Bernoulli **16**(3), 2010, 825–857
DOI: 10.3150/09-BEJ234

A self-similar process arising from a random walk with random environment in random scenery

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In this article, we merge celebrated results of Kesten and Spitzer [*Z. Wahrsch. Verw. Gebiete* **50** (1979) 5–25] and Kawazu and Kesten [*J. Stat. Phys.* **37** (1984) 561–575]. A random walk performs a motion in an i.i.d. environment and observes an i.i.d. scenery along its path. We assume that the scenery is in the domain of attraction of a stable distribution and prove that the resulting observations satisfy a limit theorem. The resulting limit process is a self-similar stochastic process with non-trivial dependencies.

Keywords: birth–death process; random environment; random scenery; random walk; self-similar process

1. Introduction

The following model for a random walk in a random environment can be found in the physics literature; see Anshelevic and Vologodskii (1981), Alexander *et al.* (1981), Kawazu and Kesten (1984). Let $\{\lambda_j; j \in \mathbb{Z}\}$ be a family of positive i.i.d. random variables and $\mathcal{A}$ the σ-algebra generated by those random variables. Let $\{X(t); t \geq 0\}$ be a continuous-time random walk on $\mathbb{Z}$ having the following asymptotic transition rates for $h \to 0$:

$$\mathbb{P}\big(X(t+h) = j+1 | X(t) = j, \mathcal{A}\big) = \lambda_j h + \mathrm{o}(h), \tag{1}$$

$$\mathbb{P}\big(X(t+h) = j-1 | X(t) = j, \mathcal{A}\big) = \lambda_{j-1} h + \mathrm{o}(h), \tag{2}$$

$$\mathbb{P}\big(X(t+h) = j | X(t) = j, \mathcal{A}\big) = 1 - (\lambda_j + \lambda_{j-1})h + \mathrm{o}(h). \tag{3}$$

In other words, the process $\{X(t); t \geq 0\}$ is a birth–death process with possibly negative population size, where, for a population with j individuals, birth occurs at rate λ_j and death at rate λ_{j-1}. We will assume that the process $\{X(t); t \geq 0\}$ starts at zero at time zero. The resulting process is symmetric, in the sense that the permeability of the edge connecting the vertices j and $j+1$ does not depend on the direction of the motion. This physical background motivates the name 'random environment' for the sequence $\{\lambda_j; j \in \mathbb{Z}\}$. In what follows, we denote the distribution

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of the random environment on the sequence space by P_λ. The following convergence results are described in Kawazu and Kesten (1984).

KK1. *If* $c := \mathbb{E}[\lambda_0^{-1}] < \infty$, *then for* P_λ*-almost all environments, the distributions (after conditioning on the environment) of the processes*

$$X_n(t) := \frac{1}{n} X(n^2 t), \qquad t \geq 0,$$

converge weakly with respect to the Skorohod topology toward the distribution of the process $\{c^{-1/2} B(t); t \geq 0\}$, *where* $\{B(t); t \geq 0\}$ *is standard Brownian motion on* $\mathbb{R}$.

(See also Papanicolaou and Varadhan (1981) for some related results.)

KK2. *If there exists a slowly varying function* L_1 *such that*

$$\frac{1}{n L_1(n)} \sum_{j=1}^{n} \frac{1}{\lambda_j} \longrightarrow 1 \qquad \textit{in probability},$$

then the distributions of the processes

$$X_n(t) := \frac{1}{n} X(n^2 L_1(n) t)$$

converge weakly with respect to the Skorohod topology toward the distribution of standard Brownian motion.

KK3. *If there exists a slowly varying function* L_2 *such that the sequence of random variables*

$$R_n := \frac{1}{n^{1/\alpha} L_2(n)} \sum_{j=1}^{n} \frac{1}{\lambda_j}$$

converges in distribution toward a one-sided stable distribution ϑ_α *with index* $\alpha \in (0, 1)$, *then the distributions of the processes*

$$X_n(t) := \frac{1}{n} X\big(n^{(1+\alpha)/\alpha} L_2(n) t\big)$$

converge weakly with respect to the Skorohod topology toward the distribution of a continuous self-similar process $\{X_*(t); t \geq 0\}$ *with scaling exponent* $\eta = \frac{\alpha}{\alpha+1}$.

Remarks. (1) In the next section, we will give a representation for the process X_* in terms of a standard Brownian motion and a stable subordinator associated with the measure ϑ_α.

(2) We note that the results from Kawazu and Kesten (1984) are generalized in Kawazu (1989).

He considered random walks in random environments defined by the following transition asymptotics:

$$\mathbb{P}\big(X(t+h) = j+1 | X(t) = j, \mathcal{A}\big) = (\lambda_j/\eta_j)h + \mathrm{o}(h),$$
$$\mathbb{P}\big(X(t+h) = j-1 | X(t) = j, \mathcal{A}\big) = (\lambda_{j-1}/\eta_j)h + \mathrm{o}(h),$$
$$\mathbb{P}\big(X(t+h) = j | X(t) = j, \mathcal{A}\big) = 1 - \big((\lambda_j + \lambda_{j-1})/\eta_j\big)h + \mathrm{o}(h),$$

where $\{\eta_j, j \in \mathbb{N}\}$ is an i.i.d. family of positive random variables satisfying suitable assumptions. Similarly to the situation studied in Kawazu and Kesten (1984), the resulting random walks converge toward appropriate continuous processes after scaling.

In Kesten and Spitzer (1979), new classes of continuous self-similar processes are described. Moreover, it was proven therein that those processes are weak limits of random walks in random scenery. Those random walks are defined as follows.

Let $\{\xi(x); x \in \mathbb{Z}\}$ and $\{Z_i; i \in \mathbb{N}\}$ be two independent families of i.i.d. random variables, where the random variables Z_i are assumed to be $\mathbb{Z}$-valued. One can think of the sequence $\{Z_i; i \in \mathbb{N}\}$ as increments of a classical $\mathbb{Z}$-valued random walk $S_k := \sum_{i=1}^{k} Z_i$. The stationary sequence $\{\xi(S_k); k \in \mathbb{N}\}$ has some non-trivial long-range dependencies if the underlying random walk $\{S_k; k \in \mathbb{N}\}$ is recurrent. This is the case, for example, if Z_1 is in the domain of attraction of an α-stable distribution with $\alpha \in (1, 2]$. The random sequence $D(n) := \sum_{k=1}^{n} \xi(S_k)$ is called a *random walk in random scenery*. In Kesten and Spitzer (1979), the following convergence result was proven for those processes.

KS1. *If* $\xi(0)$ *is in the domain of attraction of a* β*-stable distribution with* $\beta \in (0, 2]$ *and if* Z_1 *is in the domain of attraction of an* α*-stable distribution with* $\alpha \in (0, 1)$, *then the distributions of the processes*

$$D_n(t) := n^{-1/\beta} \sum_{k=1}^{\lfloor nt \rfloor} \xi(S_k)$$

converge weakly with respect to the Skorohod topology toward β*-stable Lévy motion.*

(See also Spitzer (1976) for a special case.)

KS2. *If* $\xi(0)$ *is in the domain of attraction of a* β*-stable distribution with* $\beta \in (0, 2]$ *and if* Z_1 *is in the domain of attraction of an* α*-stable distribution with* $\alpha \in (1, 2]$, *then the distributions of the processes*

$$D_n(t) := n^{-\delta} \sum_{k=1}^{\lfloor nt \rfloor} \xi(S_k)$$

converge weakly with respect to the Skorohod topology toward a continuous self-similar process D_* *with scaling exponent* $\delta = 1 - \frac{1}{\alpha} + \frac{1}{\alpha\beta}$.

Remark. The statement in KS1 corresponds to the transient case and is not difficult to prove since, in that case, the sequence $\{\xi(S_k); k \in \mathbb{N}\}$ has only weak dependencies. This is the reason why one obtains β-stable Lévy noise in the limit. We also mention that the case $\beta = 1$ is still open.

Remark. There exist various generalizations of the results of Kesten and Spitzer (1979). We will only mention Shieh (1995), where the limiting process is generalized to higher dimensions, Lang and Nguyen (1983), which deals with multidimensional random walks and some special random scenery, Maejima (1996), where the random scenery belongs to the domain of attraction of an operator-stable distribution, Arai (2001), where the random scenery belongs to the domain of partial attraction of a semi-stable distribution, and Saigo and Takahashi (2005), where the random scenery and the random walk belong to the domain of partial attraction of semi-stable and operator semi-stable distributions.

In this article, we investigate whether it is possible to substitute the classical random walk in the result of Kesten and Spitzer (1979) by the random walk in random environment which was introduced in Kawazu and Kesten (1984). We will restrict our attention to the result KK3 since this is the case where a new type of self-similar process arises at the end. For simplicity and in order to avoid complicating notation, we will assume that the slowly varying function L_2 which appears in KK3 is constant and equal to one. The general case involving non-constant L_2 can be treated in a similar way.

We now fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ which is sufficiently large to support a family of i.i.d. random variables $\{\lambda_j; j \in \mathbb{Z}\}$, a birth–death process $\{X(t); t \geq 0\}$ with asymptotic transition rates given by equations (1)–(3) and a family of i.i.d. random variables $\{\xi(k), k \in \mathbb{Z}\}$.

We assume that the families $\{\xi(k), k \in \mathbb{Z}\}$ and $\{X(t); t \geq 0\}$ are independent and that $t \mapsto X(t)$ is cadlag $\mathbb{P}$-almost surely.

Further, we assume that λ_1^{-1} is in the domain of normal attraction of a one-sided α-stable distribution ϑ_α with $\alpha \in (0, 1)$.

Moreover, we assume that $\xi(0)$ is in the domain of normal attraction of a β-stable distribution ϑ_β with $\beta \in (0, 2]$. Its characteristic function is given by

$$\psi(\theta) = \exp\bigl(-|\theta|^\beta \bigl(A_1 + \mathrm{i}A_2 \operatorname{sgn}(\theta)\bigr)\bigr),$$

where $0 < A_1 < \infty$ and $|A_1^{-1}A_2| \leq \tan(\pi\beta/2)$. For $\beta > 1$, it follows from those assumptions that $\mathbb{E}[\xi(0)] = 0$.

For $\beta = 1$, we make the further assumption that there exists a $K > 0$ such that

$$\bigl|\mathbb{E}\bigl[\xi(0)\mathbb{1}_{[-\rho,\rho]}(\xi(0))\bigr]\bigr| \leq K \qquad \text{for all } \rho > 0.$$

We can now define the following continuous-time version of the random walk in random scenery:

$$\Xi(t) := \int_0^t \xi(X(s))\,\mathrm{d}s.$$

In the following, we will use the space

$$D[0,\infty) := \{\gamma : [0,\infty) \to \mathbb{R} : \gamma \text{ is cadlag}\}$$

with the Skorohod topology. We will prove the following theorem.

Theorem 1. *For $\kappa := \frac{1}{\alpha} + \frac{1}{\beta}$ and $k_n := n^{(1+\alpha)/\alpha}$, the distributions of the processes*

$$\Xi_n(t) := n^{-\kappa} \int_0^{k_n t} \xi(X(s))\,\mathrm{d}s$$

converge weakly with respect to the Skorohod topology toward the distribution of a self-similar stochastic process $\{\Xi_*(t); t \geq 0\}$ with scaling exponent $\mu = 1 - \frac{\alpha}{\alpha+1} + \frac{\alpha}{(\alpha+1)\beta}$.

Remark. The stochastic process $\{\Xi_*(t); t \geq 0\}$ can be constructed as follows. Let Z_+ and Z_- be two independent copies of the β-stable Lévy process which can be associated with the characteristic function

$$\psi(\theta) = \exp\bigl(-|\theta|^\beta \bigl(A_1 + \mathrm{i}A_2 \operatorname{sgn}(\theta)\bigr)\bigr).$$

Further, let $\{L_*(\tau, x); \tau \geq 0, x \in \mathbb{R}\}$ be the local time of the stochastic process $\{X_*(\tau); \tau \geq 0\}$; that is, the random variable $L_*(\tau, x)$ is the derivative with respect to x of the occupation time

$$\Gamma_*(\tau, (-\infty, x]) := \int_0^\tau \mathbb{1}_{(-\infty,x]}(X_*(\sigma))\,\mathrm{d}\sigma.$$

We will see in the next section that the local time exists for all but a countable number of points $x \in \mathbb{R}$. Moreover, for all $\tau \geq 0$, the processes

$$\{L_*(\tau, x-); x \geq 0\} \quad \text{and} \quad \{L_*(\tau, -(x-)); x \geq 0\}$$

are predictable with respect to the natural filtrations of Z_+ (resp., Z_-). The following integral representation of the process Ξ_* can be given:

$$\Xi_*(\tau) := \int_0^\infty L_*(\tau, x-)\,\mathrm{d}Z_+(x) + \int_0^\infty L_*(\tau, -(x-))\,\mathrm{d}Z_-(x).$$

2. The convergence of the birth–death process

The goal of this section is to prove Corollary 2, which is the main ingredient needed to show that the finite-dimensional distributions of Ξ_n converge toward the finite-dimensional distributions of Ξ_*. This corollary contains a statement on the weak convergence of certain functionals of the occupation times of the rescaled processes X_n. A result corresponding to Corollary 2 is also proved in Kesten and Spitzer (1979); however, we have to adopt a totally different approach since we do not have such precise information on the potential theory related to the random walk X. Instead, we will understand the occupation times of X_n and prove that they converge in an appropriate sense toward the local time of the limit process X_*.

We describe some of the main arguments from the proof in Kawazu and Kesten (1984) for the convergence of the processes

$$X_n(t) := \frac{1}{n} X\bigl(n^{(1+\alpha)/\alpha} t\bigr)$$

toward the self-similar process X_* defined in Kawazu and Kesten (1984). We can enlarge our underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$ in such a way that it contains a standard Brownian motion $\{B(t); t \geq 0\}$ and a cadlag version of the stable Lévy subordinator $\{W(x); x \in \mathbb{R}\}$ which can be associated with the one-sided α-stable distribution ϑ_α.

Furthermore, we assume that $\{B(t); t \geq 0\}$, $\{W(x); x \in \mathbb{R}\}$, $\{X(t); t \geq 0\}$ and $\{\xi(n); n \in \mathbb{Z}\}$ are independent. Moreover, we assume that $W(0) = 0$ and $B(0) = 0$ hold $\mathbb{P}$-almost surely.

In the future, we will denote by $\{L(t, x); t \geq 0, x \in \mathbb{R}\}$ the local time of the Brownian motion $\{B(t); t \geq 0\}$. The process

$$V_*(t) := \int_{\mathbb{R}} L(t, W(x))\, \mathrm{d}x$$

is non-decreasing $\mathbb{P}$-almost surely. Therefore, we can define the following pseudo-inverse:

$$W^{-1}(y) := \inf\{x \in \mathbb{R}; W(x) > y\} \quad \text{and} \quad V_*^{-1}(\tau) := \inf\{t \geq 0; V_*(t) > \tau\}.$$

In Kawazu and Kesten (1984), the following representation for the self-similar process X_* is given:

$$X_*(\tau) := W^{-1}\bigl(B\bigl(V_*^{-1}(\tau)\bigr)\bigr).$$

We now sketch the main arguments from the proof in Kawazu and Kesten (1984). We will need some of those ideas in our proof of the convergence of Ξ_n toward Ξ_*. Their approach is based on the natural scale of the birth–death process. One defines

$$S(j) := \begin{cases} \sum_{k=0}^{j-1} \lambda_k^{-1} & \text{for } j > 0, \\ 0 & \text{for } j = 0, \\ -\sum_{k=j}^{-1} \lambda_k^{-1} & \text{for } j < 0. \end{cases}$$

This implies that conditioned on $\mathcal{A} := \{\lambda_j; j \in \mathbb{Z}\}$, the process $S(X(t))$ is on natural scale (see Kawazu and Kesten (1984), page 565). This means that for all $a, b, x \in \mathbb{R}$ with $a < x < b$, one has

$$\mathbb{P}\bigl(S(X(t)) \text{ hits } \{a, b\} \text{ first at } a \mid S(X(0)) = x, \mathcal{A}\bigr) = \frac{b - x}{b - a}.$$

It is then possible to represent the process $S(X(t))$ as the time change of standard Brownian motion $\{B(t); t \geq 0\}$ as follows.

One defines $m(\mathrm{d}x) := \sum_{i\in\mathbb{Z}} \delta_{S(i)}(\mathrm{d}x)$ and

$$V(t) := \int_{\mathbb{R}} L(t,x)m(\mathrm{d}x) = \sum_{i\in\mathbb{Z}} L(t, S(i)),$$

where $\{L(t,x); t \geq 0, x \in \mathbb{R}\}$ is again the local time of the standard Brownian motion B. One can see that $\{B(V^{-1}(t)); t \geq 0\}$ and $\{S(X(t)); t \geq 0\}$ are both cadlag and have the same distribution (see Kawazu and Kesten (1984), page 566).

One then has to scale the above constructions.

$$S_n(x) := n^{-1/\alpha} S(\lfloor nx \rfloor), \qquad n \in \mathbb{N}, x \in \mathbb{R},$$

where, for a positive real number x, we denote by $\lfloor x \rfloor$ its integer part. It follows from the assumptions on the environment $\{\lambda_j; j \in \mathbb{Z}\}$ that for $n \to \infty$, the processes $\{S_n(x); x \in \mathbb{R}\}$ converge in distribution toward an α-stable Lévy process $\{W(x); x \in \mathbb{R}\}$. Moreover, the process W is strictly increasing $\mathbb{P}$-almost surely since ϑ_α is a one-sided stable distribution and $\alpha \in (0, 1)$. By a method given in Skorohod (1956) and Dudley (1968), it is possible to construct a suitable probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ with suitable D-valued random variables $\tilde{S}_n$ and $\tilde{W}$ having the properties that $\tilde{S}_n$ converges toward $\tilde{W}$ almost surely with respect to $\tilde{\mathbb{P}}$ and that $\tilde{S}_n$ and $\tilde{W}$ have the same distributions as S_n (resp., W) (see Kawazu and Kesten (1984), page 567). One then defines

$$\tilde{V}_n(t) := \int_{\mathbb{R}} L(t,x)\tilde{m}_n(\mathrm{d}x) \quad \text{and} \quad \tilde{V}_*(t) := \int_{\mathbb{R}} L(t,x)\tilde{m}_*(\mathrm{d}x)$$

with

$$\int_{\mathbb{R}} f(x)\tilde{m}_n(\mathrm{d}x) := \int_{\mathbb{R}} f(\tilde{S}_n(x))\,\mathrm{d}x \quad \text{and} \quad \int_{\mathbb{R}} f(x)\tilde{m}_*(\mathrm{d}x) := \int_{\mathbb{R}} f(\tilde{W}(x))\,\mathrm{d}x$$

for all measurable $f \geq 0$. We then define $\tilde{S}_n^{-1}$, $\tilde{W}^{-1}$, $\tilde{V}_n^{-1}$ and $\tilde{V}_*^{-1}$ in the same way as W^{-1} (resp., V_*^{-1}) above.

In Kawazu and Kesten (1984) (see page 568) they prove that $\{B(\tilde{V}_n^{-1}(t)); t \geq 0\}$ converges $\tilde{\mathbb{P}}$-almost surely toward $\{B(\tilde{V}_*^{-1}(t)); t \geq 0\}$ in the J_1-topology. For convenience, we define

$$\tilde{X}_n(t) := \tilde{S}_n^{-1}(B(\tilde{V}_n^{-1}(t))), \qquad \tilde{X}_*(t) := \tilde{W}^{-1}(B(\tilde{V}_*^{-1}(t))).$$

We note that the process $\{\tilde{X}_n(t); t \geq 0\}$ is defined on $(\Omega \times \tilde{\Omega}, \mathcal{F} \times \tilde{\mathcal{F}}, \mathbb{P} \times \tilde{\mathbb{P}})$. It is proved in Kawazu and Kesten (1984) that $\{\tilde{X}_n(t); t \geq 0\}$ converges toward $\{\tilde{X}_*(t); t \geq 0\}$ with respect to the J_1-topology almost surely with respect to $\mathbb{P} \times \tilde{\mathbb{P}}$ (see page 569).

Moreover, for $B_n(t) := n^{-1/2}B(nt)$ one has that (see Kawazu and Kesten (1984), page 572)

$$|X_n(t) - S_n^{-1}(B_n(V_n^{-1}(t)))| \leq 1/n$$

and

$$\{S_n^{-1}(B_n(V_n^{-1}(t))); t \geq 0\} \stackrel{\mathcal{D}}{=} \{\tilde{S}_n^{-1}(B(\tilde{V}_n^{-1}(t))); t \geq 0\} = \{\tilde{X}_n(t); t \geq 0\}.$$

If we define $\hat{X}_n(t) := S_n^{-1}(B_n(V_n^{-1}(t)))$, then the previous observations imply that both processes $\{X_n(t); t \geq 0\}$ and $\{\hat{X}_n(t); t \geq 0\}$ converge in distribution toward $\{\tilde{X}_*(t); t \geq 0\}$, which has the same distribution as $\{X_*(t); t \geq 0\}$.

In the rest of this section, we analyze the distributional behavior of the occupation times for the process X_n (see Proposition 6). In order to obtain this result, we prove an analogous result for the process $\tilde{X}_n$ (see Lemma 5), which can be reduced to Proposition 4. The advantage of this detour is that we can prove almost sure convergence for the occupation times of the process $\tilde{X}_n$ toward the local time of $\tilde{X}_*$ (see Proposition 3). This result is based on the fact that we have explicit formulas for the occupation times of $\tilde{X}_n$ and the local time of $\tilde{X}_*$ (see Proposition 2 and Corollary 1). The explicit expression of the occupation time of $\tilde{X}_n$ and the local time of $\tilde{X}_*$ reveals that in order to prove Proposition 3, it is sufficient to prove the almost sure convergence of $\tilde{S}_n$ and $\tilde{V}_n^{-1}$ toward $\tilde{W}_*$ (resp., $\tilde{V}_*^{-1}$). The convergence of $\tilde{S}_n$ toward $\tilde{W}_*$ holds by construction. The convergence of $\tilde{V}_n$ toward $\tilde{V}_*$ is obtained in Lemma 1 and then used to obtain the convergence of $\tilde{V}_n^{-1}$ toward $\tilde{V}_*^{-1}$ in Lemma 2.

2.1. The local times of X_* and $\tilde{X}_*$

We define the time that the processes $\tilde{X}_*$ and X_* spend in the measurable set A until time τ as

$$\Gamma_*(\tau, A) := \int_0^\tau \mathbb{1}_A(X_*(\sigma))\,\mathrm{d}\sigma \qquad \left(\text{resp., } \tilde{\Gamma}_*(\tau, A) := \int_0^\tau \mathbb{1}_A(\tilde{X}_*(\sigma))\,\mathrm{d}\sigma\right).$$

We denote by $\{L_*(\tau, x); \tau \geq 0, x \in \mathbb{R}\}$ and $\{\tilde{L}_*(\tau, x); \tau \geq 0, x \in \mathbb{R}\}$ the local times of X_* (resp., $\tilde{X}_*$) if they exist. In this subsection, we prove that both local times exist almost surely and relate them to the local time $\{L(t, x); t \geq 0, x \in \mathbb{R}\}$ of the underlying Brownian motion $\{B(t); t \geq 0\}$.

Proposition 1. *One has $\mathbb{P}$-almost surely that for $\tau \geq 0$ and all $x \in \mathbb{R}$,*

$$\Gamma_*(\tau, (-\infty, x)) = \int_{-\infty}^x L(V_*^{-1}(\tau), W(y))\,\mathrm{d}y.$$

Further, $\mathbb{P} \times \tilde{\mathbb{P}}$-almost surely for all $\tau \geq 0$ and all $x \in \mathbb{R}$,

$$\tilde{\Gamma}_*(\tau, (-\infty, x)) = \int_{-\infty}^x L(\tilde{V}_*^{-1}(\tau), \tilde{W}(y))\,\mathrm{d}y.$$

Proof. We have $\mathbb{P}$-almost surely that $x \mapsto W(x)$ is increasing. It follows that the set $\mathcal{N}_1$ of $x \in \mathbb{R}$ where W is not continuous is countable. We define the set

$$\mathcal{N}_2 := \left\{x \in \mathbb{R} : \ell\left(\sigma; B(V_*^{-1}(\sigma)) = W(x)\right) > 0\right\},$$

where ℓ denotes the Lebesgue measure on $\mathbb{R}$. The set $\mathcal{N}_2$ is countable since for $x_1 \neq x_2$, one has that the sets $\{\sigma; B(V_*^{-1}(\sigma)) = W(x_1)\}$ and $\{\sigma; B(V_*^{-1}(\sigma)) = W(x_2)\}$ are disjoint. The statement then follows since there cannot be an uncountable number of disjoint subsets of $\mathbb{R}$

with positive Lebesgue measure. Thus the set $\mathcal{N} := \mathcal{N}_1 \cup \mathcal{N}_2$ is countable. Since the function $x \mapsto \Gamma_*(\tau, (-\infty, x))$ is increasing and since

$$x \mapsto \int_{-\infty}^{x} L(V_*^{-1}(\tau), W(y))\,\mathrm{d}y$$

is continuous, it is sufficient to prove the statement of the proposition for $x \in \mathcal{N}^c$.

The fact that W is increasing and continuous in x implies the equivalence of the statement $W(x) > y$ with the statement $\exists z_0 < x : W(z_0) > y$.

The latter statement is then equivalent to the statement $W^{-1}(y) := \inf\{z : W(z) > y\} < x$.

This then implies that $\mathbb{1}_{(-\infty,x)}(X_*(\sigma)) = \mathbb{1}_{(-\infty,W(x))}(B(V_*^{-1}(\sigma)))$.

We also note that $t \mapsto V(t)$ is continuous and non-decreasing. This implies that $V_* \circ V_*^{-1} = \mathrm{id}_{\mathbb{R}}$.

In the following, we want to compute the derivative of the non-decreasing function

$$M : \sigma \mapsto \int_{-\infty}^{x} L(V_*^{-1}(\sigma), W(y))\,\mathrm{d}y.$$

Since W is increasing and continuous in x, we have that $B(V_*^{-1}(\sigma_0)) < W(x)$ implies that

$$\sigma \mapsto \int_{x}^{\infty} L(V_*^{-1}(\sigma), W(y))\,\mathrm{d}y$$

is locally constant, say equal to c_0, in a neighborhood of σ_0.

Thus

$$\sigma \mapsto \int_{-\infty}^{x} L(V_*^{-1}(\sigma), W(y))\,\mathrm{d}y = V_*(V_*^{-1}(\sigma)) - c_0 = \sigma - c_0$$

in a neighborhood of σ_0.

Moreover, since W is increasing and continuous in x, we have that $B(V_*^{-1}(\sigma_0)) > W(x)$ implies

$$\sigma \mapsto \int_{-\infty}^{x} L(V_*^{-1}(\sigma), W(y))\,\mathrm{d}y$$

is locally constant in a neighborhood of σ_0.

It therefore turns out that

$$M'(\sigma) = \begin{cases} 1, & \text{if } B(V_*^{-1}(\sigma)) < W(x), \\ 0, & \text{if } B(V_*^{-1}(\sigma)) > W(x). \end{cases}$$

Moreover, for all $\sigma_1, \sigma_2 \in \mathbb{R}^+$ with $\sigma_1 \le \sigma_2$, we have that

$$\int_{-\infty}^{x} L(V_*^{-1}(\sigma_1), W(y))\,\mathrm{d}y \le \int_{-\infty}^{x} L(V_*^{-1}(\sigma_2), W(y))\,\mathrm{d}y$$

and

$$\int_{x}^{\infty} L(V_*^{-1}(\sigma_1), W(y))\,\mathrm{d}y \le \int_{x}^{\infty} L(V_*^{-1}(\sigma_2), W(y))\,\mathrm{d}y.$$

This implies that

$$\int_{-\infty}^{x} L(V_*^{-1}(\sigma_2), W(y))\,\mathrm{d}y - \int_{-\infty}^{x} L(V_*^{-1}(\sigma_1), W(y))\,\mathrm{d}y$$
$$\leq V_*(V_*^{-1}(\sigma_2)) - V_*(V_*^{-1}(\sigma_1)) = \sigma_2 - \sigma_1.$$

It follows that

$$\sigma \mapsto \int_{-\infty}^{x} L(V_*^{-1}(\sigma), W(y))\,\mathrm{d}y$$

is Lipschitz continuous with Lipschitz constant smaller than one.

Since the set $\{\sigma : B(V_*^{-1}(\sigma)) = W(x)\}$ is a zero set with respect to the Lebesgue measure ℓ for all $x \in \mathcal{N}^c$, it follows that

$$\int_0^{\tau} \mathbb{1}_{(-\infty,x)}(X_*(\sigma))\,\mathrm{d}\sigma = \int_0^{\tau} \mathbb{1}_{(-\infty,W(x))}(B(V_*^{-1}(\sigma)))\,\mathrm{d}\sigma = \int_0^{\tau} M'(\sigma)\,\mathrm{d}\sigma = M(\tau).$$

The second statement is proved in the same way. □

Corollary 1. *One has $\mathbb{P}$-almost surely that the local time $L_*(\tau, x)$ is defined for all $\tau \geq 0$ and all x, where $x \mapsto W(x)$ is continuous. Further, one has $\mathbb{P} \times \tilde{\mathbb{P}}$-almost surely that the local time $\tilde{L}_*(\tau, x)$ is defined for all $\tau \geq 0$ and all x, where $x \mapsto \tilde{W}(x)$ is continuous. In those points, one has*

$$L_*(\tau, x) = L(V_*^{-1}(\tau), W(x)) \qquad \big(\textit{resp.},\ \tilde{L}_*(\tau, x) = L(\tilde{V}_*^{-1}(\tau), \tilde{W}(x))\big).$$

Proof. Differentiation in Proposition 1 proves this corollary. □

2.2. The occupation time of $\tilde{X}_n$

For a measurable set $A \subset \mathbb{R}$, we define

$$\hat{\Gamma}_n(t, A) := \int_0^t \mathbb{1}_A(\hat{X}_n(\sigma))\,\mathrm{d}\sigma, \qquad \tilde{\Gamma}_n(t, A) := \int_0^t \mathbb{1}_A(\tilde{X}_n(\sigma))\,\mathrm{d}\sigma$$

and

$$\Gamma_n(t, A) := \int_0^t \mathbb{1}_A(X_n(\sigma))\,\mathrm{d}\sigma.$$

These are the respective times that the processes $\hat{X}_n$, $\tilde{X}_n$ and X_n spend in the set A until time t. In this section, we give an explicit expression for the occupation time of $\tilde{X}_n$ in terms of the local time $\{L(t, x); t \geq 0, x \in \mathbb{R}\}$ of the underlying Brownian motion $\{B(t); t \geq 0\}$.

Proposition 2. *One has $\mathbb{P}\times\tilde{\mathbb{P}}$-almost surely for all $\tau\geq 0$ and all $x\in\mathbb{R}$ that*

$$\tilde{\Gamma}_n(\tau,\{x\})=\begin{cases}\frac{1}{n}L\left(\tilde{V}_n^{-1}(\tau),\tilde{S}_n\left(x-\frac{1}{n}\right)\right), & \text{if } nx\in\mathbb{Z},\\ 0, & \text{if } nx\notin\mathbb{Z}.\end{cases}$$

Proof. First, we note that

$$S_n^{-1}(S_n(x))=x+1/n \qquad \text{for all } x \text{ satisfying } nx\in\mathbb{Z}.$$

If we use the fact that $\{B_n(V_n^{-1}(t));t\geq 0\}\}\overset{\mathcal{D}}{=}\{S_n(X_n(t));t\geq 0\}$, then we can see that $\{\hat{X}_n(t);t\geq 0\}\overset{\mathcal{D}}{=}\{X_n(t)+1/n;t\geq 0\}$. Therefore, we see that $\hat{X}_n$ only takes values in the lattice $\frac{1}{n}\mathbb{Z}$. Moreover, we have that $\tilde{S}_n$ and $\tilde{V}_n$ have the same joint distribution as S_n and V_n. Therefore, $\hat{X}_n=S_n^{-1}(B_n(V_n^{-1}(\cdot)))$ has the same distribution as $\tilde{X}_n=\tilde{S}_n^{-1}(B(\tilde{V}_n^{-1}(\cdot)))$. From this, it also follows that $\tilde{X}_n$ stays for all time in the countable state space $\{x\in\mathbb{R};nx\in\mathbb{Z}\}$. This implies that $\tilde{\Gamma}_n(\tau,\{x\})=0$ for $nx\notin\mathbb{Z}$. This proves one part of the statement.

For the proof of the other part of the statement, we will need the derivative of the function

$$\tilde{M}(\sigma):=\frac{1}{n}L\big(\tilde{V}_n^{-1}(\sigma),\tilde{S}_n(x-1/n)\big).$$

We first collect some useful facts which help to compute the derivative of $\tilde{M}$.

Since $\tilde{S}_n$ is constant on the intervals $[\frac{k}{n},\frac{k+1}{n})$ for all $k\in\mathbb{Z}$, we have

$$\tilde{V}_n(t)=\int_{\mathbb{R}}L(t,\tilde{S}_n(x))\,\mathrm{d}x=\frac{1}{n}\sum_{i\in\mathbb{Z}}L\big(t,\tilde{S}_n(i/n)\big). \tag{4}$$

Since the $(t,x)\mapsto L(t,x)$ is jointly continuous and non-decreasing $\mathbb{P}$-almost surely (see Boylan (1964) or Getoor and Kesten (1972)), it follows that $t\mapsto\tilde{V}_n(t)$ is continuous and non-decreasing $\mathbb{P}\times\tilde{\mathbb{P}}$-almost surely. This then gives rise to

$$\tilde{V}_n\circ\tilde{V}_n^{-1}=\mathrm{id}_{\mathbb{R}+} \qquad \mathbb{P}\times\tilde{\mathbb{P}}\text{-almost surely.} \tag{5}$$

By construction, one has for all $b\in\{\tilde{S}_n(x);x\in\mathbb{R}\}$ that $\tilde{S}_n^{-1}(b)=x$ is equivalent to $b=\tilde{S}_n(x-\frac{1}{n})$. Moreover, one has that $B(\tilde{V}_n^{-1}(\sigma))\in\{\tilde{S}_n(x);x\in\mathbb{R}\}$ for all $\sigma\geq 0$ almost surely with respect to $\mathbb{P}\times\tilde{\mathbb{P}}$. Hence,

$$\tilde{X}_n(\sigma)=\tilde{S}_n^{-1}(B(\tilde{V}_n^{-1}(\sigma)))=x \text{ is equivalent to } B(\tilde{V}_n^{-1}(\sigma))=\tilde{S}_n\left(x-\frac{1}{n}\right). \tag{6}$$

Moreover, the random variables $\{\lambda_i^{-1};i\in\mathbb{N}\}$ are positive $\mathbb{P}$-almost surely and therefore

$$\text{the restriction of } x\mapsto\tilde{S}_n(x) \text{ to the set } \frac{1}{n}\mathbb{Z} \text{ is injective almost surely with respect to } \tilde{\mathbb{P}}. \tag{7}$$

Since, conditioned on $\mathcal{A}=\sigma\{\lambda_j;\ j\in\mathbb{N}\}$, the process X is a Markov process, it follows that for $nx\in\mathbb{Z}$, there exist non-negative random variables $a_1<b_1<a_2<b_2<\cdots$ with the property

$$\{\sigma\geq 0;\ \tilde{X}_n(\sigma)=x\}=\bigcup_{i\in\mathbb{N}}[a_i,b_i)\qquad \mathbb{P}\times\tilde{\mathbb{P}}\text{-a.s.}$$

This implies that for all $\sigma_0\notin\{a_i;\ i\in\mathbb{N}\}$, there exists a neighborhood $\mathcal{U}(\sigma_0)$ containing σ_0 with the property that $\sigma\mapsto\tilde{X}_n(\sigma)=\tilde{S}_n^{-1}(B(\tilde{V}_n^{-1}(\sigma)))$ is constant on $\mathcal{U}(\sigma_0)$. Equations (6) and (7) then imply that $\sigma\mapsto B(\tilde{V}_n^{-1}(\sigma))$ must be constant on $\mathcal{U}(\sigma_0)$.

Therefore, for $\sigma_0\notin\{a_i;\ i\in\mathbb{N}\}$ and $B(\tilde{V}_n^{-1}(\sigma_0))\neq\tilde{S}_n(x-\frac{1}{n})$, we have $B(\tilde{V}_n^{-1}(\sigma))\neq\tilde{S}_n(x-\frac{1}{n})$ for all σ in a neighborhood of σ_0. Hence

$$\sigma\mapsto L\big(\tilde{V}_n^{-1}(\sigma),\tilde{S}_n(x-1/n)\big)$$

is constant in a neighborhood of σ_0. The previous argument and the fact that $\tilde{X}_n$ only jumps to nearest neighbors in $\frac{1}{n}\mathbb{Z}$ leads to the fact that $\sigma_0\notin\{a_i;\ i\in\mathbb{N}\}$ and $B(\tilde{V}_n^{-1}(\sigma_0))=\tilde{S}_n(x-\frac{1}{n})$ imply the existence of a suitable $c_0>0$ with the property

$$\sigma\mapsto\frac{1}{n}\sum_{z\neq nx-1}L\big(\tilde{V}_n^{-1}(\sigma),\tilde{S}_n(z/n)\big)=c_0$$

in a neighborhood of σ_0. Therefore, we can use (5) to see that $B(\tilde{V}_n^{-1}(\sigma_0))=\tilde{S}_n(x-\frac{1}{n})$ implies that

$$\sigma\mapsto\frac{1}{n}L\big(\tilde{V}_n^{-1}(\sigma),\tilde{S}_n(x-1/n)\big)=\tilde{V}_n(\tilde{V}_n^{-1}(\sigma))-c_0=\sigma-c_0$$

in a neighborhood of σ_0. Consequently, the function

$$\tilde{M}(\sigma):=\frac{1}{n}L\big(\tilde{V}_n^{-1}(\sigma),\tilde{S}_n(x-1/n)\big)$$

is differentiable for all $\sigma\notin\{a_i;\ i\in\mathbb{N}\}$, and for $nx\in\mathbb{Z}$, we have

$$\tilde{M}'(\sigma)=\begin{cases}1, & \text{if } B(\tilde{V}_n^{-1}(\sigma))=\tilde{S}_n\left(x-\dfrac{1}{n}\right),\\ 0, & \text{if } B(\tilde{V}_n^{-1}(\sigma))\neq\tilde{S}_n\left(x-\dfrac{1}{n}\right).\end{cases}$$

Moreover, it is possible to prove that the function $\tilde{M}$ is Lipschitz continuous with Lipschitz constant one. From those properties, it follows that

$$\int_0^\tau\mathbb{1}_{\{x\}}(\tilde{X}_n(\sigma))\,\mathrm{d}\sigma=\int_0^\tau\mathbb{1}_{\{\tilde{S}_n(x-1/n)\}}(B(\tilde{V}_n^{-1}(\sigma)))\,\mathrm{d}\sigma=\int_0^\tau\tilde{M}'(\sigma)\,\mathrm{d}\sigma=\tilde{M}(\tau).\qquad\square$$

2.3. The convergence of the occupation times

In this section, we investigate whether the occupation times of $\tilde{X}_n$ converge toward the local time of $\tilde{X}_*$ in an appropriate way as $n \to \infty$. For this, we first need some auxiliary results.

Lemma 1. *One has $\mathbb{P} \times \tilde{\mathbb{P}}$-almost surely that $\tilde{V}_n(t)$ converges toward $\tilde{V}_*(t)$ for all $t \in \mathbb{R}$.*

Proof. We fix a $T > 0$ and define $w_o := \sup\{x : L(T, x) > 0\}$ and $w_u := \inf\{x : L(T, x) > 0\}$. Those two random variables are independent of $\tilde{\mathbb{P}}$. We know that $\{\tilde{S}_n(x); x \in \mathbb{R}\}$ converges toward $\{\tilde{W}(x); x \in \mathbb{R}\}$ with respect to the J_1-topology $\tilde{\mathcal{F}}$-almost surely. We note that the local time of Brownian motion $(x, t) \mapsto L(t, x)$ is jointly continuous $\mathbb{P}$-almost surely (see Boylan (1964) or Getoor and Kesten (1972)).

It follows that $\mathbb{P} \times \tilde{\mathbb{P}}$-almost surely $\{L(t, \tilde{S}_n(x)); x \in \mathbb{R}\}$ converges toward $\{L(t, \tilde{W}(x)); x \in \mathbb{R}\}$ with respect to the J_1-topology for all $t \in [0, T]$.

We fix a pair $(\omega, \tilde{\omega}) \in \Omega \times \tilde{\Omega}$ with the property that $\{L(t, \tilde{S}_n(x))(\omega, \tilde{\omega}); x \in \mathbb{R}\}$ converges toward $\{L(t, \tilde{W}(x))(\omega, \tilde{\omega}); x \in \mathbb{R}\}$ with respect to the J_1-topology for all $t \in [0, T]$.

There then exist suitable $x_u, x_o \in \mathbb{R}$ with $\tilde{W}(x_u) \le w_u$ and $\tilde{W}(x_o) \ge w_o$, and there exists a sequence of increasing, absolutely continuous, surjective Lipschitz maps $\lambda_n : [x_u, x_o] \to [x_u, x_o]$ with the properties

$$\sup_{x \in [x_u, x_o]} |L(t, \tilde{W}(x)) - L(t, \tilde{S}_n(\lambda_n(x)))| \longrightarrow 0 \qquad \text{as } n \to \infty$$

and

$$\operatorname*{esssup}_{x \in [x_u, x_o]} |\lambda_n'(x) - 1| \longrightarrow 0 \qquad \text{as } n \to \infty.$$

We should emphasize that the derivative of the function λ_n may not exist everywhere. However, those points where it does not exist form a zero set since λ_n is an absolutely continuous Lipschitz function.

By a change of variables for all $t \in [0, T]$, one then has

$$\begin{aligned} &\int_{x_u}^{x_o} L(t, \tilde{S}_n(x))\,\mathrm{d}x - \int_{x_u}^{x_o} L(t, \tilde{S}_n(\lambda_n(x)))\,\mathrm{d}x \\ &\quad = \int_{x_u}^{x_o} L(t, \tilde{S}_n(x))\left(1 - \frac{1}{\lambda_n'(\lambda_n^{-1}(x))}\right)\mathrm{d}x + \mathrm{O}\Bigl(\sup_{x \in [x_u, x_o]} |\lambda_n(x) - x|\Bigr). \end{aligned}$$

It follows from the assumptions on the sequence λ_n that the above difference converges toward zero. Further, for all $t \in [0, T]$, we have that

$$\int_{\mathbb{R}} L(t, \tilde{S}_n(\lambda_n(x)))\,\mathrm{d}x \longrightarrow \int_{\mathbb{R}} L(t, \tilde{W}(x))\,\mathrm{d}x \qquad \text{as } n \to \infty.$$

Hence, one has $\mathbb{P} \times \tilde{\mathbb{P}}$-almost surely that $\tilde{V}_n(t)$ converges toward $\tilde{V}_*(t)$ for all $t \in [0, T]$. Thus, for every $T > 0$, we obtain an zero set N_T in $\Omega \times \tilde{\Omega}$ where this convergence does not hold. The lemma now follows since the union

$$N_\infty := \bigcup_{T\in\mathbb{N}} U_T$$

is also a zero set with respect to $\mathbb{P}\times\tilde{\mathbb{P}}$. □

Let $f:\mathbb{R}\to\mathbb{R}$ be a function. We call $\tau\in f(\mathbb{R})$ a *critical value* for f if there exist at least two distinct points $t_1, t_2\in\mathbb{R}$ such that $f(t_1)=f(t_2)=\tau$. Further, we call a point $\tau\in f(\mathbb{R})$ a *regular value* for f if it is not a critical value. It is straightforward to see that the preimages of critical values contain an open interval if the function f is non-decreasing. This implies that the set of critical values of a non-decreasing function is at most countable.

Lemma 2. *One has $\mathbb{P}\times\tilde{\mathbb{P}}$-almost surely that $\tilde{V}_n^{-1}(\tau)$ converges toward $\tilde{V}_*^{-1}(\tau)$ for all regular values τ of $\tilde{V}_*$.*

Proof. We note that $\mathbb{P}$-almost surely the local time $L(t,x)$ of the Brownian motion B is continuous and non-decreasing in t for all $x\in\mathbb{R}$ (see Boylan (1964) or Getoor and Kesten (1972)) for the continuity). It follows that $\mathbb{P}\times\tilde{\mathbb{P}}$-almost surely the function

$$t\mapsto \tilde{V}_*(t) := \int_{\mathbb{R}} L(t,x)m_*(\mathrm{d}x)$$

is continuous and non-decreasing.

Therefore, $\mathbb{P}\times\tilde{\mathbb{P}}$-almost surely the function $\tilde{V}_*^{-1}(\tau) := \inf\{t;\tilde{V}(t)>\tau\}$ is strictly increasing and right-continuous.

We use Lemma 1 to fix a pair $(\omega,\tilde{\omega})\in\Omega\times\tilde{\Omega}$ with the properties that:

(i) $\tau\mapsto\tilde{V}_*^{-1}(\tau)$ is strictly increasing and right-continuous;
(ii) $\tilde{V}_n(t)$ converges toward $\tilde{V}_*(t)$ for all $t\ge 0$.

Since the set where $\tilde{V}_*$ is not continuous is countable, the set where $\tilde{V}_*$ is continuous is dense in $[0,\infty)$.

We denote by K the set of critical values of $\tilde{V}_*$. As was pointed out before, K is at most countable. For an arbitrary point $\tau\in[0,\infty)\cap K^c$ and for any $\varepsilon>0$, one can find points $t_{\varepsilon,0}, t_{\varepsilon,1}\in(\tilde{V}_*^{-1}(\tau)-\varepsilon, \tilde{V}_*^{-1}(\tau))$ and $t_{\varepsilon,2}, t_{\varepsilon,3}\in(\tilde{V}_*^{-1}(\tau), \tilde{V}_*^{-1}(\tau)+\varepsilon)$ with the property

$$\tilde{V}_*(t_{\varepsilon,0})<\tilde{V}_*(t_{\varepsilon,1})<\tau<\tilde{V}_*(t_{\varepsilon,2})<\tilde{V}_*(t_{\varepsilon,3}).$$

We can now choose a $\delta>0$ such that

$$\tilde{V}_*(t_{\varepsilon,0})+\delta<\tilde{V}_*(t_{\varepsilon,1})-\delta<\tilde{V}_*(t_{\varepsilon,1})+\delta<\tau<\tilde{V}_*(t_{\varepsilon,2})-\delta<\tilde{V}_*(t_{\varepsilon,2})+\delta<\tilde{V}_*(t_{\varepsilon,3})-\delta.$$

Since $\tilde{V}_n$ converges toward $\tilde{V}_*$ in all points where $\tilde{V}_*$ is continuous, there exists an $n_0\in\mathbb{N}$ such that for all $n\ge n_0$, we have

$$\tilde{V}_n(t_{\varepsilon,0})<\tilde{V}_*(t_{\varepsilon,0})+\delta<\tilde{V}_*(t_{\varepsilon,1})-\delta<\tilde{V}_n(t_{\varepsilon,1})<\tilde{V}_*(t_{\varepsilon,1})+\delta<\tau$$

and

$$\tau < \tilde{V}_*(t_{\varepsilon,2}) - \delta < \tilde{V}_n(t_{\varepsilon,2}) < \tilde{V}_*(t_{\varepsilon,2}) + \delta < \tilde{V}_*(t_{\varepsilon,3}) - \delta < \tilde{V}_n(t_{\varepsilon,3}).$$

By definition of $t_{\varepsilon,0}$, we have that $z \le \tilde{V}_*^{-1}(\tau) - \varepsilon$ implies $z \le t_{\varepsilon,0}$. From monotonicity and the first of both inequalities above, it follows that

$$\tilde{V}_n(z) \le \tilde{V}_n(t_{\varepsilon,0}) \le \tilde{V}_*(t_{\varepsilon,0}) + \delta < \tilde{V}_*(t_{\varepsilon,1}).$$

We have thus seen that $z \le \tilde{V}_*^{-1}(\tau) - \varepsilon$ implies $\tilde{V}_n(z) < \tilde{V}_*(t_{\varepsilon,1})$. If we reverse the implication, then we obtain that $\tilde{V}_n(z) \ge \tilde{V}_*(t_{\varepsilon,1})$ implies $z > \tilde{V}_*^{-1}(\tau) - \varepsilon$. From this implication, it follows that

$$\tilde{V}_n^{-1}(\tilde{V}_*(t_{\varepsilon,1})) = \inf\{z : \tilde{V}_n(z) > \tilde{V}_*(t_{\varepsilon,1})\} > \tilde{V}_*^{-1}(\tau) - \varepsilon.$$

For $z = t_{\varepsilon,3}$, we have $\tilde{V}_n(z) = \tilde{V}_n(t_{\varepsilon,3}) > \tilde{V}_*(t_{\varepsilon,2})$. In other words, there exists a $z < \tilde{V}_*^{-1}(\tau) + \varepsilon$ with $\tilde{V}_n(z) > \tilde{V}_*(t_{\varepsilon,2})$. This proves that

$$\tilde{V}_*^{-1}(\tau) + \varepsilon > \tilde{V}_n^{-1}(\tilde{V}_*(t_{\varepsilon,2})).$$

Altogether, we have proven that for all $n \ge n_0$,

$$\tilde{V}_*^{-1}(\tau) - \varepsilon < \tilde{V}_n^{-1}(\tilde{V}_*(t_{\varepsilon,1})) < \tilde{V}_n^{-1}(\tilde{V}_*(t_{\varepsilon,2})) < \tilde{V}_*^{-1}(\tau) + \varepsilon.$$

By monotonicity, for all $n \ge n_0$ and all $\tau' \in [\tilde{V}_*(t_{\varepsilon,1}), \tilde{V}_*(t_{\varepsilon,2})]$, one has

$$\tilde{V}_*^{-1}(\tau) - \varepsilon < \tilde{V}_n^{-1}(\tau') < \tilde{V}_*^{-1}(\tau) + \varepsilon.$$

Since $\tau \in [\tilde{V}_*(t_{\varepsilon,1}), \tilde{V}_*(t_{\varepsilon,2})]$, the proof is complete. □

Lemma 3. *For all $\tau \ge 0$, one has that τ is a regular value of $\tilde{V}_*$ almost surely with respect to $\mathbb{P} \times \tilde{\mathbb{P}}$.*

Proof. By the invariance properties of Brownian motion, we have that for all $\gamma > 0$,

$$\{L(t, w); w \in \mathbb{R}, t \ge 0\} \stackrel{\mathcal{D}}{=} \{\gamma^{-1} L(\gamma^2 t, \gamma w); w \in \mathbb{R}, t \ge 0\}.$$

By the invariance of the α-stable Lévy process, we have that

$$\begin{aligned}\{L(t, \tilde{W}(x)); x \in \mathbb{R}, t \ge 0\} &\stackrel{\mathcal{D}}{=} \{\gamma^{-1} L(\gamma^2 t, \gamma \tilde{W}(x)); x \in \mathbb{R}, t \ge 0\} \\ &\stackrel{\mathcal{D}}{=} \{\gamma^{-1} L(\gamma^2 t, \tilde{W}(\gamma^\alpha x)); x \in \mathbb{R}, t \ge 0\}.\end{aligned}$$

Substitution then yields

$$\begin{aligned}\left\{\int_{\mathbb{R}} L(t, \tilde{W}(x))\,\mathrm{d}x; t \ge 0\right\} &\stackrel{\mathcal{D}}{=} \left\{\gamma^{-1} \int_{\mathbb{R}} L(\gamma^2 t, \tilde{W}(\gamma^\alpha x))\,\mathrm{d}x; t \ge 0\right\} \\ &\stackrel{\mathcal{D}}{=} \left\{\gamma^{-1-\alpha} \int_{\mathbb{R}} L(\gamma^2 t, \tilde{W}(x))\,\mathrm{d}x; t \ge 0\right\}.\end{aligned}$$

By definition, this means that

$$\{\tilde{V}_*(t); t \geq 0\} \stackrel{\mathcal{D}}{=} \{\gamma^{-1-\alpha} \tilde{V}_*(\gamma^2 t); t \geq 0\}.$$

We define ℓ_* to be the image measure of the Lebesgue measure ℓ with respect $\tilde{V}_*$. The previous considerations imply that

$$\ell_*(\mathrm{d}t) \stackrel{\mathcal{D}}{=} \gamma^2 \ell_*(\gamma^{-1-\alpha}\,\mathrm{d}t).$$

This identity implies that no $\tau > 0$ satisfies $\ell_*(\{\tau\}) > 0$ with a positive probability with respect to $\mathbb{P} \times \tilde{\mathbb{P}}$. To a critical value τ corresponds an interval where $t \mapsto \tilde{V}_*$ is constant, which implies that $\ell_*(\{\tau\}) > 0$. For a particular point $\tau > 0$, this cannot happen with positive probability. This finishes the proof of the statement. □

Proposition 3. *For all $\tau \geq 0$, the sequence of functions $x \mapsto L(\tilde{V}_n^{-1}(\tau), \tilde{S}_n(x + 1/n))$ converges toward the function $x \mapsto L(\tilde{V}_*^{-1}(\tau), \tilde{W}(x))$ in the J_1-topology $\mathbb{P} \times \tilde{\mathbb{P}}$-almost surely.*

Proof. It is known that $\tilde{S}_n$ converges toward $\tilde{W}$ in the J_1-topology almost surely with respect to $\tilde{\mathbb{P}}$. Moreover, by Lemmas 2 and 3, for all $\tau \geq 0$, the sequence $\tilde{V}_n^{-1}(\tau)$ converges toward $\tilde{V}_*^{-1}(\tau)$ almost surely with respect to $\mathbb{P} \times \tilde{\mathbb{P}}$. The proposition follows since it is well known that $(t, x) \mapsto L(t, x)$ is jointly continuous $\mathbb{P}$-almost surely; see Boylan (1964) or Getoor and Kesten (1972). □

Lemma 4. *For all $k \in \mathbb{N}$, $\theta_1, \ldots, \theta_k \in \mathbb{R}$ and all $\tau_1, \ldots, \tau_k \geq 0$, the set*

$$\mathcal{C} := \left\{ c > 0 : \ell\left(x \in \mathbb{R}; \left| \sum_{i=1}^{k} \theta_i L(\tilde{V}_*^{-1}(\tau_i), \tilde{W}(x)) \right| = c \right) > 0 \right\}$$

is countable $\mathbb{P} \times \tilde{\mathbb{P}}$-almost surely, where ℓ denotes the Lebesgue measure on $\mathbb{R}$.

Proof. It is well known that $x \mapsto \tilde{W}(x)$ is strictly increasing $\tilde{\mathbb{P}}$-almost surely. For $c > 0$, we define the level-sets

$$\mathcal{N}_c := \left\{ w \in \mathbb{R}; \left| \sum_{i=1}^{k} \theta_i L(\tilde{V}_*^{-1}(\tau_i), w) \right| = c \right\}.$$

Fix a strictly increasing path $f : x \mapsto \tilde{W}(x)$ and assume that there exist an uncountable number of $c > 0$ with the property that $\ell(f^{-1}(\mathcal{N}_c)) > 0$. For $c \neq c'$, the sets $f^{-1}(\mathcal{N}_c)$ and $f^{-1}(\mathcal{N}_{c'})$ are disjoint. We would obtain an uncountable number of disjoint sets with positive Lebesgue measure. This is, of course, not possible. □

Proposition 4. *For all $k \in \mathbb{N}$, $\theta_1, \ldots, \theta_k \in \mathbb{R}$ and all $\tau_1, \ldots, \tau_k \geq 0$, one has $\mathbb{P} \times \tilde{\mathbb{P}}$-almost surely that*

$$\frac{1}{n}\,\mathrm{card}\left\{x \in \mathbb{Z} : n\left|\sum_{i=1}^{k} \theta_i \tilde{\Gamma}_n(\tau_i, \{x/n\})\right| > c\right\}$$

$$\longrightarrow \ell\left(x \in \mathbb{R} : \left|\sum_{i=1}^{k} \theta_i \tilde{L}_*(\tau_i, x)\right| > c\right) \qquad \text{as } n \to \infty$$

for all but a countable number of $c > 0$.

Proof. We can find a $K > 0$ such that $\{y \in \mathbb{R} : L(\tau_i, y) \neq 0 \text{ for all } i = 1, \ldots, k\}$ is a subset of the interval $(\tilde{W}(-K), \tilde{W}(K))$. By Propositions 2, 3 and Corollary 1, the sequence

$$\begin{aligned}\tilde{A}_n(x) &:= n\left|\sum_{i=1}^{k} \theta_i \tilde{\Gamma}_n(\tau_i, \{x\})\right| \\ &= \left|\sum_{i=1}^{k} \theta_i L\big(\tilde{V}_n^{-1}(\tau_i), \tilde{S}_n(x - 1/n)\big)\right|\end{aligned}$$

converges $\mathbb{P} \times \tilde{\mathbb{P}}$-almost surely in the J_1-topology toward

$$\tilde{A}_*(x) := \left|\sum_{i=1}^{k} \theta_i \tilde{L}_*(\tau_i, x)\right| = \left|\sum_{i=1}^{k} \theta_i L(\tilde{V}_*^{-1}(\tau_i), \tilde{W}(x))\right|.$$

There then exists a sequence of continuous increasing maps $\lambda_n : [-K, K] \to [-K, K]$ such that

$$\sup_{x \in [-K,K]} |\tilde{A}_*(x) - \tilde{A}_n \circ \lambda_n(x)| \longrightarrow 0 \qquad \text{as } n \to \infty$$

and such that each λ_n is Lipschitz continuous and satisfies

$$\operatorname*{esssup}_{x \in [-K,K]} |\lambda_n'(x) - 1| \longrightarrow 0.$$

We should emphasize that the derivative of the function λ_n may not exist everywhere. However, those points where the derivative does not exist form a zero set since λ_n is an absolutely continuous Lipschitz function. We note that for suitably large $n \in \mathbb{N}$, one has

$$\frac{1}{n}\,\mathrm{card}\left\{x \in \mathbb{R};\left|\sum_{i=1}^{k} \theta_i L\big(\tilde{V}_n^{-1}(\tau_i), \tilde{S}_n(x - 1/n)\big)\right| > c\right\}$$

$$= \ell\big(x \in [-K, K];\, \tilde{A}_n(x) > c\big) = \int_{-K}^{K} \mathbb{1}_{(c,\infty)}(\tilde{A}_n(x))\,\mathrm{d}x.$$

It then follows that

$$\frac{1}{n}\operatorname{card}\left\{x \in [-K,K]; n\left|\sum_{i=1}^{k}\theta_i\tilde{\Gamma}_n(\tau_i,\{x\})\right| > c\right\} - \int_{-K}^{K}\mathbb{1}_{(c,\infty)}(\tilde{A}_n(\lambda_n(x)))\,\mathrm{d}x$$
$$= \int_{-K}^{K}\mathbb{1}_{(c,-\infty)}(\tilde{A}_n(x))\,\mathrm{d}x\left(1-\frac{1}{\lambda_n'(\lambda_n^{-1}(x))}\right)\mathrm{d}x + \mathrm{O}\Big(\sup_{x\in[-K,K]}|\lambda_n(x)-x|\Big).$$

By the assumptions on the sequence $\{\lambda_n; n \in \mathbb{N}\}$, the previous difference converges toward zero. Furthermore,

$$\int_{-K}^{K}\mathbb{1}_{(c,\infty)}(\tilde{A}_n(\lambda_n(x)))\,\mathrm{d}x \longrightarrow \int_{-K}^{K}\mathbb{1}_{(c,\infty)}(\tilde{A}_*(x))\,\mathrm{d}x \qquad \text{as } n\to\infty$$

whenever the set $\{x \in [-K,K]; \tilde{A}_*(s) = c\}$ is a zero set with respect to the Lebesgue measure ℓ on $\mathbb{R}$. Since this was proven in Lemma 4, the statement of the proposition follows. □

Subsequently, we will make use of the following notation:

$$A_n^+ := \left\{x \in \mathbb{Z}: \sum_{i=1}^{k}\theta_i\tilde{\Gamma}_n(\tau_i,\{x/n\}) > 0\right\}, \qquad A_n^- := \left\{x \in \mathbb{Z}: \sum_{i=1}^{k}\theta_i\tilde{\Gamma}_n(\tau_i,\{x/n\}) < 0\right\}$$

and

$$A^+ := \left\{x \in \mathbb{R}: \sum_{i=1}^{k}\theta_i\tilde{L}_*(\tau_i,x) > 0\right\}, \qquad A^- := \left\{x \in \mathbb{R}: \sum_{i=1}^{k}\theta_i\tilde{L}_*(\tau_i,x) < 0\right\}.$$

Later, we will need the following version of Proposition 4.

Proposition 5. *For all $k \in \mathbb{N}$, $\theta_1,\dots,\theta_k \in \mathbb{R}$ and all $\tau_1,\dots,\tau_k \geq 0$, one has $\mathbb{P}\times\tilde{\mathbb{P}}$-almost surely that*

$$\frac{1}{n}\operatorname{card}\left\{x \in \mathbb{Z}\cap A_n^{\pm}: n\left|\sum_{i=1}^{k}\theta_i\tilde{\Gamma}_n(\tau_i,\{x/n\})\right| > c\right\} \longrightarrow \ell\left(x \in \mathbb{R}\cap A^{\pm}: \left|\sum_{i=1}^{k}\theta_i\tilde{L}_*(\tau_i,x)\right| > c\right)$$

for all but a countable number of $c > 0$.

Proof. The proof uses essentially the same arguments as the proof of Proposition 4. □

Remark. With the same proof as for Proposition 4, we can show that $\mathbb{P}\times\tilde{\mathbb{P}}$-almost surely

$$\frac{1}{n}\operatorname{card}\big\{x \in \mathbb{Z}: n^2\tilde{\Gamma}_n^2(\tau_i,\{x/n\}) > c\big\} \longrightarrow \ell\big(x \in \mathbb{R}: \tilde{L}_*^2(\tau_i,x) > c\big) \qquad \text{as } n\to\infty$$

for all but a countable number of $c > 0$.

2.4. A useful lemma on integrated powers of local time

Lemma 5. *For $\tau_1, \dots, \tau_k \geq 0$ and $\theta_1, \dots, \theta_k \in \mathbb{R}$, the two sequences of random variables*

$$n^{\beta-1} \sum_{x\in\mathbb{Z}} \left| \sum_{i=1}^{k} \theta_i \tilde{\Gamma}_n(\tau_i, \{x/n\}) \right|^{\beta} \quad \textit{and}$$

$$n^{\beta-1} \sum_{x\in\mathbb{Z}} \left(\left| \sum_{i=1}^{k} \theta_i \tilde{\Gamma}_n(\tau_i, \{x/n\}) \right|^{\beta} \operatorname{sgn}\left(\sum_{i=1}^{k} \theta_i \tilde{\Gamma}_n(\tau_i, \{x/n\}) \right) \right)$$

converge $\mathbb{P} \times \tilde{\mathbb{P}}$-almost surely toward the respective random variables

$$\int_{-\infty}^{\infty} \left| \sum_{i=1}^{k} \theta_i \tilde{L}_*(\tau_i, x) \right|^{\beta} \mathrm{d}x \quad \textit{and} \quad \int_{-\infty}^{\infty} \left(\left| \sum_{i=1}^{k} \theta_i \tilde{L}_*(\tau_i, x) \right|^{\beta} \operatorname{sgn}\left(\sum_{i=1}^{k} \theta_i \tilde{L}_*(\tau_i, x) \right) \right) \mathrm{d}x.$$

Proof. We use the layer cake representation of the integrals (see Lieb and Loss (2001)) to write

$$\sum_{x\in\mathbb{Z}} \left| \sum_{i=1}^{k} \theta_i n \tilde{\Gamma}_n(\tau_i, \{x/n\}) \right|^{\beta} = \beta \int_0^{\infty} c^{\beta-1} \operatorname{card}\left\{ x \in \mathbb{Z} : n \left| \sum_{i=1}^{k} \theta_i \tilde{\Gamma}_n(\tau_i, \{x/n\}) \right| > c \right\} \mathrm{d}c$$

and

$$\int_{-\infty}^{\infty} \left| \sum_{i=1}^{k} \theta_i \tilde{L}_*(\tau_i, x) \right|^{\beta} \mathrm{d}x = \beta \int_0^{\infty} c^{\beta-1} \ell\left(x \in \mathbb{R} : \left| \sum_{i=1}^{k} \theta_i \tilde{L}_*(\tau_i, x) \right| > c \right) \mathrm{d}c.$$

We note that the convergence of $\tilde{V}_n^{-1}(\tau_i)$ toward $\tilde{V}_*^{-1}(\tau_i)$ and the fact that $t \mapsto L(t, y)$ is increasing for every $y \in \mathbb{R}$ imply that there exists an $n_0 \in \mathbb{N}$ with

$$L(\tilde{V}_n^{-1}(\tau_i), y) \leq L\big(\tilde{V}_*^{-1}(\tau_i) + 1, y\big) \qquad \text{for all } y \in \mathbb{R}, 1 \leq i \leq k, n \geq n_0.$$

Moreover, for all $i \in \{1, \dots, k\}$, the functions $y \mapsto L(\tilde{V}_*^{-1}(\tau_i) + 1, y)$ are continuous and their supports are contained in $[-K, K]$ for a suitable $K > 0$. Hence, there exists a $C > 0$ such that for $n \geq n_0$, one has

$$\begin{aligned} n \left| \sum_{i=1}^{k} \theta_i \tilde{\Gamma}_n(\tau_i, \{x/n\}) \right| &\leq \left| \sum_{i=1}^{k} \theta_i L\big(\tilde{V}_n^{-1}(\tau_i), \tilde{S}_n\big((x-1)/n\big)\big) \right| \\ &\leq \sum_{i=1}^{k} \theta_i \sup_{y\in\mathbb{R}} L\big(\tilde{V}_*^{-1}(\tau_i) + 1, y\big) \leq C. \end{aligned}$$

This implies that all of the functions

$$c \mapsto \operatorname{card}\left\{x \in \mathbb{Z} : n\left|\sum_{i=1}^{k} \theta_i \tilde{\Gamma}_n(\tau_i, \{x/n\})\right| > c\right\}$$

have support contained in $[0, C]$. Moreover, for all $c > 0$, we have

$$\operatorname{card}\left\{x \in \mathbb{Z} : n\left|\sum_{i=1}^{k} \theta_i \tilde{\Gamma}_n(\tau_i, \{x/n\})\right| > c\right\} \leq \operatorname{card}\big\{x \in \mathbb{Z} : -K \leq \tilde{S}_n\big((x-1)/n\big) \leq K\big\}.$$

Since

$$\ell\big(x;\, \tilde{W}(x) \in \{-K, K\}\big) = 0$$

and since $\tilde{S}_n$ converges toward $\tilde{W}$ with respect to the Skorohod metric, we have that

$$\frac{1}{n}\operatorname{card}\big\{x \in \mathbb{Z} : -K \leq \tilde{S}_n\big((x-1)/n\big) \leq K\big\} \longrightarrow \ell\big(x \in \mathbb{R} : -K \leq \tilde{W}(x) \leq K\big).$$

This implies that there exists an $R > 0$ such that for all $n \in \mathbb{N}$ and all $c > 0$, we have

$$\frac{1}{n}\operatorname{card}\left\{x \in \mathbb{Z} : n\left|\sum_{i=1}^{k} \theta_i \tilde{\Gamma}_n(\tau_i, \{x/n\})\right| > c\right\} \leq R.$$

The first statement of the lemma then follows from dominated convergence and Proposition 4. The second statement is proved in the same way by separating the positive and the negative parts of the integrals and using the statements from Proposition 5 instead of Proposition 4. □

Proposition 6. *For $\tau_1, \dots, \tau_k \geq 0$ and $\theta_1, \dots, \theta_k \in \mathbb{R}$, the two sequences of random variables*

$$n^{\beta-1}\sum_{x\in\mathbb{Z}}\left|\sum_{i=1}^{k} \theta_i \Gamma_n(\tau_i, \{x/n\})\right|^{\beta} \quad \textit{and}$$

$$n^{\beta-1}\sum_{x\in\mathbb{Z}}\left(\left|\sum_{i=1}^{k} \theta_i \Gamma_n(\tau_i, \{x/n\})\right|^{\beta} \operatorname{sgn}\left(\sum_{i=1}^{k} \theta_i \Gamma_n(\tau_i, \{x/n\})\right)\right)$$

converge jointly in distribution toward the respective random variables

$$\int_{-\infty}^{\infty}\left|\sum_{i=1}^{k} \theta_i L_*(\tau_i, x)\right|^{\beta} \mathrm{d}x \quad \textit{and} \quad \int_{-\infty}^{\infty}\left(\left|\sum_{i=1}^{k} \theta_i L_*(\tau_i, x)\right|^{\beta} \operatorname{sgn}\left(\sum_{i=1}^{k} \theta_i L_*(\tau_i, x)\right)\right) \mathrm{d}x.$$

Proof. We know that

$$\{L_*(t, x);\, t \geq 0, x \in \mathbb{R}\} \stackrel{\mathcal{D}}{=} \{\tilde{L}_*(t, x);\, t \geq 0, x \in \mathbb{R}\}$$

and

$$\{S_n^{-1}(B_n(V_n^{-1}(t)));\, t \geq 0\} \stackrel{\mathcal{D}}{=} \{\tilde{S}_n^{-1}(B(\tilde{V}_n^{-1}(t)));\, t \geq 0\}.$$

Therefore, by Lemma 5, the sequences of random variables

$$n^{\beta-1}\sum_{x\in\mathbb{Z}}\left|\sum_{i=1}^{k}\theta_i\hat{\Gamma}_n(\tau_i,\{x/n\})\right|^{\beta} \quad \text{and}$$

$$n^{\beta-1}\sum_{x\in\mathbb{Z}}\left(\left|\sum_{i=1}^{k}\theta_i\hat{\Gamma}_n(\tau_i,\{x/n\})\right|^{\beta}\operatorname{sgn}\left(\sum_{i=1}^{k}\theta_i\hat{\Gamma}_n(\tau_i,\{x/n\})\right)\right)$$

converge jointly in distribution toward the respective random variables

$$\int_{-\infty}^{\infty}\left|\sum_{i=1}^{k}\theta_i L_*(\tau_i,x)\right|^{\beta}\mathrm{d}x \quad \text{and} \quad \int_{-\infty}^{\infty}\left(\left|\sum_{i=1}^{k}\theta_i L_*(\tau_i,x)\right|^{\beta}\operatorname{sgn}\left(\sum_{i=1}^{k}\theta_i L_*(\tau_i,x)\right)\right)\mathrm{d}x.$$

Moreover, $S_n^{-1}(S_n(x/n)) = (x+1)/n$ for all $x \in \mathbb{Z}$. This implies that

$$\hat{X}_n(\tau) \stackrel{\mathcal{D}}{=} S_n^{-1}(S_n(X_n(\tau))) = X_n(\tau) + 1/n.$$

Hence, we have $\hat{\Gamma}_n(\tau,\{x/n\}) \stackrel{\mathcal{D}}{=} \Gamma_n(\tau,\{(x+1)/n\})$ for all $x \in \mathbb{Z}$. Therefore,

$$n^{\beta-1}\sum_{x\in\mathbb{Z}}\left|\sum_{i=1}^{k}\theta_i\hat{\Gamma}_n(\tau_i,\{x/n\})\right|^{\beta} \stackrel{\mathcal{D}}{=} n^{\beta-1}\sum_{x\in\mathbb{Z}}\left|\sum_{i=1}^{k}\theta_i\Gamma_n(\tau_i,\{x/n\})\right|^{\beta}$$

and

$$n^{\beta-1}\sum_{x\in\mathbb{Z}}\left(\left|\sum_{i=1}^{k}\theta_i\hat{\Gamma}_n(\tau_i,\{x/n\})\right|^{\beta}\operatorname{sgn}\left(\sum_{i=1}^{k}\theta_i\hat{\Gamma}_n(\tau_i,\{x/n\})\right)\right)$$
$$\stackrel{\mathcal{D}}{=} n^{\beta-1}\sum_{x\in\mathbb{Z}}\left(\left|\sum_{i=1}^{k}\theta_i\Gamma_n(\tau_i,\{x/n\})\right|^{\beta}\operatorname{sgn}\left(\sum_{i=1}^{k}\theta_i\Gamma_n(\tau_i,\{x/n\})\right)\right).$$

This proves the proposition. □

For the sequel, we define the occupation time

$$\Gamma(t,A) := \int_0^t \mathbb{1}_A(X(s))\,\mathrm{d}s$$

of the process X in the measurable set $A \subset \mathbb{R}$. Consequently, we have

$$\Xi(t) = \sum_x \Gamma(t,\{x\})\xi(x).$$

We will use this fact and the following corollary in the proofs of the next section.

Corollary 2. *For $\tau_1, \dots, \tau_k \geq 0$ and $\theta_1, \dots, \theta_k \in \mathbb{R}$, the two sequences of random variables*

$$n^{-1-\beta/\alpha} \sum_{x \in \mathbb{Z}} \left| \sum_{i=1}^{k} \theta_i \Gamma(k_n \tau_i, \{x\}) \right|^{\beta} \quad \textit{and}$$

$$n^{-1-\beta/\alpha} \sum_{x \in \mathbb{Z}} \left(\left| \sum_{i=1}^{k} \theta_i \Gamma(k_n \tau_i, \{x\}) \right|^{\beta} \operatorname{sgn}\left(\sum_{i=1}^{k} \theta_i \Gamma(k_n \tau_i, \{x\}) \right) \right)$$

converge jointly in distribution toward the respective random variables

$$\int_{-\infty}^{\infty} \left| \sum_{i=1}^{k} \theta_i L_*(\tau_i, x) \right|^{\beta} \mathrm{d}x \quad \textit{and}$$

$$\int_{-\infty}^{\infty} \left(\left| \sum_{i=1}^{k} \theta_i L_*(\tau_i, x) \right|^{\beta} \operatorname{sgn}\left(\sum_{i=1}^{k} \theta_i L_*(\tau_i, x) \right) \right) \mathrm{d}x.$$

Proof. If we let $k_n := n^{(1+\alpha)/\alpha}$, then for all $n \in \mathbb{N}$ and $x \in \mathbb{Z}$, we have that

$$\Gamma_n(\tau, x/n) = \int_0^{\tau} \mathbb{1}_{\{x/n\}}(X_n(t))\, \mathrm{d}t = k_n^{-1} \int_0^{k_n \tau} \mathbb{1}_{\{x\}}(X(t))\, \mathrm{d}t = n^{-(\alpha+1)/\alpha} \Gamma(k_n \tau, \{x\}).$$

The result then follows from Proposition 6. □

3. The finite-dimensional distributions

In this section, we prove the convergence of the finite-dimensional distributions of Ξ_n toward the finite-dimensional distributions of Ξ_*. In order to do so, we first compute the exact expression of the finite-dimensional distributions of Ξ_*. The proofs in this section follow the ideas given in Kesten and Spitzer (1979).

In the Introduction, we defined

$$\Xi_*(\tau) := \int_0^{\infty} L_*(\tau, x-)\, \mathrm{d}Z_+(x) + \int_0^{\infty} L_*(\tau, -(x-))\, \mathrm{d}Z_-(x),$$

where $\{Z_+(t); t \geq 0\}$ and $\{Z_-(t); t \geq 0\}$ are independent copies of the β-stable Lévy process, which can be associated with the stable distribution ϑ_β with characteristic function given by

$$\psi(\theta) = \exp\bigl(-|\theta|^{\beta}\bigl(A_1 + \mathrm{i}A_2 \operatorname{sgn}(\theta)\bigr)\bigr).$$

Lemma 6. *For $t_1, \dots, t_k \geq 0$ and $\theta_1, \dots, \theta_k \in \mathbb{R}$, we have that*

$$\mathbb{E}\left[\exp\left(\mathrm{i}\sum_{j=1}^{k}\theta_j\,\Xi_*(t_j)\right)\right]$$
$$= \mathbb{E}\left[\exp\left(-A_1\int_{-\infty}^{\infty}\left|\sum_{j=1}^{k}\theta_j L_*(t_j,x)\right|^{\beta}\mathrm{d}x\right)\right.$$
$$\left.\times\exp\left(-\mathrm{i}A_2\int_{-\infty}^{\infty}\left|\sum_{j=1}^{k}\theta_j L_*(t_j,x)\right|^{\beta}\mathrm{d}x\,\mathrm{sgn}\left(\sum_{j=1}^{k}\theta_j L_*(t_j,x)\right)\right)\right].$$

Proof. The proof is similar to that given in Kesten and Spitzer (1979) (see page 16ff). Let ν be the Lévy measure of Z_+. One can truncate the Lévy measure as follows:

$$\nu_1(B) = \nu(B \cap \{y \in \mathbb{R}; |y| \leq 1\}) \quad \text{and} \quad \nu_2(B) = \nu(B \cap \{y \in \mathbb{R}; |y| > 1\}).$$

Let $M(t)$ and $A(t)$ be independent Lévy processes, with respective characteristic functions

$$\mathbb{E}\big[\mathrm{e}^{\mathrm{i}\theta M(t)}\big] = \exp\left(t\int_{|y|\leq 1}(\mathrm{e}^{\mathrm{i}\theta y} - 1 - \mathrm{i}\theta y)\nu_1(\mathrm{d}y)\right)$$

and

$$\mathbb{E}\big[\mathrm{e}^{\mathrm{i}\theta A(t)}\big] = \exp\left(t\int_{|y|\leq 1}(\mathrm{e}^{\mathrm{i}\theta y} - 1)\nu_2(\mathrm{d}y)\right),$$

such that

$$Z^+(t) = M(t) + A(t) + Dt,$$

where D is a suitable real constant. This decomposition exists and is called the Lévy–Itô representation of Z^+. The advantage of this representation is that $M(t)$ is a martingale and has all moments and $A(t)$ is a process with bounded variation. Since the process $\{L_*(t,x-); x \geq 0\}$ is left-continuous and independent with respect to the filtration $\mathcal{F}_t$ generated by $Z^+(t)$, the process $\{L_*(t,x-); x \geq 0\}$ is $\mathcal{F}_t$-predictable. Moreover, $\{L_*(t,x-); x \geq 0\}$ has bounded support $\mathbb{P}$-almost surely. Therefore, we can find a suitable sequence of partitions $\{x_l^{(n)}; l \in \mathbb{N}\}$, $n \in \mathbb{N}$, with $x_l^{(n)} < x_{l+1}^{(n)}$ for all $l, n \in \mathbb{N}$ satisfying

$$\lim_{l\to\infty} x_l^{(n)} = \infty \quad \text{and} \quad \lim_{n\to\infty}\max_{l\in\mathbb{N}}\big(x_{l+1}^{(n)} - x_l^{(n)}\big) = 0$$

such that

$$\int_0^{\infty} L_*(t,x-)\,\mathrm{d}M(x) = \lim_{n\to\infty}\sum_{l=1}^{\infty} L_*\big(t,x_l^{(n)}-\big)\big(M\big(x_{l+1}^{(n)}\big) - M\big(x_l^{(n)}\big)\big)$$

with probability 1 (see Meyer (1976), Chapter II, Section 23). Moreover, we can also assume that

$$\int_0^\infty L_*(t,x-)\,\mathrm{d}A(x) = \lim_{n\to\infty}\sum_{l=1}^{\infty} L_*\big(t,x_l^{(n)}-\big)\big(A\big(x_{l+1}^{(n)}\big)-A\big(x_l^{(n)}\big)\big)$$

with probability 1.

From those considerations, it follows that there exists a sequence of partitions $(x_l^{(n)})_{l\in\mathbb{N}}$ such that

$$\int_0^\infty L_*(t,x-)\,\mathrm{d}Z_+(x) = \lim_{n\to\infty}\sum_{l=1}^{\infty} L_*\big(t,x_l^{(n)}-\big)\big(Z_+\big(x_{l+1}^{(n)}\big)-Z_+\big(x_l^{(n)}\big)\big)$$

with probability 1. Since the increments $D_l^{(n)} := Z_+(x_{l+1}^{(n)}) - Z_+(x_l^{(n)}),\ l\in\mathbb{N}$, are independent and have characteristic function

$$\mathbb{E}\big[\mathrm{e}^{\mathrm{i}\theta D_l^{(n)}}\big] = \exp\big(-\big(x_{l+1}^{(n)}-x_l^{(n)}\big)|\theta|^\beta\big(A_1+\mathrm{i}A_2\cdot\operatorname{sgn}(\theta)\big)\big)$$

by dominated convergence, we have

$$\begin{aligned}
&\mathbb{E}\left[\exp\left(\mathrm{i}\sum_{j=1}^{k}\theta_j\int_0^\infty L_*(t_j,x-)\,\mathrm{d}Z_+(x)\right)\right]\\
&\quad=\lim_{n\to\infty}\mathbb{E}\left[\exp\left(\sum_{l=1}^{\infty}\sum_{j=1}^{k}\mathrm{i}\theta_j L_*\big(t_j,x_l^{(n)}-\big)\big(Z_+\big(x_{l+1}^{(n)}\big)-Z_+\big(x_l^{(n)}\big)\big)\right)\right]\\
&\quad=\lim_{n\to\infty}\mathbb{E}\left[\exp\left(-\sum_{l=1}^{\infty}\big(x_{l+1}^{(n)}-x_l^{(n)}\big)\left|\sum_{j=1}^{k}\theta_j L_*\big(t_j,x_l^{(n)}-\big)\right|^\beta\right.\right.\\
&\qquad\qquad\left.\left.\times\left(A_1+\mathrm{i}A_2\cdot\operatorname{sgn}\left(\sum_{j=1}^{k}\theta_j L_*\big(t_j,x_l^{(n)}-\big)\right)\right)\right)\right].\\
&\quad=\mathbb{E}\left[\exp\left(-A_1\int_0^\infty\left|\sum_{j=1}^{k}\theta_j L_*\big(t_j,x_l^{(n)}\big)\right|^\beta\mathrm{d}x\right.\right.\\
&\qquad\qquad\left.\left.-\mathrm{i}A_2\int_0^\infty\left|\sum_{j=1}^{k}\theta_j L_*\big(t_j,x_l^{(n)}\big)\right|^\beta\operatorname{sgn}\left(\sum_{j=1}^{k}\theta_j L_*\big(t_j,x_l^{(n)}\big)\right)\mathrm{d}x\right)\right].
\end{aligned}$$

For Z_-, one can proceed with similar arguments. □

Proposition 7. *The finite-dimensional distributions of the processes* $\{\Xi_n(t);t\ge 0\}$ *converge toward the finite-dimensional distributions of the process* $\{\Xi_*(t);t\ge 0\}$.

Proof. As in the previous sections, we define $k_n := n^{(1+\alpha)/\alpha}$ and $\kappa := \frac{1}{\alpha} + \frac{1}{\beta}$. We already saw that we can use the occupation time $\{\Gamma(t, \{x\}); t \geq 0, x \in \mathbb{R}\}$ of the process $\{X(t); t \geq 0\}$ to represent the process $\{\Xi(t); t \geq 0\}$ as follows:

$$\Xi(t) = \sum_{x \in \mathbb{Z}} \Gamma(t, \{x\}) \xi(x).$$

It follows that

$$\Xi_n(t) = n^{-\kappa} \Xi(k_n t) = n^{-\kappa} \sum_{x \in \mathbb{Z}} \Gamma(k_n t, \{x\}) \xi(x).$$

Let $\varphi(\theta) := \mathbb{E}[\exp(\mathrm{i}\theta\xi(1))]$ be the characteristic function of the scenery random variable $\xi(1)$. It then follows from the above representation that

$$\sum_{j=1}^{k} \theta_j \Xi_n(t_j) = n^{-\kappa} \sum_{x \in \mathbb{Z}} \sum_{j=1}^{k} \theta_j \Gamma(k_n t_j, \{x\}) \xi(x)$$

and

$$R_n := \mathbb{E}\left[\exp\left(\mathrm{i} \sum_{j=1}^{k} \theta_j \Xi_n(t_j)\right)\right] = \mathbb{E}\left[\prod_{x \in \mathbb{Z}} \varphi\left(n^{-\kappa} \sum_{j=1}^{k} \theta_j \Gamma(k_n t_j, \{x\})\right)\right].$$

The random scenery $\{\xi(z); z \in \mathbb{Z}\}$ is in the domain of attraction of a β-stable distribution with characteristic function given by

$$\psi(\theta) = \exp\bigl(-|\theta|^{\beta}\bigl(A_1 + \mathrm{i}A_2 \cdot \mathrm{sgn}(\theta)\bigr)\bigr).$$

This implies that

$$1 - \varphi(\theta) \sim |\theta|^{\beta}\bigl(A_1 + \mathrm{i}A_2 \cdot \mathrm{sgn}(\theta)\bigr) \qquad \text{as } \theta \to 0.$$

Thus

$$\log(\varphi(\theta)) \sim \log(\psi(\theta)) \qquad \text{as } \theta \to 0.$$

Therefore, for $|\theta| \leq 1$, we have that

$$\left|\frac{\log(\varphi(\theta)) - \log(\psi(\theta))}{\log(\psi(\theta))}\right| = \mathrm{o}(\theta).$$

If we define

$$\varphi_{x,n} := \varphi\left(n^{-\kappa} \sum_{j=1}^{k} \theta_j \Gamma(k_n t_j, \{x\})\right)$$

and

$$\psi_{x,n} := \exp\left(-n^{-\kappa\beta}\left|\sum_{j=1}^{k}\theta_j\Gamma(k_nt_j,\{x\})\right|^{\beta}\left(A_1 + \mathrm{i}A_2\cdot\operatorname{sgn}\left(\sum_{j=1}^{k}\theta_j\Gamma(k_nt_j,\{x\})\right)\right)\right)$$

for all $x \in \mathbb{Z}$, one has

$$\left|\frac{\log(\varphi_{x,n}) - \log(\psi_{x,n})}{\log(\psi_{x,n})}\right| = \mathrm{o}\left(n^{-\kappa}\sum_{j=1}^{k}\theta_j\Gamma(k_nt_j,\{x\})\right).$$

This implies that

$$\begin{aligned}\left|\log\left(\prod_{x\in\mathbb{Z}}\varphi_{x,n}\right) - \log\left(\prod_{x\in\mathbb{Z}}\psi_{x,n}\right)\right| &= \left|\sum_{x\in\mathbb{Z}}\log(\varphi_{x,n}) - \sum_{x\in\mathbb{Z}}\log(\psi_{x,n})\right| \\ &\le \sum_{x\in\mathbb{Z}}\log(\psi_{x,n})\mathrm{o}\left(n^{-\kappa}\sum_{j=1}^{k}\theta_j\Gamma(k_nt_j,\{x\})\right).\end{aligned}$$

By Corollary 2, the right-hand side of the previous inequality converges toward zero in probability. The continuity of the logarithm then implies that

$$\left|\prod_{x\in\mathbb{Z}}\varphi_{x,n} - \prod_{x\in\mathbb{Z}}\psi_{x,n}\right| \longrightarrow 0 \qquad \text{in probability as } n \to \infty.$$

We use this and dominated convergence to prove that the limit of the sequence $\{R_n; n \in \mathbb{N}\}$ exists and is equal to the limit of the sequence

$$Q_n := \mathbb{E}\left[\exp\left(-\sum_{x\in\mathbb{Z}}n^{-\kappa\beta}\left|\sum_{j=1}^{k}\theta_j\Gamma(k_nt_j,\{x\})\right|^{\beta}\left(A_1 + \mathrm{i}A_2\cdot\operatorname{sgn}\left(\sum_{j=1}^{k}\theta_j\Gamma(k_nt_j,\{x\})\right)\right)\right)\right].$$

By Corollary 2 and Lemma 6, the sequence $\{Q_n; n \in \mathbb{N}\}$ converges toward

$$\begin{aligned}Q_* &:= \mathbb{E}\left[\exp\left(-\int_{-\infty}^{\infty}\left|\sum_{j=1}^{k}\theta_jL_*(t_j,x)\right|^{\beta}\left(A_1 + \mathrm{i}A_2\cdot\operatorname{sgn}\left(\sum_{j=1}^{k}\theta_jL_*(t_j,x)\right)\right)\mathrm{d}x\right)\right] \\ &= \mathbb{E}\left[\exp\left(\mathrm{i}\sum_{j=1}^{k}\theta_j\,\Xi_*(t_j)\right)\right].\end{aligned}$$

As we have seen in Lemma 6, Q_* is the characteristic function for the finite-dimensional distributions of $\{\Xi_*(t); t \ge 0\}$. This completes the proof of the proposition. $\square$

4. The tightness

In this section, we prove that the sequence $\{\Xi_n(t); t \geq 0\}$ is tight. The proof of Theorem 1 then follows since we have already obtained the convergence of the finite-dimensional distributions in the previous section. The main proof of tightness also follows the ideas given in Kesten and Spitzer (1979). We first need some suitable inequalities for the occupation times of X_*. However, the proofs of those inequalities differ from those given in Kesten and Spitzer (1979).

Lemma 7. *There exists a function* $\varepsilon : \mathbb{R}^+ \to \mathbb{R}^+$ *with the properties* $\varepsilon(A) \to 0$ *as* $A \to \infty$ *and*

$$\mathbb{P}\big(\Gamma(s, \{x\}) > 0 \textit{ for some } x \textit{ with } |x| > As^{\alpha/(1+\alpha)}\big) \leq \varepsilon(A) \qquad \textit{for all } s \geq 0.$$

Proof. For a positive real number x, we denote by $\lceil x \rceil$ the smallest integer which is greater or equal to x. Obviously, for all $s \geq 0$, we have

$$\begin{aligned}
&\mathbb{P}\big(\Gamma(s, \{x\}) > 0 \text{ for some } x \text{ with } |x| > As^{\alpha/(1+\alpha)}\big)\\
&\quad \leq \mathbb{P}\big(|X(r)| > As^{\alpha/(1+\alpha)} \text{ for some } r \leq s\big)\\
&\quad \leq \mathbb{P}\big(|X(r)| > A\big(\big\lceil s^{\alpha/(1+\alpha)}\big\rceil - 1\big) \text{ for some } r \leq \big\lceil s^{\alpha/(1+\alpha)}\big\rceil^{(1+\alpha)/\alpha}\big)\\
&\quad = \mathbb{P}\big(\big|X\big(\big\lceil s^{\alpha/(1+\alpha)}\big\rceil^{(1+\alpha)/\alpha} u\big)\big| > A\big\lceil s^{\alpha/(1+\alpha)}\big\rceil - A \text{ for some } u \leq 1\big)\\
&\quad \leq \mathbb{P}\Big(\sup_{r\leq 1}\big|X_{n(s)}(r)\big| > A/2\Big) \qquad \text{for } s > 1,
\end{aligned}$$

with $n(s) := \lceil s^{\alpha/(1+\alpha)} \rceil \to \infty$ as $s \to \infty$. Since

$$\mathbb{P}\Big(\sup_{r\leq 1}|X_n(r)| > A/2\Big) \longrightarrow \mathbb{P}\Big(\sup_{r\leq 1}|X_*(r)| > A/2\Big) \qquad \text{as } n \to \infty,$$

we can define

$$\varepsilon(A) := \sup_{s\geq 0} \mathbb{P}\Big(\sup_{r\leq 1}\big|X_{n(s)}(r)\big| > A/2\Big) \qquad \text{for all } A > 0.$$

This proves the statement of the lemma. □

Lemma 8. *There exists a* $C > 0$ *such that for all* $s \geq 0$, *one has*

$$\sum_{x\in\mathbb{Z}} \mathbb{E}[\Gamma^2(s, \{x\})] \sim Cs^{2-\alpha/(1+\alpha)}.$$

Proof. For a positive real number x, we denote by $\lfloor x \rfloor$ its integer part. We know that for $w(s) := \lfloor s^{\alpha/(\alpha+1)} \rfloor$, one has

$$\frac{(w(s))^{2(\alpha+1)/\alpha}}{s^2} \sum_{x\in\mathbb{Z}} \Gamma^2_{w(s)}\big(1, \{x/w(s)\}\big) = s^{-2} \sum_{x\in\mathbb{Z}} \Gamma^2\big((w(s))^{(\alpha+1)/\alpha}, \{x\}\big) \leq s^{-2} \sum_{x\in\mathbb{Z}} \Gamma^2(s, \{x\})$$

and

$$s^{-2}\sum_{x\in\mathbb{Z}}\Gamma^2(s,\{x\}) \le s^{-2}\sum_{x\in\mathbb{Z}}\Gamma^2\big(\big(w(s)+1\big)^{(\alpha+1)/\alpha},\{x\}\big)$$
$$= \frac{(w(s)+1)^{2(\alpha+1)/\alpha}}{s^2}\sum_{x\in\mathbb{Z}}\Gamma^2_{w(s)+1}\big(1,\big\{x/\big(w(s)+1\big)\big\}\big).$$

Consequently, one has

$$s^{-2}\sum_{x\in\mathbb{Z}}\mathbb{E}[\Gamma^2(s,\{x\})] \sim \sum_{x\in\mathbb{Z}}\mathbb{E}\big[\Gamma^2_{w(s)}\big(1,\{x/w(s)\}\big)\big] = \sum_{x\in\mathbb{Z}}\mathbb{E}\big[\tilde{\Gamma}^2_{w(s)}\big(1,\{x/w(s)\}\big)\big].$$

It follows from the layer cake representation and the remark after the proof of Proposition 5 that

$$w(s)\sum_{x\in\mathbb{Z}}\tilde{\Gamma}^2_{w(s)}\big(1,\{x/w(s)\}\big) = \frac{1}{w(s)}\int_0^\infty \operatorname{card}\big\{x\in\mathbb{Z}: w^2(s)\tilde{\Gamma}^2_{w(s)}\big(1,\{x/w(s)\}\big) > c\big\}\,\mathrm{d}c$$

converges $\mathbb{P}\times\tilde{\mathbb{P}}$-almost surely toward

$$\int_0^\infty \ell\big(x\in\mathbb{R}: \tilde{L}^2(1,x) > c\big)\,\mathrm{d}c = \int_{\mathbb{R}} \tilde{L}^2_*(1,x)\,\mathrm{d}x.$$

Dominated convergence and Fubini's theorem imply that

$$w(s)\sum_{x\in\mathbb{Z}}\mathbb{E}\big[\tilde{\Gamma}^2_{w(s)}\big(1,\{x/w(s)\}\big)\big] \longrightarrow \int_{\mathbb{R}}\mathbb{E}[\tilde{L}^2_*(1,x)]\,\mathrm{d}x \qquad \text{as } s\to\infty.$$

Therefore,

$$w(s)s^{-2}\sum_{x\in\mathbb{Z}}\mathbb{E}[\Gamma^2(s,\{x\})] \longrightarrow \int_{\mathbb{R}}\mathbb{E}[\tilde{L}^2_*(1,x)]\,\mathrm{d}x \qquad \text{as } s\to\infty.$$

This proves the statement of the lemma. □

Lemma 9. (1) *For all $\beta\in(0,2]$ and $\rho>0$, there exists a $C_1>0$ such that as $n\to\infty$, we have*

$$\big|\mathbb{E}\big[\xi(0)\mathbb{1}_{[-\rho,\rho]}(n^{-1/\beta}\xi(0))\big]\big| \sim C_1 n^{(1-\beta)/\beta}.$$

(2) *For all $\beta\in(0,2)$ and $\rho>0$, there exists a $C_2>0$ such that as $n\to\infty$, we have*

$$\big|\mathbb{E}\big[\xi^2(0)\mathbb{1}_{[-\rho,\rho]}\big(n^{-1/\beta}\xi(0)\big)\big]\big| \sim C_2 n^{(2-\beta)/\beta}.$$

Proof. The random variable $\xi(0)$ is in the domain of attraction of a β-stable random variable with characteristic function given by

$$\psi(\theta) = \exp\big(-|\theta|^\beta\big(A_1 + \mathrm{i}A_2\operatorname{sgn}(\theta)\big)\big),$$

with $0 < A_1 < \infty$ and $|A_1^{-1}A_2| \le \tan(\pi\beta/2)$. A consequence of this setting is that for $\beta > 1$, we have $\mathbb{E}[\xi(0)] = 0$. Further, if $\beta \in (0, 2]$, then there exist $B_1, B_2 \ge 0$ such that

$$\lim_{\rho\to\infty} \rho^\beta \mathbb{P}\big(\xi(0) \ge \rho\big) = B_1 \quad \text{and} \quad \lim_{\rho\to\infty} \rho^\beta \mathbb{P}\big(\xi(0) \le -\rho\big) = B_2.$$

For $\beta = 2$, we have $B_1 = B_2 = 0$ since the decay of the tail probabilities is exponential in that case. For $\beta \ne 1$, we then have that

$$\begin{aligned}\big|\mathbb{E}\big[\xi(0)\mathbb{1}_{[-\rho,\rho]}(n^{-1/\beta}\xi(0))\big]\big| &= \int_0^{\rho n^{1/\beta}} \mathbb{P}\big(|\xi(0)| \ge c\big)\,\mathrm{d}c \\ &\sim (B_1 + B_2)\int_0^{\rho n^{1/\beta}} c^{-\beta}\,\mathrm{d}c \\ &= (B_1 + B_2)(1-\beta)^{-1}\rho^{1-\beta}n^{(1/\beta)(1-\beta)}.\end{aligned}$$

This proves the first statement for $\beta \ne 1$. For $\beta = 1$, the statement is just our assumption from the Introduction.

Moreover, by similar arguments for $\beta \ne 2$, we have that

$$\begin{aligned}\big|\mathbb{E}\big[\xi^2(0)\mathbb{1}_{[-\rho,\rho]}(n^{-1/\beta}\xi(0))\big]\big| &\sim (B_1 + B_2)\int_0^{\rho n^{1/\beta}} c^{1-\beta}\,\mathrm{d}c \\ &= (B_1 + B_2)(2-\beta)^{-1}\rho^{2-\beta}n^{(1/\beta)(2-\beta)}.\end{aligned}$$

This completes the proof of the second statement. □

Proposition 8. *The distributions of the sequence $\{\Xi_n; n \in \mathbb{N}\}$ are tight with respect to the Skorohod topology.*

Proof. We follow the method given in Kesten and Spitzer (1979). Let $\varepsilon > 0$ be given. By Lemma 7, there exists an $A > 0$ such that $\varepsilon(AT^{-\alpha/(1+\alpha)}) \le \varepsilon/4$. This implies that

$$\begin{aligned}&\mathbb{P}\Bigg(\Xi_n(t) \ne n^{-\kappa}\sum_{|x|\le An}\Gamma(k_n t, \{x\})\xi(x) \text{ for some } t \le T\Bigg) \\ &\quad \le \mathbb{P}\big(\Gamma(k_n T, \{x\}) > 0 \text{ for some } x \text{ with } |x| > Ak_n^{\alpha/(1+\alpha)}\big) \\ &\quad \le \varepsilon\big(AT^{-\alpha/(1+\alpha)}\big) \\ &\quad \le \varepsilon/4.\end{aligned}$$

There exists a $\rho_0 > 0$ with the property that for all $\rho > \rho_0$ and all $n \in \mathbb{N}$, we have

$$3An\big(1 - \mathbb{P}\big(-\rho n^{1/\beta} \le \xi(0) \le \rho n^{1/\beta}\big)\big) \le \varepsilon/4.$$

This is valid since for suitable $B_1, B_2 \geq 0$, we have

$$\lim_{\rho\to\infty} \rho^\beta \mathbb{P}\big(\xi(0) \geq \rho\big) = B_1 \quad \text{and} \quad \lim_{\rho\to\infty} \rho^\beta \mathbb{P}\big(\xi(0) \leq -\rho\big) = B_2.$$

For all $x \in \mathbb{Z}$, we have the random variables

$$\begin{aligned}\bar{\xi}_n(x) &:= \xi(x)\mathbb{1}_{[-\rho,\rho]}(n^{-1/\beta}\xi(x)),\\ E_n &:= n^{-\kappa}\frac{1}{T}\mathbb{E}\Big[\sum_{x\in\mathbb{Z}}\Gamma(k_n t,\{x\})\bar{\xi}_n(x)\Big] = n^{-\kappa}\frac{1}{T}\mathbb{E}\Big[\sum_{x\in\mathbb{Z}}\Gamma(k_n t,\{x\})\mathbb{E}[\bar{\xi}_n(x)]\Big]\end{aligned}$$

and

$$\bar{\Xi}_n(t) := n^{-\kappa}\sum_{x\in\mathbb{Z}}\Gamma(k_n t,\{x\})\big(\bar{\xi}_n(x) - \mathbb{E}[\bar{\xi}_n(x)]\big).$$

Claim 1. The family of random variables $\{E_n(t); n \in \mathbb{N}\}$ is bounded. This is true since, by Lemma 9, we have

$$\begin{aligned}\Big|\sum_{x\in\mathbb{Z}}\Gamma(k_n t,\{x\})\mathbb{E}[\bar{\xi}_n(x)]\Big| &= |\mathbb{E}[\bar{\xi}_n(0)]|\sum_{x\in\mathbb{Z}}\Gamma(k_n t,\{x\})\\ &= k_n t|\mathbb{E}[\bar{\xi}_n(0)]| \leq Ctn^{(\alpha+1)/\alpha}n^{(1/\beta)(1-\beta)}\end{aligned}$$

and $\frac{\alpha+1}{\alpha} + \frac{1}{\beta}(1-\beta) - \kappa = 0$.

Claim 2. For all $\eta > 0$, there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, we have

$$\mathbb{P}\Big(\sup_{t\leq T}|\Xi_n(t) - \bar{\Xi}_n(t) - E_n t| > \frac{\eta}{2}\Big) \leq \frac{\varepsilon}{2}.$$

To see this, we first note that

$$\Xi_n(t) - \bar{\Xi}_n(t) - E_n t = n^{-\kappa}\sum_{x\in\mathbb{Z}}\Gamma(k_n t,\{x\})\big(\xi(x) - \bar{\xi}_n(x)\big)$$

since

$$\begin{aligned}&\Xi_n(t) - \bar{\Xi}_n(t) - E_n t - n^{-\kappa}\sum_{x\in\mathbb{Z}}\Gamma(k_n t,\{x\})\big(\xi(x) - \bar{\xi}_n(x)\big)\\ &= n^{-\kappa}\Big(\sum_{x\in\mathbb{Z}}\Gamma(k_n t,\{x\})\mathbb{E}[\bar{\xi}(x)] - \frac{t}{T}\mathbb{E}\Big[\sum_{x\in\mathbb{Z}}\Gamma(k_n t,\{x\})\mathbb{E}[\bar{\xi}(x)]\Big]\Big)\\ &= n^{-\kappa}\mathbb{E}[\bar{\xi}(0)]\Big(\sum_{x\in\mathbb{Z}}\Gamma(k_n t,\{x\}) - \frac{t}{T}\mathbb{E}\Big[\sum_{x\in\mathbb{Z}}\Gamma(k_n t,\{x\})\Big]\Big)\\ &= n^{-\kappa}\mathbb{E}[\bar{\xi}(0)]\Big(k_n t - \frac{t}{T}k_n T\Big)\\ &= 0.\end{aligned}$$

Lemma 9 implies that

$$\begin{aligned}
&\mathbb{P}\bigg(n^{-\kappa}\sum_{x\in\mathbb{Z}}\Gamma(k_nt,\{x\})\big(\xi(x)-\bar{\xi}_n(x)\big)\neq 0\text{ for some }t\le T\bigg)\\
&\quad\le\mathbb{P}\big(\Gamma(k_nT,\{x\})>0\text{ for some }x\text{ with }|x|>Ak_n^{\alpha/(1+\alpha)}\big)\\
&\qquad+\mathbb{P}\big(\xi(x)\neq\bar{\xi}_n(x)\text{ for some }|x|\le Ak_n^{\alpha/(1+\alpha)}\big)\\
&\quad\le\varepsilon\big(AT^{-\alpha/(1+\alpha)}\big)+3Ak_n^{\alpha/(1+\alpha)}\mathbb{P}\big(\xi(0)\neq\bar{\xi}_n(0)\big)\\
&\quad\le\frac{\varepsilon}{4}+3An\big(1-\mathbb{P}\big(-\rho n^{1/\beta}\le\xi(0)\le\rho n^{1/\beta}\big)\big)\\
&\quad\le\frac{\varepsilon}{2}.
\end{aligned}$$

Claim 3. There exists a $K_0>0$ such that for all $n\in\mathbb{N}$, we have

$$\mathbb{E}[|\bar{\Xi}_n(t_2)-\bar{\Xi}_n(t_1)|^2]\le C_0(t_2-t_1)^{2-(1+\alpha)/\alpha}.$$

We define the σ-field $\mathcal{X}=\{X(t);t\ge 0\}$. It then follows from the independence of $\{X(t);t\ge 0\}$ and $\{\xi(x);x\in\mathbb{Z}\}$ that

$$\begin{aligned}
&\mathbb{E}\bigg[\bigg(\sum_{x\in\mathbb{Z}}\big(\Gamma(k_nt_2,\{x\})-\Gamma(k_nt_1,\{x\})\big)\bar{\xi}_n(x)\bigg)^2\bigg]\\
&\quad=\mathbb{E}\bigg[\mathbb{E}\bigg[\bigg(\sum_{x\in\mathbb{Z}}\big(\Gamma(k_nt_2,\{x\})-\Gamma(k_nt_1,\{x\})\big)\bar{\xi}_n(x)\bigg)^2\bigg|\mathcal{X}\bigg]\bigg]\\
&\quad=\mathbb{E}\bigg[\sum_{x\in\mathbb{Z}}\big(\Gamma(k_nt_2,\{x\})-\Gamma(k_nt_1,\{x\})\big)^2\mathbb{E}[\bar{\xi}_n^2(x)|\mathcal{X}]\bigg]\\
&\quad=\sum_{x\in\mathbb{Z}}\mathbb{E}\big[\big(\Gamma(k_nt_2,\{x\})-\Gamma(k_nt_1,\{x\})\big)^2\big]\mathbb{E}[\bar{\xi}_n^2(x)].
\end{aligned}$$

This implies that

$$\begin{aligned}
\mathbb{E}[|\bar{\Xi}_n(t_2)-\bar{\Xi}_n(t_1)|^2]&\le n^{-2\kappa}\sum_{x\in\mathbb{Z}}\mathbb{E}\big[\big(\Gamma(k_nt_2,\{x\})-\Gamma(k_nt_1,\{x\})\big)^2\big]\mathbb{E}[\bar{\xi}_n^2(x)]\\
&=n^{-2\kappa}\mathbb{E}\bigg[\sum_{x\in\mathbb{Z}}\big(\Gamma(k_nt_2,\{x\})-\Gamma(k_nt_1,\{x\})\big)^2\bigg]\mathbb{E}[\bar{\xi}_n^2(0)].
\end{aligned}$$

Conditioned on $\mathcal{A}:=\{\lambda_i;i\in\mathbb{Z}\}$, the process X has the strong Markov property. Using this, we can prove that for $t_1\le t_2$, the conditional distribution of $\sum_x(\Gamma(t_2,\{x\})-\Gamma(t_1,\{x\}))^2$ with respect to $\mathcal{A}$ equals the conditional distribution of $\sum_x \Gamma^2(t_2-t_1,\{x\})$ with respect to $\mathcal{A}$. Hence,

$$\begin{aligned}\mathbb{E}\Big[\sum_{x\in\mathbb{Z}}\big(\Gamma(t_2,\{x\})-\Gamma(t_1,\{x\})\big)^2\Big] &= \mathbb{E}\Big[\mathbb{E}\Big[\sum_{x\in\mathbb{Z}}\big(\Gamma(t_2,\{x\})-\Gamma(t_1,\{x\})\big)^2\big|\mathcal{A}\Big]\Big]\\ &= \mathbb{E}\Big[\mathbb{E}\Big[\sum_{x\in\mathbb{Z}}\Gamma^2(t_2-t_1,\{x\})\big|\mathcal{A}\Big]\Big]\\ &= \mathbb{E}\Big[\sum_{x\in\mathbb{Z}}\Gamma^2(t_2-t_1,\{x\})\Big].\end{aligned}$$

By Lemma 8, it follows that

$$\begin{aligned}\mathbb{E}\Big[\sum_{x\in\mathbb{Z}}\big(\Gamma(k_nt_2,\{x\})-\Gamma(k_nt_1,\{x\})\big)^2\Big] &\le Ck_n^{2-\alpha/(1+\alpha)}(t_2-t_1)^{2-\alpha/(1+\alpha)}\\ &= Cn^{2(1+\alpha)/\alpha-1}(t_2-t_1)^{2-\alpha/(1+\alpha)}.\end{aligned}$$

Moreover, we know that

$$\mathbb{E}[\bar{\xi}_n^2(0)] \le \tilde{C}n^{(2-\beta)(1/\beta)}.$$

Putting this all together, we obtain

$$\mathbb{E}[|\bar{\Xi}_n(t_2)-\bar{\Xi}_n(t_1)|^2] \le C_0 n^{(2-\beta)(1/\beta)}n^{-2\kappa}n^{2(1+\alpha)/\alpha-1}(t_2-t_1)^{2-\alpha/(1+\alpha)}.$$

Since $(2-\beta)\frac{1}{\beta}-2\kappa+2\frac{1+\alpha}{\alpha}-1=0$, Claim 3 follows.

Since $2-\frac{\alpha}{1+\alpha}>1$, the tightness in the Skorohod topology of the family $\{\Xi_n; n\in\mathbb{N}\}$ now follows from Claims 1–3 and a theorem of Billingsley (1968) (see page 95). □

Acknowledgements

The authors wish to express their deepest gratitude toward the probability group and the staff of Academia Sinica and National Taiwan University for mathematical and administrational help during their visit to Taiwan. Special thanks go to Shieh Narn-Rueih, Hwang Chii-Ruey and Sheu Shuenn-Jyi for many interesting discussions on probability theory. Moreover, the authors would like to thank the referee for his very detailed report which helped to improve the manuscript.

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Received January 2009 and revised May 2009