simulation.

In particular, (39) implies that $\Upsilon(B)$ with $B \in \mathcal{B}_0$ has the p.g.f.

$$\mathbf{E}u^{\Upsilon(B)} = 1 - \mu^\alpha(B)(1 - u)^\alpha.$$

Note that $\Upsilon(B)$ has infinite expectation if $\mu(B)$ does not vanish. Furthermore,

$$\mathbf{P}\{\Upsilon(B) = 0\} = 1 - \mu^\alpha(B),$$

$$\mathbf{P}\{\Upsilon(B) = 1\} = \alpha\mu^\alpha(B) = q_1(\alpha)\mu^\alpha(B),$$

$$\mathbf{P}\{\Upsilon(X) = n\} = (1 - \alpha)\left(1 - \frac{\alpha}{2}\right) \cdots \left(1 - \frac{\alpha}{n-1}\right) \frac{\alpha}{n} \mu^\alpha(B) \stackrel{\text{def}}{=} q_n(\alpha)\mu^\alpha(B), \quad n \geq 2.$$

Therefore, conditioned to be non-zero, $\Upsilon(B)$ has the Sibuya distribution with parameter α justifying the chosen name for this process Υ . In the terminology of [4], $\Upsilon(B)$ has $\mu^\alpha(B)$ -scaled Sibuya distribution. In particular, $\Upsilon(X)$ is non-zero a.s. and follows $\text{Sib}(\alpha)$ distribution.

Developing the p.g.fl. (39) makes it possible to get insight into the structure of a Sibuya process:

$$\begin{aligned} G_\Upsilon[u] &= \mathbf{E} \prod_{x_i \in \Upsilon} u(x_i) = 1 - (1 - (u, \mu))^\alpha = \sum_{n=1}^\infty q_n(\alpha) (u, \mu)^n \\ &= \sum_{n=1}^\infty q_n(\alpha) \int_{X^n} u(x_1) \cdots u(x_n) \mu(dx_1) \cdots \mu(dx_n). \end{aligned}$$

Therefore, as we have already seen, the total number of points of Υ follows Sibuya distribution and, given this total number, the points are independently identically distributed according to the distribution μ . This also justifies the fact that (39) indeed is a p.g.fl. of a point process constructed this way. This type of processes is called *purely random* in [19], page 104.

Assume now that the Lévy measure Λ of a St α S random measure ζ is supported by finite measures. Then (15) holds with the spectral measure σ defined on \mathbb{M}_1 , so that the p.g.fl. of the corresponding D α S process takes the form

$$G_\Phi[u] = \exp \left\{ \int_{\mathbb{M}_1} (G_{\Upsilon(\mu)}[u] - 1) \sigma(d\mu) \right\} \tag{40}$$

with $G_{\Upsilon(\mu)}[u]$ given by (39). Thus we have shown the following result.

Theorem 24. *A D α S point process Φ with the Lévy measure supported by finite measures (equivalently, with a spectral measure σ supported by \mathbb{M}_1) can be represented as a cluster process with a Poisson centre process on \mathbb{M}_1 driven by intensity measure σ and daughter processes being Sibuya processes $\text{Sib}(\alpha, \mu)$, $\mu \in \mathbb{M}_1$. Its p.g.fl. is given by (40).*

As a by-product, we have established the following fact: Since Sibuya processes are finite with probability 1, the cluster process is finite or infinite depending on the finiteness of the Poisson processes of centres.

Corollary 25. *A D α S point process is finite if and only if its spectral measure σ is finite and is supported by finite measures.*

If $\alpha = 1$, then the Sibuya process consists of one point distributed according to μ and the cluster process represents a Poisson process with intensity measure $\bar{\mu}(\cdot) = \int_{\mathbb{M}_1} \mu(\cdot) \sigma(d\mu)$, which is clearly discrete 1-stable.

By Corollary 18, ξ is discrete α -stable if and only if ξ can be represented as a sum of a Poisson number of independent Sibuya distributed random variables, which is proved for discrete stable laws in [11].

Later on we make use of Theorem 24 for the case of an infinite countable phase space X . The finite case is considered below.

Definition 26. *Let $\mu = (\mu_1, \dots, \mu_d)$ be a d -dimensional probability distribution. A random vector Υ has multivariate Sibuya distribution with parameter measure μ and exponent $\alpha \in (0, 1]$ if its p.g.fl. has the following form:*

$$\mathbf{E} \prod_{n=1}^d z_n^{\Upsilon_n} = 1 - \left(\sum_{n=1}^d (1 - z_n) \mu_n \right)^\alpha. \tag{41}$$

If $d = 1$ and $\mu = 1$, then the multivariate Sibuya distribution becomes the ordinary Sibuya distribution with exponent α . For $d \geq 2$, the marginals Υ_n of a multivariate Sibuya vector $\Upsilon = (\Upsilon_1, \dots, \Upsilon_d)$ have the p.g.fl. given by

$$\mathbf{E} z_n^{\Upsilon_n} = 1 - \mu_n (1 - z_n)^\alpha, \quad n = 1, \dots, d.$$

Thus Υ_n takes value 0 with probability $1 - \mu_n$ but, conditional on being non-zero, it is Sib(α)-distributed.

Example 27 (D α S random vectors). Let $\xi = (\xi_1, \dots, \xi_d)$ be a D α S vector. By Theorem 24, it can be represented as a sum of multivariate Sibuya Sib(α, μ_i) vectors, where μ_i are chosen from a finite Poisson point process on Δ_d with some intensity measure σ .

Example 28 (Stationary D α S processes). As in Example 7, let $X = \mathbb{R}^d$ with ν being proportional to the Lebesgue measure and μ being the uniform distribution on the unit ball centred at the origin. Then the corresponding D α S process is a cluster process that can be described by the following procedure. First, the Boolean model of unit balls in \mathbb{R}^d is generated with the centres following the Poisson point process with intensity measure ν ; see [30]. Then a Sib(α) number of points is thrown into each such ball uniformly and independently from the other balls. The set of thus generated points is a realisation of the D α S process.

In a more general model, the uniform distribution on the unit ball can be replaced by any probability distribution kernel $P(dy, x)$, $x \in X$. A typical realisation of such a process in \mathbb{R}^2 with ν being proportional to the Lebesgue measure on $[0, 1]^2$ and $P(dy, x)$ being the Gaussian distribution centred at x with i.i.d. components of a certain variance s^2 is presented in Figure 1. The avoidance probabilities of the obtained point process are given by

$$\mathbf{P}\{\Phi(B) = 0\} = \exp\left\{-\int_{\mathbb{R}^d} (\mu_0(s^{-1}(B+x)))^\alpha dx\right\},$$

where μ_0 is the standard Gaussian measure in \mathbb{R}^2 .

Figure 1 shows that realisations of such a process are highly irregular. Many clusters with small numbers of points appear alongside huge clusters, so that the resulting point process has an infinite intensity measure. In view of this, D α S point processes can help to model point patterns with point counts exhibiting heavy-tail behaviour.

5. Regular and singular D α S processes

We have shown in the proof of Theorem 15 that a D α S point process is necessarily infinitely divisible. It is known (see, e.g., [6], Theorem 10.2.V) that the p.g.fl. of an infinitely divisible point process admits the following representation

$$G_\Phi[u] = \exp\left\{\int_{\mathcal{N}_0} [e^{(\log u, \varphi)} - 1] \mathbf{Q}(d\varphi)\right\}, \tag{42}$$

where \mathcal{N}_0 is the space of locally finite non-empty point configurations on X and \mathbf{Q} is a locally finite measure on it satisfying

$$\mathbf{Q}(\{\varphi \in \mathcal{N}_0 : \varphi(B) > 0\}) < \infty \quad \text{for all } B \in \mathcal{B}_0(X). \tag{43}$$

This measure \mathbf{Q} is called the *KLM measure* (*canonical measure* in the terminology of [19]) and it is uniquely defined for a given infinitely divisible point process Φ . Such point process Φ is called *regular* if its KLM measure is supported by the set $\{\varphi \in \mathcal{N}_0 : \varphi(X) < \infty\}$; otherwise it is called *singular*.

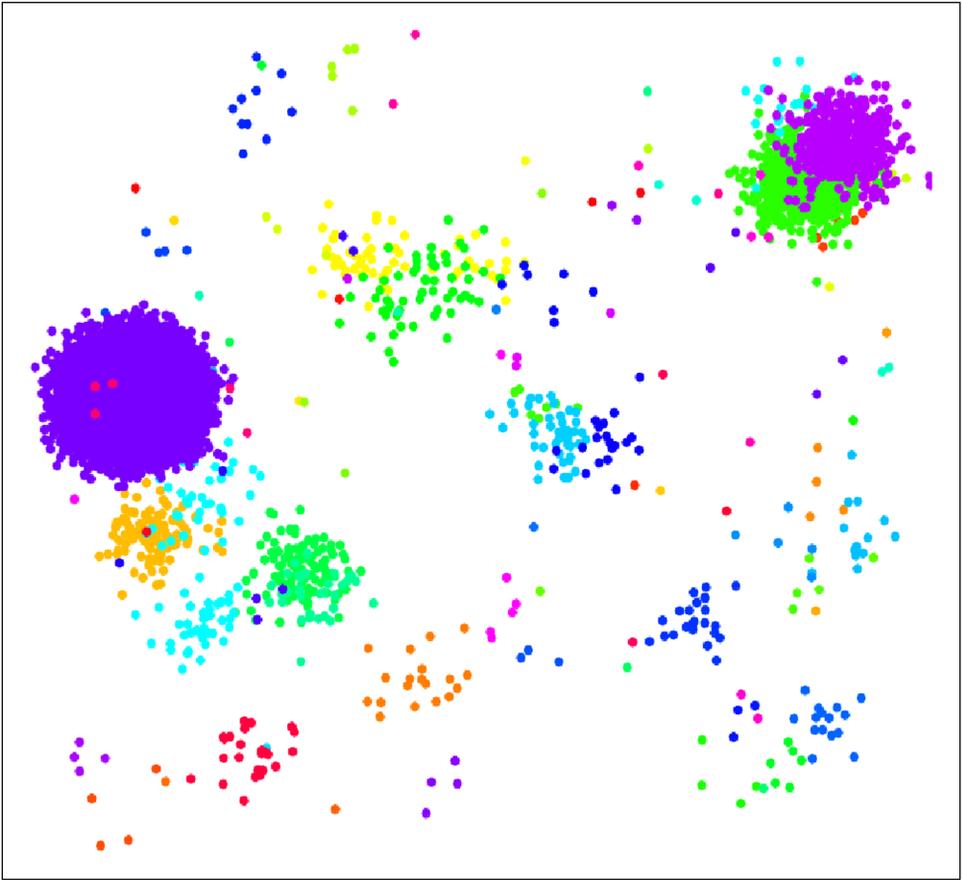


Figure 1. A realisation of a D α S stationary process with the Gaussian kernel. Different shades of gray correspond to different clusters.

It is easy to see that the expression (30) for the p.g.fl. of D α S process combined with (25) has the form (42) with

$$\mathbf{Q}(\cdot) = \int_{\mathbb{M} \setminus \{0\}} \mathbf{P}_\mu(\cdot) \Lambda(d\mu), \quad (44)$$

where \mathbf{P}_μ is the distribution of a Poisson point process driven by intensity measure μ . Moreover, the requirement (43) is exactly the property (12) of the Lévy measure with $h = \mathbb{1}_B$. The Sibuya cluster representation of D α S processes from Section 4 arises from (44) by integrating out the radial component in the polar decomposition $\mathbb{M}_1 \times \mathbb{R}_+$, where Λ is concentrated in the regular case.

The following decomposition result is inherited from the corresponding decomposition known for infinitely divisible processes; see, for example, [6], Theorem 10.2.VII. Let $\mathbb{M}_f = \{\mu \in \mathbb{M} \setminus \{0\} : \mu(X) < \infty\}$ and $\mathbb{M}_\infty = \{\mu \in \mathbb{M} : \mu(X) = \infty\}$.

Theorem 29. *A D α S point process with Lévy measure Λ and spectral measure σ can be represented as a superposition of two independent infinitely divisible processes: a regular and a singular one. The regular process is a cluster process having a p.g.fl. given by (40) with spectral measure $\sigma|_{\mathbb{M}_1} = \sigma(\cdot \mathbb{1}_{\mathbb{M}_1})$ and the singular process is a cluster process with a p.g.fl. given by (30) corresponding to Lévy measure $\Lambda|_{\mathbb{M}_\infty}$ (or by (34) with spectral measure $\sigma|_{\mathbb{S} \setminus \mathbb{M}_1}$).*

Proof. Represent Λ as the sum $\Lambda|_{\mathbb{M}_f} + \Lambda|_{\mathbb{M}_\infty}$ of two orthogonal measures: Λ restricted on the set \mathbb{M}_f and on \mathbb{M}_∞ . Then \mathbf{Q} also decomposes into two orthogonal measures: the one concentrated on finite configurations and the one concentrated on infinite point configurations (because the corresponding Poisson process Π_μ is finite or infinite with probability 1, correspondingly). Finally, $G_\Phi[u]$ in (42) separates into the product of p.g.fl.'s corresponding to the cluster process studied in the previous section and the second one with p.g.fl. (30) with Λ replaced by $\Lambda|_{\mathbb{M}_\infty}$. \square

Recall that a measure μ is called *diffuse* if $\mu(\{x\}) = 0$ for any $x \in X$. Representation (44) of the KLM measure immediately gives rise to the following results analogous to the properties of infinitely divisible processes.

Theorem 30. *A D α S point process Φ with Lévy measure Λ and spectral measure σ is*

- (i) *Simple if and only if Λ (resp., σ) is supported by diffuse measures, cf. [19], Proposition 2.2.9.*
- (ii) *Independently scattered if and only if σ is supported by the set $\{\delta_x : x \in X\} \subset \mathbb{S}$ of Dirac measures, cf. [19], Proposition 2.2.13.*
- (iii) *The distribution \mathbf{P}_1 of a D α S process Φ_1 with spectral measure σ_1 is absolutely continuous with respect to the distribution \mathbf{P}_2 of another D α S process Φ_2 with spectral measure σ_2 and the same α if and only if there exists a measurable set $A \subseteq \mathbb{M}_\infty \cap \mathbb{S} = \mathbb{S}_\infty$ such that $\sigma_1|_{\mathbb{S}_\infty} = \sigma_2|_{A \cap \mathbb{S}_\infty}$ and $\sigma_1|_{\mathbb{M}_1} \ll \sigma_2|_{\mathbb{M}_1}$.*

If X is compact, then all D α S processes are regular. In the settings of Example 32, Φ is singular if and only if μ is infinite.

Consider the practically important case of stationary D α S processes.

Theorem 31. *A stationary D α S process Φ in \mathbb{R}^d with spectral measure σ is:*

- (i) *Mixing if and only if*

$$\sigma\{\mu : \mu(B+x) \not\rightarrow 0 \text{ as } \|x\| \rightarrow \infty \text{ for some } B \in \mathcal{B}_0\} = 0.$$

- (ii) *Ergodic (or weak mixing) if and only if*

$$n^{-d} \int_{[-n/2, n/2]^d} (1 - e^{-\mu(B+x)}) dx \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

for all $B \in \mathcal{B}_0$ and all μ from the support of σ .

Regular stationary D α S processes are mixing and ergodic.

Proof. According to [6], Proposition 12.4.V, Φ is mixing if and only if for all $A, B \in \mathcal{B}_0$

$$\mathbf{Q}\{\varphi \in \mathcal{N}_0 : \varphi(A) > 0 \text{ and } \varphi(B + x) > 0\} \rightarrow 0 \quad \text{as } \|x\| \rightarrow \infty.$$

In view of (44), choosing sufficiently large $\|x\|$ such that $A \cap (B + x) = \emptyset$, this condition writes

$$\int_{\mathbb{M} \setminus \{0\}} (1 - e^{-\mu(A)})(1 - e^{-\mu(B+x)}) \Lambda(d\mu) \rightarrow 0.$$

By (12) and the dominated convergence theorem, Φ is mixing if and only if $\mu(B + x) \rightarrow 0$ as $\|x\| \rightarrow \infty$ for all $B \in \mathcal{B}_0$ and all μ from the support of the spectral measure σ . This is clearly the case if σ is supported by finite measures, that is, for regular D α S processes. Similarly, the ergodicity condition follows from [6], Proposition 12.4.V. \square

Example 32 (Regular stationary D α S processes on \mathbb{R}^d). Consider a St α S random measure ζ from Example 8. Using the notation from this example, the corresponding D α S process Φ has the p.g.fl.

$$G_\Phi[u] = \exp \left\{ - \int_{\mathbb{M}_1^0} \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}} (1 - u(x + y)) \mu(dx) \right)^\alpha dy \sigma^0(d\mu) \right\},$$

where \mathbb{M}_1^0 is the set of ‘centred’ probability measures and σ^0 is the measure on it arising from the Haar factorisation of the spectral measure $\sigma = \ell \times \sigma^0$.

The avoidance probabilities for Φ are given by

$$\mathbf{P}\{\Phi(B) = 0\} = \exp \left\{ - \int_{\mathbb{M}_1^0} \int_{\mathbb{R}^d} (\mu(B - y))^\alpha dy \sigma^0(d\mu) \right\}.$$

The simplest case is when σ^0 is concentrated on a single measure μ . We then obtain a D α S process corresponding to the St α S process from Examples 7 and 28 with ν being the Lebesgue measure ℓ .

6. Discrete stability for semigroups

Let (S, \oplus) be an Abelian semigroup with binary operation \oplus and the neutral element \mathbf{e} . We require that S possesses a (Hamel) basis $X \subset S \setminus \{\mathbf{e}\}$ such that each element $s \in S \setminus \{\mathbf{e}\}$ is *uniquely* represented by a finite linear combination with positive integer coefficients

$$s = n_1 x_1 \oplus \cdots \oplus n_k x_k, \tag{45}$$

where $\{x_1, \dots, x_k\} \subseteq X$. Here and below, ns stands for the sum $s \oplus s \oplus \dots \oplus s$ of $n \geq 0$ identical elements $s \in S$ with convention $0s = \mathbf{e}$. Equip X with the topology that makes it a locally compact second countable space; for instance the discrete topology if X is at most countable. Note that $\{x_1, \dots, x_k\}$ with multiplicities $\{n_1, \dots, n_k\}$ can be regarded as a finite counting measure on X , so that (45) establishes a bijection \mathcal{I} between S and the family of finite counting measures on X . Define a σ -algebra on S as the inverse image of the σ -algebra on the space of counting measures under the map \mathcal{I} . An S -valued random element ξ is defined with respect to the constructed σ -algebra. Then the corresponding counting measure $\Phi(\xi)$ becomes a point process on X .

The sum of random elements in S is defined directly by the \oplus -addition. The multiplication $t \circ \xi$ of a random element ξ by $t \in (0, 1]$ is defined as the random element that has the basis decomposition $t \circ \Phi(\xi)$, the latter obtained by independent thinning of $\Phi(\xi)$. By Theorem 13 and the uniqueness of the representation, the corresponding operation satisfies $t_1 \circ (t_2 \circ \xi) \stackrel{\mathcal{D}}{=} (t_1 t_2) \circ \xi$ for $t_1, t_2 \in (0, 1]$ and

$$t \circ (\xi \oplus \xi') \stackrel{\mathcal{D}}{=} (t \circ \xi) \oplus (t \circ \xi') \tag{46}$$

for independent ξ and ξ' . Note also that $t \circ \mathbf{e} = \mathbf{e}$ for all $t \in (0, 1]$.

The imposed distributivity property (46) is essential to define the stability property on a semigroup (even for deterministic ξ and ξ'), and so it also rules out the possibility of extending the scaling operation to the cases where the decomposition (45) is not unique or the coefficients are allowed to be negative. For instance, assume that $a \oplus b = \mathbf{e}$ for some non-trivial $a, b \in S$, which is the case if S is a group. Then $t \circ (a \oplus b) = \mathbf{e}$, while $t \circ a \oplus t \circ b$ is the sum of two non-trivial independent random elements and so it cannot be \mathbf{e} almost surely. This observation explains why the classical notion of discrete stability cannot be extended to the class of all integer-valued random variables.

Similar difficulties arise when defining discrete stability for the max-scheme being an example of an idempotent semigroup. Indeed, if S is the family of non-negative integers with maximum as the semigroup operation, then $a \oplus a = a$. After scaling by $t \in (0, 1)$ we obtain that $t \circ a$ is the maximum of two independent random elements distributed as $t \circ a$, which is impossible.

Definition 33. An S -valued random element ξ is called discrete stable with exponent α (notation: $D\alpha S$) if

$$t^{1/\alpha} \circ \xi' \oplus (1 - t)^{1/\alpha} \circ \xi'' \stackrel{\mathcal{D}}{=} \xi \tag{47}$$

for any $t \in [0, 1]$, where ξ', ξ'' and ξ are independent identically distributed.

Thus, ξ is $D\alpha S$ if and only if the corresponding point process $\Phi(\xi)$ is $D\alpha S$; see Definition 14. Thus, the distributions of $D\alpha S$ random elements in S are characterised by Theorem 15. Since the basis decomposition (45) is finite, $\Phi(\xi)$ is a finite point process, meaning that the spectral measure σ is finite and supported by the set \mathbb{M}_1 of probability measures on X ; see Corollary 25.

A random element ν in S is said to have Sibuya distribution if $\Phi(\nu)$ is a Sibuya point process on X . Theorem 24 and the LePage representation from Corollary 17 immediately imply the following result.

Theorem 34. *A random element ξ in S is $\text{D}\alpha\text{S}$ if and only if ξ can be represented as the sum of a Poisson number of i.i.d. Sibuya random elements. Alternatively, ξ is $\text{D}\alpha\text{S}$ if and only if it can be represented as an a.s.-finite sum $\sum_{i \geq 1} b \gamma_k^{-1/\alpha} \circ \varepsilon_i$, where γ_k , ε_k and b are defined in (23).*

When the basis X is finite, (45) establishes a homomorphism between S and \mathbb{Z}_+^d , that is, the family of d -dimensional vectors with non-negative integer components and addition as the semigroup operation. Thus, all random elements in semigroups with a finite basis can be treated as random vectors in \mathbb{Z}_+^d ; see Example 27.

Now consider the case of a discrete semigroup with an infinite countable basis X .

Example 35 (Natural numbers with multiplication). Consider the semigroup of natural numbers with the multiplication operation. Its basis X is the family \mathcal{P} of prime numbers. A random natural number ξ corresponds to a point process on \mathcal{P} ; for example, $\xi = 1$ if this point process is empty. Otherwise ξ is the multiple of prime numbers raised to their multiplicities as

$$\xi_\Phi = \prod_{p \in \mathcal{P}} p^{\Phi(p)}.$$

Then $t \circ \xi$ is obtained by independently taking out the prime divisors of ξ with probability $1 - t$. Consequently, a class of *discrete multiplicatively stable distributions* can be defined as the distributions of G -valued random variables ξ satisfying

$$(t^{1/\alpha} \circ \xi') \cdot ((1 - t)^{1/\alpha} \circ \xi'') = \xi,$$

where, as before, ξ' , ξ'' are independent copies of ξ . Then $\xi = \xi_\Phi$ is multiplicatively α -stable if the corresponding process Φ on \mathcal{P} has p.g.fl. given by (34).

Extend the domain of the p.g.fl. to the class of monotone pointwise limits of the functions u such that $1 - u \in \text{BM}(\mathcal{P})$, allowing for infinite values of the p.g.fl., and consider $h_s : \mathcal{P} \mapsto (0, 1)$ such that $h(p) = p^s$ for some s . Then

$$\mathbf{E} \xi_\Phi^s = \mathbf{E} \prod_{p \in \mathcal{P}} p^{s\Phi(p)} = G_\Phi[h_s] = \exp \left\{ - \int_{\mathbb{M}_1} \left(1 - \sum_{p \in \mathcal{P}} p^s \mu_p \right)^\alpha \sigma(d\mu) \right\},$$

where \mathbb{M}_1 is the set of probability distributions on \mathcal{P} . The expression above is finite at least for all $s < 0$ and can be used to numerically evaluate the distribution of ξ_Φ by the inverse Mellin transform.

Now consider the case of independently scattered Φ , where σ is concentrated on degenerated distributions (see Theorem 30(ii)) and thus can be identified with a sequence $\{\sigma_p, p \in \mathcal{P}\}$. Then Φ is a sequence indexed by $p \in \mathcal{P}$ of doubly stochastic Poisson random variables v_{ζ_p} with parameters ζ_p , which are independent positive $\text{St}\alpha\text{S}$ random variables with the Laplace transforms $\mathbf{E} e^{-h_p \zeta_p} = \exp\{-h_p^\alpha \sigma_p\}$. Now the distribution of ξ_Φ can be explicitly characterised: if $n = \prod_{p \in \mathcal{P}} p^{k_p}$, then

$$\mathbf{P}\{\xi_\Phi = n\} = \mathbf{P}\{v_{\zeta_p} = k_p \ \forall p \in \mathcal{P}\} = \prod_{p \in \mathcal{P}} \mathbf{E} \left[\frac{\zeta_p^{k_p}}{k_p!} e^{-\zeta_p} \right].$$

In particular, for any $q \in \mathcal{P}$ we have that

$$\mathbf{P}\{\xi_\Phi = q\} = \prod_{p \neq q} \mathbf{P}\{v_{\zeta_p} = 0\} \mathbf{E}[\zeta_q e^{-\zeta_q}] = e^{-\sigma(\mathcal{P}) + \sigma_q} \mathbf{E}[\zeta_q e^{-\zeta_q}].$$

If $\alpha = 1/2$, then the density of ζ_p is given by

$$f_{\zeta_p}(t) = \frac{\sigma_p}{2\sqrt{\pi t^{3/2}}} \exp\{-\sigma_p/(4t)\}, \quad t \geq 0,$$

leading to

$$\mathbf{P}\{\xi_\Phi = q\} = e^{-\sigma(\mathcal{P}) + \sigma_q} \frac{1}{2} \sigma_q e^{-\sigma_q} = \frac{1}{2} \sigma_q e^{-\sigma(\mathcal{P})}, \quad q \in \mathcal{P}.$$

The above construction can be extended for an *arithmetical semigroup* generated by a countable subset $\mathcal{P} = \{p_1, p_2, \dots\}$ called the *generalised primes* – for example, Beurling’s generalised prime numbers with the so-called Delone property, which implies the uniqueness of the factorisation; see [2,9].

Example 36. Consider the family S of all finite Abelian groups with the semigroup operation being the direct product. The main theorem on Abelian groups states that each such group can be uniquely decomposed into the direct product of cyclic groups with orders being prime numbers and their natural powers; see, for example, [12], Theorem 7.2. Thus, the basis X is the family of prime numbers and all powers of prime numbers. The multiplication of a cyclic group of order p by real number $t \in (0, 1]$ is defined as a random group where each factor from its decomposition is eliminated with probability $1 - t$. A spectral measure defined on the set \mathbb{M}_1 of probability distributions on X then determines the distribution of a random stable finite group.

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