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Explicit identities for Lévy processes associated to symmetric stable processes

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In this paper, we introduce a new class of Lévy processes which we call hypergeometric-stable Lévy processes because they are obtained from symmetric stable processes through several transformations, where the Gauss hypergeometric function plays an essential role. We characterize the Lévy measure of this class and obtain several useful properties such as the Wiener–Hopf factorization, the characteristic exponent and some associated exit problems.

Keywords: first exit time; first hitting time; Lamperti representation; positive self-similar Markov processes; symmetric stable Lévy processes

1. Introduction and preliminaries

Let $Z = (Z_t = \{Z_t^{(1)}, \ldots, Z_t^{(d)}\}, t \geq 0)$ be a symmetric stable Lévy process of index $\alpha \in (0, 2)$ in $\mathbb{R}^d$ $(d \geq 1)$, that is, a process with stationary independent increments, whose sample paths are càdlàg and

$$\mathbb{E}_0(\exp\{\mathrm{i}\langle\lambda, Z_t\rangle\}) = \exp\{-t\|\lambda\|^\alpha\}$$

for all $t \geq 0$ and $\lambda \in \mathbb{R}^d$. Here, $\mathbb{P}_z$ denotes the law of the process Z started from $z \in \mathbb{R}^d$, $\|\cdot\|$ the norm in $\mathbb{R}^d$ and $\langle\cdot,\cdot\rangle$ the Euclidean inner product.

The process $Z^{(k)} = (Z_t^{(k)}, t \geq 0)$ will be called the *kth coordinate process of* Z. Of course, $Z^{(k)}$ is a real symmetric stable process whose characteristic exponent is given by

$$\mathbb{E}_0\big(\exp\big\{\mathrm{i}\theta Z_t^{(k)}\big\}\big) = \exp\{-t|\theta|^\alpha\}$$

for all $t \geq 0$ and $\theta \in \mathbb{R}$.

According to Bertoin [2], Chapter I, the process Z is transient for $\alpha < d$, that is,

$$\lim_{t\to\infty} \|Z_t\| = \infty \qquad \text{a.s.}$$

and it oscillates otherwise, that is, for $\alpha \in [1, 2)$ and $d = 1$, we have

$$\limsup_{t\to\infty} Z_t = \infty \quad \text{and} \quad \liminf_{t\to\infty} Z_t = -\infty \qquad \text{a.s.}$$

When $d \geq 2$, we have that single points are polar, that is, for every $x, z \in \mathbb{R}^d$,

$$\mathbb{P}_x(Z_t = z \text{ for some } t > 0) = 0.$$

In the one-dimensional case, points are polar for $\alpha \in (0, 1]$ and when $\alpha \in (1, 2)$, the process Z makes infinitely many jumps across a point, say z, before the first hitting time of z (see, e.g., [2], Proposition VIII.8).

One of the main properties of the process Z is that it satisfies the scaling property with index α, that is, for every $b > 0$,

$$\text{the law of } (bZ_{b^{-\alpha}t}, t \geq 0) \text{ under } \mathbb{P}_x \text{ is } \mathbb{P}_{bx}. \tag{1.1}$$

This implies that the radial process $R = (R_t, t \geq 0)$ defined by $R_t = \|Z_t\|$ satisfies the same scaling property (1.1). Since Z is isotropic, its radial part R is a strong Markov process (see [10]). When $d \geq 2$, the radial process R hits points if and only if $Z^{(1)}$ hits points, that is, when $\alpha \in (1, 2)$ (see, e.g., [10], Theorem 3.1). Finally, we note that when points are polar for Z, the radial process R will never hit the point 0.

In what follows, we will assume that $\alpha \leq d$, so the radial process R will be a positive self-similar Markov process (pssMp) with index α and infinite lifetime. A natural question arises: can we characterize the Lévy process ξ associated to the pssMp $(R_t, t \geq 0)$ via the Lamperti transformation?

We briefly recall the main features of the Lamperti transformation between pssMp and Lévy processes. A positive self-similar Markov processes $(X, \mathbb{Q}_x)$, $x > 0$, is a strong Markov processes with càdlàg paths, which fulfills a scaling property. Well-known examples of this kind of process are Bessel processes, stable subordinators, stable processes conditioned to stay positive, etc.

According to Lamperti [9], any pssMp up to its first hitting time of 0 may be expressed as the exponential of a Lévy process, time changed by the inverse of its exponential functional. More formally, let $(X, \mathbb{Q}_x)$ be a pssMp with index $\beta > 0$, starting from $x > 0$, set

$$S = \inf\{t > 0 : X_t = 0\}$$

and write the canonical process X in the following form:

$$X_t = x \exp\{\xi_{\tau(tx^{-\beta})}\}, \qquad 0 \leq t < S, \tag{1.2}$$

where, for $t < S$,

$$\tau(t) = \inf\left\{s \geq 0 : \int_0^s \exp\{\beta \xi_u\} \,\mathrm{d}u \geq t\right\}.$$

Then, under $\mathbb{Q}_x$, $\xi = (\xi_t, t \geq 0)$ is a Lévy process started from 0 whose law does not depend on $x > 0$ and such that:

(i) if $\mathbb{Q}_x(S = +\infty) = 1$, then ξ has an infinite lifetime and $\limsup_{t \to +\infty} \xi_t = +\infty$, $\mathbb{P}_x$-a.s.;
(ii) if $\mathbb{Q}_x(S < +\infty, X(S-) = 0) = 1$, then ξ has an infinite lifetime and $\lim_{t \to \infty} \xi_t = -\infty$, $\mathbb{P}_x$-a.s.;

(iii) if $\mathbb{Q}_x(S < +\infty, X(S-) > 0) = 1$, then ξ is killed at an independent exponentially distributed random time with parameter $\lambda > 0$.

As mentioned in [9], the probabilities $\mathbb{Q}_x(S = +\infty)$, $\mathbb{Q}_x(S < +\infty, X(S-) = 0)$ and $\mathbb{Q}_x(S < +\infty, X(S-) > 0)$ are 0 or 1 independently of x, so the three classes presented above are exhaustive. Moreover, for any $t < \int_0^\infty \exp\{\beta\xi_s\}\,\mathrm{d}s$,

$$\tau(t) = \int_0^{x^\beta t} \frac{\mathrm{d}s}{(X_s)^\beta}, \qquad \mathbb{Q}_x\text{-a.s.} \tag{1.3}$$

Therefore, (1.2) is invertible and yields a one-to-one relation between the class of pssMp's killed at time S and the class of Lévy processes.

Another important result of Lamperti [9] provides the explicit form of the generator of any pssMp $(X, \mathbb{Q}_y)$ in terms of its underlying Lévy process. Let ξ be the underlying Lévy process associated to $(X, \mathbb{Q}_y)$ via (1.2) and denote by $\mathcal{L}$ and $\mathcal{M}$ their respective infinitesimal generators. Let $\mathcal{D}_\mathcal{L}$ be the domain of the generator $\mathcal{L}$ and recall that it contains all the functions with continuous second derivatives on $[-\infty, \infty]$, and that if f is such a function, then $\mathcal{L}$ acts as follows for $x \in \mathbb{R}$, where $\mu \in \mathbb{R}$ and $\sigma > 0$:

$$\mathcal{L}f(x) = \mu f'(x) + \frac{\sigma^2}{2} f''(x) + \int_\mathbb{R} \big(f(x+y) - f(x) - f'(x)h(y)\big)\Pi(\mathrm{d}y) - bf(x). \tag{1.4}$$

The measure $\Pi(\mathrm{d}x)$ is the so-called Lévy measure of ξ, which satisfies

$$\Pi(\{0\}) = 0 \quad \text{and} \quad \int_\mathbb{R} (1 \wedge |x|^2)\Pi(\mathrm{d}x) < \infty.$$

The function $h(\cdot)$ is any bounded Borel function such that $h(y) \sim y$ as $y \to 0$. The positive constant b represents the killing rate of ξ ($b = 0$ if ξ has infinite lifetime). It is important to note that *in* (1.4)*, the choice of the function h is arbitrary and the coefficient μ is the only one which depends on this choice*.

Lamperti establishes the following result in [9].

Theorem 1. *If g is such that g, yg' and $y^2 g''$ are continuous on* $[0, \infty]$, *then they belong to the domain $\mathcal{D}_\mathcal{M}$ of the infinitesimal generator of* $(X, \mathbb{Q}_y)$, *which acts as follows for* $a > 0$:

$$\begin{aligned}\mathcal{M}g(a) &= \mu a^{1-\beta} g'(a) + \frac{\sigma^2}{2} a^{2-\beta} g''(a) - ba^{-\beta} g(a) \\ &\quad + a^{-\beta} \int_0^\infty \big(g(au) - g(a) - ag'(a)h(\log u)\big)G(\mathrm{d}u),\end{aligned}$$

where $G(\mathrm{d}u) = \Pi(\mathrm{d}u) \circ \log u$ *for* $u > 0$. *This expression determines the law of the process* $(X_t, 0 \le t \le T)$ *under* $\mathbb{Q}_y$.

Previous work on this subject appears in [5], where the authors study the radial part of a Cauchy process $C = (C_t, t \ge 0)$ (i.e., $\alpha = d = 1$) and obtain the infinitesimal generator of its associated Lévy process $\xi = (\xi_t, t \geq 0)$ via the Lamperti transformation. More precisely, the infinitesimal generator of ξ is given as

$$\mathcal{L}g(\xi) = \frac{1}{\pi} \int \frac{\cosh \eta}{(\sinh \eta)^2} \big(g(\xi + \eta) - g(\xi) - \eta g'\big(\xi \mathbb{1}_{|\eta| \leq 1}\big)\big) \, d\eta$$

and its characteristic exponent satisfies

$$\mathbb{E}(\exp\{i\lambda \xi_t\}) = e^{-i\lambda \tanh \pi\lambda/2}.$$

As we will see in Sections 2 and 5, this example is a particular case of the results obtained in this paper by very different methods. As expected, the formulae obtained in both papers coincide for $\alpha = d = 1$.

It is important to point out that in [5], it is announced that the authors intend to continue this line of research by studying the case of the norm of a multidimensional Cauchy process, but, to the best of our knowledge, this has not be done.

This paper is organized as follows. In Section 2, we compute the infinitesimal generator of the radial process R and, using Theorem 1, we obtain the characteristics of its associated Lévy process ξ. The Lévy measure obtained has a rather complicated form since it is expressed in terms of the Gauss hypergeometric function ${}_2\mathcal{F}_1$. When $d = 1$, we show that the process ξ can be expressed as the sum of a Lamperti stable process (see [4] for a proper definition) and an independent Poisson process.

In Section 3, we study one-sided exit problems of the Lévy process ξ, using well-known results of Blumenthal *et al.* [3] for the symmetric α-stable process Z. When $\alpha < d$, a straightforward computation allows us to deduce the law of the random variable $\underline{\xi}_\infty = \inf_{t \geq 0} \xi_t$.

In Section 4, we study the special case $1 < \alpha < d$. Using the work of Port [11] on the radial processes of Z, we compute the probability that the Lévy process ξ hits points.

Finally, in Section 5, we obtain the Wiener–Hopf factorization of ξ and deduce the explicit form of the characteristic exponent. Concluding remarks in Section 6 show how to obtain n-tuple laws for ξ and R following [8].

2. The underlying Lévy process of *R*

In this section, we compute the generator of the radial process R and the characteristics of the underlying Lévy process ξ in the Lamperti representation (1.2) of the latter.

To this end, we will use the expression of Z as a subordinated Brownian motion. More precisely, if $B = (B_t, t \geq 0)$ is a d-dimensional Brownian motion started from $x \in \mathbb{R}^d$ and we let $\sigma = (\sigma_t, t \geq 0)$ be an independent stable subordinator with index $\alpha/2$ initiated from 0, then the process $(B_{2\sigma_t}, t \geq 0)$ is a standard symmetric α-stable process.

Let us define the so-called Pochhammer symbol by

$$(z)_\alpha = \frac{\Gamma(z+\alpha)}{\Gamma(z)} \qquad \text{for } z \in \mathbb{C}$$

and the Gauss hypergeometric function by

$${}_2\mathcal{F}_1(a,b;c;z) = \sum_{k=0}^{\infty} z^k \frac{(a)_k (b)_k}{(c)_k k!} \qquad \text{for } \|z\| < 1,$$

where $a, b, c > 0$.

Theorem 2. *Let* $g : \mathbb{R}_+ \to \mathbb{R}$ *be such that* $g \in C_0^2(\mathbb{R}_+)$. *The infinitesimal generator of* $R = (R_t, t \geq 0)$, *denoted by* M, *acts as follows for* $a > 0$:

$$Mg(a) = a^{-\alpha} \int_0^\infty \big(g(ya) - g(a) - ag'(a)\ell(\log y)\big) \frac{y^{d-1}}{(1+y^2)^{(\alpha+d)/2}} \overline{F}\left(\left(\frac{2y}{1+y^2}\right)^2\right) \mathrm{d}y,$$

where

$$\overline{F}(z) = \frac{2^\alpha \alpha (d/2)_{\alpha/2}}{\Gamma(1-\alpha/2)} {}_2\mathcal{F}_1\big((\alpha+d)/4, (\alpha+d)/4 + 1/2; d/2; z\big) \qquad \textit{for } z \in (-1,1) \tag{2.1}$$

and the function ℓ *is given by*

$$\ell(y) = \frac{y}{1+y^2} \mathrm{e}^{(1-d)y} (1+\mathrm{e}^{2y})^{(\alpha+d)/2-1} \mathbb{1}_{\{|y|<1\}}. \tag{2.2}$$

Remark 1. Following the original notation of Lamperti in [9], the generator can also be written as

$$Mg(a) = a^{-\alpha} \int_0^\infty \left(g(ya) - g(a) - ag'(a) \frac{\log y}{(1+\log^2 y)}\right) \times \frac{(1+\log^2 y)}{\log^2 y} \mathrm{d}G^*(\mathrm{d}y) + a^{1-\alpha} \nu g'(a),$$

where G^* is a finite measure given by

$$G^*(\mathrm{d}y) = \frac{y^{d-1}}{(1+y^2)^{(\alpha+d)/2}} \frac{\log^2 y}{1+\log^2 y} \overline{F}\left(\left(\frac{2y}{1+y^2}\right)^2\right) \mathrm{d}y$$

and

$$\nu = \int_0^\infty \left(\ell(\log y) - \frac{\log y}{(1+\log^2 y)}\right) \frac{1+\log^2 y}{\log y} G^*(\mathrm{d}y).$$

It is easy to prove that the integral defining ν converges.

Proof of Theorem 2. From [13], Theorem 32.1, and the fact that Z can be seen as a subordinated Brownian motion, the infinitesimal generator M of $R = (R_t, t \geq 0)$ is given as

$$Mh = \int_0^\infty (P_s h - h)\rho(\mathrm{d}s),$$

where ρ is the Lévy measure of the stable subordinator 2σ, given by

$$\rho(\mathrm{d}s) = \frac{2^{\alpha/2-1}\alpha}{\Gamma(1-\alpha/2)} s^{-(1+\alpha/2)} \mathbb{1}_{\{s>0\}}\,\mathrm{d}s,$$

P_s is the semigroup of the d-dimensional Bessel process and h is any function in the domain of the infinitesimal generator of $(P_t, t \geq 0)$.

Let g be as in the statement and recall that for $a > 0$, the semigroup for the d-dimensional Bessel process satisfies

$$P_s g(a) = \int_0^\infty \mathrm{d}y\, \frac{g(y)}{s} \left(\frac{y}{a}\right)^{d/2-1} y \exp\left(-\frac{y^2+a^2}{2s}\right) I_{d/2-1}\left(\frac{ay}{s}\right),$$

where $I_{d/2-1}$ is the modified Bessel function of index $d/2-1$ (see, e.g., [12]). Therefore, putting the pieces together, it follows that

$$\begin{aligned} Mg(a) = \frac{2^{\alpha/2-1}\alpha}{\Gamma(1-\alpha/2)} \int_0^\infty \int_0^\infty & y\big(g(y)-g(a)\big)\left(\frac{y}{a}\right)^{d/2-1} \\ & \times \frac{1}{s^{2+\alpha/2}} \exp\left(-\frac{a^2+y^2}{2s}\right) I_{d/2-1}\left(\frac{ay}{s}\right) \mathrm{d}y\,\mathrm{d}s. \end{aligned} \tag{2.3}$$

Now, recall the identity for the modified Bessel function $I_{d/2-1}$,

$$I_{d/2-1}(x) = \sum_{k=0}^\infty \frac{(x/2)^{2k+d/2-1}}{\Gamma(d/2+k)k!},$$

and note that, for $a \neq y$,

$$\begin{aligned} &\int_0^\infty \frac{\mathrm{d}s}{s^{2+\alpha/2}} \exp\left(-\frac{a^2+y^2}{2s}\right) I_{d/2-1}\left(\frac{ay}{s}\right) \\ &= \sum_{k=0}^\infty \int_0^\infty \mathrm{d}s \left(\frac{ay}{2s}\right)^{2k+d/2-1} \frac{s^{-2-\alpha/2}}{\Gamma(d/2+k)k!} \exp\left(-\frac{a^2+y^2}{2s}\right) \\ &= \sum_{k=0}^\infty \frac{1}{\Gamma(d/2+k)k!} \left(\frac{ay}{a^2+y^2}\right)^{2k+(\alpha+d)/2} \left(\frac{2}{ay}\right)^{1+\alpha/2} \int_0^\infty \mathrm{d}u\, u^{2k+(\alpha+d)/2-1} \mathrm{e}^{-u} \\ &= 2^{1+\alpha/2} \frac{(ay)^{d/2-1}}{(a^2+y^2)^{(\alpha+d)/2}} \sum_{k=0}^\infty \left(\frac{ay}{a^2+y^2}\right)^{2k} \frac{\Gamma(2k+(\alpha+d)/2)}{\Gamma(k+1)\Gamma(d/2+k)}. \end{aligned} \tag{2.4}$$

Next, we consider the property of the Gamma function,

$$\Gamma(2z) = (2\pi)^{-1/2} 2^{2z-1/2} \Gamma(z)\Gamma(z+1/2), \tag{2.5}$$

and deduce that

$$\begin{aligned}\Gamma\big(2k+(\alpha+d)/2\big) &= (2\pi)^{-1/2} 2^{2k+(\alpha+d)/2-1/2} \Gamma\big(k+(\alpha+d)/4\big)\Gamma\big(k+(\alpha+d)/4+1/2\big)\\ &= 2^{2k}\Gamma\big((\alpha+d)/2\big)\big((\alpha+d)/4\big)_k\big((\alpha+d)/4+1/2\big)_k.\end{aligned}$$

Therefore, using the above identity, we see that (2.4) is equal to

$$\frac{2^{\alpha/2+1}(ay)^{d/2-1}}{(a^2+y^2)^{(\alpha+d)/2}} \frac{\Gamma((\alpha+d)/2)}{\Gamma(d/2)} \sum_{k=0}^{\infty} \left(\left(\frac{2ay}{a^2+y^2}\right)^2\right)^k \frac{((\alpha+d)/4)_k((\alpha+d)/4+1/2)_k}{(d/2)_k k!},$$

where the series above is the Gauss hypergeometric function

$${}_2\mathcal{F}_1\left((\alpha+d)/4, (\alpha+d)/4+1/2; d/2; \left(\frac{2ay}{a^2+y^2}\right)^2\right).$$

We note that we cannot use Fubini's theorem in (2.3) because the expression inside the integral with respect to the product measure is not integrable. This is easily seen by observing that

$$\left|{}_2\mathcal{F}_1\left((\alpha+d)/4, (\alpha+d)/4+1/2; d/2; \left(\frac{2ay}{a^2+y^2}\right)^2\right)\right| \sim |y-a|^{-(\alpha+1)} \qquad \text{as } y \to a.$$

So, let us instead consider $\varepsilon, a \geq 0$ and introduce the sets

$$A_\varepsilon(a) = \left(0, \frac{a^2}{(a+\varepsilon)}\right) \cup (a+\varepsilon, \infty), \qquad C_{a,\varepsilon} = \left(0, \frac{a}{(a+\varepsilon)}\right) \cup \left(\frac{a+\varepsilon}{a}, \infty\right).$$

We then study the integral

$$\int_0^\infty \int_{A_\varepsilon(a)} y\big(g(y)-g(a)\big)\left(\frac{y}{a}\right)^{d/2-1} \frac{1}{s^{2+\alpha/2}} \exp\left(-\frac{a^2+y^2}{2s}\right) I_{d/2-1}\left(\frac{ay}{s}\right) \mathrm{d}y\, \mathrm{d}s. \tag{2.6}$$

We would like to use Fubini's theorem in the expression above and, to this end, we now prove the integrability of the integrand in (2.6) with respect to the product measure. For simplicity, we use the notation established in (2.1) and, using Tonelli's theorem and (2.4), we have

$$\begin{aligned}&\int_0^\infty \int_{A_\varepsilon(a)} y|g(y)-g(a)|\left(\frac{y}{a}\right)^{d/2-1} \frac{1}{s^{2+\alpha/2}} \exp\left(-\frac{a^2+y^2}{2s}\right) I_{d/2-1}\left(\frac{ay}{s}\right) \mathrm{d}y\, \mathrm{d}s\\ &\leq 2\|g\|_\infty \int_{A_\varepsilon(a)} \frac{y^{d-1}}{(a^2+y^2)^{(\alpha+d)/2}} \overline{F}\left(\left(\frac{2ay}{a^2+y^2}\right)^2\right) \mathrm{d}y,\end{aligned}$$

which is finite. So, let us now return to (2.6). Applying Fubini's theorem and (2.4), we obtain

$$\begin{aligned}\frac{2^{\alpha/2-1}\alpha}{\Gamma(1-\alpha/2)}&\int_0^\infty\int_{A_\varepsilon(a)} y\big(g(y)-g(a)\big)\left(\frac{y}{a}\right)^{d/2-1}\frac{1}{s^{2+\alpha/2}}\exp\left(-\frac{a^2+y^2}{2s}\right)I_{d/2-1}\left(\frac{ay}{s}\right)\mathrm{d}y\,\mathrm{d}s\\ &=\int_{A_\varepsilon(a)}\big(g(y)-g(a)\big)\frac{y^{d-1}}{(a^2+y^2)^{(\alpha+d)/2}}\overline{F}\left(\left(\frac{2ay}{a^2+y^2}\right)^2\right)\mathrm{d}y \qquad (2.7)\\ &=a^{-\alpha}\int_{C(a,\varepsilon)}\big(g(ay)-g(a)\big)\frac{y^{d-1}}{(1+y^2)^{(\alpha+d)/2}}\overline{F}\left(\left(\frac{2y}{1+y^2}\right)^2\right)\mathrm{d}y.\end{aligned}$$

In order to get the result, we first show that if $B(a,\varepsilon)=C(a,\varepsilon)\cap(1/\mathrm{e},\mathrm{e})$, then

$$\int_{B(a,\varepsilon)}\frac{\log y}{1+\log^2 y}\frac{1}{1+y^2}\overline{F}\left(\left(\frac{2y}{1+y^2}\right)^2\right)\mathrm{d}y=0. \qquad (2.8)$$

To do so, we note that the integral in (2.8) is equal to

$$\begin{aligned}&\int_{1+a^{-1}\varepsilon}^{\mathrm{e}}\frac{\log y}{1+\log^2 y}\frac{1}{1+y^2}\overline{F}\left(\left(\frac{2y}{1+y^2}\right)^2\right)\mathrm{d}y\\ &\quad+\int_{1/\mathrm{e}}^{a/(a+\varepsilon)}\frac{\log y}{1+\log^2 y}\frac{1}{1+y^2}\overline{F}\left(\left(\frac{2y}{1+y^2}\right)^2\right)\mathrm{d}y.\end{aligned}$$

Making the change of variable $y=z^{-1}$ in the first integral above, we get that

$$\begin{aligned}&\int_{1+a^{-1}\varepsilon}^{\mathrm{e}}\frac{\log y}{1+\log^2 y}\frac{1}{1+y^2}\overline{F}\left(\left(\frac{2y}{1+y^2}\right)^2\right)\mathrm{d}y\\ &\quad=-\int_{1/\mathrm{e}}^{a/(a+\varepsilon)}\frac{\log z}{1+\log^2 z}\frac{1}{1+z^2}\overline{F}\left(\left(\frac{2z}{1+z^2}\right)^2\right)\mathrm{d}z\end{aligned}$$

and the identity (2.8) follows. It is easy to see, using (2.4), that

$$\begin{aligned}&\int_{B(a,\varepsilon)}\frac{\log y}{1+\log^2 y}\frac{1}{1+y^2}\overline{F}\left(\left(\frac{2y}{1+y^2}\right)^2\right)\mathrm{d}y\\ &\quad=\frac{a^\alpha 2^{\alpha/2-1}\alpha}{\Gamma(1-\alpha/2)}\int_0^\infty\int_0^\infty y\ell(\log y/a)\left(\frac{y}{a}\right)^{d/2-1}\mathbb{1}_{A_\varepsilon(a)}(y) \qquad (2.9)\\ &\qquad\qquad\times\frac{1}{s^{2+\alpha/2}}\exp\left(-\frac{a^2+y^2}{2s}\right)I_{d/2-1}\left(\frac{ay}{s}\right)\mathrm{d}y\,\mathrm{d}s,\end{aligned}$$

where ℓ is defined as in (2.2). Finally, we add the null term

$$a^{1-\alpha}\int_0^\infty g'(a)\frac{\log y}{1+\log^2 y}\frac{1}{1+y^2}\overline{F}\left(\left(\frac{2y}{1+y^2}\right)^2\right)\mathbb{1}_{B(a,\varepsilon)}(y)\,\mathrm{d}y$$

to the identity (2.7) and, after some calculations using (2.9), we obtain

$$\begin{aligned}\frac{2^{\alpha/2-1}\alpha}{\Gamma(1-\alpha/2)}\int_0^\infty\int_{A_\varepsilon(a)} y\big(g(y)-g(a)-ag'(a)\ell\big(\log(y/a)\big)\big)\left(\frac{y}{a}\right)^{d/2-1}\\ \times\frac{1}{s^{2+\alpha/2}}\exp\left(-\frac{a^2+y^2}{2s}\right)I_{d/2-1}\left(\frac{ay}{s}\right)\mathrm{d}y\,\mathrm{d}s \qquad (2.10)\\ = a^{-\alpha}\int_{C(a,\varepsilon)}\big(g(ya)-g(a)-ag'(a)\ell(\log y)\big)\frac{y^{d-1}}{(1+y^2)^{(\alpha+d)/2}}\overline{F}\left(\left(\frac{2y}{1+y^2}\right)^2\right)\mathrm{d}y.\end{aligned}$$

Using the dominated convergence theorem and (2.10), we can therefore conclude that

$$\begin{aligned}Mg(a)&=\frac{2^{\alpha/2-1}\alpha}{\Gamma(1-\alpha/2)}\int_0^\infty\int_0^\infty y\big(g(y)-g(a)\big)\left(\frac{y}{a}\right)^{d/2-1}\\ &\qquad\times\frac{1}{s^{2+\alpha/2}}\exp\left(-\frac{a^2+y^2}{2s}\right)I_{d/2-1}\left(\frac{ay}{s}\right)\mathrm{d}y\,\mathrm{d}s\\ &=\lim_{\varepsilon\to0}\frac{2^{\alpha/2-1}\alpha}{\Gamma(1-\alpha/2)}\int_0^\infty\int_{A_\varepsilon(a)} y\big(g(y)-g(a)-ag'(a)\ell\big(\log(y/a)\big)\big)\left(\frac{y}{a}\right)^{d/2-1}\\ &\qquad\times\frac{1}{s^{2+\alpha/2}}\exp\left(-\frac{a^2+y^2}{2s}\right)I_{d/2-1}\left(\frac{ay}{s}\right)\mathrm{d}y\,\mathrm{d}s\\ &=a^{-\alpha}\int_0^\infty\big(g(ya)-g(a)-ag'(a)\ell(\log y)\big)\frac{y^{d-1}}{(1+y^2)^{(\alpha+d)/2}}\overline{F}\left(\left(\frac{2y}{1+y^2}\right)^2\right)\mathrm{d}y.\end{aligned}$$ □

Using Lamperti's result (recalled in Theorem 1), we may now give the explicit form of the generator of the Lévy process ξ associated to the pssMp R. We will call this new class of Lévy processes *hypergeometric-stable*.

Corollary 1. *Let ξ be the Lévy process in the Lamperti representation (1.2) of the radial process R. The infinitesimal generator $\mathcal{L}$ of ξ, with domain $\mathcal{D}_\mathcal{L}$, is given, in the polar case, by*

$$\mathcal{L}f(x)=\int_{\mathbb{R}}\big(f(x+y)-f(x)-f'(x)\ell(y)\big)\Pi(\mathrm{d}y) \qquad (2.11)$$

for any $f\in\mathcal{D}_\mathcal{L}$ and $x\in\mathbb{R}$, where

$$\Pi(\mathrm{d}y)=\frac{\mathrm{e}^{dy}}{(1+\mathrm{e}^{2y})^{(\alpha+d)/2}}\overline{F}\left(\frac{4\mathrm{e}^{2y}}{(\mathrm{e}^{2y}+1)^2}\right)\mathrm{d}y.$$

Equivalently, the characteristic exponent of ξ is given by

$$\Psi(\lambda)=\mathrm{i}\lambda\mu+\int_{\mathbb{R}}\big(1-\mathrm{e}^{\mathrm{i}\lambda y}+\mathrm{i}\lambda y\mathbb{1}_{\{|y|<1\}}\big)\Pi(\mathrm{d}y),$$

where

$$\mu = \int_{\mathbb{R}} \big(\ell(y) - y\mathbb{1}_{\{|y|\le 1\}}\big) \frac{e^{dy}}{(1+e^{2y})^{(\alpha+d)/2}} \overline{F}\left(\frac{4e^{2y}}{(e^{2y}+1)^2}\right) dy.$$

We finish this section with a remarkable result on the decomposition of the Lévy measure of the process ξ when the dimension is $d = 1$ and $\alpha \in (0, 1]$ (polar case). Such a decomposition describes the structure of ξ in terms of two independent Lévy processes, each with a different type of path behaviour.

Recall, in this case, that the symmetric stable process Z is of bounded variation and so its radial part R and the associated Lévy process ξ will also be of bounded variation. Hence, the characteristic exponent of ξ is given by

$$\Psi(\lambda) = \int_{\mathbb{R}} (e^{i\lambda y} - 1)\Pi(dy).$$

Proposition 1. *If we assume that* $d = 1$, *then we have*

$$\Psi(\lambda) = \int_{\mathbb{R}} (e^{i\lambda y} - 1)\Pi_1(dy) + \int_{\mathbb{R}} (e^{i\lambda y} - 1)\Pi_2(dy),$$

where Π_1 *is the Lévy measure of a Lamperti Lévy process with characteristics* $(0, 1, \alpha)$ (*see*, *e.g.*, [4]), *that is*,

$$\Pi_1(dy) = \frac{2^{\alpha-1}\alpha(1/2)_{\alpha/2}}{\Gamma(1-\alpha/2)} \left(\frac{e^y}{(e^y-1)^{\alpha+1}} 1_{\{y>0\}} + \frac{e^y}{(1-e^y)^{\alpha+1}} 1_{\{y<0\}}\right) dy$$

and

$$\Pi_2(dy) = \frac{2^{\alpha-1}\alpha(1/2)_{\alpha/2}}{\Gamma(1-\alpha/2)} \frac{e^y}{(e^y+1)^{\alpha+1}} dy$$

is the Lévy measure of a compound Poisson process.

Proof. Let $x \in [0, 1)$. Using identity (2.5) twice, we deduce that

$$\begin{aligned}
&{}_2\mathcal{F}_1\big((\alpha+1)/4, (\alpha+1)/4 + 1/2; 1/2; x^2\big) \\
&\quad = \sum_{k=0}^{\infty} x^{2k} \frac{((\alpha+1)/4)_k((\alpha+1)/4+1/2)_k}{k!(1/2)_k} \\
&\quad = \frac{\Gamma(1/2)}{\Gamma((\alpha+1)/4+1/2)} \frac{2^{1/2-\alpha/2}}{\Gamma((\alpha+1)/4)} \\
&\qquad \times \sum_{k=0}^{\infty} x^{2k} \frac{\Gamma((\alpha+1)/2+2k)}{\Gamma(2k+1)}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{2^{1/2-\alpha/2}\Gamma(1/2)}{(2\pi)^{1/2}2^{1/2-(\alpha+1)/2}\Gamma((\alpha+1)/2)} \\
&\quad\times \frac{1}{2}\left(\sum_0^\infty x^k \frac{\Gamma((\alpha+1)/2+k)}{\Gamma(1+k)} + \sum_0^\infty (-x)^k \frac{\Gamma((\alpha+1)/2+k)}{\Gamma(1+k)}\right) \\
&= \frac{1}{2}\left(\sum_{k=0}^\infty x^k \frac{((\alpha+1)/2)_k}{k!} + \sum_{k=0}^\infty (-x)^k \frac{((\alpha+1)/2)_k}{k!}\right) \\
&= 2^{-1}\bigl((1-x)^{-(\alpha+1)/2} + (1+x)^{-(\alpha+1)/2}\bigr).
\end{aligned}
$$

Now, from the identity above, we deduce that the Lévy measure of the process ξ satisfies

$$
\begin{aligned}
\Pi(\mathrm{d}y) &= \frac{2^{\alpha-1}\alpha(1/2)_{\alpha/2}}{\Gamma(1-\alpha/2)} \frac{\mathrm{e}^y}{(1+\mathrm{e}^{2y})^{(\alpha+1)/2}} \left(\left(1-\frac{2\mathrm{e}^y}{\mathrm{e}^{2y}+1}\right)^{-(\alpha+1)/2} \right.\\
&\qquad\qquad \left. + \left(1+\frac{2\mathrm{e}^y}{\mathrm{e}^{2y}+1}\right)^{-(\alpha+1)/2}\right)\mathrm{d}y \\
&= \frac{2^{\alpha-1}\alpha(1/2)_{\alpha/2}}{\Gamma(1-\alpha/2)} \mathrm{e}^y \left(\frac{1}{|\mathrm{e}^y-1|^{\alpha+1}} + \frac{1}{(\mathrm{e}^y+1)^{\alpha+1}}\right)\mathrm{d}y
\end{aligned}
$$

and the statement follows. □

3. Entrance laws for the process ξ: Intervals

In this section, we will work in the case $\alpha < d$, which is the transient case, and will obtain some explicit identities for the one-sided exit problem.

In what follows, P will be a reference probability measure on $\mathcal{D}$ (the Skorokhod space of $\mathbb{R}$-valued càdlàg paths), under which ξ is the hypergeometric-stable Lévy process described in Corollary 1 starting from 0. For any $y \in \mathbb{R}$, let

$$T_y^+ = \inf\{t \ge 0 : \xi_t > y\} \quad \text{and} \quad T_y^- = \inf\{t \ge 0 : \xi_t < y\},$$

and, for any $x > 0$, let

$$\sigma_x^+ = \inf\{t \ge 0 : R_t > x\} \quad \text{and} \quad \sigma_x^- = \inf\{t \ge 0 : R_t < x\}.$$

Lemma 1. *Fix $-\infty < v < 0 < u < \infty$. Suppose that A is any interval in $[u, \infty)$ and B is any interval in $(-\infty, v]$. Then*

$$P(\xi_{T_u^+} \in A; T_u^+ < \infty) = \mathbb{P}_x(R_{\sigma_{\mathrm{e}^u}^+} \in \mathrm{e}^A; \sigma_{\mathrm{e}^u}^+ < \infty)$$

and

$$P(\xi_{T_v^-} \in B; T_v^- < \infty) = \mathbb{P}_x(R_{\sigma_{\mathrm{e}^v}^-} \in \mathrm{e}^B; \sigma_{\mathrm{e}^v}^- < \infty),$$

where x is such that $\|x\| = 1$.

The proof is a consequence of the Lamperti representation and is left as an exercise. Although somewhat obvious, this lemma indicates that in order to understand the exit problem for the process ξ, we need to study how the radial process R exits a positive interval around $x > 0$. Fortunately, this is possible, thanks to a result of Blumenthal *et al.* [3], who established the following for the symmetric α-stable process Z.

Define

$$f(y,z) = \pi^{-(d/2+1)}\Gamma\left(\frac{d}{2}\right)\sin\left(\frac{\pi\alpha}{2}\right)|1-\|y\|^2|^{\alpha/2}|1-\|z\|^2|^{-\alpha/2}\|y-z\|^{-d}.$$

Theorem 3 ([3]). *Suppose that $\alpha < d$ and that $(Z, \mathbb{P}_x)$ is a symmetric α-stable process with values in $\mathbb{R}^d$, started from x. For $\|y\| < 1$ and $\|z\| \geq 1$, we have*

$$\mathbb{P}_y(Z_{\sigma_1^+} \in \mathrm{d}z; \sigma_1^+ < \infty) = f(y,z)\,\mathrm{d}z. \tag{3.1}$$

Similarly, for $\|y\| > 1$ and $\|z\| \leq 1$, we have

$$\mathbb{P}_y(Z_{\sigma_1^-} \in \mathrm{d}z; \sigma_1^- < \infty) = f(y,z)\,\mathrm{d}z. \tag{3.2}$$

The one-sided exit problem for ξ can be solved using Lemma 1 and Theorem 3, as follows.

Theorem 4. *Suppose that $\alpha < d$ and fix $\theta \geq 0$ and $-\infty < v < 0 < u < \infty$. Then*

$$\begin{aligned} &P(\xi_{T_u^+} - u \in \mathrm{d}\theta, T_u^+ < \infty) \\ &\quad = \frac{2}{\pi}\sin\left(\frac{\pi\alpha}{2}\right)\mathrm{e}^{2(u+\theta)}(1-\mathrm{e}^{-2u})^{\alpha/2}(\mathrm{e}^{2\theta}-1)^{-\alpha/2}\left(\mathrm{e}^{2(\theta+u)}-1\right)^{-1}\mathrm{d}\theta \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} &P(v - \xi_{T_v^-} \in \mathrm{d}\theta, T_v^- < \infty) \\ &\quad = \frac{2}{\pi}\sin\left(\frac{\pi\alpha}{2}\right)\mathrm{e}^{d(v-\theta)}(\mathrm{e}^{-2v}-1)^{\alpha/2}(1-\mathrm{e}^{-2\theta})^{-\alpha/2}\left(1-\mathrm{e}^{2(v-\theta)}\right)^{-1}\mathrm{d}\theta. \end{aligned} \tag{3.4}$$

Proof. Since Z is a symmetric α-stable process, we have, for any $x \in \mathbb{R}^d$ and $b > 0$,

$$\mathbb{P}_x(b^{-1}Z_{\sigma_b^+} \in \mathrm{d}y; \sigma_b^+ < \infty) = \mathbb{P}_{x/b}(Z_{\sigma_1^+} \in \mathrm{d}y; \sigma_1^+ < \infty),$$

which implies that

$$\mathbb{P}_x(R_{\sigma_{\mathrm{e}^u}^+} \in [\mathrm{e}^u, \mathrm{e}^{u+\theta}]; \sigma_{\mathrm{e}^u}^+ < \infty) = \mathbb{P}_{\mathrm{e}^{-u}x}(R_{\sigma_1^+} \in [1, \mathrm{e}^\theta]; \sigma_1^+ < \infty). \tag{3.5}$$

We first study the case $d=1$. Here, we assume that $x=1$. From (3.1), (3.5) and Lemma 1, we have, for $u,\theta\geq 0$,

$$P(\xi_{T_u^+}\leq u+\theta; T_u^+<\infty)=\mathbb{P}_{\mathrm{e}^{-u}}(R_{\sigma_1^+}\in[1,\mathrm{e}^{\theta}];\sigma_1^+<\infty)$$
$$=\frac{1}{\pi}\sin\left(\frac{\pi\alpha}{2}\right)(1-\mathrm{e}^{-2u})^{\alpha/2}\int_{1\leq|y|\leq \mathrm{e}^{\theta}}|1-|y|^2|^{-\alpha/2}|\mathrm{e}^{-u}-y|^{-1}\,\mathrm{d}y,$$

from which (3.3) follows.

We now study the case $d\geq 2$. To this end, we fix $x\in\mathbb{R}^d$ such that $\|x\|=1$ and $w_d=2\pi^{d/2}(\Gamma(d/2))^{-1}$. Hence, using identity (3.1) and polar coordinates in $\mathbb{R}^d$, we have, for $u,\theta\geq 0$,

$$\mathbb{P}_{\mathrm{e}^{-u}x}(R_{\sigma_1^+}\in[1,\mathrm{e}^{\theta}];\sigma_1^+<\infty)$$
$$=\pi^{-(d/2+1)}\Gamma\left(\frac{d}{2}\right)\sin\left(\frac{\pi\alpha}{2}\right)(1-\mathrm{e}^{-2u})^{\alpha/2}\int_{1\leq\|y\|\leq \mathrm{e}^{\theta}}|1-\|y\|^2|^{-\alpha/2}\|\mathrm{e}^{-u}x-y\|^{-d}\,\mathrm{d}y$$
$$=\pi^{-(d/2+1)}\Gamma\left(\frac{d}{2}\right)\sin\left(\frac{\pi\alpha}{2}\right)(1-\mathrm{e}^{-2u})^{\alpha/2}$$
$$\times\int_1^{\mathrm{e}^{\theta}}\mathrm{d}r\,\frac{r^{d-1}}{(r^2-1)^{\alpha/2}}\int_0^{\pi}\mathrm{d}\phi\,\frac{w_{d-1}\sin^{d-2}\phi}{(r^2-2r\mathrm{e}^{-u}\cos\phi+\mathrm{e}^{-2u})^{d/2}}.$$

On the other hand, from [7], formula (3.665), we get, for $r>1$,

$$\int_0^{\pi}\mathrm{d}\phi\,\frac{\sin^{d-2}\phi}{(r^2-2r\mathrm{e}^{-u}\cos\phi+\mathrm{e}^{-2u})^{d/2}}=\frac{\pi^{1/2}\Gamma((d-1)/2)}{\Gamma(d/2)}\mathrm{e}^{2u}r^{2-d}(r^2\mathrm{e}^{2u}-1)^{-1},$$

which implies that

$$\mathbb{P}_{\mathrm{e}^{-u}x}(R_{\sigma_1^+}\in[1,\mathrm{e}^{\theta}];\sigma_1^+<\infty)$$
$$=\frac{2}{\pi}\sin\left(\frac{\pi\alpha}{2}\right)(1-\mathrm{e}^{-2u})^{\alpha/2}\mathrm{e}^{2u}\int_1^{\mathrm{e}^{\theta}}\mathrm{d}r\,r(r^2-1)^{-\alpha/2}(r^2-1)^{-1}.$$

Therefore, from Lemma 1 and (3.5), we conclude that

$$P(\xi_{T_u^+}\leq u+\theta;T_u^+<\infty)$$
$$=\frac{2}{\pi}\sin\left(\frac{\pi\alpha}{2}\right)(1-\mathrm{e}^{-2u})^{\alpha/2}\mathrm{e}^{2u}\int_1^{\mathrm{e}^{\theta}}\mathrm{d}r\,r(r^2-1)^{-\alpha/2}(r^2-1)^{-1},$$

which proves (3.3) for the case $d\geq 2$.

The second part of the theorem can be proven in a similar way. Indeed, from the scaling property of Z, we have, for $\theta\geq 0$ and $v\leq 0$, that

$$\mathbb{P}_x(R_{\sigma^-_{\mathrm{e}^v}}\in[\mathrm{e}^{v-\theta},\mathrm{e}^v];\sigma^-_{\mathrm{e}^v}<\infty)=\mathbb{P}_{\mathrm{e}^{-v}x}(R_{\sigma_1^-}\in[\mathrm{e}^{-\theta},1];\sigma_1^-<\infty).\tag{3.6}$$

Assume that $d=1$ and take $x=1$. From (3.2), (3.6) and Lemma 1, we have

$$\begin{aligned}
&P(\xi_{T_v^-} \geq \theta - v; T_v^- < \infty)\\
&= \mathbb{P}_{\mathrm{e}^{-v}}(R_{\sigma_1^-} \in [\mathrm{e}^{-\theta}, 1]; \sigma_1^- < \infty)\\
&= \frac{1}{\pi} \sin\left(\frac{\pi\alpha}{2}\right)(\mathrm{e}^{-2v} - 1)^{\alpha/2} \int_{\mathrm{e}^{-\theta} \leq |y| \leq 1} \left|1 - |y|^2\right|^{-\alpha/2} |\mathrm{e}^{-v} - y|^{-1}\, \mathrm{d}y,
\end{aligned}$$

from which (3.4) follows.

We now study the case $d \geq 2$. To this end, we fix $x \in \mathbb{R}^d$ such that $\|x\| = 1$ and set $w_d = 2\pi^{d/2}(\Gamma(d/2))^{-1}$. Hence, using (3.2), polar coordinates and [7], formula (3.665), we get, for $\theta \geq 0$ and $v \leq 0$,

$$\begin{aligned}
&\mathbb{P}_{\mathrm{e}^{-v}x}(R_{\sigma_1^-} \in [\mathrm{e}^{-\theta}, 1]; \sigma_1^- < \infty)\\
&\quad = \pi^{-(d/2+1)} \Gamma\left(\frac{d}{2}\right) \sin\left(\frac{\pi\alpha}{2}\right)(\mathrm{e}^{-2v} - 1)^{\alpha/2}\\
&\qquad \times \int_{\mathrm{e}^{-\theta} < \|y\| \leq 1} |1 - \|y\|^2|^{-\alpha/2} \|\mathrm{e}^{-v}x - y\|^{-d}\, \mathrm{d}y\\
&\quad = \pi^{-(d/2+1)} \Gamma\left(\frac{d}{2}\right) \sin\left(\frac{\pi\alpha}{2}\right)(\mathrm{e}^{-2v} - 1)^{\alpha/2}\\
&\qquad \times \int_{\mathrm{e}^{-\theta}}^{1} \mathrm{d}r\, \frac{r^{d-1}}{(1-r^2)^{-\alpha/2}} \int_0^{\pi} \mathrm{d}\theta\, \frac{w_d \sin^{d-2}\theta}{(r^2 + \mathrm{e}^{-2v} - 2r\mathrm{e}^{-v}\cos\theta)^{d/2}}\\
&\quad = \frac{2}{\pi} \sin\left(\frac{\pi\alpha}{2}\right)(\mathrm{e}^{-2v} - 1)^{\alpha/2} \mathrm{e}^{-(2-d)v}\\
&\qquad \times \int_{\mathrm{e}^{-\theta}}^{1} \mathrm{d}r\, r^{d-1}(1-r^2)^{-\alpha/2}(\mathrm{e}^{-2v} - r^2)^{-1}.
\end{aligned}$$

Therefore, from Lemma 1 and (3.6), we conclude that

$$\begin{aligned}
&P(v - \xi_{T_v^-} \leq \theta; T_u^- < \infty)\\
&= \frac{2}{\pi} \sin\left(\frac{\pi\alpha}{2}\right)(\mathrm{e}^{-2v} - 1)^{\alpha/2} \mathrm{e}^{-(2-d)v} \int_{\mathrm{e}^{-\theta}}^{1} \mathrm{d}r\, r^{d-1}(1-r^2)^{-\alpha/2}(\mathrm{e}^{-2v} - r^2)^{-1}.
\end{aligned}$$

This complete the proof. □

Additional computations yield the following corollary.

Corollary 2. *Suppose that $\alpha < d$ and let $\underline{\xi}_\infty = \inf_{t \geq 0} \xi_t$. For $z \geq 0$,*

$$P(-\underline{\xi}_\infty \in \mathrm{d}z) = 2 \frac{\Gamma(d/2)}{\Gamma((d-\alpha)/2)\Gamma(\alpha/2)} \mathrm{e}^{-(d-2)z}(\mathrm{e}^{2z} - 1)^{\alpha/2 - 1}\, \mathrm{d}z.$$

Proof. We first note that

$$\int_0^r u^{d-\alpha-1}(r^2-u^2)^{(\alpha-2)/2}\,\mathrm{d}u = \frac{r^{d-2}}{2}\frac{\Gamma(\alpha/2)\Gamma((d-\alpha)/2)}{\Gamma(d/2)}$$

and that for $u \in [0, 1]$ and $z > 0$,

$$\int_0^{1-u^2} \mathrm{d}y\, y^{-\alpha/2}(\mathrm{e}^{2z}-1+y)^{-1}(1-y-u^2)^{\alpha/2-1} = \frac{\pi}{\sin(\pi\alpha/2)}\frac{(\mathrm{e}^{2z}-u^2)^{\alpha/2-1}}{(\mathrm{e}^{2z}-1)^{\alpha/2}}.$$

Thus, we have

$$\begin{aligned}
&\int_0^1 \mathrm{d}r\, r^{d-1}(1-r^2)^{-\alpha/2}(\mathrm{e}^{2z}-r^2)^{-1}\\
&= \frac{2\Gamma(d/2)}{\Gamma(\alpha/2)\Gamma((d-\alpha)/2)}\int_0^1 \mathrm{d}r\, r(1-r^2)^{-\alpha/2}(\mathrm{e}^{2z}-r^2)^{-1}\\
&\qquad\qquad \times \int_0^r u^{d-\alpha-1}(r^2-u^2)^{(\alpha-2)/2}\,\mathrm{d}u\\
&= \frac{\Gamma(d/2)}{\Gamma(\alpha/2)\Gamma((d-\alpha)/2)}\int_0^1 \mathrm{d}u\, u^{d-\alpha-1}\int_0^{1-u^2}\mathrm{d}y\, y^{-\alpha/2}(\mathrm{e}^{2z}-1+y)^{-1}(1-y-u^2)^{\alpha/2-1}\\
&= \frac{\Gamma(d/2)}{\Gamma(\alpha/2)\Gamma((d-\alpha)/2)}\frac{\pi}{\sin(\pi\alpha/2)}(\mathrm{e}^{2z}-1)^{-\alpha/2}\int_0^1 \mathrm{d}u\, u^{d-\alpha-1}(\mathrm{e}^{2z}-u^2)^{\alpha/2-1}\\
&= \frac{\Gamma(d/2)}{\Gamma(\alpha/2)\Gamma((d-\alpha)/2)}\frac{\pi}{2\sin(\pi\alpha/2)}(\mathrm{e}^{2z}-1)^{-\alpha/2}\mathrm{e}^{(d-2)z}\int_{\mathrm{e}^{2z}-1}^{\infty}\mathrm{d}r\,\frac{r^{\alpha/2-1}}{(r+1)^{d/2}}.
\end{aligned}$$

Therefore, from the above computations and (3.4), we get, for $z > 0$,

$$\begin{aligned}
&P(\underline{\xi}_\infty \le -z)\\
&= P(T^-_{-z} < \infty)\\
&= \frac{2}{\pi}\sin\left(\frac{\pi\alpha}{2}\right)\mathrm{e}^{-dz}(\mathrm{e}^{2z}-1)^{\alpha/2}\int_0^\infty \mathrm{e}^{-d\theta}(1-\mathrm{e}^{-2\theta})^{-\alpha/2}\big(1-\mathrm{e}^{-2(z+\theta)}\big)^{-1}\,\mathrm{d}\theta\\
&= \frac{2}{\pi}\sin\left(\frac{\pi\alpha}{2}\right)\mathrm{e}^{-(d-2)z}(\mathrm{e}^{2z}-1)^{\alpha/2}\int_0^1 \mathrm{d}r\, r^{d-1}(1-r^2)^{-\alpha/2}(\mathrm{e}^{2z}-r^2)^{-1}\\
&= \frac{\Gamma(d/2)}{\Gamma(\alpha/2)\Gamma((d-\alpha)/2)}\int_{\mathrm{e}^{2z}-1}^{\infty}\mathrm{d}r\,\frac{r^{\alpha/2-1}}{(r+1)^{d/2}}.
\end{aligned}$$

This completes the proof. □

4. Entrance laws: Points

For any $y \in \mathbb{R}$ and $r > 0$, let

$$T_y = \inf\{t > 0 : \xi_t = y\} \quad \text{and} \quad \sigma_r = \inf\{t > 0 : R_t = r\}.$$

We also introduce

$$\mathrm{P}_\nu^\mu(z) = \frac{1}{\Gamma(1-\mu)} \left(\frac{z+1}{z-1}\right)^{\mu/2} {}_2\mathcal{F}_1\left(-\nu, \nu+1; 1-\mu; \frac{1-z}{2}\right), \qquad z > 1,$$

the so-called Legendre function of the first kind.

The purpose of this section is to explicitly compute the probability that the process ξ hits a point, that is, $P(T_r < \infty)$, as well as some related quantities. Our study is based on the work of Port [11], where the author computes the probability that the radial process R hits a given point when $\alpha \in (1, 2)$. We recall that the radial process R only hits points when $\alpha \in (1, 2)$.

The one-point hitting probability for R, presented in [11], is given by the formula

$$\begin{aligned}\mathbb{P}_x(\sigma_r < \infty) = {} & \frac{2^{2-\alpha}\pi^{1/2}\Gamma((d+\alpha)/2-1)}{\Gamma((\alpha-1)/2)} \\ & \times r^{d/2+1-\alpha}|1-r^2|^{\alpha/2-1}\mathrm{P}_{-\alpha/2}^{1-d/2}\left(\frac{1+r^2}{|1-r^2|}\right),\end{aligned} \tag{4.1}$$

where $r > 0$ and $x \in \mathbb{R}^d$ such that $\|x\| = 1$. From the Lamperti representation (1.2) and identity (4.1), we obtain the one-point hitting problem for ξ as follows.

Theorem 5. *Let* $1 < \alpha < d$. *Then, for* $y \in \mathbb{R}$,

$$P(T_y < \infty) = \frac{2^{2-\alpha}\pi^{1/2}\Gamma((d+\alpha)/2-1)}{\Gamma((\alpha-1)/2)} \mathrm{e}^{(d/2-1)y}|\mathrm{e}^{-2y}-1|^{\alpha/2-1}\mathrm{P}_{-\alpha/2}^{1-d/2}\left(\frac{1+\mathrm{e}^{2y}}{|1-\mathrm{e}^{2y}|}\right).$$

Now, we explore more elaborate hitting probabilities (n-point hitting problem) for the Lévy process ξ when $1 < \alpha < d$. This is possible thanks to a result of Port [11] and the Lamperti representation (1.2) of the process R. Let $B = \{r_1, r_2, \ldots, r_n\}$, where $r_1 < r_2 < \cdots < r_n$.

Recall, from [11], that the potential density $u(\cdot,\cdot)$ of the radial process R which is specified by

$$\begin{aligned}&\mathbb{E}_z\left(\int_0^\infty \mathbb{1}_{\{R_t \in A\}}\,\mathrm{d}t\right) \\ &\quad = \frac{1}{2^{d/2}\Gamma(d/2+1)} \int_A \mathrm{d}y\, y^d u(\|z\|, y) \qquad \text{for } z \in \mathbb{R}^d,\ A \in \mathcal{B}(\mathbb{R}_+),\end{aligned}$$

satisfies (see [11], Lemmas 2.1 and 2.2), for $x, y > 0$,

$$u(x,y) = \frac{2^{(d/2)-\alpha}\Gamma(d/2)\Gamma((d-\alpha)/2)}{\Gamma(\alpha/2)}(xy)^{1-d/2}|x^2-y^2|^{\alpha/2-1}\mathrm{P}^{1-d/2}_{-\alpha/2}\left(\frac{x^2+y^2}{|x^2-y^2|}\right)$$

and

$$u(x,x) = \frac{\pi^{-1/2}2^{d/2-2}\Gamma((\alpha-1)/2)}{\Gamma((\alpha+d)/2-1)}\frac{\Gamma(d/2)\Gamma((d-\alpha)/2)}{\Gamma(\alpha/2)}x^{\alpha-d},$$

and that the matrix $U = [u(r_i, r_j)]_{n\times n}$ is invertible. Let us denote its inverse by $K_B = [K_B(i,j)]_{n\times n}$ and set $\sigma_B = \inf\{t > 0 : R_t \in B\}$.

According to Port, the probability that the process R hits the set B at a finite time is given by

$$\mathbb{P}_z(\sigma_B < \infty) = \sum_{i=1}^{n}\sum_{j=1}^{n} u(\|z\|, r_j)K_B(i,j) \tag{4.2}$$

and the probability that it first hits the point r_j is given by

$$\mathbb{P}_z(R_{\sigma_B} = r_j; \sigma_B < \infty) = \sum_{i=1}^{n} u(\|z\|, r_i)K_B(i,j). \tag{4.3}$$

For a two-point set $B = \{r_1, r_2\}$, we have that

$$K_B = \frac{1}{\Delta}\begin{pmatrix} U_{22} & -U_{12} \\ -U_{12} & U_{11}\end{pmatrix},$$

where $\Delta = U_{11}U_{22} - U_{12}^2$. Then, from (4.2) and (4.3), we have

$$\begin{aligned}\mathbb{P}_z(\sigma_B < \infty) = {} & \frac{u(\|z\|, r_1)u(r_2,r_2) + u(\|z\|, r_2)u(r_1,r_1)}{u(r_1,r_1)u(r_2,r_2) - u(r_1,r_2)^2} \\ & - \frac{u(r_1,r_2)[u(\|z\|, r_1) + u(\|z\|, r_2)]}{u(r_1,r_1)u(r_2,r_2) - u(r_1,r_2)^2}\end{aligned}$$

and

$$\mathbb{P}_z(\sigma_{r_1} < \sigma_{r_2}) = \frac{u(\|z\|, r_1)u(r_2,r_2) - u(\|z\|, r_2)u(r_2,r_1)}{u(r_1,r_1)u(r_2,r_2) - u(r_1,r_2)^2},$$

$$\mathbb{P}_z(\sigma_{r_2} < \sigma_{r_1}) = \frac{u(\|z\|, r_2)u(r_1,r_1) - u(\|z\|, r_1)u(r_1,r_2)}{u(r_1,r_1)u(r_2,r_2) - u(r_1,r_2)^2}.$$

Hence, the two-point hitting probabilities for the Lévy process ξ are as follows.

Theorem 6. *Suppose that* $1 < \alpha < d$ *and fix* $-\infty < v < 0 < u < \infty$. *Define*

$$T_{\{v,u\}} = \inf\big\{t > 0 : \xi_t \in \{v,u\}\big\}.$$

We have

$$P\big(T_{\{v,u\}} < \infty\big) = \frac{u(1, \mathrm{e}^v)u(\mathrm{e}^u, \mathrm{e}^u) + u(1, \mathrm{e}^u)u(\mathrm{e}^v, \mathrm{e}^v)}{u(\mathrm{e}^v, \mathrm{e}^v)u(\mathrm{e}^u, \mathrm{e}^u) - u(\mathrm{e}^v, \mathrm{e}^u)^2} - \frac{u(\mathrm{e}^v, \mathrm{e}^u)[u(1, \mathrm{e}^v) + u(1, \mathrm{e}^u)]}{u(\mathrm{e}^v, \mathrm{e}^v)u(\mathrm{e}^u, \mathrm{e}^u) - u(\mathrm{e}^v, \mathrm{e}^u)^2},$$

$$P\big(\xi_{T_{\{v,u\}}} = v\big) = f(1, \mathrm{e}^v, \mathrm{e}^u) \quad \textit{and} \quad P\big(\xi_{T_{\{v,u\}}} = u\big) = f(1, \mathrm{e}^u, \mathrm{e}^v),$$

where

$$f(x, a, b) = \frac{u(x, a)/u(b, a) - u(x, b)/u(b, b)}{u(a, a)/u(b, a) - u(a, b)/u(b, b)}.$$

5. Wiener–Hopf factorization

In this section, we work in the polar case (recall that this happens when $d \geq 2$ or when $d = 1$ and $\alpha \in (0, 1)$) and explicitly compute the characteristic exponent of the process ξ using its Wiener–Hopf factorization. Denote by $\{(L_t^{-1}, H_t) : t \geq 0\}$ and $\{(\widehat{L}_t^{-1}, \widehat{H}_t) : t \geq 0\}$ the (possibly killed) bivariate subordinators representing the ascending and descending ladder processes of ξ, respectively (see [2] for a proper definition). Write $\kappa(\theta, \lambda)$ and $\widehat{\kappa}(\theta, \lambda)$ for their joint Laplace exponents for $\theta, \lambda \geq 0$. For convenience, we will write

$$\hat{\kappa}(0, \lambda) = \hat{q} + \hat{c}\lambda + \int_{(0,\infty)} (1 - \mathrm{e}^{-\lambda x})\Pi_{\widehat{H}}(\mathrm{d}x),$$

where $\hat{q} \geq 0$ is the killing rate of $\widehat{H}$ so that $\widehat{q} > 0$ if and only if $\lim_{t\uparrow\infty} \xi_t = \infty$, $\hat{c} \geq 0$ is the drift of $\widehat{H}$ and $\Pi_{\widehat{H}}$ is its jump measure. Similar notation will also be used for $\kappa(0, \lambda)$ by replacing $\hat{q}$, $\hat{\xi}$, $\hat{c}$ and $\Pi_{\widehat{H}}$ by q, ξ, c and Π_H, respectively. Note that, necessarily, $q = 0$ since $\lim_{t\uparrow\infty} \xi_t = \infty$.

Associated with the ascending and descending ladder processes are the bivariate renewal functions V and $\widehat{V}$, respectively. The former is defined by

$$V(\mathrm{d}s, \mathrm{d}x) = \int_0^\infty \mathrm{d}t \cdot P(L_t^{-1} \in \mathrm{d}s, H_t \in \mathrm{d}x)$$

and taking double Laplace transforms shows that

$$\int_0^\infty \int_0^\infty \mathrm{e}^{-\theta s - \lambda x} V(\mathrm{d}s, \mathrm{d}x) = \frac{1}{\kappa(\theta, \lambda)} \qquad \text{for } \theta, \lambda \geq 0. \tag{5.1}$$

A similar definition and relation holds for $\widehat{V}$. These bivariate renewal measures are essentially the Green measures of the ascending and descending ladder processes. With an abuse of notation, we shall also write $V(\mathrm{d}x)$ and $\widehat{V}(\mathrm{d}x)$ for the marginal measures $V([0, \infty), \mathrm{d}x)$ and $\widehat{V}([0, \infty), \mathrm{d}x)$, respectively. (Since we shall never use the marginals $V(\mathrm{d}s, [0, \infty))$ and $\widehat{V}(\mathrm{d}s, [0, \infty))$, there should be no confusion.) Note that local time at the maximum is defined only up to a multiplicative constant. For this reason, the exponent κ can only be defined up to a multiplicative constant and hence the same is true of the measure V (and then this argument obviously applies to $\widehat{V}$).

The main result of this section is the Wiener–Hopf factorization of the characteristic exponent of the Lévy process ξ.

Theorem 7. *Let $\alpha < d$ and let ξ be the hypergeometric-stable Lévy process. Its characteristic exponent Ψ then enjoys the following Wiener–Hopf factorization*:

$$\begin{aligned}\Psi(\lambda) &= 2^{\alpha}\frac{\Gamma((-\mathrm{i}\lambda+\alpha)/2)}{\Gamma(-\mathrm{i}\lambda/2)}\frac{\Gamma((\mathrm{i}\lambda+d)/2)}{\Gamma((\mathrm{i}\lambda+d-\alpha)/2)}\\ &= 2^{\alpha}\frac{\Gamma(d/2)\Gamma((-\mathrm{i}\lambda+\alpha)/2)}{\Gamma((d-\alpha)/2)\Gamma(-\mathrm{i}\lambda/2)}\times\frac{\Gamma((d-\alpha)/2)\Gamma((\mathrm{i}\lambda+d)/2)}{\Gamma(d/2)\Gamma((\mathrm{i}\lambda+d-\alpha)/2)},\end{aligned} \tag{5.2}$$

where the first equality holds up to a multiplicative constant.

The proof of Theorem 7 relies on the computation of the Laplace exponents of the ascending ladder height and descending ladder height processes of ξ.

Lemma 2. *Let $\alpha < d$ and let ξ be the hypergeometric-stable Lévy process. The Laplace exponent of its descending ladder height process $\widehat{H}$ is given by*

$$\hat{\kappa}(0,\lambda) = \frac{\Gamma((d+\lambda)/2)\Gamma((d-\alpha)/2)}{\Gamma(d/2)\Gamma((d-\alpha+\lambda)/2)}. \tag{5.3}$$

Proof. Recall from the proof of Corollary 2 that

$$P\Big(-\inf_{t\geq 0}\xi_t \leq z\Big) = \frac{\Gamma(d/2)}{\Gamma((d-\alpha)/2)\Gamma(\alpha/2)}\int_0^{\mathrm{e}^{2z}-1}(u+1)^{-d/2}u^{\alpha/2-1}\,\mathrm{d}u.$$

Also, recall that $\widehat{V}$ denotes the renewal function associated with $\widehat{H}$. From [2], Proposition VI.17, we know that

$$\widehat{V}(z) := \widehat{V}([0,z]) = \widehat{V}([0,\infty))P\Big(-\inf_{t\geq 0}\xi_t \leq z\Big) \qquad \text{for all } z \geq 0.$$

As mentioned before, it is well known that $\widehat{V}$ is unique up to a multiplicative constant which depends on the normalization of the local time of ξ at its infimum. Without loss of generality, we may therefore assume in the forthcoming analysis that $\widehat{V}(\infty)$ may be taken to be identically equal to 1. Hence,

$$\widehat{V}(z) = \frac{\Gamma(d/2)}{\Gamma((d-\alpha)/2)\Gamma(\alpha/2)}\int_0^{\mathrm{e}^{2z}-1}(u+1)^{-d/2}u^{\alpha/2-1}\,\mathrm{d}u.$$

Now, let $K(\alpha, d) = \Gamma(d/2)(\Gamma((d-\alpha)/2)\Gamma(\alpha/2))^{-1}$ and note that

$$\lambda \int_0^\infty e^{-\lambda x} \widehat{V}(x)\,dx = \lambda K(\alpha, d) \int_0^\infty dx\, e^{-\lambda x} \int_0^{e^{2x}-1} du\,(u+1)^{-d/2} u^{\alpha/2-1}$$
$$= K(\alpha, d) \int_0^\infty (u+1)^{-(d+\lambda)/2} u^{\alpha/2-1}\,du$$
$$= K(\alpha, d) \int_0^1 u^{(d-\alpha+\lambda)/2-1}(1-u)^{\alpha/2-1}\,du = \frac{\Gamma(d/2)\Gamma((d+\lambda-\alpha)/2)}{\Gamma((d+\lambda)/2)\Gamma((d-\alpha)/2)}.$$

Finally, from (5.1), we deduce that

$$\hat{\kappa}(0, \lambda) = \frac{\Gamma((d+\lambda)/2)\Gamma((d-\alpha)/2)}{\Gamma(d/2)\Gamma((d-\alpha+\lambda)/2)}.$$

This completes the proof. □

For the computation of the Laplace exponent of the ascending ladder height process H, we will make use of an important identity obtained by Vigon [14] that relates Π_H, the Lévy measure of the ascending ladder height process H, to that of the Lévy process ξ and $\widehat{V}$, the potential measure of the descending ladder height process $\widehat{H}$. Specifically, defining $\overline{\Pi}_H(x) = \Pi_H(x, \infty)$, the identity states that

$$\overline{\Pi}_H(r) = \int_0^\infty \widehat{V}(dl)\overline{\Pi}^+(l+r), \qquad r > 0, \tag{5.4}$$

where $\overline{\Pi}^+(u) = \Pi(u, \infty)$ for $u > 0$.

Now, recall the property of the hypergeometric function ${}_2\mathcal{F}_1$ (see, e.g., [1], identity (3.1.9)),

$${}_2\mathcal{F}_1(a, b; a-b+1; x) = (1+x)^{-a}{}_2\mathcal{F}_1\left(a/2, (a+1)/2; a-b+1; \frac{4x}{(1+x)^2}\right), \tag{5.5}$$

and note that the Lévy measure of the process ξ can be written as follows:

$$\Pi(dy) = \frac{e^{-\alpha y}}{(1+e^{-2y})^{(\alpha+d)/2}} \overline{F}\left(\frac{4e^{-2y}}{(1+e^{-2y})^2}\right)\mathbb{1}_{\{y>0\}}\,dy$$
$$+ \frac{e^{dy}}{(1+e^{2y})^{\alpha+d/2}} \overline{F}\left(\frac{4e^{2y}}{(1+e^{2y})^2}\right)\mathbb{1}_{\{y<0\}}\,dy.$$

Therefore,

$$\Pi(dy) = \frac{2^\alpha \alpha (d/2)_{\alpha/2}}{\Gamma(1-\alpha/2)} e^{-\alpha y}{}_2\mathcal{F}_1\big((\alpha+d)/2, \alpha/2+1; d/2; e^{-2y}\big)\mathbb{1}_{\{y>0\}}\,dy$$
$$+ \frac{2^\alpha \alpha (d/2)_{\alpha/2}}{\Gamma(1-\alpha/2)} e^{dy}{}_2\mathcal{F}_1(\alpha+d/2, \alpha/2+1; d/2; e^{2y})\mathbb{1}_{\{y<0\}}\,dy. \tag{5.6}$$

Lemma 3. *Let $\alpha < d$ and let ξ be the hypergeometric-stable Lévy process. The Laplace exponent of its ascending ladder height process H is given by*

$$\kappa(0,\lambda) = \frac{2^{\alpha}\Gamma(d/2)\Gamma((\lambda+\alpha)/2)}{\Gamma((d-\alpha)/2)\Gamma(\lambda/2)}. \tag{5.7}$$

Proof. We first note, from the proof of Lemma 2, that the renewal measure $\widehat{V}(\mathrm{d}y)$ associated with $\widehat{H}$ satisfies

$$\widehat{V}(\mathrm{d}y) = \frac{2\Gamma(d/2)}{\Gamma((d-\alpha)/2)\Gamma(\alpha/2)} \mathrm{e}^{(2-d)y}(\mathrm{e}^{2y}-1)^{\alpha/2-1}\,\mathrm{d}y. \tag{5.8}$$

We also recall the following property of the Gamma function:

$$\Gamma(1-\alpha/2)\Gamma(\alpha/2) = \frac{\pi}{\sin(\pi\alpha/2)}.$$

From Vigon's formula (5.4) and identity (5.6), we have

$$\begin{aligned}\overline{\Pi}_H(x) = {} & \frac{2^{\alpha+1}\alpha\sin(\alpha\pi/2)}{\pi}\frac{\Gamma((d+\alpha)/2)}{\Gamma((d-\alpha)/2)}\int_0^\infty \mathrm{d}y\,\mathrm{e}^{(2-d)y}(\mathrm{e}^{2y}-1)^{\alpha/2-1} \\ & \times \int_{x+y}^\infty \mathrm{d}u\,\mathrm{e}^{-\alpha u}{}_2\mathcal{F}_1\big((\alpha+d)/2, \alpha/2+1; d/2; \mathrm{e}^{-2u}\big).\end{aligned}$$

On the other hand, from the definition of ${}_2\mathcal{F}_1$, we get

$$\begin{aligned}&\int_{x+y}^\infty \mathrm{d}u\,\mathrm{e}^{-\alpha u}{}_2\mathcal{F}_1\big((\alpha+d)/2, \alpha/2+1; d/2; \mathrm{e}^{-2u}\big) \\ &\quad = \frac{1}{2}\int_0^{\mathrm{e}^{-2(x+y)}} \mathrm{d}z\, z^{\alpha/2-1}{}_2\mathcal{F}_1\big((\alpha+d)/2, \alpha/2+1; d/2; z\big) \\ &\quad = \frac{\mathrm{e}^{-\alpha(x+y)}}{\alpha}{}_2\mathcal{F}_1\big((d+\alpha)/2, \alpha/2; d/2; \mathrm{e}^{-2(x+y)}\big).\end{aligned}$$

Set

$$C(\alpha,d) = \frac{2^{\alpha+1}\sin(\alpha\pi/2)}{\pi}\frac{\Gamma((d+\alpha)/2)}{\Gamma((d-\alpha)/2)}.$$

Hence, putting the pieces together, we obtain

$$\begin{aligned}\overline{\Pi}_H(x) &= C(\alpha,d)\mathrm{e}^{-\alpha x}\int_0^\infty {}_2\mathcal{F}_1\big((d+\alpha)/2, \alpha/2; d/2; \mathrm{e}^{-2(x+y)}\big)\mathrm{e}^{y(2-d-\alpha)}(\mathrm{e}^{2y}-1)^{\alpha/2-1}\,\mathrm{d}y \\ &= C(\alpha,d)\sum_{k=0}^\infty \mathrm{e}^{-2x(\alpha/2+k)}\frac{((d+\alpha)/2)_k(\alpha/2)_k}{(d/2)_k k!}\int_0^\infty \mathrm{e}^{-2y(d/2+k)}(1-\mathrm{e}^{-2y})^{\alpha/2-1}\,\mathrm{d}y\end{aligned}$$

$$= \frac{C(\alpha,d)}{2}\sum_{k=0}^{\infty} \mathrm{e}^{-2x(\alpha/2+k)} \frac{((d+\alpha)/2)_k(\alpha/2)_k}{(d/2)_k k!} \int_0^1 u^{d/2+k-1}(1-u)^{\alpha/2-1}\,\mathrm{d}u$$

$$= \frac{C(\alpha,d)}{2}\sum_{k=0}^{\infty} \mathrm{e}^{-2x(\alpha/2+k)} \frac{((d+\alpha)/2)_k(\alpha/2)_k}{(d/2)_k k!} \frac{\Gamma(d/2+k)\Gamma(\alpha/2)}{\Gamma((d+\alpha)/2+k)}$$

$$= \frac{C(\alpha,d)}{2} \frac{\Gamma(d/2)\Gamma(\alpha/2)}{\Gamma((d+\alpha)/2)} \mathrm{e}^{-\alpha x} \sum_{k=0}^{\infty} \mathrm{e}^{-2kx} \frac{(\alpha/2)_k}{k!}$$

$$= \frac{2^\alpha \sin(\alpha\pi/2)}{\pi} \frac{\Gamma(d/2)\Gamma(\alpha/2)}{\Gamma((d-\alpha)/2)} \mathrm{e}^{-\alpha x}(1-\mathrm{e}^{-2x})^{-\alpha/2}.$$

From Theorem 3, we deduce that the process ξ does not creep upward. Hence, by [2], Theorem VI.19, the ascending ladder height process H has no drift. Also, recall that the process ξ drifts to ∞, which implies that the process H has no killing term. Therefore, the Laplace exponent $\kappa(0,\lambda)$ of H is given by

$$\frac{\kappa(0,\lambda)}{\lambda} = \frac{2^\alpha \sin(\alpha\pi/2)}{\pi} \frac{\Gamma(d/2)\Gamma(\alpha/2)}{\Gamma((d-\alpha)/2)} \int_0^\infty \mathrm{e}^{-\lambda x}\mathrm{e}^{-\alpha x}(1-\mathrm{e}^{-2x})^{-\alpha/2}\,\mathrm{d}x.$$

By integrating by parts and a change of variable, we get

$$\kappa(0,\lambda) = \frac{\alpha 2^\alpha \sin(\alpha\pi/2)}{\pi} \frac{\Gamma(d/2)\Gamma(\alpha/2)}{\Gamma((d-\alpha)/2)} \int_0^\infty \big(1-\mathrm{e}^{-(\lambda/2)x}\big) \frac{\mathrm{e}^x}{(\mathrm{e}^x-1)^{\alpha/2+1}}\,\mathrm{d}x.$$

According to [4], Theorem 3.1, the previous integral satisfies

$$\int_0^\infty \big(1-\mathrm{e}^{-(\lambda/2)x}\big) \frac{\mathrm{e}^x}{(\mathrm{e}^x-1)^{\alpha/2+1}}\,\mathrm{d}x = -\frac{\Gamma(-\alpha/2)\Gamma((\lambda+\alpha)/2)}{\Gamma(\lambda/2)},$$

where $\Gamma(-\alpha/2) = -\alpha^{-1}\Gamma(1-\alpha/2)$. Therefore,

$$\kappa(0,\lambda) = \frac{2^\alpha \Gamma(d/2)\Gamma((\lambda+\alpha)/2)}{\Gamma((d-\alpha)/2)\Gamma(\lambda/2)}.$$

This completes the proof. □

Proof of Theorem 7. From the fluctuation theory of Lévy processes, it is known that Wiener–Hopf factorization of the characteristic exponent of ξ is given by

$$\psi(\lambda) = \kappa(0,-\mathrm{i}\lambda) \times \hat{\kappa}(0,\mathrm{i}\lambda),$$

up to a multiplicative constant. Hence, the result follows from Lemmas 2 and 3. □

Remark 2. We have obtained the characteristic exponent for the process ξ in the case where $\alpha < d$, using the Wiener–Hopf factorization. We will now see that the same formula holds true in the example studied in [5]: $\alpha = d = 1$.

Recall that they obtained the following characteristic exponent of ξ:

$$E[\exp\{\mathrm{i}\lambda\xi_t\}] = \exp\left\{-t\lambda\tanh\left(\frac{\pi\lambda}{2}\right)\right\}, \qquad t \geq 0, \lambda \in R.$$

We have

$$\begin{aligned}\psi(\lambda) = \lambda\tanh\left(\frac{\pi\lambda}{2}\right) &= \frac{\pi/\cosh(\pi\lambda/2)}{\pi/((\lambda/2)\sinh(\pi\lambda/2))} \\ &= \frac{|\Gamma((\mathrm{i}\lambda+1)/2)|^2}{|\Gamma(\mathrm{i}\lambda/2)|^2} = \left(\frac{\mathrm{i}\lambda+1}{2}\right)_{1/2}\left(-\frac{\mathrm{i}\lambda}{2}\right)_{1/2}.\end{aligned}$$

Recall that, in the case $\alpha < d$, the characteristic exponent is given by (5.2). From the above computation, we note that this formula still holds for the case $\alpha = d = 1$.

From the uniqueness of the Wiener–Hopf factorization, we deduce that the characteristic exponents of the subordinators $\hat{H}$ and H are, respectively,

$$\hat{\kappa}(0, \mathrm{i}\lambda) = \left(\frac{\mathrm{i}\lambda+1}{2}\right)_{1/2}, \qquad \kappa(0, -\mathrm{i}\lambda) = \left(-\frac{\mathrm{i}\lambda}{2}\right)_{1/2}.$$

6. *n*-tuple laws at first and last passage times

Recall that the renewal measure $\widehat{V}(\mathrm{d}y)$ associated with $\widehat{H}$ satisfies

$$\widehat{V}(\mathrm{d}y) = \frac{2\Gamma(d/2)}{\Gamma((d-\alpha)/2)\Gamma(\alpha/2)}\mathrm{e}^{(2-d)y}(\mathrm{e}^{2y}-1)^{\alpha/2-1}\,\mathrm{d}y.$$

From the form of the Laplace exponent of H and (5.1), we get that the renewal measure $V(\mathrm{d}y)$ associated with H satisfies

$$V(\mathrm{d}y) = \frac{\Gamma((d-\alpha)/2)}{2^{\alpha-1}\Gamma(d/2)\Gamma(\alpha/2)}(1-\mathrm{e}^{-2y})^{\alpha/2-1}\,\mathrm{d}y.$$

Since we have explicit expressions for the renewal functions V and $\widehat{V}$, we can get, from the main results of Doney and Kyprianou [6] and Kyprianou *et al.* [8], n-tuple laws at first and last passage times for the Lévy process ξ and the radial part of the symmetric stable Lévy process Z.

Marginalizing the quintuple law at first passage of Doney and Kyprianou [6] (see Theorem 3) and by the Lamperti representation (1.2), we now obtain the following new identities.

Proposition 2. *Let* $\overline{\xi}_t = \sup_{0\leq s\leq t}\xi_s$. *For* $y \in [0, x]$, $v \geq y$ *and* $u > 0$,

$$\begin{aligned}&P(\xi_{T_x^+} - x \in \mathrm{d}u, x - \xi_{T_x^+-} \in \mathrm{d}v, x - \overline{\xi}_{T_x^+-} \in \mathrm{d}y) \\ &\quad = \frac{4\alpha\Gamma((\alpha+d)/2)}{\Gamma(d/2)\Gamma(\alpha/2)}\frac{\sin(\alpha\pi/2)}{\pi}\big(1-\mathrm{e}^{-2(x-y)}\big)^{\alpha/2-1}\mathrm{e}^{(2-d)(v-y)}\big(\mathrm{e}^{2(v-y)}-1\big)^{\alpha/2-1} \\ &\qquad \times \mathrm{e}^{-\alpha(u+v)}{}_2\mathcal{F}_1\big((\alpha+d)/2, \alpha/2+1; d/2; \mathrm{e}^{-2(u+v)}\big)\,\mathrm{d}y\,\mathrm{d}v\,\mathrm{d}u.\end{aligned}$$

For $z \in [x, 1]$, $w \in [0, z]$ *and* $\theta > 1$,

$$\mathbb{P}_x\Big(\sup_{0\le s<\sigma_1^+} R_s \in \mathrm{d}z,\, R_{\sigma_1^+-} \in \mathrm{d}w,\, R_{\sigma_1^+} \in \mathrm{d}\theta\Big)$$
$$= \frac{4\alpha\Gamma((\alpha+d)/2)}{\Gamma(d/2)\Gamma(\alpha/2)} \frac{\sin(\alpha\pi/2)}{\pi} z^{3-d-\alpha} w^{d-1}\theta^{-\alpha-2}(z^2-x^2)^{\alpha/2-1}$$
$$\times (z^2-w^2)^{\alpha/2-1} {}_2\mathcal{F}_1\big((\alpha+d)/2, \alpha/2+1; d/2; (w/\theta)^2\big)\,\mathrm{d}z\,\mathrm{d}w\,\mathrm{d}\theta.$$

Note that the normalizing constant above is chosen to make the densities on the right-hand side into distributions. It is also important to point out that the triple law for the Lévy process ξ extends the identity in (3.3).

Let us define the last passage time and the future infimum for the processes ξ and R as, respectively,

$$U_x = \sup\{t : \xi_t < x\}, \qquad L_x = \sup\{t : R_t < x\},$$
$$J_t = \inf_{s\ge t} \xi_s \quad \text{and} \quad F_t = \inf_{s\ge t} R_s.$$

From [10], Proposition 2.3, we know that if $z > 0$, then the radial process R of the symmetric stable Lévy process is regular for both (z, ∞) and $[0, z)$. Hence, from the Lamperti representation (1.2), we deduce that the Lévy process ξ is regular for both $(-\infty, 0)$ and $(0, \infty)$. Now, applying [8], Corollaries 2 and 5, we obtain quadruple laws at last passage times for ξ and R.

Proposition 3. *For* $x, v > 0$, $0 \le y < x + v$ *and* $w \ge v > 0$,

$$P(-J_0 \in \mathrm{d}v, J_{U_x} - x \in \mathrm{d}u, x - \xi_{U_x-} \in \mathrm{d}y, \xi_{U_x} - x \in \mathrm{d}w)$$
$$= \frac{8\alpha\Gamma((\alpha+d)/2)}{\Gamma((d-\alpha)/2)\Gamma^2(\alpha/2)} \frac{\sin(\alpha\pi/2)}{\pi} \mathrm{e}^{(2-d)(v+w-u)}(\mathrm{e}^{2v}-1)^{\alpha/2-1}\big(\mathrm{e}^{2(w-u)}-1\big)^{\alpha/2-1}$$
$$\times \big(1-\mathrm{e}^{-2(x+v-y)}\big)^{\alpha/2-1}\mathrm{e}^{-\alpha(w+y)} {}_2\mathcal{F}_1\big((\alpha+d)/2, \alpha/2+1; d/2; \mathrm{e}^{-2(w+y)}\big)\,\mathrm{d}w\,\mathrm{d}y\,\mathrm{d}u\,\mathrm{d}v.$$

For $x, b > 0$, *we have, on* $v \ge x^{-1} \vee b^{-1}$, $v^{-1} < y < b$ *and* $b < u \le w < \infty$,

$$\mathbb{P}_x(1/F_0 \in \mathrm{d}v, R_{L_b-} \in \mathrm{d}y, R_{L_b} \in \mathrm{d}w, F_{L_b} \in \mathrm{d}u)$$
$$= \frac{8\alpha\Gamma((\alpha+d)/2)}{\Gamma((d-\alpha)/2)\Gamma^2(\alpha/2)} \frac{\sin(\alpha\pi/2)}{\pi} b^{d-2\alpha} v^{1-d} y w^{1-d-\alpha} u^{d-\alpha-1}(v^2-1)\big(y^2-(bv)^{-2}\big)^{\alpha/2-1}$$
$$\times \big(w^2-(bu)^2\big)^{\alpha/2-1} {}_2\mathcal{F}_1\big((\alpha+d)/2, \alpha/2+1; d/2; (y/bw)^2\big)\,\mathrm{d}v\,\mathrm{d}y\,\mathrm{d}w\,\mathrm{d}u.$$

We conclude this section with a nice formula for the potential kernel of the Lévy process ξ killed as it enters $(-\infty, 0)$, that follows from Bertoin [2], Theorem VI.20.

Proposition 4. *There exist a constant $k > 0$ such that for every measurable function $f : [0,\infty) \to [0,\infty)$ and $x \geq 0$, we have*

$$E_x\left(\int_0^{T_0^-} f(\xi_t)\,\mathrm{d}t\right) = k\frac{2^{2-\alpha}}{\Gamma^2(\alpha/2)}\int_0^\infty \mathrm{d}y\,(1-\mathrm{e}^{-2y})^{\alpha/2-1}\int_0^x \mathrm{d}z\,\mathrm{e}^{(2-d)z}(\mathrm{e}^{2z}-1)^{\alpha/2-1} f(x+y-z).$$

In particular, the potential measure of the Lévy process ξ killed as it enters $(-\infty,0)$ has a density which is given by

$$r(x,u) = k\frac{2^{2-\alpha}}{\Gamma^2(\alpha/2)}\int_{(u-x)\vee 0}^u (1-\mathrm{e}^{-2y})^{\alpha/2-1}\mathrm{e}^{(2-d)(x+y-u)}\left(\mathrm{e}^{2(x+y-u)}-1\right)^{\alpha/2-1}\mathrm{d}y.$$

Note that, from the previous proposition, we can obtain the potential kernel of the radial process R killed as it enters $(0,1)$. If we let $x > 1$, then

$$\begin{aligned}\mathbb{E}_x\left(\int_0^{\sigma_1^-} f(R_t)\,\mathrm{d}t\right) &= E_{\log x}\left(\int_0^{T_0^-} f(\mathrm{e}^{\xi_t})\mathrm{e}^{\alpha\xi_t}\,\mathrm{d}t\right)\\ &= k\frac{2^{2-\alpha}}{\Gamma^2(\alpha/2)}\int_0^\infty \mathrm{d}y(1-\mathrm{e}^{-2y})^{\alpha/2-1}\\ &\quad\times\int_0^{\log x}\mathrm{d}z\,\mathrm{e}^{(2-d)z}(\mathrm{e}^{2z}-1)^{\alpha/2-1}x^\alpha\mathrm{e}^{\alpha(y-z)}f(x\mathrm{e}^{y-z}).\end{aligned}$$

In particular,

$$\begin{aligned}\mathbb{E}_x(\sigma_1^-) &= E_{\log x}\left(\int_0^{T_0^-}\mathrm{e}^{\alpha\xi_t}\,\mathrm{d}t\right)\\ &= k\frac{2^{2-\alpha}}{\Gamma^2(\alpha/2)}\int_0^\infty \mathrm{d}y(1-\mathrm{e}^{-2y})^{\alpha/2-1}\int_0^{\log x}\mathrm{d}z\,\mathrm{e}^{(2-d)z}(\mathrm{e}^{2z}-1)^{\alpha/2-1}x^\alpha\mathrm{e}^{\alpha(y-z)}\\ &= k\frac{x^\alpha}{2\Gamma(\alpha)}\int_{x^{-2}}^1 \mathrm{d}u\,u^{d/2-1}(1-u)^{\alpha/2-1}.\end{aligned}$$

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