

Relative log-concavity and a pair of triangle inequalities

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The *relative log-concavity* ordering \leq_{lc} between probability mass functions (pmf's) on non-negative integers is studied. Given three pmf's f, g, h that satisfy $f \leq_{lc} g \leq_{lc} h$, we present a pair of (reverse) triangle inequalities: if $\sum_i i f_i = \sum_i i g_i < \infty$, then

$$D(f|h) \geq D(f|g) + D(g|h)$$

and if $\sum_i i g_i = \sum_i i h_i < \infty$, then

$$D(h|f) \geq D(h|g) + D(g|f),$$

where $D(\cdot|\cdot)$ denotes the Kullback–Leibler divergence. These inequalities, interesting in themselves, are also applied to several problems, including maximum entropy characterizations of Poisson and binomial distributions and the best binomial approximation in relative entropy. We also present parallel results for continuous distributions and discuss the behavior of \leq_{lc} under convolution.

Keywords: Bernoulli sum; binomial approximation; Hoeffding's inequality; maximum entropy; minimum entropy; negative binomial approximation; Poisson approximation; relative entropy

1. Introduction and main result

A non-negative sequence $u = \{u_i, i \geq 0\}$ is *log-concave* if (a) the support of u is an interval in $\mathbf{Z}_+ = \{0, 1, \dots\}$ and (b) $u_i^2 \geq u_{i+1}u_{i-1}$ for all i or, equivalently, $\log(u_i)$ is concave in $\text{supp}(u)$. Such sequences occur naturally in combinatorics, probability and statistics, for example, as probability mass functions (pmf's) of many discrete distributions. Given two pmf's $f = \{f_0, f_1, \dots\}$ and $g = \{g_0, g_1, \dots\}$ on \mathbf{Z}_+ , we say that f is log-concave relative to g , written as $f \leq_{lc} g$, if

1. each of f and g is supported on an interval on \mathbf{Z}_+ ;
2. $\text{supp}(f) \subset \text{supp}(g)$;
3. $\log(f_i/g_i)$ is concave in $\text{supp}(f)$.

We have $f \leq_{lc} f$ (assuming interval support) and $f \leq_{lc} g, g \leq_{lc} h \implies f \leq_{lc} h$. In other words, \leq_{lc} defines a pre-order among discrete distributions with interval supports on \mathbf{Z}_+ . When g is a geometric pmf, $f \leq_{lc} g$ simply means that f is log-concave; when g is a binomial or Poisson pmf and $f \leq_{lc} g$, then f is *ultra log-concave* [23] (see Section 2).

Whitt [27] discusses this particular ordering and illustrates its usefulness with a queueing theory example. Yu [30] uses \leq_{lc} to derive simple conditions that imply other stochastic orders such as the usual stochastic order, the hazard rate order and the likelihood ratio order. Stochastic

orders play an important role in diverse areas, including reliability theory and survival analysis ([2,7]); see Shaked and Shanthikumar [24] for a book-length treatment. In this paper, we are concerned with entropy relations between distributions under \leq_{lc} . The investigation is motivated by maximum entropy characterizations of binomial and Poisson distributions (see Section 2). For a random variable X on \mathbf{Z}_+ with pmf f , the Shannon entropy is defined as

$$H(X) = H(f) = - \sum_{i=0}^{\infty} f_i \log(f_i).$$

By convention, $0 \log(0) = 0$. The relative entropy (Kullback and Leibler [19]; Kullback [18]; Csiszár and Shields [5]) between pmf's f and g on \mathbf{Z}_+ is defined as

$$D(f|g) = \begin{cases} \sum_{i=0}^{\infty} f_i \log(f_i/g_i), & \text{if } \text{supp}(f) \subset \text{supp}(g), \\ \infty, & \text{otherwise.} \end{cases}$$

By convention, $0 \log(0/0) = 0$. We state our main result.

Theorem 1. *Let f, g, h be pmf's on \mathbf{Z}_+ such that $f \leq_{lc} g \leq_{lc} h$. If f and g have finite and equal means, then $D(f|h) < \infty$ and*

$$D(f|h) \geq D(f|g) + D(g|h); \tag{1.1}$$

if h and g have finite and equal means, then

$$D(h|f) \geq D(h|g) + D(g|f). \tag{1.2}$$

Theorem 1 has an appealing geometric interpretation. (With a slight abuse of notation, we write the mean of a pmf g as $E(g) = \sum_i i g_i$.) If g and h satisfy $E(g) < \infty$ and $g \leq_{lc} h$, then (1.1) gives

$$D(g|h) = \inf_{f \in F} D(f|h), \quad F = \{f: f \leq_{lc} g, E(f) = E(g)\}.$$

That is, g is the *I-projection* of h onto F . Relation (1.2) can be interpreted similarly. See Csiszár and Shields [5] for general definitions and properties of the I-projection and the related *reverse I-projection*.

While Theorem 1 is interesting in itself, it can also be used to derive several classical and new entropy comparison results. We therefore defer its proof to Section 3, after considering these applications. We conclude in Section 4 with extensions to continuous distributions. Throughout, we also discuss the behavior of \leq_{lc} under convolution, as this becomes relevant in a few places.

2. Some implications of Theorem 1

Theorem 1 is used to unify and generalize classical results on maximum entropy characterizations of Poisson and binomial distributions in Section 2.1 and to determine the best binomial

approximation to a sum of independent Bernoulli random variables (in relative entropy) in Section 2.2. Section 2.3 contains analogous results for the negative binomial. Theorem 1 also implies monotonicity (in terms of relative entropy) in certain Poisson limit theorems.

2.1. Maximum entropy properties of binomial and Poisson distributions

Throughout this subsection (and in Section 2.2), let X_1, \dots, X_n be independent Bernoulli random variables with $\Pr(X_i = 1) = 1 - \Pr(X_i = 0) = p_i, 0 < p_i < 1$. Define $S = \sum_{i=1}^n X_i$ and $\bar{p} = (1/n) \sum_{i=1}^n p_i$.

A theorem of Shepp and Olkin [25] (see also [22] and [10]) states that

$$H(S) \leq H(\text{bi}(n, \bar{p})), \tag{2.1}$$

where $\text{bi}(n, p)$ denotes the binomial pmf with n trials and probability p for success. In other words, subject to a fixed mean $n\bar{p}$, the entropy of S is maximized when all p_i are equal. Karlin and Rinott [16] (see also Harremoës [10]) note the corresponding result

$$H(S) \leq H(\text{po}(n\bar{p})), \tag{2.2}$$

where $\text{po}(\lambda)$ denotes the Poisson pmf with mean λ .

Johnson [13] gives a generalization of (2.2) to *ultra log-concave* (ULC) distributions. The notion of ultra log-concavity was introduced by Pemantle [23] in the study of negative dependence. A pmf f on \mathbf{Z}_+ is ULC of order k if $f_i / \binom{k}{i}$ is log-concave in i ; it is ULC of order ∞ , or simply ULC, if $i! f_i$ is log-concave. Equivalently, these definitions can be stated with the \leq_{lc} notation:

1. f is ULC of order k if $f \leq_{\text{lc}} \text{bi}(k, p)$ for some $p \in (0, 1)$ (the value of p does not affect the definition);
2. f is ULC of order ∞ if $f \leq_{\text{lc}} \text{po}(\lambda)$ for some $\lambda > 0$ (the value of λ does not affect the definition).

An example is the distribution of S in (2.2) and (2.1). Denoting the pmf of S by f^S , we have

$$f^S \leq_{\text{lc}} \text{bi}(n, \bar{p}), \tag{2.3}$$

which can be shown to be a reformulation of Newton’s inequalities (Hardy *et al.* [9]). Also, note that, as can be verified using the definition, f being ULC of order k means that it is also ULC of orders $k + 1, k + 2, \dots, \infty$. Another notable property of ULC distributions, expressed in our notation, is due to Liggett [21].

Theorem 2 ([21]). *If $f \leq_{\text{lc}} \text{bi}(k, p)$ and $g \leq_{\text{lc}} \text{bi}(m, p)$, $p \in (0, 1)$, then*

$$f * g \leq_{\text{lc}} \text{bi}(k + m, p),$$

where $f * g = \{\sum_{i=0}^j f_i g_{j-i}, j = 0, \dots, k + m\}$ denotes the convolution of f and g .

This is a strong result; it implies (2.3) trivially. Simply observe that $\text{bi}(1, p_i) \leq_{\text{lc}} \text{bi}(1, \bar{p}), i = 1, \dots, n$, and apply Theorem 2 to obtain $f^S = \text{bi}(1, p_1) * \dots * \text{bi}(1, p_n) \leq_{\text{lc}} \text{bi}(n, \bar{p})$, that is, f^S is ULC of order n . A limiting case of Theorem 2 also holds: for pmf's f and g on \mathbf{Z}_+ , we have

$$f \leq_{\text{lc}} \text{po}(\lambda), \quad g \leq_{\text{lc}} \text{po}(\mu) \implies f * g \leq_{\text{lc}} \text{po}(\lambda + \mu).$$

The following generalization of (2.2) is proved by Johnson [13].

Theorem 3 ([13]). *If a pmf f on \mathbf{Z}_+ is ULC, then*

$$H(f) \leq H(\text{po}(E(f))).$$

Johnson's proof uses two operations, namely convolution with a Poisson pmf and binomial thinning, to construct a semigroup action on the set of ULC distributions with a fixed mean. The entropy is then shown to be monotone along this semigroup. A corresponding generalization of (2.1) appears in Yu [28]. The proof adopts the idea of Johnson [13] and is likewise non-trivial.

Theorem 4 ([28]). *If a pmf f is ULC of order n , then*

$$H(f) \leq H(\text{bi}(n, E(f)/n)).$$

We point out that Theorems 3 and 4 can be deduced from Theorem 1; in fact, both are special cases of the following result.

Theorem 5. *Any log-concave pmf g on \mathbf{Z}_+ is the unique maximizer of entropy in the set $F = \{f: f \leq_{\text{lc}} g, E(f) = E(g)\}$.*

Proof. The log-concavity of g ensures that $\lambda \equiv E(g) < \infty$. Letting $f \in F$ and using the geometric pmf $\text{ge}(p) = \{p(1-p)^i, i = 0, 1, \dots\}$, we get

$$D(f | \text{ge}(p)) = -H(f) - \log(p) - \lambda \log(1-p),$$

$$D(g | \text{ge}(p)) = -H(g) - \log(p) - \lambda \log(1-p),$$

which also shows that $H(f) < \infty$ and $H(g) < \infty$. Since $f \leq_{\text{lc}} g \leq_{\text{lc}} \text{ge}(p)$, Theorem 1 yields

$$-H(f) \geq D(f|g) - H(g) \geq -H(g)$$

so that $H(f) \leq H(g)$ for all $f \in F$, with equality if and only if $D(f|g) = 0$, that is, $f = g$. \square

Theorems 3 and 4 are obtained by noting that both $\text{po}(\lambda)$ and $\text{bi}(n, p)$ are log-concave. For recent extensions of Theorems 3 and 4 to compound distributions, see [14] and [31].

2.2. Best binomial approximations in relative entropy

Recall that $S = \sum_{i=1}^n X_i$ is a sum of independent Bernoulli random variables, each with success probability p_i . Let $\lambda = \sum_{i=1}^n p_i$ and let f^S denote the pmf of S . Approximating S with a Poisson distribution $\text{Po}(\lambda)$ is an old problem (Le Cam [20], Chen [3], Barbour *et al.* [1]). Approximating S with a binomial $\text{Bi}(n, \bar{p})$, $\bar{p} = (1/n) \sum_{i=1}^n p_i$, has also been considered (Stein [26], Ehm [8]). The results are typically stated in terms of the total variation distance, defined for pmf's f and g as $V(f, g) = \frac{1}{2} \sum_i |f_i - g_i|$. For example, Ehm [8] applies the method of Stein and Chen to derive the bound ($\bar{q} = 1 - \bar{p}$)

$$V(f^S, \text{bi}(n, \bar{p})) \leq (1 - \bar{p}^{n+1} - \bar{q}^{n+1})[(n + 1)\bar{p}\bar{q}]^{-1} \sum_{i=1}^n (p_i - \bar{p})^2.$$

Here, we are concerned with the following problem: what is the best m , $m \geq n$, and $p \in (0, 1)$ for approximating S with $\text{Bi}(m, p)$? Intuition says $\text{Bi}(n, \bar{p})$. Indeed, Choi and Xia [4] study this in terms of the total variation distance $d_m = V(f^S, \text{bi}(m, \lambda/m))$ and prove that under certain conditions, for large enough m , d_m increases with m .

Theorem 6 ([4]). *Let $r = \lfloor \lambda \rfloor$ be the integer part of λ and let $\delta = \lambda - r$. If $r > 1 + (1 + \delta)^2$ and*

$$m \geq \max\{n, \lambda^2 / (r - 1 - (1 + \delta)^2)\},$$

then $d_m < d_{m+1} < V(f^S, \text{po}(\lambda))$.

The derivation of Theorem 6 is somewhat involved. However, if we consider this problem in terms of relative entropy rather than total variation, then Theorem 7 below gives a definite and equally intuitive answer. Similar results (see Section 2.3) hold for the negative binomial approximation of a sum of independent geometric random variables.

Theorem 7. *Suppose that $m' \geq m \geq n$, $p' \in (0, 1)$. Then,*

$$D(f^S | \text{bi}(m', p')) \geq D(f^S | \text{bi}(m, \lambda/m)) + D(\text{bi}(m, \lambda/m) | \text{bi}(m', p')) \tag{2.4}$$

and therefore

$$D(f^S | \text{bi}(m', p')) \geq D(f^S | \text{bi}(m, \lambda/m)) \geq D(f^S | \text{bi}(n, \bar{p})).$$

Proof. Let $f = f^S$, $g = \text{bi}(m, \lambda/m)$ and $h = \text{bi}(m', p')$ in Theorem 1. By (2.3), we have $f \leq_{\text{lc}} \text{bi}(n, \bar{p}) \leq_{\text{lc}} g \leq_{\text{lc}} h$. The claim follows from (1.1). □

Theorem 7 shows that, for approximating S in the sense of relative entropy,

1. $\text{Bi}(m, \lambda/m)$, which has the same mean as S , is preferable to $\text{Bi}(m, p')$, $p' \neq \lambda/m$;
2. $\text{Bi}(n, \bar{p})$ is preferable to $\text{Bi}(m, \lambda/m)$, $m > n$.

Obviously, the proof of (2.4) still applies when $\text{bi}(m', p')$ is replaced by $\text{po}(\lambda)$. Hence,

$$D(f^S | \text{po}(\lambda)) \geq D(f^S | \text{bi}(n, \bar{p})) + D(\text{bi}(n, \bar{p}) | \text{po}(\lambda)), \tag{2.5}$$

that is, $\text{Po}(\lambda)$ is worse than $\text{Bi}(n, \bar{p})$ by at least $D(\text{bi}(n, \bar{p}) | \text{po}(\lambda))$.

We conclude this subsection with another interesting result in the form of a corollary of Theorem 1. Writing $b_m = \text{bi}(m, \lambda/m)$ for simplicity, we have

$$D(b_m | \text{po}(\lambda)) \geq D(b_m | b_{m+1}) + D(b_{m+1} | \text{po}(\lambda))$$

and, therefore,

$$D(b_m | \text{po}(\lambda)) > D(b_{m+1} | \text{po}(\lambda)), \quad m > \lambda. \tag{2.6}$$

That is, the limit $\text{Bi}(m, \lambda/m) \rightarrow \text{Po}(\lambda), m \rightarrow \infty$, is monotone in relative entropy. As simple as (2.6) may seem, it is difficult to derive it directly without Theorem 1, which perhaps explains why (2.6) appears new, even though the binomial-to-Poisson limit is common knowledge.

2.3. Analogous results for the negative binomial

Let T be a sum of geometric random variables, $T = \sum_{i=1}^n Y_i$, where $Y_i \sim \text{Ge}(r_i)$ independently, $r_i \in (0, 1)$. Denote the mean of T by $\mu = \sum_{i=1}^n (1 - r_i)/r_i$ and denote the pmf of T by f^T . Let $\text{nb}(n, r) = \{\binom{n+i-1}{i} r^n (1-r)^i, i = 0, 1, \dots\}$ denote the pmf of the negative binomial $\text{NB}(n, r)$.

The counterpart of (2.1) appears in Karlin and Rinott [16].

Theorem 8 ([16]). $H(T) \geq H(\text{nb}(n, n/(n + \mu)))$.

In other words, subject to a fixed mean μ , the entropy of T is minimized when all r_i are equal. Theorem 8 can be generalized as follows.

Theorem 9. Any log-concave pmf f is the unique minimizer of entropy in the set $G = \{g: f \leq_{\text{lc}} g \leq_{\text{lc}} \text{ge}(p), E(g) = E(f)\}, p \in (0, 1)$.

We realize that Theorem 9 is just a reformulation of Theorem 5, which follows from Theorem 1. To show that Theorem 9 indeed implies Theorem 8, we need the following inequality of Hardy *et al.* [9], written in our notation as

$$\text{nb}(n, n/(n + \mu)) \leq_{\text{lc}} f^T. \tag{2.7}$$

We also need f^T to be log-concave, but this holds because convolutions of log-concave sequences are also log-concave.

Next, we consider the problem of selecting the best $m, m \geq n$, and $r \in (0, 1)$ for approximating T with $\text{NB}(m, r)$.

Theorem 10. *Suppose $m' \geq m \geq n$ and $r' \in (0, 1)$. Write $\text{nb}_m = \text{nb}(m, m/(m + \mu))$ as shorthand. Then,*

$$D(f^T | \text{nb}(m', r')) \geq D(f^T | \text{nb}_m) + D(\text{nb}_m | \text{nb}(m', r'))$$

and, therefore,

$$D(f^T | \text{nb}(m', r')) \geq D(f^T | \text{nb}_m) \geq D(f^T | \text{nb}(n, n/(n + \mu))).$$

Proof. The relations

$$\text{nb}(m', r') \leq_{\text{lc}} \text{nb}_m \leq_{\text{lc}} \text{nb}(n, n/(n + \mu))$$

are easy to verify. We also have (2.7). The claim follows from (1.2). □

Theorem 10 implies that for approximating T in the sense of relative entropy, $\text{NB}(n, n/(n + \mu))$ is no worse than $\text{NB}(m', r')$ whenever $m' \geq n$. The counterpart of (2.5) also holds ($\text{nb}_n = \text{nb}(n, n/(n + \mu))$):

$$D(f^T | \text{po}(\mu)) \geq D(f^T | \text{nb}_n) + D(\text{nb}_n | \text{po}(\mu)),$$

that is, $\text{Po}(\mu)$ is worse than $\text{NB}(n, n/(n + \mu))$ by at least $D(\text{nb}_n | \text{po}(\mu))$.

In addition, parallel to (2.6), we have

$$D(\text{nb}_m | \text{po}(\mu)) > D(\text{nb}_{m'} | \text{po}(\mu)), \quad m' > m > 0, \tag{2.8}$$

that is, the limit $\text{NB}(m, m/(m + \mu)) \rightarrow \text{Po}(\mu)$, $m \rightarrow \infty$, is monotone in relative entropy. Note that in (2.8), m and m' need not be integers; similarly in Theorem 10.

We conclude this subsection with a problem on the behavior of \leq_{lc} under convolution. Analogous to Theorem 2 is the following result of Davenport and Pólya ([6], Theorem 2), rephrased in terms of \leq_{lc} .

Theorem 11 ([6]). *Suppose that pmf's f and g on \mathbf{Z}_+ satisfy $\text{nb}(k, r) \leq_{\text{lc}} f$, $\text{nb}(m, r) \leq_{\text{lc}} g$ for $k, m > 0, r \in (0, 1)$. Their convolution $f * g$ then satisfies*

$$\text{nb}(k + m, r) \leq_{\text{lc}} f * g.$$

Actually, Davenport and Pólya [6] assume that $k + m = 1$, so their conclusion is the log-convexity of $f * g$, but it is readily verified that the same proof works for all positive k and m . The limiting case also holds, that is,

$$\text{po}(\lambda) \leq_{\text{lc}} f, \quad \text{po}(\mu) \leq_{\text{lc}} g \implies \text{po}(\lambda + \mu) \leq_{\text{lc}} f * g.$$

An open problem is to determine general conditions that ensure

$$f \leq_{\text{lc}} f', \quad g \leq_{\text{lc}} g' \implies f * g \leq_{\text{lc}} f' * g'. \tag{2.9}$$

Theorem 2 simply says that (2.9) holds if $f' = \text{bi}(k, p)$ and $g' = \text{bi}(m, p)$ with the same p and Theorem 11 says that (2.9) holds if $f = \text{nb}(k, r)$ and $g = \text{nb}(m, r)$ with the same r . The proofs of

Theorems 11 and 2 (Theorem 2 especially) are non-trivial. It is reasonable to ask whether there exist other interesting and non-trivial instances of (2.9).

3. Proof of Theorem 1

The proof of Theorem 1 hinges on the following lemma that dates back to Karlin and Novikoff [15] and Karlin and Studden [17]. Our assumptions are slightly different from those of Karlin and Studden [17], Lemma XI. 7.2. In the proof (included for completeness), the number of sign changes of a sequence is counted discarding zero terms.

Lemma 1 ([17]). *Let $a_i, i = 0, 1, \dots$, be a real sequence such that $\sum_{i=0}^{\infty} a_i = 0$ and $\sum_{i=0}^{\infty} i \times a_i = 0$. Suppose that the set $C = \{i: a_i > 0\}$ is an interval on \mathbf{Z}_+ . For any concave function $w(i)$ on \mathbf{Z}_+ , we then have*

$$\sum_{i=0}^{\infty} w(i)a_i \geq 0. \tag{3.1}$$

Proof. Karlin and Studden ([17], Lemma XI. 7.2) assume that $a_i, i = 0, 1, \dots$, changes sign exactly twice, with sign sequence $-, +, -$. However, it also suffices to assume that C is an interval. Suppose that a_i changes sign exactly once, with sign sequence $+, -$, that is, there exists $0 \leq k < \infty$ such that $a_i \geq 0, 0 \leq i \leq k$, with strict inequality for at least one $i \leq k$, and $a_i \leq 0, i > k$. Then,

$$\sum_{i=0}^{\infty} i a_i \leq \sum_{i=0}^k k a_i + \sum_{i=k+1}^{\infty} (k+1) a_i = - \sum_{i=0}^k a_i < 0,$$

a contradiction. Similarly, the sign sequence cannot be $-, +$ either. Assuming that C is an interval, this shows that, except for the trivial case $a_i \equiv 0$, the sequence a_i changes sign exactly twice, with sign sequence $-, +, -$.

The rest of the argument is well known. We proceed to show that the sequence $A_j = \sum_{i=0}^j a_i$ has exactly one sign change, with sign sequence $-, +$. Similarly, $\sum_{i=0}^j A_i \leq 0$ for all $j = 0, 1, \dots$, which implies (3.1) for every concave function $w(i)$ upon applying summation by parts. □

Theorem 12 below is a consequence of Lemma 1. Although not phrased as such, the basic idea is implicit in Karlin and Studden [17] in their analyses of special cases; see also Whitt [27]. When f is the pmf of a sum of n independent Bernoulli random variables and $g = \text{bi}(n, E(f)/n)$, as discussed in Section 2, Theorem 12 reduces to an inequality of Hoeffding [11].

Theorem 12. *Suppose that two pmf's f and g on \mathbf{Z}_+ satisfy $f \leq_{lc} g$ and $E(f) = E(g) < \infty$. For any concave function $w(i)$ on \mathbf{Z}_+ , we then have*

$$\sum_{i=0}^{\infty} f_i w(i) \geq \sum_{i=0}^{\infty} g_i w(i).$$

Proof. Since $E(g) < \infty$ and w is concave, $\sum_{i=0}^{\infty} g_i w(i)$ either converges absolutely or diverges to $-\infty$. Assume the former. Since $\log(f_i/g_i)$ is concave and hence unimodal, the set $C = \{i: f_i - g_i > 0\}$ must be an interval. The result then follows from Lemma 1. \square

Theorem 1 is a consequence of Theorem 12. Actually, we prove a slightly more general “quadrangle inequality,” which may be of interest. Theorem 1 corresponds to the special case $g = g'$ in Theorem 13.

Theorem 13. Let f, g, g', h be pmf's on \mathbf{Z}_+ such that $f \leq_{lc} g \leq_{lc} g' \leq_{lc} h$. If $E(f) = E(g) < \infty$, then $D(f|h) < \infty$ and

$$D(f|h) + D(g|g') \geq D(f|g') + D(g|h); \tag{3.2}$$

if $E(g') = E(h) < \infty$, then

$$D(h|f) + D(g'|g) \geq D(g'|f) + D(h|g). \tag{3.3}$$

Proof. The concavity of $\log(f_i/h_i)$ and $E(f) < \infty$ imply $D(f|h) < \infty$. Likewise for $D(g|h)$. Thus, (3.2) can be written as

$$D(f|h) - D(f|g') \geq D(g|h) - D(g|g')$$

or, equivalently,

$$\sum_i f_i \log(g'_i/h_i) \geq \sum_i g_i \log(g'_i/h_i). \tag{3.4}$$

Since $\log(g'_i/h_i)$ is concave in $\text{supp}(g')$, and $\text{supp}(f) \subset \text{supp}(g) \subset \text{supp}(g')$, (3.4) follows directly from Theorem 12.

To prove (3.3), we may assume $D(h|f) < \infty$ and $D(g'|g) < \infty$. These imply, in particular, that $\text{supp}(f) = \text{supp}(g') = \text{supp}(h)$. We get

$$\sum_i g'_i \log(f_i/g_i) \geq \sum_i h_i \log(f_i/g_i)$$

and (3.3) follows as before. \square

4. The continuous case

For probability density functions (pdf's) f and g with respect to Lebesgue measure on \mathbf{R} , the differential entropy of f and the relative entropy between f and g are defined, respectively, as

$$H(f) = \int_{-\infty}^{\infty} -f(x) \log(f(x)) dx \quad \text{and} \quad D(f|g) = \int_{-\infty}^{\infty} f(x) \log(f(x)/g(x)) dx.$$

Parallel to the discrete case, let us write $f \leq_{lc} g$ if

1. $\text{supp}(f)$ and $\text{supp}(g)$ are both intervals on \mathbf{R} ;
2. $\text{supp}(f) \subset \text{supp}(g)$; and
3. $\log(f(x)/g(x))$ is concave in $\text{supp}(f)$.

There then holds a continuous analog of Theorem 1 (with its first phrase replaced by “Let f, g, h be pdf’s on \mathbf{R} ”); the proof is similar and is hence omitted.

The following maximum/minimum entropy result parallels Theorems 5 and 9.

Theorem 14. *If a pdf g on \mathbf{R} is log-concave, then it maximizes the differential entropy in the set $F = \{f: f \leq_{\text{lc}} g, E(f) = E(g)\}$. Alternatively, if a pdf f on \mathbf{R} is log-concave, then it minimizes the differential entropy in the set $G = \{g: f \leq_{\text{lc}} g, g \text{ is log-concave and } E(g) = E(f)\}$.*

We illustrate Theorem 14 with a minimum entropy characterization of the gamma distribution. This parallels Theorem 8 for the negative binomial. Denote by $\text{gam}(\alpha, \beta)$ the pdf of the gamma distribution $\text{Gam}(\alpha, \beta)$, that is,

$$\text{gam}(x; \alpha, \beta) = \beta^{-\alpha} x^{\alpha-1} e^{-x/\beta} / \Gamma(\alpha), \quad x > 0.$$

Theorem 15. *Let $\alpha_i \geq 1, \beta_i > 0$ and let $X_i \sim \text{Gam}(\alpha_i, 1), i = 1, \dots, n$, independently. Define $S = \sum_{i=1}^n \beta_i X_i$. Then, subject to a fixed mean $ES = \sum_{i=1}^n \alpha_i \beta_i$, the differential entropy of S (as a function of $\beta_i, i = 1, \dots, n$) is minimized when all β_i are equal.*

Note that Theorem 3.1 of Karlin and Rinott ([16]; see also Yu [29]) implies that Theorem 15 holds when all α_i are equal. We use \leq_{lc} to give an extension to general $\alpha_i \geq 1$. A useful result is Lemma 2, which reformulates Theorem 4 of Davenport and Pólya [6]. As in Theorem 11, Davenport and Pólya assume $\alpha_1 + \alpha_2 = 1$, but the proof works for all positive α_1, α_2 .

Lemma 2 ([6], Theorem 4). *Let $\alpha_1, \alpha_2 > 0$ and let f and g be pdf’s on $(0, \infty)$ such that $\text{gam}(\alpha_1, 1) \leq_{\text{lc}} f$ and $\text{gam}(\alpha_2, 1) \leq_{\text{lc}} g$. Then,*

$$\text{gam}(\alpha_1 + \alpha_2, 1) \leq_{\text{lc}} f * g,$$

where $(f * g)(x) = \int_0^x f(y)g(x - y) dy$.

Proof of Theorem 15. Repeated application of Lemma 2 yields

$$\text{gam}(\alpha_+, 1) \leq_{\text{lc}} f^S, \tag{4.1}$$

where $\alpha_+ = \sum_{i=1}^n \alpha_i$ and f^S denotes the pdf of S . Alternatively, we can show (4.1) by noting that f^S is a mixture of $\text{gam}(\alpha_+, \beta)$, where β has the distribution of $S / \sum_{i=1}^n X_i$ (see, e.g., [27] and [30]). Since $\alpha_i \geq 1$, each X_i is log-concave and so is f^S . The claim follows from Theorem 14. \square

Weighted sums of gamma variates, as in Theorem 15, arise naturally in statistical contexts, for example, as quadratic forms in normal variables, but their distributions can be non-trivial to compute (Imhof [12]). When comparing different gamma distributions as convenient approximations, we obtain a result similar to Theorems 7 and 10. The proof, also similar, is omitted.

Theorem 16. Fix $\alpha_i > 0, \beta_i > 0$ and let $X_i \sim \text{Gam}(\alpha_i, 1), i = 1, \dots, n$, independently. Define $S = \sum_{i=1}^n \beta_i X_i$, with pdf f^S . Write $g_a = \text{gam}(a, \sum_{i=1}^n \beta_i \alpha_i / a)$ as shorthand. For $b > 0$ and $a' \geq a \geq \alpha_+$, where $\alpha_+ = \sum_{i=1}^n \alpha_i$, we then have

$$D(f^S | \text{gam}(a', b)) \geq D(f^S | g_a) + D(g_a | \text{gam}(a', b))$$

and, consequently,

$$D(f^S | \text{gam}(a', b)) \geq D(f^S | g_a) \geq D(f^S | g_{\alpha_+}).$$

In other words, to approximate S in the sense of relative entropy, $\text{Gam}(\alpha_+, \sum_{i=1}^n \beta_i \alpha_i / \alpha_+)$, which has the same mean as S , is no worse than $\text{Gam}(a, b)$ whenever $a \geq \alpha_+$. Note that, unlike in Theorem 15, we do not require here that $\alpha_i \geq 1$.

Overall, there is a remarkable parallel between the continuous and discrete cases.

Acknowledgments

The author would like to thank three referees for their constructive comments.

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Received March 2008 and revised May 2009