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Approximating a geometric fractional Brownian motion and related processes via discrete Wick calculus

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We approximate the solution of some linear systems of SDEs driven by a fractional Brownian motion B^H with Hurst parameter $H \in (\frac{1}{2}, 1)$ in the Wick–Itô sense, including a geometric fractional Brownian motion. To this end, we apply a Donsker-type approximation of the fractional Brownian motion by disturbed binary random walks due to Sottinen. Moreover, we replace the rather complicated Wick products by their discrete counterpart, acting on the binary variables, in the corresponding systems of Wick difference equations. As the solutions of the SDEs admit series representations in terms of Wick powers, a key to the proof of our Euler scheme is an approximation of the Hermite recursion formula for the Wick powers of B^H.

Keywords: discrete Wick calculus; fractional Brownian motion; weak convergence; Wick–Itô integral

1. Introduction

A fractional Brownian motion B^H with Hurst parameter $H \in (0, 1)$ is a continuous zero-mean Gaussian process in $\mathbb{R}$ with stationary increments and covariance function

$$\mathbf{E}[B_t^H B_s^H] = \tfrac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}).$$

The process $B^{1/2}$ is a standard Brownian motion, but a fractional Brownian motion is not a semimartingale for $H \neq \frac{1}{2}$. In this paper, we restrict ourselves to the case $H > 1/2$, in which the corresponding fractional Gaussian noise $(B_{n+1}^H - B_n^H)_{n\in\mathbb{N}}$ exhibits long-range dependence.

In recent years, a lively interest in integration theory with respect to fractional Brownian motion has emerged (see, e.g., the monographs by Mishura or Biagini *et al.* [4,15]). One of the extensions of the Itô integral beyond semimartingales is the fractional Wick–Itô integral. It is based on the Wick product $\diamond$, which has its origin as a renormalization operator in quantum physics. In probability theory, the Wick product with ordinary differentiation rule imitates the situation of ordinary multiplication with Itô differentiation rule (cf. Holden *et al.* [11]). Actually, this makes it a natural tool to apply for extending the Itô integral.

We first consider the fractional Doléans–Dade SDE $\mathrm{d}S_t = S_t \mathrm{d}^\diamond B_t^H$, $S_0 = 1$, in terms of the fractional Wick–Itô integral. The well-known solution, $\exp(B_t^H - \frac{1}{2}t^{2H})$, is the geometric fractional Brownian motion, also known as the Wick exponential of fractional Brownian motion. Note that the Wick exponential has expectation equal to one and can therefore be interpreted as a multiplicative noise. Moreover, the ordinary exponential can be obtained from the Wick exponential by a deterministic scaling. Neither process is a semimartingale for $H \neq \frac{1}{2}$. The name "Wick exponential" is justified by the fact that it exhibits a power series expansion with Wick powers $(B_t^H)^{\diamond k}$ instead of ordinary powers.

More generally, we consider a linear system of SDEs,

$$\begin{aligned} \mathrm{d}X_t &= (A_1 X_t + A_2 Y_t)\, \mathrm{d}^{\diamond} B_t^H, \qquad X_0 = x_0, \\ \mathrm{d}Y_t &= (B_1 X_t + B_2 Y_t)\, \mathrm{d}^{\diamond} B_t^H, \qquad Y_0 = y_0. \end{aligned} \tag{1}$$

One can obtain Wick power series expansions for the solution of this system, too. Our goal is to approximate these Wick analytic functionals of a fractional Brownian motion. To this end, we require an approximation of a fractional Brownian motion and an approximation of the Wick product.

There are several ways to approximate a fractional Brownian motion. One of the first approximations was given by Taqqu [22] in terms of stationary Gaussian sequences. We refer to Mishura [15], Section 1.15.3, for further approaches to weak convergence to a fractional Brownian motion. Sottinen constructed a simple approximation of a fractional Brownian motion on an interval for $H > \frac{1}{2}$ by sums of square-integrable random variables in [21]. He used the Wiener integral representation of a fractional Brownian motion on an interval, $B_t^H = \int_0^t z_H(t,s)\, \mathrm{d}B_s$, for a suitable deterministic kernel $z_H(t,s)$, due to Molchan and Golosov, and Norros *et al.* [16–18]. For this purpose, he combined a pointwise approximation of the kernel $z_H(t,s)$ with Donsker's theorem. This approach was extended by Nieminen [19] to weak convergence of perturbed martingale differences to fractional Brownian motion. We shall utilize Sottinen's approximation with binary random variables throughout this paper.

The main problem of applying the Wick product on random variables with continuous distributions is that it is not a pointwise operation. Thus, an explicit computation of the Wick–Itô integral is only possible in rare special cases. But this is precisely the advantage of the binary random walks. In such a purely discrete setup, we apply the discrete counterpart of the Wick product as introduced in Holden *et al.* [10]. Starting from the binary random walk, one can build up a discrete Wiener space, and the discrete Wick product depends on this discretization. This Wiener chaos gives the analogy to the continuous Wick products. For a survey on discrete Wiener chaos, we refer to Gzyl [9]. However, we will introduce the discrete Wick product in a self-contained way in Section 3.

We can now formulate a weak Euler scheme of the linear system of SDEs (1) in the Wick–Itô sense,

$$\begin{aligned} X_l^n &= X_{l-1}^n + (A_1 X_{l-1}^n + A_2 Y_{l-1}^n) \diamond_n \big(B_{l/n}^{H,n} - B_{(l-1)/n}^{H,n}\big), \\ X_0^n &= x_0, \qquad l = 1, \ldots, n, \\ Y_l^n &= Y_{l-1}^n + (B_1 X_{l-1}^n + B_2 Y_{l-1}^n) \diamond_n \big(B_{l/n}^{H,n} - B_{(l-1)/n}^{H,n}\big), \\ Y_0^n &= y_0, \qquad l = 1, \ldots, n, \end{aligned} \tag{2}$$

where $\diamond_n$ is the discrete Wick product and $(B_{l/n}^{H,n} - B_{(l-1)/n}^{H,n})$ are the increments of the disturbed binary random walk. As a main result, we show that the piecewise constant interpolation of the solution of (2) converges weakly in the Skorokhod space to the solution of (1). This is the first rigorous convergence result connecting discrete and continuous Wick calculus of which we are aware. As a special case, (2) contains the Wick difference equation

$$X_l^n = X_{l-1}^n + X_{l-1}^n \diamond_n \big(B_{l/n}^{H,n} - B_{(l-1)/n}^{H,n}\big), \qquad X_0^n = 1, \qquad l = 1, \ldots, n. \tag{3}$$

As a consequence, the piecewise constant interpolation of (3) converges weakly to a geometric fractional Brownian motion, the solution of the fractional Doléans–Dade SDE. This was conjectured by Bender and Elliott [3] in their study of the Wick fractional Black–Scholes market.

In [21], Sottinen considered the corresponding difference equation in the pathwise sense, that is, with ordinary multiplication instead of the discrete Wick product:

$$\hat{X}_l^n = \hat{X}_{l-1}^n + \hat{X}_{l-1}^n \big(B_{l/n}^{H,n} - B_{(l-1)/n}^{H,n}\big), \qquad \hat{X}_0^n = 1, \qquad l = 1, \ldots, n. \tag{4}$$

The solution is explicitly given by the multiplicative expression

$$\hat{X}_l^n = \prod_{j=1}^{l} \big(1 + \big(B_{j/n}^{H,n} - B_{(j-1)/n}^{H,n}\big)\big). \tag{5}$$

By the logarithmic transform of ordinary products into sums and a Taylor expansion, one obtains an additive expression for $\ln(\hat{X}_l^n)$ which converges weakly to a fractional Brownian motion. In this way, Sottinen proved the convergence of $\hat{X}$ to the ordinary exponential of a fractional Brownian motion [21], Theorem 3. This approach fails for the solution of (3) since, in a product representation, analogous to (5), the discrete Wick product $\diamond_n$ appears instead of ordinary multiplication. There is, however, no straightforward way to transform discrete Wick products into sums by application of a continuous functional.

However, the solution of (2) exhibits an expression which is closely related to a discrete Wick power series representation. Therefore, the convergence can be initiated explicitly for the Wick powers of a fractional Brownian motion, which fulfill the Hermite recursion formula. We obtain a discrete analog to this recursion formula for discrete Wick powers of disturbed binary random walks. Actually, the weak convergence of these discrete Wick powers is the key to the proof for our Euler scheme.

The paper is organized as follows. In Section 2, we give some preliminaries on the Wick–Itô integral with respect to a fractional Brownian motion and introduce the Wick exponential and other Wick analytic functionals. We then define the approximating sequences and state the main results in Section 3. Section 4 is devoted to some L^2- estimates of the approximating sequences. We prove convergence in finite-dimensional distributions in Section 5 and tightness in Section 6.

2. Wick exponential and Wick analytic functionals

In this section, we introduce the Wick product and the Wick–Itô integral, and describe the Hermite recursion formula for Wick powers of a zero-mean Gaussian random variable. We then obtain the Wick power series expansions for the solutions of SDEs (1).

We consider a *geometric fractional Brownian motion* or the so-called *Wick exponential* of a fractional Brownian motion $\exp(B_t^H - \frac{1}{2}t^{2H})$. For $H = \frac{1}{2}$, this is exactly a geometric Brownian motion, also known as the stochastic exponential of a standard Brownian motion. For all $H \in (0, 1)$ and $t \geq 0$, it holds that $t^{2H} = \mathbf{E}[(B_t^H)^2]$ and thus the Wick exponential generalizes the stochastic exponential. It is well known that $\exp(B_t - \frac{1}{2}t)$ solves the Doléans–Dade equation

$$\mathrm{d}S_t = S_t \,\mathrm{d}B_t, \qquad S_0 = 1,$$

where the integral is an ordinary Itô integral. Actually, the Wick exponential of fractional Brownian motion solves the corresponding fractional Doléans–Dade equation

$$\mathrm{d}S_t = S_t \,\mathrm{d}^\diamond B_t^H, \qquad S_0 = 1,$$

in terms of a fractional Wick–Itô integral (cf. Mishura [15], Theorem 3.3.2). We want to approximate solutions of similar SDEs.

Let Φ and Ψ be two zero-mean Gaussian random variables. The *Wick exponential* is then defined as

$$\exp^\diamond(\Phi) := \exp\bigl(\Phi - \tfrac{1}{2}\mathbf{E}[|\Phi|^2]\bigr).$$

For a standard Brownian motion $(B_t)_{t\geq 0}$ and $s < t < u$, it holds that

$$\exp^\diamond(B_u - B_t)\exp^\diamond(B_t - B_s) = \exp^\diamond(B_u - B_s).$$

Forcing this renormalization property to hold for all, possibly correlated, Φ and Ψ, leads to the definition of the *Wick product* $\diamond$ of two Wick exponentials:

$$\exp^\diamond(\Phi) \diamond \exp^\diamond(\Psi) := \exp^\diamond(\Phi + \Psi).$$

The Wick product can be extended to larger classes of random variables by density arguments (cf. [2,6,20]). For a general introduction to the Wick product, we refer to the monographs by Kuo and Holden *et al.* [11,14] and Hu and Yan [13]. Note that the Wick product is not a pointwise operation. If we suppose that $\Phi \sim \mathcal{N}(0, \sigma)$, then we have, by definition, $\Phi^{\diamond 0} = 1$, $\Phi^{\diamond 1} = \Phi$ and the recursion

$$\Phi^{\diamond n+1} = \Phi^{\diamond n} \diamond \Phi.$$

Observe that it holds that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\exp^\diamond(x\Phi)\bigg|_{x=0} &= \frac{\mathrm{d}}{\mathrm{d}x}\exp\biggl(x\Phi - \frac{1}{2}\mathbf{E}[|x\Phi|^2]\biggr)\bigg|_{x=0}\\ &= (\Phi - x\sigma^2)\exp\biggl(x\Phi - \frac{1}{2}\mathbf{E}[|x\Phi|^2]\biggr)\bigg|_{x=0} = \Phi.\end{aligned}$$

Suppose we have

$$\Phi^{\diamond k} = \frac{\mathrm{d}^k}{\mathrm{d}w^k}\exp^\diamond(w\Phi)\bigg|_{w=0}$$

for all positive integers $k \le n$. Then, with $z = w + x$, $\frac{\mathrm{d}z}{\mathrm{d}w} = \frac{\mathrm{d}z}{\mathrm{d}x} = 1$, we get

$$\begin{aligned}\Phi^{\diamond(n+1)} &= \frac{\mathrm{d}^n}{\mathrm{d}w^n} \exp^{\diamond}(w\Phi)\bigg|_{w=0} \diamond \frac{\mathrm{d}}{\mathrm{d}x} \exp^{\diamond}(x\Phi)\bigg|_{x=0} \\ &= \frac{\mathrm{d}^n}{\mathrm{d}w^n} \frac{\mathrm{d}}{\mathrm{d}x} \exp^{\diamond}\big((w+x)\Phi\big)\bigg|_{w=0,x=0} = \frac{\mathrm{d}^{n+1}}{\mathrm{d}z^{n+1}} \exp^{\diamond}(z\Phi)\bigg|_{z=0}.\end{aligned}$$

We now obtain, by differentiation and the Leibniz rule, the following Wick recursion formula:

$$\begin{aligned}\Phi^{\diamond n+1} &= \frac{\mathrm{d}^n}{\mathrm{d}w^n}\left((\Phi - w\sigma^2)\exp\left(w\Phi - \frac{1}{2}\mathbf{E}[|w\Phi|^2]\right)\right)\bigg|_{w=0} \\ &= (\Phi - w\sigma^2)\frac{\mathrm{d}^n}{\mathrm{d}w^n}\exp\left(w\Phi - \frac{1}{2}\mathbf{E}[|x\Phi|^2]\right)\bigg|_{w=0} \\ &\quad + n(-\sigma^2)\frac{\mathrm{d}^{n-1}}{\mathrm{d}w^{n-1}}\exp\left(w\Phi - \frac{1}{2}\mathbf{E}[|w\Phi|^2]\right)\bigg|_{w=0} \\ &= \Phi\Phi^{\diamond n} - n\sigma^2\Phi^{\diamond n-1}.\end{aligned} \tag{6}$$

Define the *Hermite polynomial of degree $n \in \mathbb{N}$ with parameter σ^2* as

$$h^n_{\sigma^2}(x) := (-\sigma^2)^n \exp\left(\frac{x^2}{2\sigma^2}\right)\frac{\mathrm{d}^n}{\mathrm{d}x^n}\exp\left(\frac{-x^2}{2\sigma^2}\right).$$

The series expansion

$$\exp\left(x - \frac{1}{2}\sigma^2\right) = \sum_{n=0}^{\infty}\frac{1}{n!}h^n_{\sigma^2}(x) \tag{7}$$

then holds true. The first Hermite polynomials are $h^0_{\sigma^2}(x) = 1$, $h^1_{\sigma^2}(x) = x$. By the Leibniz rule, we obtain the Hermite recursion formula

$$h^{n+1}_{\sigma^2}(x) = xh^n_{\sigma^2}(x) - n\sigma^2 h^{n-1}_{\sigma^2}(x). \tag{8}$$

By the equivalent first terms and recursions (6) and (8), we can conclude that for any Gaussian random variable $\Phi \sim \mathcal{N}(0,\sigma)$ and all $n \in \mathbb{N}$, we have

$$\Phi^{\diamond n} = h^n_{\sigma^2}(\Phi). \tag{9}$$

By (7), we additionally have

$$\exp^{\diamond}(\Phi) = \sum_{n=0}^{\infty}\frac{1}{n!}\Phi^{\diamond n}. \tag{10}$$

The fractional Wick–Itô integral, introduced by Duncan *et al.* [6], is an extension of the Itô integral beyond the semimartingale framework. There are several approaches to the fractional

Wick–Itô integral. Essentially, these approaches are via white noise theory, as in Elliott and von der Hoek [7], and Hu and Øksendal [12], by Malliavin calculus in Alòs *et al.* [1], or by an S-transform approach in Bender [2]. In contrast to the forward integral, the fractional Wick–Itô integral has zero mean in general. This is the crucial property for an additive noise. The Wick–Itô integral is based on the Wick product. For a sufficiently good process $(X_s)_{s\in[0,t]}$, the fractional Wick–Itô integral with respect to fractional Brownian motion $(B_s^H)_{[0,t]}$ can be easily defined by Wick–Riemann sums (cf. Duncan *et al.* [6] or Mishura [15], Theorem 2.3.10). If we suppose that $\pi_n = \{0 = t_0 < t_1 < \cdots < t_n = t\}$ with $\max_{t_i\in\pi_n} |t_i - t_{i-1}| \to 0$ for $n \to \infty$, then

$$\int_0^t X_s \,\mathrm{d}^\diamond B_s^H := \lim_{n\to\infty} \sum_{t_i\in\pi_n} X_{t_{i-1}} \diamond (B_{t_i}^H - B_{t_{i-1}}^H),$$

if the Wick products and the $L^2(\Omega)$-limit exist. For more information on Wick–Itô integrals with respect to fractional Brownian motion, we refer to Mishura [15], Chapter 2.

By the fractional Itô formula (cf. [2], Theorem 5.3 or [4], Theorem 3.7.2), we have

$$\mathrm{d}(B_t^H)^{\diamond k} = k(B_t^H)^{\diamond k-1}\,\mathrm{d}^\diamond B_t^H, \qquad (B_0^H)^{\diamond k} = \mathbf{1}_{\{k=0\}}. \tag{11}$$

For the Wick exponential

$$\exp^\diamond(B_t^H) = \sum_{k=0}^{\infty} \frac{1}{k!}(B_t^H)^{\diamond k}, \tag{12}$$

we obtain, by summing up the identity (11), the fractional Doléans–Dade equation,

$$\mathrm{d}S_t = S_t\,\mathrm{d}^\diamond B_t^H, \qquad S_0 = 1. \tag{13}$$

For any analytic function $F(x) = \sum_{k=0}^{\infty} \frac{a_k}{k!} x^k$, we define the *Wick version* as

$$F^\diamond(\Phi) = \sum_{k=0}^{\infty} \frac{a_k}{k!} \Phi^{\diamond k}.$$

From the recursive system of SDEs (11), we obtain SDEs for other *Wick analytic functionals* of a fractional Brownian motion

$$F^\diamond(B_t^H) = \sum_{k=0}^{\infty} \frac{a_k}{k!} (B_t^H)^{\diamond k}.$$

Recall the linear system of SDEs (1),

$$\mathrm{d}X_t = (A_1 X_t + A_2 Y_t)\,\mathrm{d}^\diamond B_t^H, \qquad X_0 = x_0,$$
$$\mathrm{d}Y_t = (B_1 X_t + B_2 Y_t)\,\mathrm{d}^\diamond B_t^H, \qquad Y_0 = y_0.$$

The coefficients of the solution,

$$X_t = \sum_{k=0} \frac{a_k}{k!}(B_t^H)^{\diamond k}, \qquad Y_t = \sum_{k=0} \frac{b_k}{k!}(B_t^H)^{\diamond k}, \tag{14}$$

can be obtained recursively via (11) to be

$$a_0 = x_0, \qquad b_0 = y_0, \qquad a_k = A_1 a_{k-1} + A_2 b_{k-1}, \qquad b_k = B_1 a_{k-1} + B_2 b_{k-1}.$$

Note that it holds that $|a_k|, |b_k| \le C^k$ for a $C \in \mathbb{R}_+$. This is according to the recursive derivation of the coefficients and it ensures that the Wick analytic functionals X_t and Y_t are square-integrable (cf. the proof of Proposition 6).

3. The approximation results

Here, we present the approximating sequences and discuss the main results. More precisely, we introduce the Donsker-type approximation of a fractional Brownian motion and the discrete Wick product, and obtain Wick difference equations, which correspond to the SDEs. We shall work with a fractional Brownian motion on the interval [0, 1], but all results extend to any compact interval $[0, T]$.

We first consider the following kernel representation of a fractional Brownian motion on the interval [0, 1], based on works by Molchan and Golosov [16,17],

$$B_t^H = \int_0^t z_H(t, s)\, \mathrm{d}B_s. \tag{15}$$

For $H > \frac{1}{2}$, the deterministic kernel takes the form

$$z_H(t, s) = \mathbf{1}_{\{t \ge s\}} c_H \left(H - \frac{1}{2}\right) s^{1/2-H} \int_s^t u^{H-1/2}(u - s)^{H-3/2}\, \mathrm{d}u \tag{16}$$

with the constant

$$c_H = \sqrt{\frac{2H\Gamma(3/2 - H)}{\Gamma(H + 1/2)\Gamma(2 - 2H)}},$$

where Γ is the Gamma function (Norros *et al.* [18] or Nualart [20], Section 5.1.3). In order to simplify the notation, we think of $H \in (\frac{1}{2}, 1)$ as fixed from now on and omit the subscript H in the notation of the kernel. For an introduction to some elementary properties of fractional Brownian motion, we refer to Nualart [20], Chapter 5, Mishura [15] or Biagini *et al.* [4].

We apply Sottinen's approximation of a fractional Brownian motion by *disturbed binary random walks*. Suppose $(\Omega, \mathcal{F}, P)$ is a probability space and, for all $n \in \mathbb{N}$ and $i = 1, \dots, n$, we have independent and identically distributed symmetric Bernoulli random variables $\xi_i^n : \Omega \to \{-1, 1\}$ with $P(\xi_i^n = 1) = P(\xi_i^n = -1)$. By Donsker's theorem, the sequence of random walks $B_t^{(n)} = \frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor nt \rfloor} \xi_i^{(n)}$ converges weakly to a standard Brownian motion $B = (B_t)_{t \in [0,1]}$ [5], Theorem 16.1. The idea of Sottinen [21] is to combine these random walks with a pointwise approximation of the kernel in representation (15). Define the pointwise approximation of $z(t, s)$ as

$$z^{(n)}(t, s) := n \int_{s-1/n}^{s} z\left(\frac{\lfloor nt \rfloor}{n}, u\right) \mathrm{d}u.$$

The sequence of binary random walks

$$B_t^{H,n} := \int_0^t z^{(n)}(t,s)\,\mathrm{d}B_s^{(n)} = \sum_{i=1}^{\lfloor nt \rfloor} n \int_{(i-1)/n}^{i/n} z\left(\frac{\lfloor nt \rfloor}{n}, s\right) \mathrm{d}s \frac{1}{\sqrt{n}} \xi_i^{(n)}$$

then converges weakly to a fractional Brownian motion $(B_t^H)_{t\in[0,1]}$ in the Skorokhod space $D([0,1],\mathbb{R})$ [21], Theorem 1.

A major advantage of the binary random walks is that we can avoid the difficult Wick product for random variables with continuous distributions. We approximate this operator on the binary random walks by *discrete Wick products*.

For any $n \in \mathbb{N}$, let $(\xi_1^n, \xi_2^n, \dots, \xi_n^n)$ be the n-tuple of independent and identically distributed symmetric Bernoulli random variables for the binary random walk $B_t^{H,n}$. The *discrete Wick product* is defined as

$$\prod_{i\in A} \xi_i^n \diamond_n \prod_{i \in B} \xi_i^n := \begin{cases} \prod_{i \in A\cup B} \xi_i^n, & \text{if } A \cap B = \emptyset, \\ 0, & \text{otherwise}, \end{cases}$$

where $A, B \subseteq \{1, \dots, n\}$. We denote by

$$\mathcal{F}_n := \sigma(\xi_1^n, \xi_2^n, \dots, \xi_n^n)$$

the σ-field generated by the Bernoulli variables. Define

$$\Xi_A^n := \prod_{i\in A} \xi_i^n.$$

Clearly, the family of functions $\{\Xi_A^n : A \subseteq \{1, \dots, n\}\}$ is an orthonormal set in $L^2(\Omega, \mathcal{F}_n, P)$. Since its cardinality is equal to the dimension of $L^2(\Omega, \mathcal{F}_n, P)$, it constitutes a basis. Thus, every $X \in L^2(\Omega, \mathcal{F}_n, P)$ has a unique expansion, called the *Walsh decomposition*,

$$X = \sum_{A\subseteq\{1,\dots,n\}} x_A^n \, \Xi_A^n,$$

where $x_A^n \in \mathbb{R}$. The Walsh decomposition can be regarded as a discrete version of the chaos expansion. By algebraic rules, one obtains, for $X = \sum_{A\subseteq\{1,\dots,n\}} x_A^n \, \Xi_A^n$ and $Y = \sum_{B\subseteq\{1,\dots,n\}} y_B^n \, \Xi_B^n$,

$$X \diamond_n Y = \sum_{C\subseteq\{1,\dots,n\}} \left(\sum_{\substack{A\cup B = C \\ A \cap B = \emptyset}} x_A^n y_B^n \right) \Xi_C^n.$$

Furthermore, the L^2-inner product can be computed in terms of the Walsh decomposition as

$$\mathbf{E}[XY] = \sum_{A\subseteq\{1,\dots,n\}} x_A^n y_A^n. \tag{17}$$

There exists an analogous formula for the Wick product on the white noise space via chaos expansions that justifies the analogy between the discrete and ordinary Wick calculus (cf. Kuo [14]). For more information on the discrete Wick product, we refer to Holden *et al.* [10]. More generally, the introduction of a discrete Wiener chaos depends on the class of discrete random variables $(\xi_1^n, \xi_2^n, \ldots, \xi_n^n)$. We refer to Gzyl [9] for a survey of other discrete Wiener chaos approaches.

The representation

$$B_t^{H,n} = \sum_{i=1}^{\lfloor nt \rfloor} b_{t,i}^n \xi_i^n \qquad \text{with } b_{t,i}^n := \sqrt{n} \int_{(i-1)/n}^{i/n} z\left(\frac{\lfloor nt \rfloor}{n}, s\right) \mathrm{d}s$$

is the Walsh decomposition for the binary random walk approximating B^H in $L^2(\Omega, \mathcal{F}_n, P)$. Note that $b_{t,i}^n = b_{\lfloor nt \rfloor/n,i}^n$. Thus, we can consider $B_t^{H,n} = B_{\lfloor nt \rfloor/n}^{H,n}$ as a process in discrete time. We can now state our first convergence result.

Theorem 1. *Suppose that*:

1. $\lim_{n\to\infty} a_{n,k} = a_k$ *exists for all* $k \in \mathbb{N}$;
2. *there exists a* $C \in \mathbb{R}_+$ *such that* $|a_{n,k}| \le C^k$ *for all* $n, k \in \mathbb{N}$.

The sequence of processes $\sum_{k=0}^{n} \frac{a_{n,k}}{k!} (B^{H,n})^{\diamond_n k}$ *then converges weakly to the Wick power series* $\sum_{k=0}^{\infty} \frac{a_k}{k!} (B^H)^{\diamond k}$ *in the Skorokhod space* $D([0,1], \mathbb{R})$.

The proof is given in Sections 5 and 6.

Consider now the following recursive system of Wick difference equations:

$$U_l^{k,n} = U_{l-1}^{k,n} + kU_{l-1}^{k-1,n} \diamond_n \left(B_{l/n}^{H,n} - B_{(l-1)/n}^{H,n}\right), \qquad U_l^{0,n} = 1, \qquad U_0^{k,n} = 0, \tag{18}$$

for all $l = 1, \ldots, n$ and $k \in \mathbb{N}$. This is the discrete counterpart of the recursive system of SDEs in (11). We observe that $U^{0,n} = 1 = (B^{H,n})^{\diamond_n 0}$ and $U^{1,n} = (B^{H,n})^{\diamond_n 1}$, but

$$U_2^{2,n} = 2B_{1/n}^{H,n} \diamond_n B_{2/n}^{H,n} \ne B_{2/n}^{H,n} \diamond_n B_{2/n}^{H,n} = (B_{2/n}^{H,n})^{\diamond_n 2}.$$

Thus, in contrast to the continuous case in (11), the discrete Wick powers are not the solutions for (18) if $k \ge 2$.

However, we can prove a variant of Theorem 1, based on the system of recursive Wick difference equations, whose proof will also be given in Sections 5 and 6.

Theorem 2. *Under the assumptions of Theorem 1, define* $\widetilde{U}_t^{k,n} := U_{\lfloor nt \rfloor}^{k,n}$ *as the piecewise constant interpolation of* (18).

The sequence of processes $\sum_{k=0}^{n} \frac{a_{n,k}}{k!} \widetilde{U}^{k,n}$ *then converges weakly to the Wick power series* $\sum_{k=0}^{\infty} \frac{a_k}{k!} (B^H)^{\diamond k}$ *in the Skorokhod space* $D([0,1], \mathbb{R})$.

Example 1 (Wick powers of a fractional Brownian motion). For $a_{n,k} = l!\mathbf{1}_{\{k=l\}}$,

$$(B^{H,n})^{\diamond_n l} \xrightarrow{d} (B^H)^{\diamond l},$$
$$\widetilde{U}^{l,n} \xrightarrow{d} (B^H)^{\diamond l}.$$

Example 2 (Geometric fractional Brownian motion). For $a_{n,k} = a_k = 1$, we have

$$\exp^{\diamond_n}(B_t^{H,n}) := \sum_{k=0}^{\lfloor nt \rfloor} \frac{1}{k!}(B_t^{H,n})^{\diamond_n k} \xrightarrow{d} \exp^{\diamond}(B^H),$$
$$\widetilde{S}^n := \sum_{k=0}^{n} \frac{1}{k!}\widetilde{U}^{k,n} \xrightarrow{d} \exp^{\diamond}(B^H).$$

Observe that by summing up the recursive system of Wick difference equations (18), we obtain

$$S_l^n = S_{l-1}^n + S_{l-1}^n \diamond_n \left(B_{l/n}^{H,n} - B_{(l-1)/n}^{H,n}\right), \qquad S_0^n = 1, \tag{19}$$

for $l = 1, \ldots, n$, where $S_l^n = \widetilde{S}_{l/n}^n$. Hence, the piecewise constant interpolation of (19) converges weakly to the solution of the fractional Doléans–Dade equation (13).

The reasoning of the previous example can be generalized as follows.

Theorem 3 (Linear SDE with drift). *Suppose $\mu, s_0 \in \mathbb{R}$, $\sigma > 0$. Then $\widetilde{S}_t^n := S_{\lfloor nt \rfloor}^n$, where S^n is the solution of the Wick difference equation*

$$S_l^n = \left(1 + \frac{\mu}{n}\right) S_{l-1}^n + \sigma S_{l-1}^n \diamond_n \left(B_{l/n}^{H,n} - B_{(l-1)/n}^{H,n}\right), \qquad S_0^n = s_0, \qquad l = 1, \ldots, n, \tag{20}$$

converges weakly to the solution of the linear SDE with drift

$$\mathrm{d}S_t = \mu S_t \,\mathrm{d}t + \sigma S_t \,\mathrm{d}^{\diamond} B_t^H, \qquad S_0 = s_0, \tag{21}$$

in the Skorokhod space $D([0,1], \mathbb{R})$.

Proof. First, observe that for $\sigma_n \to \sigma > 0$ and $\widetilde{a}_{n,k} = a_{n,k}\sigma_n^k$, we obtain, by Theorem 2, that

$$\widetilde{V}^n := \sum_{k=0}^{n} \frac{a_{n,k}}{k!}\sigma_n^k \widetilde{U}^{k,n} \xrightarrow{d} \sum_{k=0}^{\infty} \frac{a_k}{k!}(\sigma B^H)^{\diamond k}.$$

With the choice $a_{n,k} := 1$ and

$$\sigma_n := \frac{\sigma}{1 + \mu/n} \to \sigma \qquad \text{as } n \to \infty,$$

we observe by (18) that $V_l^n := \widetilde{V}_{l/n}^n$ satisfies

$$V_l^n = V_{l-1}^n + \left(\frac{\sigma}{1+\mu/n}\right) V_{l-1}^n \diamond_n \left(B_{l/n}^{H,n} - B_{(l-1)/n}^{H,n}\right), \qquad V_0^n = 1, \qquad l = 1, \dots, n.$$

Consider now the piecewise constant function $(\widetilde{W}_t^n)_{t\in[0,1]}$ determined by $\widetilde{W}_t^n := W_{\lfloor nt \rfloor}^n$ and

$$W_l^n = \left(1 + \frac{\mu}{n}\right) W_{l-1}^n, \qquad W_0^n = s_0, \qquad l = 1, \dots, n.$$

By this well-known Euler scheme,

$$(\widetilde{W}_t^n)_{t\in[0,1]} \longrightarrow s_0(\exp(\mu t))_{t\in[0,1]} \tag{22}$$

in the sup-norm on [0, 1]. The product

$$\begin{aligned} V_l^n W_l^n &= \left(1 + \frac{\mu}{n}\right) V_{l-1}^n W_{l-1}^n + \left[\left(\frac{\sigma}{1+\mu/n}\right) V_{l-1}^n \diamond_n \left(B_{l/n}^{H,n} - B_{(l-1)/n}^{H,n}\right)\right]\left(1 + \frac{\mu}{n}\right) W_{l-1}^n \\ &= \left(1 + \frac{\mu}{n}\right) V_{l-1}^n W_{l-1}^n + \sigma V_{l-1}^n W_{l-1}^n \diamond_n \left(B_{l/n}^{H,n} - B_{(l-1)/n}^{H,n}\right), \\ V_0^n W_0^n &= s_0, \qquad l = 1, \dots, n, \end{aligned}$$

satisfies the Wick difference equation (20) for $S_l^n = V_l^n W_l^n$. The multiplication by the deterministic function $s_0 \exp(\mu t)$ is continuous on the Skorokhod space. Thus, with (22) and Billingsley [5], Theorem 4.1, we obtain

$$(\widetilde{S}_t^n)_{t\in[0,1]} = (\widetilde{V}_t^n \widetilde{W}_t^n)_{t\in[0,1]} \xrightarrow{d} s_0(\exp(\mu t) \exp^{\diamond}(\sigma B_t^H))_{t\in[0,1]}$$

in the Skorokhod space $D([0,1], \mathbb{R})$. As $s_0 \exp(\mu t) \exp^{\diamond}(\sigma B_t^H)$ solves the SDE (21) (cf. Mishura [15], Theorem 3.3.2), the proof is complete. □

Remark 1. Theorem 3 holds with additional approximations $(\sigma_n, \mu_n) \to (\sigma, \mu)$.

Remark 2. Theorem 3 was conjectured by Bender and Elliott [3] in their study of the discrete Wick-fractional Black–Scholes market. They deduced an arbitrage in this model for sufficiently large n. Although the arbitrage or no-arbitrage property is not preserved by weak convergence, this model showed that it is even possible to obtain arbitrage in this simple discrete Wick fractional market model. In a recent work [23], Valkeila shows that an alternative approximation to the exponential of a fractional Brownian motion by a superposition of some independent renewal reward processes leads to an arbitrage-free and complete model. We refer to Gaigalas and Kaj [8] for a general limit discussion for these superposition processes.

Theorem 4 (Linear system of SDEs). *The piecewise constant interpolation*

$$(\widetilde{X}_t^n, \widetilde{Y}_t^n)^{\mathrm{T}} := \left(X_{\lfloor nt \rfloor}^n, Y_{\lfloor nt \rfloor}^n\right)^{\mathrm{T}}$$

for the solution of the linear system of Wick difference equations

$$X_l^n = X_{l-1}^n + (A_1 X_{l-1}^n + A_2 Y_{l-1}^n) \diamond_n \left(B_{l/n}^{H,n} - B_{(l-1)/n}^{H,n}\right), \qquad X_0^n = x_0, \qquad l = 1, \dots, n,$$
$$Y_l^n = Y_{l-1}^n + (B_1 X_{l-1}^n + B_2 Y_{l-1}^n) \diamond_n \left(B_{l/n}^{H,n} - B_{(l-1)/n}^{H,n}\right), \qquad Y_0^n = y_0, \qquad l = 1, \dots, n,$$

converges weakly to the solution $(X, Y)^{\mathrm{T}}$ *of the corresponding linear system of SDEs* (1) *in the Skorokhod space* $D([0, 1], \mathbb{R})^2$.

Proof. Analogously to (14), we obtain, by the recursive system of Wick difference equations for $U^{k,n}$ in (18), the coefficients for the solution of the systems of difference equations

$$X_l^n = \sum_{k=0}^{\infty} \frac{a_k}{k!} U_l^{k,n}, \qquad Y_l^n = \sum_{k=0}^{\infty} \frac{b_k}{k!} U_l^{k,n},$$

recursively by

$$a_0 = x_0, \qquad b_0 = y_0, \qquad a_k = A_1 a_{k-1} + A_2 b_{k-1}, \qquad b_k = B_1 a_{k-1} + B_2 b_{k-1}.$$

We define the upper bound

$$M_{AB} := 2 \max\{|A_1|, |A_2|, |B_1|, |B_2|\}.$$

Suppose that $r_1, r_2 \in \mathbb{R}$ are arbitrary. By the linear system and (18), the sequence of processes

$$r_1 \widetilde{X}^n + r_2 \widetilde{Y}^n = \sum_{k=0}^{n} \left(\frac{r_1 a_k + r_2 b_k}{k!} \right) \widetilde{U}^{k,n}$$

fulfils the conditions in Theorem 2 with

$$|r_1 a_k + r_2 b_k| \le \max\{|x_0|, |y_0|\}(|r_1| + |r_2|) M_{AB}^k.$$

Thus, we obtain the weak convergence

$$r_1 \widetilde{X}^n + r_2 \widetilde{Y}^n \xrightarrow{d} \sum_{k=0}^{\infty} \left(\frac{r_1 a_k + r_2 b_k}{k!} \right) (B^{H,n})^{\diamond k} = r_1 X + r_2 Y.$$

The Cramér–Wold device (Billingsley [5], Theorem 7.7) can now be used to complete the proof. □

Remark 3. Theorem 4 can be extended to higher-dimensional linear cases. It also holds for an additional approximation of the coefficients $A_{n,i} \to A_i$ and $B_{n,i} \to B_i$ for $n \to \infty$.

Example 3 (Wick-sine and Wick-cosine). The piecewise constant interpolation of

$$X_l^n = X_{l-1}^n + Y_{l-1}^n \diamond_n \big(B_{l/n}^{H,n} - B_{(l-1)/n}^{H,n}\big), \qquad X_0^n = 0, \qquad l = 1, \dots, n,$$
$$Y_l^n = Y_{l-1}^n - X_{l-1}^n \diamond_n \big(B_{l/n}^{H,n} - B_{(l-1)/n}^{H,n}\big), \qquad Y_0^n = 1, \qquad l = 1, \dots, n,$$

converges weakly to the solution of the linear system

$$\mathrm{d}X_t = Y_t \, \mathrm{d}^\diamond B_t^H, \qquad X_0 = 0,$$
$$\mathrm{d}Y_t = -X_t \, \mathrm{d}^\diamond B_t^H, \qquad Y_0 = 1,$$

the process $(\sin^\diamond(B_t^H), \cos^\diamond(B_t^H))^{\mathrm{T}}$. By Theorem 1, it can also be approximated by the discrete Wick version functional $(\sin^{\diamond_n}(B_t^{H,n}), \cos^{\diamond_n}(B_t^{H,n}))^{\mathrm{T}}$.

4. Walsh decompositions and L^2-estimates

In this section, we give the Walsh decompositions for the approximating sequences and obtain some L^2-estimates. A key to the approximation results will be the convergence of the L^2-norms of the discrete Wick powers of $B_t^{H,n}$ to the corresponding L^2-norms of the Wick powers of B_t^H.

Recall the Walsh decomposition $B_t^{H,n} = \sum_{i=1}^{\lfloor nt \rfloor} b_{t,i}^n \xi_i^n$. Define

$$b_{t,A}^n := \prod_{i \in A} b_{t,i}^n, \qquad \Xi_A^n := \prod_{i \in A} \xi_i^n, \qquad d_{l,i}^n := b_{l/n,i}^n - b_{(l-1)/n,i}^n$$

for $l = 1, \dots, n$. Note that $d_{i,i}^n = b_{i/n,i}^n$, $d_{l,i}^n = 0$, for $i > l$ and that the increment has the representation

$$B_{l/n}^{H,n} - B_{(l-1)/n}^{H,n} = \sum_{i=1}^{l} d_{l,i}^n \xi_i^n .$$

Recall the recursive system of Wick difference equations,

$$U_l^{k,n} = U_{l-1}^{k,n} + k U_{l-1}^{k-1,n} \diamond_n \big(B_{l/n}^{H,n} - B_{(l-1)/n}^{H,n}\big), \qquad U_l^{0,n} = 1, \qquad U_0^{k,n} = 0, \tag{23}$$

for $l = 1, \dots, n$ and $k \in \mathbb{N}$.

Proposition 1. *For all $n, k \in \mathbb{N}$ and $l = 0, \dots, n$, we have the Walsh decompositions*

$$\frac{1}{k!} U_l^{k,n} = \sum_{\substack{C \subseteq \{1,\dots,l\} \\ |C| = k}} \Bigg(\sum_{\substack{m: C \to \{1,\dots,l\} \\ \text{injective}}} \prod_{p \in C} d_{m(p),p}^n \Bigg) \Xi_C^n, \tag{24}$$

$$\frac{1}{k!} (B_{l/n}^{H,n})^{\diamond_n k} = \sum_{\substack{C \subseteq \{1,\dots,l\} \\ |C| = k}} b_{l/n,C}^n \, \Xi_C^n, \tag{25}$$

$$\frac{1}{k!}(B^{H,n}_{l/n})^{\diamond_n k} - \frac{1}{k!}U^{k,n}_l = \sum_{\substack{C\subseteq\{1,\dots,l\}\\ |C|=k}} \left(\sum_{\substack{m:C\to\{1,\dots,l\}\\ \text{not injective}}} \prod_{p\in C} d^n_{m(p),p} \right) \Xi^n_C. \tag{26}$$

Proof. We use the conventions that an empty sum is zero, an empty product is one and that there exists exactly one map from the empty set to an arbitrary set. For these reasons, the formulas hold for $k=0$ or $l=0$. We prove (24) by induction as follows. For all $l=0,\dots,n$ and all $k\in\mathbb{N}$, it is obvious that $U^{0,n}_l=1$ and $U^{k,n}_0=0$, as in formula (24). Suppose the formula is proved for all positive integers less than or equal to a certain k and all $l=0,\dots,n$. Furthermore, for $k+1$, suppose the formula is proved for all positive integers less than or equal to a certain l. For $k+1$ and $l+1$, we compute, by the difference equation (23) and the induction hypothesis,

$$\begin{aligned}
U^{k+1,n}_{l+1} - U^{k+1,n}_l &= (k+1)k!\Bigg(\sum_{\substack{C\subseteq\{1,\dots,l\}\\ |C|=k}} \sum_{\substack{m:C\to\{1,\dots,l\}\\ \text{injective}}} \prod_{p\in C} d^n_{m(p),p}\, \Xi^n_C \Bigg) \diamond_n \sum_{i=1}^{l+1} d^n_{l+1,i}\xi^n_i \\
&= (k+1)! \sum_{\substack{C\subseteq\{1,\dots,l\}\\ i\in\{1,\dots,l+1\}\\ |C|=k,\, i\notin C}} \sum_{\substack{m:C\to\{1,\dots,l\}\\ \text{injective}}} \prod_{p\in C} d^n_{m(p),p} d^n_{l+1,i}\, \Xi^n_{C\cup\{i\}} \qquad (27)\\
&= (k+1)! \sum_{\substack{C'\subseteq\{1,\dots,l+1\}\\ |C'|=k+1}} \sum_{\substack{m':C'\to\{1,\dots,l+1\}\\ \text{injective},\ \exists q: m(q)=l+1}} \prod_{p\in C'} d^n_{m(p),p}\, \Xi^n_{C'}.
\end{aligned}$$

Note that $d_{m,p}=0$ for all $p-1\geq m$. Thus, by the induction hypothesis,

$$U^{k+1,n}_l = (k+1)! \sum_{\substack{C\subseteq\{1,\dots,l+1\}\\ |C|=k+1}} \sum_{\substack{m:C\to\{1,\dots,l+1\}\\ \text{injective},\ \forall q: m(q)<l+1}} \prod_{p\in C} d^n_{m(p),p}\, \Xi^n_C. \tag{28}$$

Thanks to equations (27) and (28), we obtain (24). In particular, $U^{k,n}_l=0$ for all $k>l$. We now compute the kth Wick power of $(B^{H,n}_{l/n})$ as follows:

$$\begin{aligned}
\left(\sum_{i=1}^n b^n_{l/n,i}\xi^n_i\right)^{\diamond_n k} &= \sum_{\substack{i_1,i_2,\dots,i_k=1\\ \text{pairwise distinct}}}^{l} \left(\prod_{j=1}^k b^n_{l/n,i_j} \prod_{j=1}^k \xi^n_{i_j} \right) \\
&= \sum_{\substack{C\subseteq\{1,\dots,l\}\\ |C|=k}} k! \left(\prod_{i\in C} b^n_{l/n,i} \prod_{i\in C} \xi^n_i \right) = \sum_{\substack{C\subseteq\{1,\dots,l\}\\ |C|=k}} k! b^n_{l/n,C}\, \Xi^n_C.
\end{aligned}$$

In particular, $(B^{H,n}_{l/n})^{\diamond_n k}=0$ for all $k>l$. This yields (25).

The telescoping sum yields

$$\sum_{m(p)=1}^{l} d^n_{m(p),p} = \sum_{m(p)=p}^{l} d^n_{m(p),p} = \sum_{m(p)=p}^{l} \big(b^n_{m(p)/n,p} - b^n_{(m(p)-1)/n,p}\big) = b^n_{l/n,p}$$

and thus we get

$$\sum_{m:C\to\{1,\dots,l\}} \prod_{p\in C} d^n_{m(p),p} = \prod_{p\in C}\left(\sum_{m(p)=1}^{l} d^n_{m(p),p}\right) = \prod_{p\in C} b^n_{l/n,p} = b^n_{l/n,C}. \tag{29}$$

Equation (26) is, thus, implied by (24) and (25). □

In the following propositions, we obtain some elementary estimates for the L^2-norm of discrete Wick powers of $B^{H,n}_t$.

Proposition 2. *For all $t \in [0,1]$ and $i \in \{1,\dots,\lfloor nt\rfloor\}$,*

$$b^n_{t,i} \le 2c_H n^{-(1-H)}.$$

Proof. We estimate

$$\begin{aligned}
b^n_{t,i} &= n^{1/2} c_H\left(H-\frac{1}{2}\right)\int_{(i-1)/n}^{i/n} s^{1/2-H}\int_s^{\lfloor nt\rfloor/n} u^{H-1/2}(u-s)^{H-3/2}\,\mathrm{d}u\,\mathrm{d}s\\
&\le n^{1/2} c_H\left(H-\frac{1}{2}\right)\int_{(i-1)/n}^{i/n} s^{1/2-H}\left(\frac{\lfloor nt\rfloor}{n}\right)^{H-1/2}\frac{1}{H-1/2}\left(\frac{\lfloor nt\rfloor}{n}-s\right)^{H-1/2}\mathrm{d}s\\
&\le n^{1/2} c_H\frac{1}{3/2-H}\left(\left(\frac{i}{n}\right)^{3/2-H} - \left(\frac{i-1}{n}\right)^{3/2-H}\right)\left(\frac{\lfloor nt\rfloor}{n}\right)^{2(H-1/2)}.
\end{aligned}$$

Since $t\le 1$, $\frac{1}{3/2-H} \le 2$ and $|x|^{3/2-H} - |y|^{3/2-H} \le |x-y|^{3/2-H}$, the assertion follows. □

Remark 4. Observe that

$$\mathbf{E}[B^{H,n}_t B^{H,n}_s] = \mathbf{E}\left[\sum_{i_1,i_2=1}^{\lfloor nt\rfloor} b^n_{t,i_1} b^n_{s,i_2}\xi^n_{i_1}\xi^n_{i_2}\right] = \sum_{i=1}^{\lfloor nt\rfloor}(b^n_{t,i} b^n_{s,i}). \tag{30}$$

By Nieminen [19], we thus get, for any $s,t\in[0,1]$, the following convergence:

$$\begin{aligned}
\mathbf{E}[B^{H,n}_t B^{H,n}_s] &= \sum_{i=1}^{\lfloor nt\rfloor} n\int_{(i-1)/n}^{i/n} z\left(\frac{\lfloor nt\rfloor}{n},u\right)\mathrm{d}u\int_{(i-1)/n}^{i/n} z\left(\frac{\lfloor ns\rfloor}{n},u\right)\mathrm{d}u\\
&\longrightarrow \int_0^1 z(t,u)z(s,u)\,\mathrm{d}u = \mathbf{E}[B^H_t B^H_s].
\end{aligned} \tag{31}$$

Moreover, we have, by the Cauchy–Schwarz inequality, the upper bound

$$\begin{aligned}\mathbf{E}[(B_t^{H,n} - B_s^{H,n})^2] &= \sum_{i=1}^{\lfloor nt\rfloor}\left(\sqrt{n}\int_{(i-1)/n}^{i/n}\left(z\left(\frac{\lfloor nt\rfloor}{n}, u\right)\mathrm{d}u - z\left(\frac{\lfloor ns\rfloor}{n}, u\right)\right)\mathrm{d}u\right)^2 \\ &\le \sum_{i=1}^{\lfloor nt\rfloor}\int_{(i-1)/n}^{i/n}\left(z\left(\frac{\lfloor nt\rfloor}{n}, u\right) - z\left(\frac{\lfloor ns\rfloor}{n}, u\right)\right)^2\mathrm{d}u \\ &= \left|\frac{\lfloor nt\rfloor}{n} - \frac{\lfloor ns\rfloor}{n}\right|^{2H}.\end{aligned} \tag{32}$$

Proposition 3. *For all $t \ge s$ in $[0,1]$ and all $N \in \mathbb{N}$ such that $\lfloor ns\rfloor \ge N$, we have*

$$\begin{aligned}0 &\le \mathbf{E}[(B_t^{H,n})^2]^N + \mathbf{E}[(B_s^{H,n})^2]^N - 2\mathbf{E}[B_t^{H,n}B_s^{H,n}]^N \\ &\quad - \frac{1}{N!}\mathbf{E}\big[\big((B_t^{H,n})^{\diamond_n N} - (B_s^{H,n})^{\diamond_n N}\big)^2\big] \\ &\le 2c_H^2 N^2 t^{2H(N-1)} n^{-(2-2H)}.\end{aligned} \tag{33}$$

In particular,

$$\lim_{n\to\infty}\mathbf{E}\big[\big((B_t^{H,n})^{\diamond_n N} - (B_s^{H,n})^{\diamond_n N}\big)^2\big] = \mathbf{E}\big[\big((B_t^{H})^{\diamond N} - (B_s^{H})^{\diamond N}\big)^2\big].$$

Proof. As $N \le \lfloor ns\rfloor$, we get, making use of Proposition 1, (17) and (30) in Remark 4,

$$\begin{aligned}&\frac{1}{N!}\mathbf{E}[(B_t^{H,n})^{\diamond_n N}(B_s^{H,n})^{\diamond_n N}] \\ &\quad = \frac{1}{N!}\mathbf{E}\Bigg[\Bigg(N!\sum_{\substack{C\subseteq\{1,\dots,\lfloor nt\rfloor\}\\ |C|=N}} b_{t,C}^n\,\Xi_C^n\Bigg)\Bigg(N!\sum_{\substack{C\subseteq\{1,\dots,\lfloor ns\rfloor\}\\ |C|=N}} b_{s,C}^n\,\Xi_C^n\Bigg)\Bigg] \\ &\quad = N!\sum_{\substack{C\subseteq\{1,\dots,\lfloor nt\rfloor\}\\ |C|=N}} b_{t,C}^n b_{s,C}^n \\ &\quad = \sum_{i_1,\dots,i_N=1}^{\lfloor nt\rfloor}\prod_{j=1}^{N}(b_{t,i_j}^n b_{s,i_j}^n) - \sum_{\substack{i_1,\dots,i_N=1\\ \exists k,l:\, i_k=i_l}}^{\lfloor nt\rfloor}\prod_{j=1}^{N}(b_{t,i_j}^n b_{s,i_j}^n) \\ &\quad = \mathbf{E}[B_t^{H,n}B_s^{H,n}]^N - \sum_{\substack{i_1,\dots,i_N=1\\ \exists k,l:\, i_k=i_l}}^{\lfloor nt\rfloor}\prod_{j=1}^{N}(b_{t,i_j}^n b_{s,i_j}^n).\end{aligned} \tag{34}$$

Thus, we have

$$\begin{aligned}
&\mathbf{E}[(B_t^{H,n})^2]^N + \mathbf{E}[(B_s^{H,n})^2]^N - 2\mathbf{E}[B_t^{H,n} B_s^{H,n}]^N \\
&\quad - \frac{1}{N!}\mathbf{E}\big[\big((B_t^{H,n})^{\diamond_n N} - (B_s^{H,n})^{\diamond_n N}\big)^2\big] \\
&= \sum_{\substack{i_1,\dots,i_N=1\\ \exists k,l:\, i_k=i_l}}^{\lfloor nt\rfloor} \left(\prod_{j=1}^N (b_{t,i_j}^n)^2 + \prod_{j=1}^N (b_{s,i_j}^n)^2 - 2\prod_{j=1}^N (b_{t,i_j}^n b_{s,i_j}^n)\right) \\
&= \sum_{\substack{i_1,\dots,i_N=1\\ \exists k,l:\, i_k=i_l}}^{\lfloor nt\rfloor} \left(\prod_{j=1}^N (b_{t,i_j}^n) - \prod_{j=1}^N (b_{s,i_j}^n)\right)^2 \geq 0.
\end{aligned} \tag{35}$$

Hence, the left-hand side of the inequality in (33) follows. By Proposition 2, (30) and (32) in Remark 4, as well as $|\frac{\lfloor nt\rfloor}{n}| \leq t$, we obtain

$$\begin{aligned}
&\sum_{\substack{i_1,\dots,i_N=1\\ \exists k,l:\, i_k=i_l}}^{\lfloor nt\rfloor} \left(\prod_{j=1}^N (b_{t,i_j}^n) - \prod_{j=1}^N (b_{s,i_j}^n)\right)^2 \\
&\leq \sum_{\substack{i_1,\dots,i_N=1\\ \exists k,l:\, i_k=i_l}}^{\lfloor nt\rfloor} \left(\prod_{j=1}^N (b_{t,i_j}^n)\right)^2 \leq \binom{N}{2}\left(\max_i (b_{t,i}^n)^2\right) \sum_{i_1,\dots,i_{N-1}=1}^{\lfloor nt\rfloor} \left(\prod_{j=1}^{N-1} (b_{t,i_j}^n)\right)^2 \\
&\leq 2c_H^2 N^2 \mathbf{E}[(B_t^{H,n})^2]^{N-1} n^{-(2-2H)} \leq 2c_H^2 N^2 t^{2H(N-1)} n^{-(2-2H)} \to 0
\end{aligned} \tag{36}$$

for $n \to \infty$. The representation of Wick powers of B_t^H by Hermite polynomials, as in (9), their orthonormality (cf. Kuo [14], page 355) and the polarization identity collectively yield $\mathbf{E}[(B_t^H)^{\diamond N}(B_s^H)^{\diamond N}] = N!\mathbf{E}[(B_t^H)(B_s^H)]^N$ (cf. also [20], Lemma 1.1.1). Thus, we have, by (35),

$$\begin{aligned}
&\mathbf{E}\big[\big((B_t^{H,n})^{\diamond_n N} - (B_s^{H,n})^{\diamond_n N}\big)^2\big] - \mathbf{E}\big[\big((B_t^H)^{\diamond N} - (B_s^H)^{\diamond N}\big)^2\big] \\
&\quad = N!\big(\mathbf{E}[(B_t^{H,n})^2]^N - \mathbf{E}[(B_t^H)^2]^N + \mathbf{E}[(B_s^{H,n})^2]^N - \mathbf{E}[(B_s^H)^2]^N \\
&\qquad - 2\mathbf{E}[B_t^{H,n} B_s^{H,n}]^N + 2\mathbf{E}[B_t^H B_s^H]^N\big) \\
&\qquad - N! \sum_{\substack{i_1,\dots,i_N=1\\ \exists k,l:\, i_k=i_l}}^{\lfloor nt\rfloor} \left(\prod_{j=1}^N (b_{t,i_j}^n) - \prod_{j=1}^N (b_{s,i_j}^n)\right)^2.
\end{aligned}$$

Applying the convergences (36) and (31) yields

$$\mathbf{E}\big[\big((B_t^{H,n})^{\diamond_n N} - (B_s^{H,n})^{\diamond_n N}\big)^2\big] - \mathbf{E}\big[\big((B_t^H)^{\diamond N} - (B_s^H)^{\diamond N}\big)^2\big] \to 0. \qquad \square$$

Remark 5. In particular, we obtain, by (34), (32) and $|\frac{\lfloor nt\rfloor}{n}| \le t$,

$$\sum_{\substack{C\subseteq\{1,\dots,\lfloor nt\rfloor\}\\ |C|=N}} (b^n_{t,C})^2 = \left(\frac{1}{N!}\right)^2 \mathbf{E}[((B^{H,n}_t)^{\diamond_n N})^2] \\ \le \frac{1}{N!}\mathbf{E}[(B^{H,n}_t)^2]^N \le \frac{1}{N!}\left|\frac{\lfloor nt\rfloor}{n}\right|^{2HN} \le \frac{1}{N!}t^{2HN}. \tag{37}$$

The next proposition estimates the difference between the approximating sequences in Theorems 1 and 2.

Proposition 4. *Under the assumptions of Theorem* 1, *there exists a constant* $K > 0$ *such that for all* $t \in [0,1]$, $n \ge 1$ *and* $k \in \mathbb{N}$,

$$\mathbf{E}\left[\left|\sum_{k=0}^{n}\frac{a_{n,k}}{k!}(B^{H,n}_t)^{\diamond_n k} - \sum_{k=0}^{n}\frac{a_{n,k}}{k!}\widetilde{U}^{k,n}_t\right|^2\right] \le Kn^{1-2H} \tag{38}$$

for the approximating processes in Theorems 1 *and* 2.

Proof. Recall that $d^n_{r,i} = b^n_{r/n,i} - b^n_{(r-1)/n,i} = \sqrt{n}\int_{(i-1)/n}^{i/n}(z(\frac{r}{n},s) - z(\frac{r-1}{n},s))\,\mathrm{d}s$. By (32) in Remark 4, we obtain

$$(d^n_{r,i})^2 \le \sum_{i=1}^{r}(d^n_{r,i})^2 \le \left|\frac{r}{n} - \frac{r-1}{n}\right|^{2H} = n^{-2H}.$$

Thus, we have $d^n_{r,i} \le n^{-H}$ for all $i, n, r \ge 1$. Hence, we obtain, as the sum in (29) telescopes, for $|C| \ge 2$,

$$\sum_{\substack{m:C\to\{1,\dots,\lfloor nt\rfloor\}\\ \text{not injective}}}\prod_{l\in C} d^n_{m(l),l} = \sum_{\substack{m:C\to\{1,\dots,\lfloor nt\rfloor\}\\ \exists u,v\in C:\, m(u)=m(v)}}\prod_{l\in C} d^n_{m(l),l} \\ = \sum_{u\in C}\sum_{m:C\setminus\{u\}\to\{1,\dots,\lfloor nt\rfloor\}}\left(\prod_{l\in C\setminus\{u\}} d^n_{m(l),l}\right)\sum_{v\in C\setminus\{u\}} d^n_{m(v),u} \\ \le \max_{i,r} d^n_{r,i}(|C|-1)\sum_{\substack{C'\subset C\\ |C'|=|C|-1}}\sum_{m:C'\to\{1,\dots,\lfloor nt\rfloor\}}\prod_{l\in C'} d^n_{m(l),l} \\ \le n^{-H}(|C|-1)\sum_{\substack{C'\subset C\\ |C'|=|C|-1}} b^n_{t,C'}. \tag{39}$$

By (26), (39), (37) and since $(\lfloor nt \rfloor - (k-1)) \leq n$, we obtain, for $k \geq 1$,

$$\begin{aligned}
&\mathbf{E}\left[\left|\frac{1}{k!}\big((B_t^{H,n})^{\diamond_n k} - \widetilde{U}_t^{k,n}\big)\right|^2\right] \\
&\quad \leq \sum_{\substack{C \subseteq \{1,\dots,\lfloor nt \rfloor\} \\ |C|=k}} \left(n^{-H}(k-1) \sum_{\substack{C' \subset C \\ |C'|=|C|-1}} b_{t,C'}^n \right)^2 \\
&\quad \leq n^{-2H}(k-1)^2 \sum_{\substack{C \subseteq \{1,\dots,\lfloor nt \rfloor\} \\ |C|=k}} (k-1) \sum_{\substack{C' \subset C \\ |C'|=|C|-1}} (b_{t,C'}^n)^2 \\
&\quad \leq n^{-2H}(k-1)^3 \big(\lfloor nt \rfloor - (k-1)\big) \sum_{\substack{C' \subseteq \{1,\dots,\lfloor nt \rfloor\} \\ |C'|=k-1}} (b_{t,C'}^n)^2 \leq \frac{(k-1)^3}{(k-1)!} t^{2H(k-1)} n^{1-2H}.
\end{aligned}$$

Since $a_{n,k} \leq C^k$ in Theorems 1 and 2, and $((B_t^{H,n})^{\diamond_n k} - \widetilde{U}_t^{k,n})$ are zero for $k = 0, 1$ and orthogonal for different k by Proposition 1 and (17), we get

$$\begin{aligned}
&\mathbf{E}\left[\left|\sum_{k=0}^{n} \frac{a_{n,k}}{k!}(B_t^{H,n})^{\diamond_n k} - \sum_{k=0}^{n} \frac{a_{n,k}}{k!}\widetilde{U}_t^{k,n}\right|^2\right] \\
&\quad = \sum_{k=2}^{\lfloor nt \rfloor} \mathbf{E}\left[\left|\frac{a_{n,k}}{k!}\big((B_t^{H,n})^{\diamond_n k} - \widetilde{U}_t^{k,n}\big)\right|^2\right] \leq \left(\sum_{k=2}^{\infty} C^{2k} \frac{(k-1)^3}{(k-1)!} t^{2H(k-1)}\right) n^{1-2H}.
\end{aligned}$$

As the series on the right-hand side converges uniformly in $t \in [0, 1]$, the assertion follows. □

5. Convergence of the finite-dimensional distributions

We first prove that Theorems 1 and 2 hold with weak convergence replaced by convergence of the finite-dimensional distributions. To this end, we first approximate the Wick powers of B_t^H by induction. We then combine these convergence results to approximate the Wick analytic functionals $F^\diamond(B_t^H) = \sum_{k=0}^{\infty} \frac{a_k}{k!}(B_t^H)^{\diamond k}$. Finally, we conclude that convergence in finite dimensions holds in Theorem 2.

We observed in Section 2 that $(B_t^H)^{\diamond N} = h_{|t|^{2H}}^N(B_t^H)$ and that the Hermite recursion formula

$$(B_t^H)^{\diamond(N+1)} = (B_t^H)(B_t^H)^{\diamond N} - |t|^{2H} N (B_t^H)^{\diamond(N-1)} \tag{40}$$

holds. For the discrete Wick powers of the discrete variables, we now obtain a discrete variant of (40).

Proposition 5 (Discrete Hermite recursion). *For all* $N \geq 1$ *and* $t \in [0, 1]$,

$$(B_t^{H,n})^{\diamond_n (N+1)} = B_t^{H,n} (B_t^{H,n})^{\diamond_n N} - N\mathbf{E}[(B_t^{H,n})^2](B_t^{H,n})^{\diamond_n (N-1)} + R(B_t^{H,n}, N), \tag{41}$$

with remainder

$$R(B_t^{H,n}, N) = N! \sum_{\substack{C\subseteq\{1,\dots\lfloor nt\rfloor\} \\ |C|=N-1}} b_{t,C}^n \, \Xi_C^n \sum_{i\in C} (b_{t,i}^n)^2 \tag{42}$$

and

$$\mathbf{E}[(R(B_t^{H,n}, N))^2] \leq 16 c_H^4 N! N^3 n^{-(4-4H)}. \tag{43}$$

In particular, we will use the fact that the discrete Hermite recursion (41) converges weakly to Hermite recursion (40) for $n \to \infty$.

Proof. By Proposition 1, we get

$$\begin{aligned}(B_t^{H,n})^{\diamond_n (N+1)} &= B_t^{H,n} \diamond_n (B_t^{H,n})^{\diamond_n N} = \left(\sum_{i=1}^{\lfloor nt\rfloor} b_{t,i}^n \xi_i^n \right) \diamond_n \left(\sum_{\substack{A\subseteq\{1,\dots,\lfloor nt\rfloor\} \\ |A|=N}} N! b_{t,A}^n \, \Xi_A^n \right) \\ &= B_t^{H,n} (B_t^{H,n})^{\diamond_n N} - \sum_{\substack{A\subseteq\{1,\dots,\lfloor nt\rfloor\} \\ |A|=N}} \sum_{i\in A} N! b_{t,i}^n b_{t,A}^n \, \Xi_A^n \xi_i^n .\end{aligned} \tag{44}$$

For the second term in the last line in equation (44), by (30) in Remark 4 and Proposition 1, we obtain

$$\begin{aligned}&\sum_{\substack{A\subseteq\{1,\dots,\lfloor nt\rfloor\} \\ |A|=N}} \sum_{i\in A} N! b_{t,i}^n b_{t,A}^n \, \Xi_A^n \xi_i^n \\ &= N! \sum_{\substack{A\subseteq\{1,\dots,\lfloor nt\rfloor\} \\ |A|=N}} \sum_{i\in A} b_{t,A\setminus\{i\}}^n \, \Xi_{A\setminus\{i\}}^n (b_{t,i}^n \xi_i^n)^2 = N! \sum_{\substack{C\subseteq\{1,\dots,\lfloor nt\rfloor\} \\ |C|=N-1}} b_{t,C}^n \, \Xi_C^n \sum_{i\notin C} (b_{t,i}^n)^2 \\ &= N(N-1)! \sum_{\substack{C\subseteq\{1,\dots,\lfloor nt\rfloor\} \\ |C|=N-1}} b_{t,C}^n \, \Xi_C^n \left(\sum_{i=1}^{\lfloor nt\rfloor} (b_{t,i}^n)^2 - \sum_{i\in C} (b_{t,i}^n)^2 \right) \\ &= N (B_t^{H,n})^{\diamond_n (N-1)} \mathbf{E}[(B_t^{H,n})^2] - N! \sum_{\substack{C\subseteq\{1,\dots,\lfloor nt\rfloor\} \\ |C|=N-1}} b_{t,C}^n \, \Xi_C^n \sum_{i\in C} (b_{t,i}^n)^2,\end{aligned}$$

which yields (41) and (42). Thus, thanks to Proposition 2, Remark 5 and $t \leq 1$, we obtain

$$\begin{aligned}\mathbf{E}[(R(B_t^{H,n}, N))^2] &= (N!)^2 \sum_{\substack{C\subseteq\{1,\dots,\lfloor nt\rfloor\}\\ |C|=N-1}} (b_{t,C}^n)^2 \left(\sum_{i\in C}(b_{t,i}^n)^2\right)^2 \\ &\leq (N!)^2 \frac{1}{(N-1)!} t^{2H(N-1)} \big((N-1)4c_H^2 n^{-(2-2H)}\big)^2 \\ &\leq 16 c_H^4 N! N^3 n^{-(4-4H)}. \end{aligned}$$

□

Theorem 5. *For all $N \in \mathbb{N}$,*

$$(1, B^{H,n}, \dots, (B^{H,n})^{\diamond_n N}) \xrightarrow{fd} (1, B^H, \dots, (B^H)^{\diamond N}). \tag{45}$$

Proof. The proof proceeds by induction on N. By Sottinen's approximation, $(1, B^{H,n}) \xrightarrow{fd} (1, B^H)$. Suppose that equation (45) is proved for some $N \geq 1$. Assume that $k \in \mathbb{N}$ and $r_i^j \in \mathbb{R}$ for $j = 0, \dots, N+1$, $i = 1, \dots, k$ and $t_1, t_2, \dots, t_k \in [0, 1]$ are chosen arbitrarily. By the pointwise convergence $\mathbf{E}[(B_t^{H,n})^2] \to |t|^{2H}$ and the generalized continuous mapping theorem (Billingsley [5], Theorem 5.5), the induction hypothesis implies that

$$\begin{aligned}&\sum_{l=0}^{N}\left(\sum_{j=1}^{k} r_j^l (B_{t_j}^{H,n})^{\diamond_n l}\right) + \sum_{j=1}^{k} r_j^{N+1}\big(B_{t_j}^{H,n}(B_{t_j}^{H,n})^{\diamond_n N} - N\mathbf{E}[(B_{t_j}^{H,n})^2](B_{t_j}^{H,n})^{\diamond_n (N-1)}\big) \\ &\xrightarrow{d} \sum_{l=0}^{N}\left(\sum_{j=1}^{k} r_j^l (B_{t_j}^{H})^{\diamond l}\right) + \sum_{j=1}^{k} r_j^{N+1}\big(B_{t_j}^{H}(B_{t_j}^{H})^{\diamond N} - N|t_j|^{2H}(B_{t_j}^{H})^{\diamond (N-1)}\big).\end{aligned}$$

Since $H > \frac{1}{2}$, (43) yields $R(B_t^{H,n}, N) \longrightarrow 0$ in $L^2(\Omega, P)$. Thus, by Slutsky's theorem [5], Theorem 4.1, and the Hermite recursions (40) and (41), we obtain

$$\sum_{l=0}^{N+1}\left(\sum_{j=1}^{k} r_j^l (B_{t_j}^{H,n})^{\diamond_n l}\right) \xrightarrow{d} \sum_{l=0}^{N+1}\left(\sum_{j=1}^{k} r_j^l (B_{t_j}^{H})^{\diamond l}\right).$$

By the Cramér–Wold device (Billingsley [5], Theorem 7.7), we have

$$(1, B^{H,n}, \dots, (B^{H,n})^{\diamond_n N+1}) \xrightarrow{fd} (1, B^H, \dots, (B^H)^{\diamond N+1})$$

and the induction is complete. □

Proposition 6. *In the context of Theorem 1, convergence holds in finite-dimensional distributions.*

Proof. By Billingsley [5], Theorem 4.2, it suffices to show that the following three conditions hold:

$$\forall m \in \mathbb{N}, \qquad \sum_{k=0}^{m} \frac{a_{n,k}}{k!} (B_t^{H,n})^{\diamond_n k} \xrightarrow{fd} \sum_{k=0}^{m} \frac{a_k}{k!} (B_t^H)^{\diamond k} \qquad \text{as } n \to \infty; \tag{46}$$

$$\forall t \in [0,1], \qquad \lim_{m\to\infty} \limsup_{n\to\infty} \mathbf{E}\left[\left|\sum_{k=0}^{n} \frac{a_{n,k}}{k!} (B_t^{H,n})^{\diamond_n k} - \sum_{k=0}^{m} \frac{a_{n,k}}{k!} (B_t^{H,n})^{\diamond_n k}\right| \wedge 1\right] = 0; \tag{47}$$

$$\sum_{k=0}^{m} \frac{a_k}{k!} (B_t^H)^{\diamond k} \xrightarrow{fd} \sum_{k=0}^{\infty} \frac{a_k}{k!} (B_t^H)^{\diamond k} \qquad \text{as } m \to \infty. \tag{48}$$

Condition (46) follows directly from Theorem 5 and the generalized continuous mapping theorem ([5], Theorem 5.5). For the second condition, we compute

$$\begin{aligned}
&\mathbf{E}\left[\left(\sum_{k=0}^{n} \frac{a_{n,k}}{k!} (B_t^{H,n})^{\diamond k} - \sum_{k=0}^{m} \frac{a_{n,k}}{k!} (B_t^{H,n})^{\diamond_n k}\right)^2\right] \\
&= \mathbf{E}\left[\left(\sum_{k=m+1}^{n} \frac{a_{n,k}}{k!} (B_t^{H,n})^{\diamond_n k}\right)^2\right] \\
&= \sum_{k=m+1}^{n} \left(\frac{a_{n,k}}{k!}\right)^2 \mathbf{E}[((B_t^{H,n})^{\diamond_n k})^2] \le \sum_{k=m+1}^{n} \left(\frac{C^k}{k!}\right)^2 k! t^{2Hk},
\end{aligned}$$

applying the estimate of Remark 5. Here, we used the fact that discrete Wick powers of different orders are orthogonal. Thus, we even obtain $\lim_{m\to\infty} \limsup_{n\to\infty} \sum_{k=m+1}^{n} \frac{C^{2k}}{k!} = 0$ and, for all $t \in [0,1]$, a stronger result than condition (47),

$$\lim_{m\to\infty} \limsup_{n\to\infty} \mathbf{E}\left[\left(\sum_{k=0}^{n} \frac{a_{n,k}}{k!} (B_t^{H,n})^{\diamond k} - \sum_{k=0}^{m} \frac{a_{n,k}}{k!} (B_t^{H,n})^{\diamond_n k}\right)^2\right] = 0.$$

By the orthogonality of the different Wick powers, we have

$$\mathbf{E}\left[\left(\sum_{k=m+1}^{\infty} \frac{a_k}{k!} (B_t^H)^{\diamond k}\right)^2\right] = \sum_{k=m+1}^{\infty} \left(\frac{a_k}{k!}\right)^2 \mathbf{E}[((B_t^H)^{\diamond k})^2] \le \sum_{k=m+1}^{\infty} \left(\frac{C^{2k}}{k!}\right) t^{2Hk} \to 0$$

for $m \to \infty$, which implies that condition (48) even holds in $L^2(\Omega, P)$. □

In view of Proposition 4 and Slutsky's theorem, the previous proposition also implies the following.

Proposition 7. *In the context of Theorem 2, convergence holds in finite-dimensional distributions.*

6. Tightness

We now show the tightness of the sequences in Theorems 1 and 2 by the following criterion, which is a variant of Theorem 15.6 in Billingsley [5].

Theorem 6. *Suppose that, for the random elements Y^n in the Skorokhod space $D([0,1],\mathbb{R})$ and $\sum_{k=0}^{\infty}\frac{a_k}{k!}(B^H)^{\diamond k}$ in $C([0,1],\mathbb{R})$,*

$$Y^n \xrightarrow{fd} \sum_{k=0}^{\infty}\frac{a_k}{k!}(B^H)^{\diamond k},$$

and for $s \le t$ in $[0,1]$,

$$\mathbf{E}[(Y_t^n - Y_s^n)^2] \le L\left|\frac{\lfloor nt\rfloor}{n} - \frac{\lfloor ns\rfloor}{n}\right|^{2H},$$

where $L > 0$ is a constant. Then Y^n converges weakly to $\sum_{k=0}^{\infty}\frac{a_k}{k!}(B^H)^{\diamond k}$ in $D([0,1],\mathbb{R})$.

Proof. Let $s < t < u$ in $[0,1]$. By the Cauchy–Schwarz inequality,

$$\begin{aligned}
&\mathbf{E}[|Y_t^n - Y_s^n||Y_u^n - Y_t^n|] \\
&\quad \le (\mathbf{E}[|Y_t^n - Y_s^n|^2])^{1/2}(\mathbf{E}[|Y_u^n - Y_t^n|^2])^{1/2} \\
&\quad \le L\left|\frac{\lfloor nt\rfloor}{n} - \frac{\lfloor ns\rfloor}{n}\right|^{H}\left|\frac{\lfloor nu\rfloor}{n} - \frac{\lfloor nt\rfloor}{n}\right|^{H} \le L\left|\frac{\lfloor nu\rfloor}{n} - \frac{\lfloor ns\rfloor}{n}\right|^{2H}.
\end{aligned}$$

If $u - s \ge \frac{1}{n}$, we have, since $\lfloor nu\rfloor \le nu$ and $-\lfloor ns\rfloor \le -ns + 1$,

$$\left|\frac{\lfloor nu\rfloor}{n} - \frac{\lfloor ns\rfloor}{n}\right|^{2H} \le \bigl(2(u-s)\bigr)^{2H}$$

and thus

$$\mathbf{E}[|Y_t^n - Y_s^n||Y_u^n - Y_t^n|] \le L2^{2H}(u-s)^{2H}. \tag{49}$$

If $u - s < \frac{1}{n}$, then we have either $\lfloor ns\rfloor = \lfloor nt\rfloor$ or $\lfloor nt\rfloor = \lfloor nu\rfloor$ and so the left-hand side in (49) is zero. Thus, the inequality (49) holds for all $s < t < u$. By the convergence of the finite-dimensional distributions and [5], Theorem 15.6, we get the weak convergence of the processes. □

For the application of this criterion to the discrete Wick powers, we need two lemmas.

Lemma 1. *Let $(X, \langle\cdot,\cdot\rangle)$ be a real inner product space and $\|x\| := \langle x, x\rangle$ the corresponding norm on X. Then, for all $x, y \in X$ and $N \ge 1$,*

$$\|x\|^{2N} + \|y\|^{2N} - 2(\langle x, y\rangle)^N \le 2^{N+1}(\|x\| + \|y\|)^{2(N-1)}\|x - y\|^2.$$

Proof. It holds that

$$\begin{aligned}
2(\langle x, y\rangle)^N &= 2\Big(\frac{1}{2}(\|x\|^2 + \|y\|^2 - \|x-y\|^2)\Big)^N \\
&= \frac{1}{2^{N-1}}\Bigg[(\|x\|^2 + \|y\|^2)^N + \sum_{k=0}^{N-1}\binom{N}{k}(\|x\|^2+\|y\|^2)^k(-1)^{N-k}\|x-y\|^{2(N-k)}\Bigg] \\
&= \frac{1}{2^{N-1}}(\|x\|^2 + \|y\|^2)^N \\
&\quad - \|x-y\|^2\frac{1}{2^{N-1}}\sum_{k=0}^{N-1}\binom{N}{k}(\|x\|^2+\|y\|^2)^k(-1)^{N-k-1}\|x-y\|^{2(N-k-1)}.
\end{aligned}$$

Hence, we get

$$\begin{aligned}
&\|x\|^{2N} + \|y\|^{2N} - 2(\langle x, y\rangle)^N \\
&\quad = \|x\|^{2N} + \|y\|^{2N} - \frac{1}{2^{N-1}}(\|x\|^2+\|y\|^2)^N \\
&\qquad + \|x-y\|^2\frac{1}{2^{N-1}}\sum_{k=0}^{N-1}\binom{N}{k}(\|x\|^2+\|y\|^2)^k(-1)^{N-k-1}\|x-y\|^{2(N-k-1)}.
\end{aligned} \tag{50}$$

Since $(\frac{1}{2})^{N-1}\sum_{k=0}^{N}\binom{N}{k} = 2$, the first line on the right-hand side of (50) can be treated as follows:

$$\begin{aligned}
&\|x\|^{2N} + \|y\|^{2N} - \frac{1}{2^{N-1}}(\|x\|^2+\|y\|^2)^N \\
&\quad = \frac{1}{2^{N-1}}\sum_{k=1}^{N-1}\binom{N}{k}\Big(\frac{\|x\|^{2N}+\|y\|^{2N}}{2} - \|x\|^{2k}\|y\|^{2(N-k)}\Big).
\end{aligned} \tag{51}$$

As $\binom{N}{k} = \binom{N}{N-k}$, we now collect the corresponding summands in sum (51) for $k \neq \frac{N}{2}$. We obtain, by the mean value theorem with

$$M_{x,y} := \max_{\lambda\in[0,1]}\big(\lambda\|x\| + (1-\lambda)\|y\|\big) = \max\{\|x\|, \|y\|\}$$

and since $k(N-k) \le \frac{N^2}{4}$,

$$\begin{aligned}
&\|x\|^{2N} + \|y\|^{2N} - \|x\|^{2k}\|y\|^{2(N-k)} - \|x\|^{2(N-k)}\|y\|^{2k} \\
&\quad = (\|x\|^{2k} - \|y\|^{2k})\big(\|x\|^{2(N-k)} - \|y\|^{2(N-k)}\big) \\
&\quad \le 2kM_{x,y}^{2k-1}\|x-y\|2(N-k)M_{x,y}^{2(N-k)-1}\|x-y\| \le N^2 M_{x,y}^{2(N-1)}\|x-y\|^2.
\end{aligned}$$

Analogously, we obtain, for $k = \frac{N}{2}$,

$$\frac{\|x\|^{2N} + \|y\|^{2N}}{2} - \|x\|^N \|y\|^N = \frac{1}{2}(\|x\|^N - \|y\|^N)^2 \le \frac{1}{2} M_{x,y}^{2(N-1)} N^2 \|x-y\|^2.$$

Plugging these estimates into (51) and recalling that $(\frac{1}{2})^{N-1} \sum_{k=1}^{N-1} \binom{N}{k} \frac{1}{2} = (1 - \frac{1}{2^{N-1}})$, we obtain

$$\|x\|^{2N} + \|y\|^{2N} - \left(\frac{1}{2}\right)^{N-1} (\|x\|^2 + \|y\|^2)^N \le \left(1 - \frac{1}{2^{N-1}}\right) N^2 M_{x,y}^{2(N-1)} \|x-y\|^2. \qquad (52)$$

For the term in the second line on the right-hand side of (50), we observe that, by the triangle inequality,

$$\begin{aligned}&(\|x\|^2 + \|y\|^2)^k (-1)^{N-k-1} \|x-y\|^{2(N-k-1)} \\ &\quad \le (\|x\| + \|y\|)^{2k} (\|x\| + \|y\|)^{2(N-k-1)} = (\|x\| + \|y\|)^{2(N-1)}.\end{aligned}$$

Applying

$$M_{x,y} \le \|x\| + \|y\|, \qquad \left(\frac{1}{2}\right)^{N-1} \sum_{k=0}^{N-1} \binom{N}{k} = 2 - \frac{1}{2^{N-1}},$$

and (52) to (50), we have

$$\begin{aligned}&\|x\|^{2N} + \|y\|^{2N} - 2(\langle x, y\rangle)^N \\ &\quad \le \left[\left(1 - \frac{1}{2^{N-1}}\right) N^2 + \left(2 - \frac{1}{2^{N-1}}\right)\right] (\|x\| + \|y\|)^{2(N-1)} \|x-y\|^2.\end{aligned}$$

By a short calculation and induction, we obtain

$$\left(1 - \frac{1}{2^{N-1}}\right) N^2 + \left(2 - \frac{1}{2^{N-1}}\right) \le \mathbf{1}_{\{N \ne 3\}} 2^N + \mathbf{1}_{\{N=3\}} \frac{17}{2} < 2^{N+1}. \qquad \square$$

Lemma 2. *For all $t > s$ in $[0, 1]$, we have*

$$\frac{1}{N!} \mathbf{E}\big[\big((B_t^{H,n})^{\diamond_n N} - (B_s^{H,n})^{\diamond_n N}\big)^2\big] \le 8^N \left| \frac{\lfloor nt \rfloor}{n} - \frac{\lfloor ns \rfloor}{n} \right|^{2H}.$$

Proof. For $N = 1$, the inequality is fulfilled by (32) in Remark 4. For $N > 1$, we consider the cases $N > \lfloor ns \rfloor$ and $\lfloor ns \rfloor \ge N$ separately. For $\lfloor nt \rfloor \ge N > \lfloor ns \rfloor$, we have $(B_s^{H,n})^{\diamond_n N} = 0$. Hence, Proposition 3 and Remark 5 imply that

$$\frac{1}{N!} \mathbf{E}\big[\big((B_t^{H,n})^{\diamond_n N} - (B_s^{H,n})^{\diamond_n N}\big)^2\big] = \frac{1}{N!} \mathbf{E}[((B_t^{H,n})^{\diamond_n N})^2] \le \left| \frac{\lfloor nt \rfloor}{n} \right|^{2HN}.$$

Since $N \geq 2$, $2H > 1$ and $\frac{\lfloor nt \rfloor}{n} \leq 1$, we obtain

$$\begin{aligned}\left|\frac{\lfloor nt \rfloor}{n}\right|^{2HN} &\leq \left|\frac{\lfloor nt \rfloor}{n}\right|^{2} = \left(\frac{1}{n}\right)^2 \big((\lfloor nt \rfloor - \lfloor ns \rfloor) + \lfloor ns \rfloor\big)^2 \\ &\leq \left(\frac{1}{n}\right)^2 \big((\lfloor nt \rfloor - \lfloor ns \rfloor) + N\big)^2 \leq \left(\frac{1}{n}\right)^2 \big((\lfloor nt \rfloor - \lfloor ns \rfloor)(N+1)\big)^2 \\ &= (N+1)^2 \left|\frac{\lfloor nt \rfloor}{n} - \frac{\lfloor ns \rfloor}{n}\right|^2.\end{aligned}$$

Since $(N+1)^2 \leq 3^N$ for $N \geq 2$ and $2H < 2$, we obtain

$$\frac{1}{N!}\mathbf{E}\big[\big((B_t^{H,n})^{\diamond_n N} - (B_s^{H,n})^{\diamond_n N}\big)^2\big] \leq 3^N \left|\frac{\lfloor nt \rfloor}{n} - \frac{\lfloor ns \rfloor}{n}\right|^{2H}$$

for all $\lfloor nt \rfloor \geq N > \lfloor ns \rfloor$. Recall that, by Proposition 3, for $\lfloor nt \rfloor > \lfloor ns \rfloor \geq N$,

$$\begin{aligned}&\frac{1}{N!}\mathbf{E}\big[\big((B_t^{H,n})^{\diamond_n N} - (B_s^{H,n})^{\diamond_n N}\big)^2\big] \\ &\quad \leq \mathbf{E}[(B_t^{H,n})^2]^N + \mathbf{E}[(B_s^{H,n})^2]^N - 2\mathbf{E}[(B_t^{H,n})(B_s^{H,n})]^N.\end{aligned}$$

For any $n \in \mathbb{N}$, we can rewrite

$$\mathbf{E}[(B_t^{H,n})(B_s^{H,n})] = \sum_{i=1}^{n} b_{t,i}^n b_{s,i}^n$$

as an ordinary inner product on $\mathbb{R}^n$ of the vectors $(b_{t,1}^n, \ldots, b_{t,n}^n)^{\mathrm{T}}$ and $(b_{s,1}^n, \ldots, b_{s,n}^n)^{\mathrm{T}}$. Thus, the application of Lemma 1 with $t, s \in [0, 1]$ gives

$$\begin{aligned}&\frac{1}{N!}\mathbf{E}\big[\big((B_t^{H,n})^{\diamond_n N} - (B_s^{H,n})^{\diamond_n N}\big)^2\big] \\ &\quad \leq 2^{N+1}\big(\mathbf{E}[(B_t^{H,n})^2]^{1/2} + \mathbf{E}[(B_s^{H,n})^2]^{1/2}\big)^{2(N-1)}\mathbf{E}\big[\big((B_t^{H,n}) - (B_s^{H,n})\big)^2\big] \\ &\quad \leq 2^{N+1}2^{2(N-1)}\left|\frac{\lfloor nt \rfloor}{n} - \frac{\lfloor ns \rfloor}{n}\right|^{2H} \leq 8^N \left|\frac{\lfloor nt \rfloor}{n} - \frac{\lfloor ns \rfloor}{n}\right|^{2H}.\end{aligned}$$

If $N > \lfloor nt \rfloor$, then the left-hand side of the assertion vanishes. □

Remark 6. The proofs for a fractional Brownian motion on some interval $[0, T] \subset \mathbb{R}$ follow by a straightforward modification. As $\mathbf{E}[(B_t^{H,n})^2] \leq T^{2H}$ for $t \in [0, T]$ and

$$\left|\frac{\lfloor nt \rfloor}{n}\right|^{2HN} \leq T^{2H(N-1)}\left|\frac{\lfloor nt \rfloor}{n}\right|^{2H},$$

we obtain the previous lemma for $t > s$ in $[0, T]$ as

$$\frac{1}{N!}\mathbf{E}\big[\big((B_t^{H,n})^{\diamond_n N} - (B_s^{H,n})^{\diamond_n N}\big)^2\big] \le (8T^{2H})^N \left|\frac{\lfloor nt\rfloor}{n} - \frac{\lfloor ns\rfloor}{n}\right|^{2H}.$$

We are now able to prove the weak convergence to the Wick analytic functionals of a fractional Brownian motion.

Proof of Theorem 1. We apply Theorem 6. The convergence of finite-dimensional distributions was shown in Proposition 6. Let $s < t$ in $[0, 1]$. Recall $a_{n,k} \le C^k$. Then, by the orthogonality of $((B_t^{H,n})^{\diamond_n k} - (B_s^{H,n})^{\diamond_n k})$ for different k and Lemma 2, we have

$$\begin{aligned}
&\mathbf{E}\left[\left(\sum_{k=0}^{n}\frac{a_{n,k}}{k!}(B_t^{H,n})^{\diamond_n k} - \sum_{k=0}^{n}\frac{a_{n,k}}{k!}(B_s^{H,n})^{\diamond_n k}\right)^2\right] \\
&\quad = \sum_{k=0}^{n}\left(\frac{a_{n,k}}{k!}\right)^2 \mathbf{E}\big[\big((B_t^{H,n})^{\diamond_n k} - (B_s^{H,n})^{\diamond_n k}\big)^2\big] \le \sum_{k=0}^{n}\frac{C^{2k}}{k!}8^k\left|\frac{\lfloor nt\rfloor}{n} - \frac{\lfloor ns\rfloor}{n}\right|^{2H}.
\end{aligned}$$

Since $0 < \sum_{k=0}^{\infty}\frac{8^k C^{2k}}{k!} = \exp(8C^2) =: L < \infty$, we have

$$\mathbf{E}\left[\left(\sum_{k=0}^{n}\frac{a_{n,k}}{k!}(B_t^{H,n})^{\diamond_n k} - \sum_{k=0}^{n}\frac{a_{n,k}}{k!}(B_s^{H,n})^{\diamond_n k}\right)^2\right] \le L\left|\frac{\lfloor nt\rfloor}{n} - \frac{\lfloor ns\rfloor}{n}\right|^{2H}. \tag{53}$$

□

The alternative approximation, stated in Theorem 2, follows similarly, as we shall now see.

Proof of Theorem 2. Let $s < t$ in $[0, 1]$. Recall that $d_{m,i}^n > 0$ only if $i \le m$. Thus, by Proposition 1, we can write

$$\sum_{k=0}^{n}\frac{a_{n,k}}{k!}\widetilde{U}_s^{k,n} = \sum_{k=0}^{\lfloor nt\rfloor} a_{n,k} \sum_{\substack{C\subseteq\{1,\dots,\lfloor nt\rfloor\}\\ |C|=k}} \sum_{\substack{m:C\to\{1,\dots,\lfloor ns\rfloor\}\\ \text{injective}}} \prod_{l\in C} d_{m(l),l}^n\, \Xi_C^n.$$

Observe that, by the telescoping sum in (29), we have

$$\begin{aligned}
\sum_{\substack{m:C\to\{1,\dots,\lfloor nt\rfloor\}\\ \text{injective}\\ \exists u:m(u)>\lfloor ns\rfloor}} \prod_{l\in C} d_{m(l),l}^n &\le \sum_{\substack{m:C\to\{1,\dots,\lfloor nt\rfloor\}\\ \exists u:\, m(u)>\lfloor ns\rfloor}} \prod_{l\in C} d_{m(l),l}^n \\
&= \sum_{m:C\to\{1,\dots,\lfloor nt\rfloor\}} \prod_{l\in C} d_{m(l),l}^n - \sum_{m:C\to\{1,\dots,\lfloor ns\rfloor\}} \prod_{l\in C} d_{m(l),l}^n \\
&= b_{t,C}^n - b_{s,C}^n.
\end{aligned}$$

Thus, due to the orthogonality of $\widetilde{U}_t^{k,n} - \widetilde{U}_s^{k,n}$ for different values of k, Proposition 1 and estimate (53), we obtain

$$\mathbf{E}\left[\left(\sum_{k=0}^{n} \frac{a_{n,k}}{k!}\widetilde{U}_t^{k,n} - \sum_{k=0}^{n} \frac{a_{n,k}}{k!}\widetilde{U}_s^{k,n}\right)^2\right]$$

$$= \sum_{k=0}^{\lfloor nt \rfloor} a_{n,k}^2 \sum_{\substack{C\subseteq\{1,\dots,\lfloor nt \rfloor\} \\ |C|=k}} \Bigg(\sum_{\substack{m:C\to\{1,\dots,\lfloor nt \rfloor\} \\ \text{injective} \\ \exists u: m(u)>\lfloor ns \rfloor}} \prod_{l\in C} d_{m(l),l}^n \Bigg)^2$$

$$\leq \sum_{k=0}^{\lfloor nt \rfloor} a_{n,k}^2 \sum_{\substack{C\subset\{1,\dots,\lfloor nt \rfloor\} \\ |C|=k}} (b_{t,C}^n - b_{s,C}^n)^2 \leq L \left| \frac{\lfloor nt \rfloor}{n} - \frac{\lfloor ns \rfloor}{n} \right|^{2H}$$

and the result follows from Proposition 7 and Theorem 6. □

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