Discrete approximation of a stable self-similar stationary increments process

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The aim of this paper is to present a result of discrete approximation of some class of stable self-similar stationary increments processes. The properties of such processes were intensively investigated, but little is known about the context in which such processes can arise. To our knowledge, discretization and convergence theorems are available only in the case of stable Lévy motions and fractional Brownian motions. This paper yields new results in this direction. Our main result is the convergence of the random rewards schema first introduced by Cohen and Samorodnitsky, which we consider in a more general setting. Strong relationships with Kesten and Spitzer's random walk in random sceneries are evidenced. Finally, we study some path properties of the limit process.

Keywords: random scenery; random walk; self-similarity; stable process

1. Introduction

1.1. Motivation

In the past few decades, many efforts were made to achieve a better comprehension of stable self-similar stationary increments processes and to achieve a classification of such processes. The Gaussian case was studied by Mandelbrot and Van Ness [8] and gives rise to an easy classification: To any index of self-similarity $H \in (0, 1)$ corresponds a unique (up to a multiplicative constant) Gaussian H-self-similar process, called *fractional Brownian motion*. Basic properties of the fractional Brownian motion were investigated by Mandelbrot and Van Ness. In the α-stable case, $0 < \alpha < 2$, the family of self-similar stationary increments processes is much larger. The feasible range of pairs (α, H) (where α is the index of stability and H the exponent of self-similarity) is

$$\begin{cases} 0 < H \leq \dfrac{1}{\alpha}, & \text{if } 0 < \alpha \leq 1, \\ 0 < H < 1, & \text{if } 1 < \alpha < 2, \end{cases}$$

and, apart from the case $0 < \alpha < 1$ and $H = 1/\alpha$, a feasible pair (α, H) does not determine the law of the process. Two of the most well-known families of stable self-similar stationary increments process are obtained by taking two of the many possible integral representations of

the fractional Brownian motion and modifying them appropriately: the linear fractional stable motion and the real harmonizable fractional stable motion.

In many of these works, the construction of such processes is based on the theoretical tool of stochastic or stable integration. This strategy allows us to construct many processes with interesting properties. However, apart from the case of Lévy motions and fractional Brownian motions, we do not know any situation where these processes naturally arise. For example, the Donsker theorem yields an approximation of Brownian motion by discrete processes that arise naturally in many cases of modeling, and hence explains why Brownian motion is so important. In the context of communication networks, the best known result of this type is perhaps the paper by Mikosch *et al.* [9], where the limiting model turns out to be either fractional Brownian motion or Lévy stable motion (depending on the relationship between the number of users and the time scale).

In [2], the authors present a discrete schema, called the *random rewards schema*, and show its convergence to a stable self-similar stationary increments process called FBM-1/2-local time fractional symmetric stable motion. This provides a rigorous framework in which a non-standard self-similar stationary increments process appears naturally as the limit of a discrete schema. Roughly speaking, the schema can be explained as follows: A simple network is modelized by the integer line. We suppose that a random reward is associated to each node of the network. A user is moving randomly on the network and is earning the rewards associated to the nodes visited. The schema deals with the total reward earned when many such users are considered. In this paper, we consider the random rewards schema in a more general setting. We suppose that the random reward is in the domain of attraction of a β-stable distribution and that the random walk performed by the user is in the domain of attraction of an α-stable Lévy motion, with $1 < \alpha \leq 2$ (in [2], only the case $\alpha = 2$ was considered). The convergence of the random reward schema is proven and some properties of the limit process are investigated. This yields a new class of β-stable H-self-similar stationary increments processes, with $H = 1 - \alpha^{-1} + (\alpha\beta)^{-1}$. With this procedure, the range of feasible pairs is:

$$
\begin{cases}
\dfrac{\beta + 1}{2\beta} \leq H < \dfrac{1}{\beta}, & \text{if } 0 < \beta < 1, \\
H = 1, & \text{if } \beta = 1, \\
\dfrac{1}{\beta} < H \leq \dfrac{\beta + 1}{2\beta} & \text{if } 1 < \beta < 2.
\end{cases}
$$

The discrete approximation can be used to simulate these stablep self-similar stationary increments processes, which are otherwise hard to simulate because their finite-dimensional distributions involve a local time process.

The next section is devoted to the presentation of the model and of the results we obtained. In the second section we prove the convergence of the generalized random reward schema. We begin the proof with some estimates on quantities associated with random walks (local times, number of self-intersections and range) and then focus on the convergence of the finite-dimensional distributions and on the tightness of the sequence. In the last section, some properties of the limit process are studied.

1.2. Model and results

To place our results in context, we first describe a result of Kesten and Spitzer [5] that shows that a stable process in random scenery can be realized as the limit in distribution of a random walk in random scenery.

Let $\xi = (\xi_x)_{x \in \mathbb{Z}}$ denote a sequence of independent, identically distributed, real-valued random variables. The sequence ξ is called a *random scenery*. Suppose that it belongs to the normal domain of attraction of a strictly stable distribution Z_β of index $\beta \in (0, 2]$. This means that the following weak convergence holds:

$$n^{1/\beta} \sum_{x=0}^{n} \xi_x \xrightarrow[n \to \infty]{\mathcal{L}} Z_\beta. \tag{1}$$

We suppose that the characteristic function $\bar{\lambda}$ of the distribution Z_β writes, for all $u \in \mathbb{R}$

$$\bar{\lambda}(u) = \mathbb{E}\exp(iu Z_\beta) = \exp\left(-\sigma^\beta |u|^\beta \left(1 - iv \tan\left(\frac{\pi\beta}{2}\right)\mathrm{sgn}(u)\right)\right), \tag{2}$$

for some constants $\sigma > 0$ and $v \in [-1, 1]$. Note that, in the case $\beta \neq 1$, this is the most general form of a strictly stable distribution. In the case $\beta = 1$, this is the general form of a strictly stable distribution satisfying the symmetry condition

$$\exists M \geq 0, \forall L > 0 \qquad \left|\mathbb{E}\left(Z_\beta 1_{\{Z_\beta < L\}}\right)\right| \leq M. \tag{3}$$

Let $S = (S_k)_{k \in \mathbb{N}}$ be a *random walk* on $\mathbb{Z}$ independent of the random scenery ξ. We suppose that

$$\begin{cases} S_0 = 0, \\ S_n = \sum_{k=1}^{n} X_k, \qquad n \geq 1, \end{cases}$$

where the X_i are i.i.d. integer-valued random variables belonging to the normal domain of attraction of a strictly stable distribution Z_α of index

$$\alpha \in (1, 2].$$

This means that the following weak convergence holds:

$$n^{-1/\alpha} S_n \xrightarrow[n \to \infty]{\mathcal{L}} Z_\alpha, \tag{4}$$

where the distribution Z_α has characteristic function

$$\mathbb{E}\exp(iu Z_\alpha) = \exp\left(-\chi^\alpha |u|^\alpha \left(1 - i\mu \tan\left(\frac{\pi\alpha}{2}\right)\mathrm{sgn}(u)\right)\right), \tag{5}$$

for all $u \in \mathbb{R}$ and for some constants $\chi > 0$ and $\mu \in [-1, 1]$.

We define the *random walk in random scenery* as the process $(Z_n)_{n \geq 0}$ given by

$$Z_n = \sum_{k=0}^{n} \xi_{S_k}. \tag{6}$$

Stated simply, a random walk in random scenery is a cumulative sum process whose summands are drawn from the scenery; the order in which the summands are drawn is determined by the path of the random walk. We extend this definition to non-integer time $s \geq 0$ by the linear interpolation

$$Z_s = Z_{[s]} + (s - [s])\big(Z_{[s]+1} - Z_{[s]}\big). \tag{7}$$

We now describe the limit theorem for the random walk in random scenery established by Kesten and Spitzer [5].

Cumulative sums of the scenery converge in $\mathcal{D}(\mathcal{R})$, the space of cadlag functions:

$$\left(n^{-1/\beta} \sum_{k=0}^{[nx]} \xi_k\right)_{x \in \mathbb{R}} \xRightarrow[n \to \infty]{\mathcal{L}} (W(x))_{x \in \mathbb{R}},$$

where W is a bilater β-stable Lévy process such that $W(0) = 0$, and $W(1)$ and $W(-1)$ are distributed according to Z_β. The random walk converges in $\mathcal{D}([0, \infty))$:

$$\left(n^{-1/\alpha} S_{[nt]}\right)_{t \geq 0} \xRightarrow[n \to \infty]{\mathcal{L}} (Y(t))_{t \geq 0},$$

where the limit process Y is an α-stable Lévy process such that $Y(0) = 0$ and $Y(1)$ is distributed according to Z_α.

To describe the limit process known as a *stable process in random scenery* we suppose that Y and W are two independent processes defined on the same probability space and distributed as above. Let $L_t(x)$ be the jointly continuous version of the local time of the process Y. Kesten and Spitzer prove the following weak convergence in the space of continuous function $\mathcal{C}([0, \infty))$:

$$(n^{-\delta} Z_{nt})_{t \geq 0} \xRightarrow[n \to \infty]{\mathcal{L}} (\Delta(t))_{t \geq 0}, \tag{8}$$

where $\delta = 1 - \alpha^{-1} + (\alpha\beta)^{-1}$ and Δ is the process defined by

$$\Delta(t) = \int_{-\infty}^{+\infty} L_t(x) \, dW(x).$$

The limit process Δ is known to be a continuous δ-self-similar stationary increments process. It can be seen as a mixture of β-stable processes, but it is not a stable process.

To get a stable process, we consider sums of independent stable processes in random scenery. Let $\Delta^{(i)}, i \geq 1$ be independent copies of the process Δ.

Theorem 1.1. *The following weak convergence holds in $\mathcal{C}([0, \infty))$:*

$$\left(n^{-1/\beta} \sum_{i=1}^{n} \Delta^{(i)}(t)\right)_{t \geq 0} \overset{\mathcal{L}}{\underset{n \to \infty}{\Longrightarrow}} (\Gamma(t))_{t \geq 0}. \tag{9}$$

Remark. In this theorem, Γ is defined as the weak limit of a sequence of processes. As we will see in Theorem 1.3 below, Γ is a continuous β-stable δ-self-similar stationary increments process. It is worth noting that in the case $\beta = 2$ (i.e., if the scenery has finite variance), Γ must hence be a fractional Brownian motion of index $\delta = 1 - \frac{1}{2\alpha}$.

Replacing the stable process in random scenery by a random walk in random scenery, we obtain the random rewards schema, which yields a discrete approximation of the process Γ. Let $\xi^{(i)} = (\xi_x^{(i)})_{x \in \mathbb{Z}}$, $i \geq 1$ be independent copies of ξ. Let $S^{(i)} = (S_n^{(i)})_{n \in \mathbb{N}}$ be independent copies of S and also independent of the $\xi^{(i)}$, $i \geq 1$. Denote by $D_n^{(i)}$ the ith random walk in random scenery defined by

$$D_n^{(i)}(t) = n^{-\delta} Z_{nt}^{(i)}, \tag{10}$$

where the definition of $Z_n^{(i)}$ is given by equations (6) and (7) with ξ and S replaced by the ith random scenery $\xi^{(i)}$ and the ith random walk $S^{(i)}$, respectively.

Theorem 1.2. *Let c_n be a sequence of integers such that $\lim c_n = +\infty$. Then, the following weak convergence holds in $\mathcal{C}([0, \infty))$:*

$$\left(c_n^{-1/\beta} \sum_{i=1}^{c_n} D_n^{(i)}(t)\right)_{t \geq 0} \overset{\mathcal{L}}{\underset{n \to \infty}{\Longrightarrow}} (\Gamma(t))_{t \geq 0}. \tag{11}$$

The limit process Γ is the same as in Theorem 1.1.

The proofs of Theorems 1.1 and 1.2 involve proof of convergence of finite-dimensional distributions (Propositions 2.1 and 3.1, respectively) followed by the proof of tightness (Propositions 2.2 and 3.2, respectively).

We then focus on the limit process Γ. Its finite-dimensional distribution is a β-stable multivariate distribution whose characteristic functions are explicited in the following theorem. Since the finite-dimensional distributions are not very convenient to describe trajectorial properties of the process Γ, we also introduce an integral representation of Γ.

Let $(\Omega', \mathcal{F}', \mathbb{P}')$ be a probability space supporting an α-stable Lévy motion $Y(t)$ and let $L_t(x)$ be its jointly continuous local time process. Let M be a β-stable random measure on the space $\Omega' \times \mathbb{R}$ with control measure $\mathbb{P}' \times \lambda$ (where λ is the Lebesgue measure on $\mathbb{R}$) and skewness intensity identically equal to ν. The random measure itself lives on some other probability space $(\Omega, \mathcal{F}, \mathbb{P})$. For the definition and properties of stable random measures and integrals with respect with these measures, we refer to the monograph of Samorodnitsky and Taqqu [11].

Theorem 1.3. *The process* Γ *is a* β*-stable* δ*-self-similar stationary increments process whose finite-dimensional distributions satisfy for all* $(\theta_1, \ldots, \theta_k) \in \mathbb{R}^k$ *and* $(t_1, \ldots, t_k) \in [0, +\infty)^k$,

$$\mathbb{E}\left[\exp\left(i \sum_{j=1}^{k} \theta_j \Gamma(t_j)\right)\right]$$

$$= \exp\left[-\sigma^\beta \mathbb{E} \int_{-\infty}^{+\infty} \left|\sum_{j=1}^{k} \theta_j L_{t_j}(x)\right|^\beta \right. \tag{12}$$

$$\left. \times \left(1 - i\nu \tan\left(\frac{\pi\beta}{2}\right) \mathrm{sgn}\left(\sum_{j=1}^{k} \theta_j L_{t_j}(x)\right)\right) \mathrm{d}x\right].$$

Furthermore, it has the following integral representation:

$$\Gamma(t) \overset{\mathcal{L}}{=} \int_{\Omega' \times \mathbb{R}} \sigma L_t(x)(\omega') M(\mathrm{d}\omega', \mathrm{d}x), \quad t \geq 0, \tag{13}$$

where $\overset{\mathcal{L}}{=}$ *stands for the equality of the finite-dimensional distributions of the two processes.*

Cohen and Samorodnitsky [2] considered the process $\Gamma(t)$ similar to the one in Theorem 1.3 but they had $\alpha = 2$ and $L_t(x)$ the jointly continuous local time of fractional Brownian motion with exponent of self-similarity $H \in (0, 1)$. (The only case where Cohen and Samorodnitsky [2] obtained this $\Gamma(t)$ as a weak limit of a random reward scheme is $H = 0.5$.) They called the process defined by (13) (with the above $L_t(x)$) FBM-H-local time fractional β-stable motion. Following their terminology one can call $\Gamma(t)$ an "α-stable Lévy motion local time fractional β-stable motion" or a "fractional β-stable motion induced by α-stable Lévy motion local time."

The integral representation (13) is a powerful tool to derive the path properties of the process Γ. It can be used to explicate the probability tails of the suprema of the process. In the following, we only consider the case $0 < \beta < 2$. As we mentioned above, in the case $\beta = 2$, the process Γ is a fractional Brownian motion and its properties were widely studied.

Theorem 1.4. *Suppose that* $0 < \beta < 2$. *Then, for any* $T > 0$,

$$\lim_{u \to \infty} u^\beta \mathbb{P}\left(\sup_{t \in [0,T]} \Gamma(t) \geq u\right) = C_\beta \frac{1+\nu}{2} \mathbb{E}\left(\int_\mathbb{R} L_T(x)^\beta \, \mathrm{d}x\right)$$

and

$$\lim_{u \to \infty} u^\beta \mathbb{P}\left(\sup_{t \in [0,T]} |\Gamma(t)| \geq u\right) = C_\beta \mathbb{E}\left(\int_\mathbb{R} L_T(x)^\beta \, \mathrm{d}x\right),$$

with

$$C_\beta = \left(\int_0^\infty x^{-\beta} \sin x \, \mathrm{d}x\right)^{-1}.$$

The last result of this paper is devoted to the Hölder regularity of sample path of the process Γ:

Theorem 1.5. *Suppose that* $0 < \beta < 2$. *If* $\beta \geq 1$, *we suppose furthermore that* $v = 0$. *Then, the process* $(\Gamma(t))_{t \geq 0}$ *satisfies almost surely*

$$\sup_{0 \leq s < t \leq 1/2} \frac{|\Gamma(t) - \Gamma(s)|}{(t-s)^{1-1/\alpha} |\log(t-s)|^{1/\alpha+\varepsilon}} < \infty,$$

with

$$\varepsilon = \begin{cases} 0, & \text{if } 1 \leq \beta < 2, \\ \dfrac{1}{2}, & \text{if } 0 < \beta < 1. \end{cases}$$

The proof of Theorem 1.5 employs techniques used in the proof of Theorem 5.1 in Cohen and Samorodnitsky [2].

2. Sums of stable processes in random scenery. Proof of Theorem 1.1

For $n \geq 1$, let Γ_n be the continuous process defined by

$$\Gamma_n(t) = n^{-1/\beta} \sum_{i=1}^{n} \Delta^{(i)}(t), \qquad t \geq 0.$$

Theorem 1.1 claims that the sequence Γ_n converges weakly in $\mathcal{C}([0, \infty))$. We prove this fact by proving the convergence of the finite-dimensional distributions and the tightness of the sequence. Theorem 1.1 is thus a consequence of Propositions 2.1 and 2.2 below.

Proposition 2.1. *The finite-dimensional distributions of* $(\Gamma_n(t))_{t \geq 0}$ *converge weakly as* $n \to \infty$.

Proof. Let $(\theta_1, \ldots, \theta_k) \in \mathbb{R}^k$ and $(t_1, \ldots, t_k) \in [0, +\infty)^k$. We prove the convergence of the characteristic functions

$$\mathbb{E}\left[\exp\left(i \sum_{j=1}^{k} \theta_j \Gamma_n(t_j)\right)\right] = \mathbb{E}\left[\exp\left(in^{-1/\beta} \sum_{i=1}^{n} \sum_{j=1}^{k} \theta_j \Delta^{(i)}(t_j)\right)\right].$$

Since the $\Delta^{(i)}$ are independent copies of Δ,

$$\mathbb{E}\left[\exp\left(i \sum_{j=1}^{k} \theta_j \Gamma_n(t_j)\right)\right] = \mathbb{E}\left[\exp\left(in^{-1/\beta} \sum_{j=1}^{k} \theta_j \Delta(t_j)\right)\right]^n. \tag{14}$$

Using Lemma 5 in [5], which gives the characteristic function of the finite-dimensional distributions of Δ, we compute

$$\mathbb{E}\left[\exp\left(in^{-1/\beta}\sum_{j=1}^{k}\theta_j\Delta(t_j)\right)\right] = \mathbb{E}[\exp(-n^{-1}X)] \tag{15}$$

with

$$X = \sigma^\beta\int_{\mathbb{R}}\left|\sum_{j=1}^{k}\theta_j L_{t_j}(x)\right|^\beta\left(1 - i\nu\tan\left(\frac{\pi\beta}{2}\right)\mathrm{sgn}\left(\sum_{j=1}^{k}\theta_j L_{t_j}(x)\right)\right)dx. \tag{16}$$

We will prove that equation (15) implies the following asymptotic as $n \to \infty$

$$\mathbb{E}\left[\exp\left(in^{-1/\beta}\sum_{j=1}^{k}\theta_j\Delta(t_j)\right)\right] = 1 - n^{-1}\mathbb{E}(X) + \mathrm{o}(n^{-1}). \tag{17}$$

The fact that X is integrable is proven in Lemma 2.1 below. We now prove equation (17). To this aim, we compute the limit as n goes to infinity of

$$n\left(\mathbb{E}\left[\exp\left(in^{-1/\beta}\sum_{j=1}^{k}\theta_j\Delta(t_j)\right)\right] - 1\right) = \mathbb{E}(f_n(X)),$$

where f_n is defined on $\mathcal{C}$ by $f_n(x) = n(\exp(-n^{-1}x) - 1)$. The sequence $f_n(X)$ converges almost surely to $-X$. Furthermore, $|f_n(X)|$ is almost surely bounded from above by $|X|$ (because $|f_n(x)| \le |x|$ if $\mathrm{Re}(x) \ge 0$) and X is integrable (see Lemma 2.1 below). Hence, using dominated convergence, we prove that $\mathbb{E}(f_n(X))$ converges to $-\mathbb{E}(X)$. This last fact is equivalent to equation (17).

Finally, equations (14) and (17) together yield

$$\lim_{n\to\infty}\mathbb{E}\left[\exp\left(i\sum_{j=1}^{k}\theta_j\Gamma_n(t_j)\right)\right] = \exp(-\mathbb{E}(X)). \tag{18}$$

$\square$

Remark. The limit process is denoted by Γ. Its finite-dimensional distributions are given by equation (12), which is a consequence of equations (16) and (18) together.

In the previous demonstration, we use the fact that X is integrable. Since

$$|X| \le \left(1 + \left|\nu\tan\left(\frac{\pi\beta}{2}\right)\right|\right)\sigma^\beta\left(\sum_{j=1}^{k}|\theta_j|\right)^\beta\int_{\mathbb{R}}L_t(x)^\beta\,dx$$

with $t = \max\{t_j, 1 \le j \le k\}$, the integrability of X is a consequence of the following lemma:

Lemma 2.1. *For any $t \geq 0$, the random variable $\int_{\mathbb{R}} L_t(x)^\beta \, \mathrm{d}x$ is integrable.*

Proof. In the case $\beta = 2$, the following estimation from Khoshnevisan and Lewis (see [7], Corollary 5.6, page 105)

$$\lim_{\lambda \to \infty} \lambda^\alpha \mathbb{P}\left(\int_{\mathbb{R}} L_1^2(x) \, \mathrm{d}x > \lambda \right) = c \in (0, \infty)$$

proves that the self-intersection local time of the α-stable Lévy process has a bounded expectation at time 1. The self-similarity of the local time implies that

$$\mathbb{E}\left(\int_{\mathbb{R}} L_t^2(x) \, \mathrm{d}x \right) = t^{2-1/\alpha} \mathbb{E}\left(\int_{\mathbb{R}} L_1^2(x) \, \mathrm{d}x \right) < \infty.$$

In the case $\beta \in (0, 2)$, we use the Hölder inequality,

$$\mathbb{E}\left(\int_{\mathbb{R}} L_t^\beta(x) \, \mathrm{d}x \right) = \mathbb{E} \int_{\mathbb{R}} L_t^\beta(x) 1_{\{\sup_{0 \leq s \leq t} |Y(s)| \geq |x|\}} \, \mathrm{d}x$$

$$\leq \left(\mathbb{E} \int_{\mathbb{R}} L_t^2(x) \, \mathrm{d}x \right)^{\beta/2} \left(\int_{\mathbb{R}} \mathbb{P}\left(\sup_{0 \leq s \leq t} |Y(s)| \geq |x| \right) \mathrm{d}x \right)^{1-\beta/2}.$$

The last integral is bounded because the probability tail of suprema of an α-stable Lévy process verifies

$$\lim_{\lambda \to \infty} \lambda^\alpha \mathbb{P}\left(\sup_{0 \leq s \leq t} |Y(s)| \geq \lambda \right) = c' \in (0, \infty)$$

(see Theorem 10.51, page 470 in [11]). □

Proposition 2.2. *The sequence of process Γ_n is tight in $\mathcal{C}([0, \infty))$.*

Proof. The case $\beta = 2$ is straightforward: the process Δ is square integrable and for all $0 \leq t_1 < t_2$

$$\mathbb{E}[|\Delta(t_2) - \Delta(t_1)|^2] = \mathbb{E}\left[\left(\int_{\mathbb{R}} \left(L_{t_2}(x) - L_{t_1}(x) \right) \mathrm{d}W_x \right)^2 \right]$$

$$= \sigma^2 \mathbb{E}\left[\int_{\mathbb{R}} \left(L_{t_2}(x) - L_{t_1}(x) \right)^2 \mathrm{d}x \right]$$

$$= \sigma^2 \mathbb{E}\left[\int_{\mathbb{R}} L_{t_2-t_1}^2(x) \, \mathrm{d}x \right]$$

$$= \sigma^2 (t_2 - t_1)^{2-1/\alpha} \mathbb{E}\left[\int_{\mathbb{R}} L_1(x)^2 \, \mathrm{d}x \right].$$

In the computation above, we first use the Itô formula to compute the second order moment of a stochastic integral with respect to Brownian motion. Then we use of the self-similarity and the

stationarity of the increments of the local time. Since Γ_n is a sum of independent copies of Δ,

$$\mathbb{E}[|\Gamma_n(t_2) - \Gamma_n(t_1)|^2] = \mathbb{E}\left[\left|n^{-1/2}\sum_{i=1}^{n}\Delta^{(i)}(t_2) - \Delta^{(i)}(t_1)\right|^2\right]$$

$$= \sigma^2(t_2 - t_1)^{2-1/\alpha}\mathbb{E}\left[\int_{\mathbb{R}} L_1(x)^2\,dx\right].$$

Using the Kolmogorov criterion, we deduce that the sequence Γ_n is tight.

In the case $0 < \beta < 2$, the proof has to be modified because the process Γ_n has infinite variance. In order to prove the tightness of Γ_n, let us introduce a truncation method. As a Lévy process, W is a semimartingale whose Lévy–Itô decomposition is given by

$$W_x = bx + \int_0^x\int_{|u|\leq 1} u(\mu - \bar{\mu})(du, ds) + \int_0^x\int_{|u|>1} u\mu(du, ds), \tag{19}$$

where μ is a Poisson random measure on $\mathbb{R}\times\mathbb{R}$ with intensity $\bar{\mu}(du, dx) = \lambda(du)\otimes dx$, b is the drift and λ stands for the stable Lévy measure on $\mathbb{R}$:

$$\lambda(du) = \left(c_-1_{\{u<0\}} + c_+1_{\{u>0\}}\right)\frac{du}{|u|^{\beta+1}}, \qquad c_-, c_+ \geq 0,\ c_- + c_+ > 0.$$

For some truncation level $R > 1$, let $W^{(R^-)}$ and $W^{(R^+)}$ be the independent Lévy processes defined by

$$W_x^{(R^-)} = \int_0^x\int_{|u|\leq R} u(\mu - \bar{\mu})(du, ds), \qquad W_x^{(R^+)} = \int_0^x\int_{|u|>R} u\mu(du, ds).$$

The first one has a compactly supported Lévy measure and is a square-integrable martingale with infinitely many jumps bounded by R on each compact time interval, whereas the second one is a compound Poisson process. The Lévy–Itô decomposition (19) rewrites as

$$W_x = b_Rx + W_x^{(R^-)} + W_x^{(R^+)},$$

where $b_R = b + \int_{1<|y|\leq R} u\lambda(du)$ is a drift depending on R.

This decomposition of the stable scenery yields the following decomposition of the stable process in random scenery:

$$\Delta(t) = b_Rt + \Delta^{(R^-)}(t) + \Delta^{(R^+)}(t),$$

with

$$\Delta^{(R^-)}(t) = \int_{\mathbb{R}} L_t(x)\,dW_x^{(R^-)}, \qquad \Delta^{(R^+)}(t) = \int_{\mathbb{R}} L_t(x)\,dW_x^{(R^+)}.$$

The process $\Delta^{(R^-)}$ is square integrable and for any $0 \le t_1 < t_2$,

$$
\begin{aligned}
\mathbb{E}\big[(\Delta^{(R^-)}(t_2) - \Delta^{(R^-)}(t_1))^2\big] &= \mathbb{E}\bigg[\bigg(\int_{\mathbb{R}} (L_{t_2}(x) - L_{t_1}(x))\,\mathrm{d}W_x^{(R^-)}\bigg)^2\bigg] \\
&= \mathbb{E}\bigg[\int_{\mathbb{R}} (L_{t_2}(x) - L_{t_1}(x))^2\,\mathrm{d}\langle W^{(R^-)}, W^{(R^-)}\rangle_x\bigg] \\
&= \frac{c_- + c_+}{2 - \beta} R^{2-\beta} \mathbb{E}\bigg[\int_{\mathbb{R}} L_{t_2-t_1}(x)^2\,\mathrm{d}x\bigg] \\
&= \frac{c_- + c_+}{2 - \beta} R^{2-\beta} (t_2 - t_1)^{2-1/\alpha} \mathbb{E}\bigg[\int_{\mathbb{R}} L_1(x)^2\,\mathrm{d}x\bigg]. \quad (20)
\end{aligned}
$$

Here we have used the following expression of the bracket angle:

$$
\begin{aligned}
\langle W^{(R^-)}, W^{(R^-)}\rangle_x &= \int_0^x \int_{|u| \le R} u^2 \bar{\mu}(\mathrm{d}u, \mathrm{d}s) \\
&= \frac{c_- + c_+}{2 - \beta} R^{2-\beta} x,
\end{aligned}
$$

and also the properties of stationarity and self-similarity of the stable process that goes through its local time.

On the other hand, since $W_x^{(R^+)}$ is a pure jump process, the stochastic integral defining $\Delta^{(R^+)}$ is constructed in the Lebesgue–Stieltjes sense and is equal to

$$
\Delta^{(R^+)}(t) = \int_{\mathbb{R}} \int_{|u| > R} u L_t(s) \mu(\mathrm{d}u,\ \mathrm{d}s).
$$

Let $T \ge 0$. Suppose that the Lévy-process W has no jump of size greater than R on the (bounded) support of L_T, then the process $\Delta^{(R^+)}$ is identically 0 on $[0, T]$. Hence,

$$
\mathbb{P}\big(\Delta^{(R^+)} \equiv 0 \text{ on } [0, T]\big) \ge \mathbb{P}\big(\mu\{(u, x) \in \mathbb{R}^2 \mid |u| \ge R \text{ and } L_T(x) \ne 0\} = 0\big).
$$

Conditionally on the support of the local time, the random variable

$$
\mu\{(u, x) \in \mathbb{R}^2 \mid |u| \ge R \text{ and } L_T(x) \ne 0\}
$$

has Poisson distribution with parameter

$$
\begin{aligned}
\bar{\mu}\big(\mathrm{supp}(L_T) \times (\mathbb{R} \setminus [-R, R])\big) &= \int_{\mathbb{R}} 1_{\{L_T(x) \ne 0\}}\,\mathrm{d}x \int_{|u| > R} \lambda(y)\,\mathrm{d}u \\
&\le 2 \sup_{0 \le t \le T} |Y(t)| \frac{c^+ + c^-}{\beta} R^{-\beta}.
\end{aligned}
$$

Hence, the probability that $\Delta^{(R^+)}$ is identically 0 on $[0, T]$ is underestimated by

$$\mathbb{P}\big(\Delta^{(R^+)} \equiv 0 \text{ on } [0, T]\big) \geq \mathbb{E}\left[\exp\left(-2\frac{c^+ + c^-}{\beta} R^{-\beta} \sup_{0 \leq t \leq T} |Y(t)|\right)\right]$$

$$\geq 1 - 2\frac{c^+ + c^-}{\beta} R^{-\beta} \mathbb{E}\left(\sup_{0 \leq t \leq T} |Y(t)|\right). \tag{21}$$

We now consider a corresponding decomposition for Γ_n. Let $R > 1$ and introduce the truncation level $R_n = Rn^{1/\beta}$. Each random walk in random scenery can be decomposed as above, which yields,

$$\Delta^{(i)}(t) = b_{R_n} t + \Delta^{(i, R_n^-)}(t) + \Delta^{(i, R_n^+)}(t).$$

The process Γ_n writes

$$\Gamma_n(t) = n^{1-1/\beta} b_{R_n} t + \Gamma_n^{(R^-)}(t) + \Gamma_n(R^+)(t), \tag{22}$$

with

$$\Gamma_n^{(R^-)}(t) = n^{-1/\beta} \sum_{i=1}^{n} \Delta^{(i, R_n^-)}(t), \qquad \Gamma_n^{(R^+)}(t) = n^{-1/\beta} \sum_{i=1}^{n} \Delta^{(i, R_n^+)}(t).$$

The tightness of the sequence Γ_n is a consequence of equation (22) and of the following assertions:

A$_1$ the sequence $n^{1-1/\beta} b_{R_n}$, $n \geq 1$ is bounded,

A$_2$ for each $R > 1$, the sequence of process $\Gamma_n^{(R^-)}$, $n \geq 1$ is tight,

A$_3$ for each $T > 0$, the probability that $\Gamma_n^{(R^+)} \equiv 0$ on $[0, T]$ satisfies

$$\lim_{R \to \infty} \limsup_{n \to \infty} \mathbb{P}\big(\Gamma_n^{(R^+)} \equiv 0 \text{ on } [0, T]\big) = 1.$$

Proof of Assertion A$_1$. We compute

$$b_R = b + \int_{1 < |y| \leq R} u\lambda(\mathrm{d}u) = b + \frac{c^+ + c^-}{1 - \beta}(R^{1-\beta} - 1),$$

and hence

$$n^{1-1/\beta} b_{R_n} = \left(b - \frac{c^+ + c^-}{1 - \beta}\right) n^{1-\frac{1}{\beta}} + \frac{c^+ + c^-}{1 - \beta} R^{1-\beta}.$$

In the case $\beta \leq 1$, this quantity is trivially bounded. In the case $\beta > 1$, this is still true because

$$\mathbb{E}(W_1) = b - \frac{c^+ + c^-}{1 - \beta} = 0.$$

Proof of Assertion A₂. Let $R > 1$. For any $0 < t_1 < t_2$, equation (20) implies

$$\mathbb{E}\big[\big(\Gamma_n^{(R^-)}(t_2) - \Gamma_n^{(R^-)}(t_1)\big)^2\big] = n^{1-2/\beta}\,\mathbb{E}\big[\big(\Delta^{(R_n^-)}(t_2) - \Delta^{(R_n^-)}(t_1)\big)^2\big]$$

$$= n^{1-2/\beta}\frac{c_- + c_+}{2 - \beta}R_n^{2-\beta}(t_2 - t_1)^{2-1/\alpha}\mathbb{E}\bigg[\int_{\mathbb{R}} L_1(x)^2\,dx\bigg]$$

$$= \frac{c_- + c_+}{2 - \beta}R^{2-\beta}(t_2 - t_1)^{2-1/\alpha}\mathbb{E}\bigg[\int_{\mathbb{R}} L_1(x)^2\,dx\bigg]. \qquad (23)$$

Using the Kolmogorov criterion, this estimation implies that $\Gamma_n^{(R^-)}$ is tight.

Proof of Assertion A₃. As $\Gamma_n^{(R^+)}$ is a sum of n independent copies of $\Delta^{(R_n^+)}$, it vanishes on $[0, T]$ as soon as each summand does. Hence, using equation (21), we get

$$\mathbb{P}\big(\Gamma_n^{(R^+)} \equiv 0 \text{ on } [0, T]\big) \geq \big[\mathbb{P}\big(\Delta^{(R_n^+)} \equiv 0 \text{ on } [0, T]\big)\big]^n$$

$$\geq \bigg[1 - 2\frac{c^+ + c^-}{\beta}R_n^{-\beta}\mathbb{E}\bigg(\sup_{0\leq t\leq T}|Y(t)|\bigg)\bigg]^n.$$

Take into account the scaling $R_n = Rn^{1/\beta}$, and let n go to infinity:

$$\limsup_{n\to\infty}\mathbb{P}\big(\Gamma_n^{(R^+)} \equiv 0 \text{ on } [0, T]\big) \geq \exp\bigg[-2\frac{c^+ + c^-}{\beta}R^{-\beta}\mathbb{E}\bigg(\sup_{0\leq t\leq T}|Y(t)|\bigg)\bigg].$$

This last estimation implies the asymptotic given in A_3 as R goes to infinity.

$$\square$$

Remark. We have proven the tightness of the sequence $\Gamma_n, n \geq 1$ by using Kolmogorov's criterion. As a by-product of estimation (23), we can deduce that the limit Γ is locally γ-Hölder for all $\gamma < \frac{\alpha-1}{2\alpha}$. However, this is not sharp and Theorem 1.5 gives a better result about the regularity of the trajectories of the process Γ.

3. The random rewards schema. Proof of Theorem 1.2

3.1. Preliminaries: local times, number of self-intersections and range

Let $x \in \mathbb{Z}$ and $n \geq 1$. The local time $N_n(x)$ of the random walk $(S_k)_{k\geq 0}$ at point x up to time n is defined by

$$N_n(x) = \sum_{k=0}^{n} 1_{\{S_k = x\}}.$$

It represents the amount of time the walk spends at state x up to time n. We extend this definition to non-integer time $s \geq 0$ by linear interpolation:

$$N_s(x) = N_{[s]}(x) + (s - [s])\big(N_{[s]+1}(x) - N_{[s]}(x)\big).$$

The random walk in random scenery defined by equations (6) and (7) writes for all $s \geq 0$

$$Z_s = \sum_{x \in Z} N_s(x)\xi_x, \tag{24}$$

where the collection of random variables $\{N_s(x), x \in \mathbb{Z}\}$ and $\{\xi_x, x \in \mathbb{Z}\}$ are independent. The number of self-intersections V_n of the random walk up to time n is defined by

$$V_n = \sum_{0 \leq i,j \leq n} 1_{\{S_i = S_j\}} = \sum_{x \in \mathbb{Z}} N_n(x)^2. \tag{25}$$

The range R_n of the random walk up to time n is defined by

$$R_n = \mathrm{card}\{S_k; 0 \leq k \leq n\} = \sum_{x \in \mathbb{Z}} 1_{\{N_n(x) \neq 0\}}. \tag{26}$$

It represents the number of different states visited by the random walk up to time n.

The definitions (25) and (26) extend obviously to noninteger time $s \geq 0$.

The following Lemma yields some estimates of these quantities.

Lemma 3.1. • *The following convergence in probability holds*

$$n^{-\delta} \sup_{x \in \mathbb{Z}} N_n(x) \xrightarrow[n \to \infty]{\mathrm{P}} 0. \tag{27}$$

• *For any $p \in [1, +\infty)$, there exists some constant C such that for all $n \geq 1$,*

$$\mathbb{E}(V_n^p) \leq Cn^{(2-1/\alpha)p}. \tag{28}$$

• *For any $p \in [1, \alpha)$, there exists some constant C such that for all $n \geq 1$*

$$\mathbb{E}(R_n^p) \leq Cn^{p/\alpha}. \tag{29}$$

Proof. • The proof is given in [5], Lemma 4, page 12.

• Since if $0 < p' < p$, $\mathbb{E}(|X|^{p'}) \leq \mathbb{E}(|X|^p) + 1$, we can suppose that p is an integer ≥ 1. The number of self-intersections up to time n is bounded from above by

$$V_n \leq 2 \sum_{0 \leq i \leq j \leq n} 1_{\{S_i = S_j\}}.$$

Using the Minkowsky inequality,

$$\|V_n\|_p \leq 2 \sum_{i=0}^{n} \left\| \sum_{j=i}^{n} 1_{\{S_i = S_j\}} \right\|_p, \tag{30}$$

where $\|X\|_p = \mathbb{E}(|X|^p)^{1/p}$. For fixed i, the stationarity of the random walk's increments implies that the distribution of $\sum_{j=i}^n 1_{\{S_i = S_j\}}$ and $\sum_{j=0}^{n-i} 1_{\{S_i = 0\}} = N_{n-i}(0)$ are equal. Since $N_{n-i}(0) \leq N_n(0)$, equation (30) yields

$$\mathbb{E}(V_n^p) \leq 2^p (n+1)^p \mathbb{E}(N_n(0)^p). \tag{31}$$

Lemma 1, page 12 of [5] states that there exists some $C > 0$ such that for all $n \geq 1$

$$\mathbb{E}(N_n(0)^p) \leq C n^{p(1-1/\alpha)}. \tag{32}$$

Equations (31) and (32) together yield equation (28).

• We suppose $p > 1$, the case $p = 1$ follows. For $n \geq 0$, define the random variable M_n by

$$M_n = \max_{0 \leq k \leq n} |S_k|. \tag{33}$$

Up to time n, the random walk $(S_k)_{k \geq 0}$ remains in the set $\{x \in \mathbb{Z}; |x| \leq M_n\}$, and hence

$$R_n \leq 2 M_n + 1 \qquad \text{a.s.} \tag{34}$$

Since $(|S_n|)_{n \geq 0}$ is a non-negative submartingale and since $\mathbb{E}(|S_n|^p) < \infty$ if $p \in (1, \alpha)$, we can apply the maximal inequality (Theorem 20, page 11 of [10]). This yields that for any $p \in (1, \alpha)$,

$$\mathbb{E}(M_n^p) \leq \left(\frac{p}{p-1} \right)^p \mathbb{E}(|S_n|^p). \tag{35}$$

We need the following estimation (equation 5s, page 684 in [6]): For any $p \in (1, \alpha)$, there exists some constant $C > 0$ such that for all $n \geq 1$,

$$\mathbb{E}(|S_n|^p) \leq C n^{p/\alpha}. \tag{36}$$

Combining equations (34)–(36), we prove equation (29) for $p > 1$. □

We recall the following lemma from Kesten and Spitzer, which is in the bulk of the proof.

Lemma 3.2. *For all* $(\theta_1, \ldots, \theta_k) \in \mathbb{R}^k$, $(t_1, \ldots, t_k) \in [0, +\infty)^k$, $\sigma > 0$, $\beta \in (0, 2]$ *and* $\nu \in [-1, 1]$ *(with* $\nu = 0$ *when* $\beta = 1$*), the distribution of*

$$X_n = \sigma^\beta n^{-\delta\beta} \sum_{x \in \mathbb{Z}} \left| \sum_{j=1}^k \theta_j N_{nt_j}(x) \right|^\beta \left(1 - i\nu \tan\left(\frac{\pi\beta}{2} \right) \mathrm{sgn}\left(\sum_{j=1}^k \theta_j N_{nt_j}(x) \right) \right)$$

converges weakly as $n \to \infty$ *to* X *defined by equation* (16).

Lemma 3.3. *Furthermore, for all* $p \in [1, \alpha(1 - \frac{\beta}{2})^{-1})$ *(if* $\beta = 2$, *read for all* $p \in [1, \infty)$*), there exists some constant* $C > 0$ *such that for all* $n \geq 1$,

$$\mathbb{E}(|X_n|^p) \leq C.$$

Proof. Let $T = \max(t_1, \ldots, t_n)$ and $\Theta = \sum_{j=1}^{k} |\theta_j|$. The random variables $|X_n|$ are bounded above by

$$\left(1 + \left| \nu \tan\left(\frac{\pi\beta}{2}\right) \right| \right) \Theta^\beta n^{-\delta\beta} \sum_{x \in \mathbb{Z}} N_{[nT]+1}^\beta(x).$$

In the case $\beta = 2$, this quantity is equal to $\Theta^2 n^{\frac{1}{\alpha}-2} V_{[nT]+1}$, and in this case, Lemma 3.3 is a consequence of equation (28).

In the case $\beta < 2$, the Hölder inequality yields

$$\sum_{x \in \mathbb{Z}} N_{[nT]+1}^\beta(x) \leq \left(\sum_{x \in \mathbb{Z}} 1_{\{N_{[nT]+1}(x) \neq 0\}} \right)^{1-\beta/2} \left(\sum_{x \in \mathbb{Z}} N_{[nT]+1}^2(x) \right)^{\beta/2}$$

$$= R_{[nT]+1}^{1-\beta/2} V_{[nT]+1}^{\beta/2}.$$

Hence, up to a multiplicative constant, the expectation $\mathbb{E}(|X_n|^p)$ is overestimated by

$$\mathbb{E}\left[\left(n^{-\delta\beta} \sum_{x \in \mathbb{Z}} N_{[nT]+1}^\beta(x) \right)^p \right] \leq \mathbb{E}\left[\left(n^{-1/\alpha} R_{[nT]+1} \right)^{p(1-\beta/2)} \left(n^{1/\alpha-2} V_{[nT]+1} \right)^{p\beta/2} \right].$$

We now apply the Hölder inequality again, where the random variables are positive and the expectations eventually $+\infty$: for all $\gamma > \frac{p\beta}{2}$,

$$\mathbb{E}\left[\left(n^{-1/\alpha} R_{[nT]+1} \right)^{p(1-\beta/2)} \left(n^{1/\alpha-2} V_{[nT]+1} \right)^{p\beta/2} \right]$$

$$\leq \mathbb{E}\left[\left(n^{-1/\alpha} R_{[nT]+1} \right)^{p(1-\beta/2)2\gamma/(2\gamma-p\beta)} \right]^{(2\gamma-p\beta)/2\gamma} \mathbb{E}\left[\left(n^{1/\alpha-2} V_{[nT]+1} \right)^\gamma \right]^{p\beta/(2\gamma)}.$$

We can choose γ such that this last term is bounded as $n \to \infty$. From Lemma 3.1, this is the case if the condition $p(1 - \frac{\beta}{2}) \frac{2\gamma}{2\gamma-p\beta} < \alpha$ is satisfied. Since $p(1 - \frac{\beta}{2}) < \alpha$, this last condition is satisfied for γ large enough. $\qquad \square$

3.2. Convergence of the finite-dimensional distributions

We define the process G_n by

$$G_n(t) = c_n^{-1/\beta} \sum_{i=1}^{c_n} D_n^{(i)}(t), \qquad t \geq 0, \tag{37}$$

where $D_n^{(i)}$ is the ith random walk in random scenery properly rescaled and defined by (10). Theorem 1.2 states that G_n converges weakly to Γ in $\mathcal{C}([0, \infty))$.

The convergence of the finite-dimensional marginals is proven by studying the asymptotic behaviour of their characteristic function.

Let λ be the characteristic function of the variables $\xi_k^{(i)}$ defined by

$$\lambda(u) = \mathbb{E}\big(\exp\big(iu\xi_1^{(1)}\big)\big).$$

Since the random variables $\xi_k^{(i)}$ are in the domain of attraction of Z_β,

$$\lambda(u) = \bar{\lambda}(u) + o(|u|^\beta), \qquad \text{as } u \to 0, \tag{38}$$

where $\bar{\lambda}$ is the characteristic function of Z_β given by equation (2).

Proposition 3.1. *The finite-dimensional distributions of $(G_n(t))_{t\geq 0}$ converge weakly as $n \to \infty$ to those of $(\Gamma(t))_{t\geq 0}$ defined in equation (9).*

Remark. It is worth noting that the properties of stability, self-similarity and stationarity can be seen in the characteristic function of the marginals.

For sake of clarity, we divide the proof of Proposition 3.1 into three lemmas. In Lemma 3.4 below, we express the characteristic function in terms of the local time of the random walks S_n and of the characteristic function λ of the scenery. In Lemma 3.5, we estimate the error committed when replacing the scenery by a stable scenery. Finally, in Lemma 3.6, we prove the proposition in the case of a stable scenery.

Lemma 3.4. *For all $(\theta_1, \ldots, \theta_k) \in \mathbb{R}^k$, $(t_1, \ldots, t_k) \in [0, +\infty)^k$,*

$$\mathbb{E}\left[\exp\left(i\sum_{j=1}^k \theta_j G_n(t_j)\right)\right] = \left(\mathbb{E}\left[\prod_{x\in\mathbb{Z}} \lambda\left(c_n^{-1/\beta} n^{-\delta} \sum_{j=1}^k \theta_j N_{nt_j}(x)\right)\right]\right)^{c_n}.$$

Proof. From equation (37) and the independence of the processes $(D_n^{(i)}(t))_{t\geq 0}$,

$$\mathbb{E}\left[\exp\left(i\sum_{j=1}^k \theta_j G_n(t_j)\right)\right] = \mathbb{E}\left[\exp\left(ic_n^{-1/\beta} \sum_{j=1}^k \theta_j D_n^{(1)}(t_j)\right)\right]^{c_n}. \tag{39}$$

Hence we have to compute the characteristic function of a single random walk in random scenery. In the sequel, we omit the superscript (1), and write $D_n(t)$ (respectively, S_k, ξ_k, ...) instead of $D_n^{(1)}(t)$ (respectively, $S_k^{(1)}$, $\xi_k^{(1)}$, ...). Using equations (10) and (24), the characteristic function writes

$$\mathbb{E}\left[\exp\left(ic_n^{-1/\beta} \sum_{j=1}^k \theta_j D_n(t_j)\right)\right]$$

$$= \mathbb{E}\left[\prod_{x\in\mathbb{Z}} \exp\left(ic_n^{-1/\beta} n^{-\delta} \sum_{j=1}^k \theta_j N_{nt_j}(x)\xi_x\right)\right].$$

Using the independence of the random walk $(S_n)_{n\geq0}$ and the random scenery $(\xi_x)_{x\in\mathbb{Z}}$, and performing integration on the scenery yields

$$
\mathbb{E}\left[\exp\left(ic_n^{-1/\beta}\sum_{j=1}^{k}\theta_j D_n(t_j)\right)\right]
$$

$$
=\mathbb{E}\left[\prod_{x\in\mathbb{Z}}\lambda\left(c_n^{-1/\beta}n^{-\delta}\sum_{j=1}^{k}\theta_j N_{nt_j}(x)\right)\right]. \tag{40}
$$

Equations (39) and (40) together yield Lemma 3.4 $\square$

Lemma 3.5. *For all* $(\theta_1,\ldots,\theta_k)\in\mathbb{R}^k$, $(t_1,\ldots,t_k)\in[0,+\infty)^k$,

$$
\mathbb{E}\left[\prod_{x\in\mathbb{Z}}\lambda\left(c_n^{-1/\beta}n^{-\delta}\sum_{j=1}^{k}\theta_j N_{nt_j}(x)\right)\right]=\mathbb{E}\left[\prod_{x\in\mathbb{Z}}\bar{\lambda}\left(c_n^{-1/\beta}n^{-\delta}\sum_{j=1}^{k}\theta_j N_{nt_j}(x)\right)\right]+o(c_n^{-1}).
$$

Proof. Denote by $U_n(x)$ the random variables defined by

$$
U_n(x)=n^{-\delta}\sum_{j=1}^{k}\theta_j N_{nt_j}(x),\qquad x\in\mathbb{Z}.
$$

With these notations, Lemma 3.5 is equivalent to

$$
\lim_{n\to\infty}\mathbb{E}\left[c_n\left(\prod_{x\in\mathbb{Z}}\lambda(c_n^{-1/\beta}U_n(x))-\prod_{x\in\mathbb{Z}}\bar{\lambda}(c_n^{-1/\beta}U_n(x))\right)\right]=0.
$$

Note that the products, although indexed by $x\in\mathbb{Z}$, have only a finite number of factors different from 1. And furthermore, all factors are complex numbers in $\bar{\mathbb{D}}=\{z\in\mathcal{C}\mid|z|\leq1\}$. We use the following inequality: Let $(z_i)_{i\in I}$ and $(z_i')_{i\in I}$ be two families of complex numbers in $\bar{\mathbb{D}}$ such that all terms are equal to one, except a finite number of them. Then

$$
\left|\prod_{i\in I}z_i'-\prod_{i\in I}z_i\right|\leq\sum_{i\in I}|z_i'-z_i|.
$$

This yields

$$
\left|\prod_{x\in\mathbb{Z}}\lambda(c_n^{-1/\beta}U_n(x))-\prod_{x\in\mathbb{Z}}\bar{\lambda}(c_n^{-1/\beta}U_n(x))\right|
$$

$$
\leq\sum_{x\in\mathbb{Z}}\left|\lambda(c_n^{-1/\beta}U_n(x))-\bar{\lambda}(c_n^{-1/\beta}U_n(x))\right|. \tag{41}
$$

We define a continuous and bounded function g by $g(0) = 0$ and

$$g(v) = |v|^{-\beta} \left| \lambda(v) - \bar{\lambda}(v) \right|, \qquad v \neq 0,$$

so that for any $x \in \mathbb{Z}$,

$$|\lambda(c_n^{-1/\beta} U_n(x)) - \bar{\lambda}(c_n^{-1/\beta} U_n(x))| = c_n^{-1} |U_n(x)|^{\beta} g(c_n^{-1/\beta} U_n(x)).$$

In order to obtain an uniform estimation, define the random variable U_n by

$$U_n = \max_{x \in \mathbb{Z}} |U_n(x)|,$$

and the function $\tilde{g} : [0, +\infty) \to [0, +\infty)$ by

$$\tilde{g}(u) = \sup_{|v| \leq u} |g(v)|.$$

Note that $\tilde{g}$ is continuous, bounded and vanishes at 0. Then, for any $x \in \mathbb{Z}$,

$$|\lambda(c_n^{-1/\beta} U_n(x)) - \bar{\lambda}(c_n^{-1/\beta} U_n(x))| \leq c_n^{-1} |U_n(x)|^{\beta} \tilde{g}(c_n^{-1/\beta} U_n). \tag{42}$$

Equations (41) and (42) together yield

$$c_n \left| \prod_{x \in \mathbb{Z}} \lambda(c_n^{-1/\beta} U_n(x)) - \prod_{x \in \mathbb{Z}} \bar{\lambda}(c_n^{-1/\beta} U_n(x)) \right| \leq \tilde{g}(c_n^{-1/\beta} U_n) \sum_{x \in \mathbb{Z}} |U_n(x)|^{\beta}.$$

From Lemma 3.1, U_n converges in probability to 0 as $n \to \infty$. Since $\tilde{g}$ is continuous and vanishes at 0, $\tilde{g}(c_n^{-1/\beta} U_n)$ converges also in probability to 0. From Lemma 3.2 (with $\nu = 0$), $\sum_{x \in \mathbb{Z}} |U_n(x)|^{\beta}$ converges in distribution. Hence, the product

$$\tilde{g}(c_n^{-1/\beta} U_n) \sum_{x \in \mathbb{Z}} |U_n(x)|^{\beta}$$

converges to 0 in probability. We finally prove that the expectation converges also to 0. We use uniform integrability and prove that the sequence of random variables $\tilde{g}(c_n^{-1/\beta} U_n) \times \sum_{x \in \mathbb{Z}} |U_n(x)|^{\beta}$ is bounded in L^p for some $p > 1$. The function $\tilde{g}$ is bounded on $[0, +\infty)$, and Lemma 3.3 states that $\sum_{x \in \mathbb{Z}} |U_n(x)|^{\beta}$ is bounded in L^p for some $p > 1$. □

Lemma 3.6. *For all $(\theta_1, \ldots, \theta_k) \in \mathbb{R}^k$, $(t_1, \ldots, t_k) \in [0, +\infty)^k$,*

$$\mathbb{E}\left[\prod_{x \in \mathbb{Z}} \bar{\lambda}\left(c_n^{-1/\beta} n^{-\delta} \sum_{j=1}^{k} \theta_j N_{nt_j}(x) \right) \right] = 1 - c_n^{-1} \mathbb{E}(X) + o(c_n^{-1}),$$

where X is defined in Lemma 3.2.

Proof. Recall the definition of the random variable X_n from Lemma 3.2 and of the characteristic function $\bar{\lambda}$ from equation (2). With this notation, Lemma 3.6 is equivalent to

$$\lim_{n \to +\infty} \mathbb{E}(f_n(X_n)) = \mathbb{E}(X),$$

where f_n is the function defined on $\mathbb{C}$ by

$$f_n(x) = c_n\big(1 - \exp(-c_n^{-1}x)\big).$$

It is easy to verify that the sequence of functions f_n satisfies the following property: For every x and for every sequence $(x_n)_{n \geq 1}$ converging to x,

$$\lim_{n \to \infty} f_n(x_n) = x.$$

Furthermore Lemma 3.2 states that the sequence $(X_n)_{n \geq 1}$ converges in distribution to X when $n \to \infty$. Using the continuous mapping theorem (Therorem 5.5 of [1]), we prove the weak convergence of the sequence of random variables $f_n(X_n)$ to X.

In order to prove the convergence of the expectation, we use uniform integrability and prove that there exists some $p > 1$ such that $\mathbb{E}(|f_n(X_n)|^p)$ is bounded. The integral representation

$$f_n(x) = x \int_0^1 \exp(-uc_n^{-1}x) \, du$$

shows that for $\mathrm{Re}(x) \geq 0$,

$$|f_n(x)| \leq |x|.$$

Hence Lemma 3.3 implies that $|f_n(X_n)| \leq |X_n|$ is bounded in L^p for some $p > 1$ and this proves the lemma. $\square$

Proof of Proposition 3.1. Using Lemmas 3.4–3.6, we have

$$\mathbb{E}\left[\exp\left(i \sum_{j=1}^k \theta_j G_n(t_j)\right)\right] = \left(\mathbb{E}\left[\prod_{x \in \mathbb{Z}} \lambda\left(c_n^{-1/\beta} n^{-\delta} \sum_{j=1}^k \theta_j N_{nt_j}(x)\right)\right]\right)^{c_n}$$

$$= \left(\mathbb{E}\left[\prod_{x \in \mathbb{Z}} \bar{\lambda}\left(c_n^{-1/\beta} n^{-\delta} \sum_{j=1}^k \theta_j N_{nt_j}(x)\right)\right] + o(c_n^{-1})\right)^{c_n}$$

$$= \big(1 - c_n^{-1}\mathbb{E}(X) + o(c_n^{-1})\big)^{c_n}.$$

This last quantity has limit $\exp(-\mathbb{E}(X))$ as $n \to \infty$. This proves Proposition 3.1. $\square$

3.3. Tightness

The weak convergence stated in Therorem 1.1 is a consequence of Proposition 3.1 and the following proposition:

Proposition 3.2. *The family of processes* $(\Gamma_n(t))_{t\geq 0}$ *is tight in* $\mathcal{C}([0,\infty))$.

The tightness will be proven by approximating G_n by processes with finite variance obtained with truncated random sceneries. Let a be a positive real. We decompose the scenery $(\xi_x^{(i)})_{x\in\mathbb{Z}, i\geq 1}$ into two parts

$$\xi_x^{(i)} = \bar{\xi}_{a,x}^{(i)} + \hat{\xi}_{a,x}^{(i)},$$

where $(\bar{\xi}_{a,x}^{(i)})$ denotes the ith truncated scenery defined by

$$\bar{\xi}_{a,x}^{(i)} = \xi_x^{(i)} 1_{\{|\xi_x^{(i)}|\leq a\}},$$

and $\hat{\xi}_{a,x}^{(i)}$ the remainder scenery

$$\hat{\xi}_{a,x}^{(i)} = \xi_x^{(i)} 1_{\{|\xi_x^{(i)}|>a\}}.$$

In the sequel, we will use intensively the following estimations that are adapted from Feller [4]:

Lemma 3.7. • *There exists C such that for any $a > 0$,*

$$\left|\mathbb{E}(\bar{\xi}_{a,x}^{(i)})\right| \leq Ca^{1-\beta},$$

$$\mathbb{E}(|\bar{\xi}_{a,x}^{(i)}|^2) \leq Ca^{2-\beta}.$$

• *There exists C such that*

$$\mathbb{P}(\hat{\xi}_{a,x}^{(i)} \neq 0) \leq Ca^{-\beta}.$$

Proof. These estimates rely on the behaviour of the tail of the random variable $\xi_x^{(i)}$: There exist some constant A_1 and A_2 such that

$$\lim_{u\to+\infty} u^\beta \mathbb{P}(\xi_x^{(i)} > u) = A_1,$$

$$\lim_{u\to+\infty} u^\beta \mathbb{P}(\xi_x^{(i)} < -u) = A_2.$$

We prove the lemma in the case $0 < \beta < 2$. In this case, we have $A_1 \neq 0$ or $A_2 \neq 0$. In the case $\beta = 2$, the constants A_1 and A_2 are zero and the demonstration is easily modified.

• The estimation of $\mathbb{E}(\bar{\xi}_{a,x}^{(i)})$ comes from the formula

$$\mathbb{E}(\bar{\xi}_{a,x}^{(i)}) = \int_0^a \left(\mathbb{P}(\xi_x^{(i)} 1_{\{|\xi_x^{(i)}|\leq a\}} > u) - \mathbb{P}(\xi_x^{(i)} 1_{\{|\xi_x^{(i)}|\leq a\}} < -u)\right) du.$$

In the case $\beta < 1$, $|\mathbb{E}(\bar{\xi}_{a,x}^{(i)})|$ is bounded from above by

$$\int_0^a \mathbb{P}(|\xi_x^{(i)}| > u) du$$

and this integral is equivalent as $a \to +\infty$ to $Ca^{1-\beta}$.

In the case $\beta > 1$, $\xi_x^{(i)}$ has expectation 0. This implies that $|\mathbb{E}(\bar{\xi}_{a,x}^{(i)})|$ is bounded from above by

$$\int_a^\infty \mathbb{P}(|\xi_x^{(i)}| > u) \, du$$

and this integral is equivalent as $a \to +\infty$ to $Ca^{1-\beta}$.

In the case $\beta = 1$, the symmetry condition (3) means exactly that $|\mathbb{E}(\bar{\xi}_{a,x}^{(i)})|$ is bounded from above by some constant.

The second moment of the truncated scenery is overestimated by

$$\mathbb{E}(|\bar{\xi}_{a,x}^{(i)}|^2) \le \mathbb{E}(|\xi_{a,x}^{(i)}|^2 \mathbf{1}_{\{|\xi_x^{(i)}| \le a\}})$$

$$= \int_0^{a^2} \mathbb{P}(|\xi_x^{(i)}| > u^{1/2}) \, du$$

and this integral is equivalent as $a \to +\infty$ to $Ca^{2-\beta}$.

- Finally, the probability

$$\mathbb{P}(\hat{\xi}_{a,x}^{(i)} \ne 0) = \mathbb{P}(|\xi_x^{(i)}| > a \ne 0)$$

is equivalent to $(A_1 + A_2)a^{-\beta}$ as $n \to +\infty$.   □

Proof of Proposition 3.2. For $a > 0$, we use truncations with $a_n = an^{1/\alpha\beta} c_n^{1/\beta}$ and write

$$G_n(t) = \bar{\Gamma}_{n,a}(t) + \hat{\Gamma}_{n,a}(t),$$

where

$$\bar{\Gamma}_{n,a}(t) = n^{-\delta} c_n^{-1/\beta} \sum_{i=1}^{c_n} \sum_{x \in \mathbb{Z}} N_{nt}^{(i)}(x) \bar{\xi}_{a_n,x}^{(i)},$$

$$\hat{\Gamma}_{n,a}(t) = n^{-\delta} c_n^{-1/\beta} \sum_{i=1}^{c_n} \sum_{x \in \mathbb{Z}} N_{nt}^{(i)}(x) \hat{\xi}_{a_n,x}^{(i)}.$$

In order to prove Proposition 3.2, we prove that for any $T > 0$,

$$\lim_{a \to \infty} \limsup_{n \to \infty} \mathbb{P}\left(\sup_{t \in [0,T]} |\hat{\Gamma}_{n,a}(t)| > 0 \right) = 0 \tag{43}$$

and that for fixed $a > 0$, the family of processes $(\bar{\Gamma}_{n,a}(t))_{t \ge 0}$ is tight.

Notice that the process $\hat{\Gamma}_{n,a}$ vanishes on $[0, T]$ as soon as the different remainder sceneries vanish on the range of the random walks up to time $[nT] + 1$. Since the range of the ith random walk is included in $[-M_{[nT]+1}^{(i)}, M_{[nT]+1}^{(i)}]$ and since the different random walks in random

sceneries are independent,

$$\mathbb{P}\left(\sup_{t\in[0,T]}|\hat{\Gamma}_{n,a}(t)|=0\right) \geq \left[\mathbb{P}\left(\text{for all } x \in \left[-M_{[nT]+1}^{(1)}, M_{[nT]+1}^{(1)}\right], \hat{\xi}_{a_n,x}^{(1)}=0\right)\right]^{c_n}$$

$$\geq \left(\mathbb{E}\left[\left(\mathbb{P}\left(\hat{\xi}_{a_n,0}^{(1)}=0\right)\right)^{2M_{[nT]+1}+1}\right]\right)^{c_n}.$$

Using Lemma 3.7, there exists a constant C such that

$$\mathbb{E}\left[\left(\mathbb{P}\left(\hat{\xi}_{a_n,0}^{(1)}=0\right)\right)^{2M_{[nT]+1}+1}\right] \geq \mathbb{E}\left[\left(1-Ca_n^{-\beta}\right)^{2M_{[nT]+1}+1}\right]$$

$$\geq \mathbb{E}\left[1+\left(2M_{[nT]+1}+1\right)\log(1-Ca_n^{-\beta})\right].$$

We majorate from above $\mathbb{E}(M_{[nT]+1})$ using equations (35) and (2), which yields

$$\mathbb{E}\left[\left(\mathbb{P}\left(\hat{\xi}_{a_n,0}^{(1)}=0\right)\right)^{2M_{[nT]+1}+1}\right] \geq 1+\log(1-Ca_n^{-\beta})C(nT)^{1/\alpha}.$$

Combining these estimates,

$$\mathbb{P}\left(\sup_{t\in[0,T]}|\hat{\Gamma}_{n,a}(t)|=0\right) \geq \left(1+\log(1-Ca_n^{-\beta})C(nT)^{1/\alpha}\right)^{c_n}.$$

Let n go to infinity,

$$\limsup_{n\to\infty}\mathbb{P}\left(\sup_{t\in[0,T]}|\hat{\Gamma}_{n,a}(t)|>0\right) \leq 1-\exp(-Ca^{-\beta}T^{1/\alpha}).$$

Letting a go to infinity, we obtain equation (43).

For fixed a, we now prove the tightness of the family of processes $(\bar{\Gamma}_{n,a}(t))_{t\geq 0}$. Using the independence of the different random walks and random sceneries, we compute

$$\mathbb{E}[|\bar{\Gamma}_{n,a}(t_2)-\bar{\Gamma}_{n,a}(t_1)|^2]$$

$$\leq n^{-2\delta}c_n^{-2/\beta}\mathbb{E}\left[\left|\sum_{1\leq i,j\leq c_n}\sum_{x,y\in\mathbb{Z}}\left(N_{nt_2}^{(i)}(x)-N_{nt_1}^{(i)}(x)\right)\left(N_{nt_2}^{(j)}(y)-N_{nt_1}^{(j)}(y)\right)|\bar{\xi}_{a_n,x}^{(i)}||\bar{\xi}_{a_n,y}^{(j)}|\right|\right]$$

$$\leq n^{-2\delta}c_n^{-2/\beta}\sum_{1\leq i,j\leq c_n}\sum_{x,y\in\mathbb{Z}}\mathbb{E}\left[\left(N_{nt_2}^{(i)}(x)-N_{nt_1}^{(i)}(x)\right)\left(N_{nt_2}^{(j)}(y)-N_{nt_1}^{(j)}(y)\right)\right]\mathbb{E}\left[|\bar{\xi}_{a_n,x}^{(i)}||\bar{\xi}_{a_n,y}^{(j)}|\right]$$

$$\leq n^{-2\delta}c_n^{-2/\beta}c_n(c_n-1)\left[\sum_{x\in\mathbb{Z}}\mathbb{E}\left(N_{nt_2}^{(1)}(x)-N_{nt_1}^{(1)}(x)\right)\right]^2\left[\mathbb{E}|\bar{\xi}_{a_n,0}^{(1)}|\right]^2$$

$$+n^{-2\delta}c_n^{-2/\beta}c_n\sum_{x\neq y\in\mathbb{Z}}\mathbb{E}\left[\left(N_{nt_2}^{(1)}(x)-N_{nt_1}^{(1)}(x)\right)\left(N_{nt_2}^{(1)}(y)-N_{nt_1}^{(1)}(y)\right)\right]\left[\mathbb{E}|\bar{\xi}_{a_n,0}^{(1)}|\right]^2$$

$$+n^{-2\delta}c_n^{-2/\beta}c_n\sum_{x\in\mathbb{Z}}\mathbb{E}\left[\left(N_{nt_2}^{(1)}(x)-N_{nt_1}^{(1)}(x)\right)^2\right]\mathbb{E}\left[|\bar{\xi}_{a_n,0}^{(1)}|^2\right].$$

Using Lemma 3.7, there exists some C such that

$$\mathbb{E}\big[\big|\bar{\xi}^{(1)}_{a_n,x}\big|\big] \leq Ca_n^{1-\beta} \leq Cn^{1/(\alpha\beta)-1/\alpha}c_n^{1/\beta-1}$$

and

$$\mathbb{E}\big[\big|\bar{\xi}^{(1)}_{a_n,x}\big|^2\big] \leq Ca_n^{2-\beta} \leq Cn^{2/(\alpha\beta)-1/\alpha}c_n^{2/\beta-1}.$$

Furthermore,

$$\mathbb{E}\left[\sum_{x\in\mathbb{Z}}\big(N^{(1)}_{nt_2}(x) - N^{(1)}_{nt_1}(x)\big)\right] = n(t_2 - t_1)$$

and

$$\mathbb{E}\left[\sum_{x\neq y\in\mathbb{Z}}\big(N^{(1)}_{nt_2}(x) - N^{(1)}_{nt_1}(x)\big)\big(N^{(1)}_{nt_2}(y) - N^{(1)}_{nt_1}(y)\big)\right]$$

$$= n^2(t_2 - t_1)^2 - \mathbb{E}\left[\sum_{x\in\mathbb{Z}}\big(N^{(1)}_{nt_2}(x) - N^{(1)}_{nt_1}(x)\big)^2\right].$$

Using the strong Markov property and Lemma 3.1, the number of self-intersection between time nt_1 and nt_2 is overestimated by

$$\mathbb{E}\left[\sum_{x\in\mathbb{Z}}\big(N^{(i)}_{nt_2}(x) - N^{(i)}_{nt_1}(x)\big)^2\right] \leq \mathbb{E}\left[\sum_{x\in\mathbb{Z}}\big(N^{(i)}_{[nt_2]+1-[nt_1]}(x)\big)^2\right]$$

$$\leq C([nt_2] - [nt_1] + 1)^{2-1/\alpha}.$$

Furthermore, in the case $|t_2 - t_1| \leq 1/n$, we can see that

$$\mathbb{E}\left[\sum_{x\in\mathbb{Z}}\big(N^{(1)}_{nt_2}(x) - N^{(1)}_{nt_1}(x)\big)^2\right] \leq 2(nt_2 - nt_1)^2,$$

since in the sum, at most two terms are not zero and those terms are bounded by $(nt_2 - nt_1)^2$.

Combining all these estimates, we get

$$\mathbb{E}[|\bar{\Gamma}_{n,a}(t_2) - \bar{\Gamma}_{n,a}(t_1)|^2] \leq C|t_2 - t_1|^2 + C|t_2 - t_1|^{2-1/\alpha}.$$

Using Theorem 12.3 in Billingsley [1], this proves the tightness of the family of processes $(\bar{\Gamma}_{n,a}(t))_{t\geq 0}$.   □

4. Properties of the process Γ. Proof of Theorems 1.3–1.5

In this section, we study the properties of the sample path of the limit process Γ. The key of this study is a representation of the process Γ as a stable integral (Theorem 1.3). For the definition and

properties of stable random measures and integrals with respect with these measures, we refer to the monograph of Samorodnitsky and Taqqu [11]. In [2], the authors study closely related processes, and we follow their methods. This allows us to prove various properties of the sample path of the limit process Γ such as the probability tail of the suprema (Theorem 1.4) and the Hölder regularity (Theorem 1.5).

4.1. Integral representation of Γ

Proof of Theorem 1.3. As we mentioned in a remark, equation (12) is a consequence of the proof of Proposition 2.1. We now prove that Γ has an integral representation given by equation (13).

Let $G(t)$ be the process defined by the stable integral

$$G(t) = \int_{\Omega' \times \mathbb{R}} \sigma L_t(x)(\omega') M(\mathrm{d}x, \mathrm{d}\omega'),$$

where M is a β-stable measure with control measure $\mathbb{P}' \times \lambda$ and skewness intensity ν. To check that G is well defined we need to prove that for all $t \geq 0$,

$$\int_{\Omega' \times \mathbb{R}} \sigma^\beta L_t^\beta(x)(\omega') \, \mathrm{d}x \mathbb{P}'(\mathrm{d}\omega') = \sigma^\beta \mathbb{E}'\left(\int_{\mathbb{R}} L_t^\beta(x) \, \mathrm{d}x\right) < \infty.$$

This was proven in Lemma 2.1, and hence the stable integral process G is well defined.

By definition of stable integration (see Section 3 in [11]), the finite-dimensional distributions of the process G are given by the following: For any $\theta_1, \ldots, \theta_k \in \mathbb{R}$ and $t_1, \ldots, t_k \geq 0$

$$\mathbb{E}\exp\left(\mathrm{i}\sum_{j=1}^k \theta_j G(t_j)\right)$$

$$= \exp\left(-\sigma^\beta \mathbb{E}'\int_{\mathbb{R}}\left|\sum_{j=1}^k \theta_j L_{t_j}(x)\right|^\beta \left(1 - \mathrm{i}\nu\,\mathrm{sgn}\left(\sum_{j=1}^k \theta_j L_{t_j}(x)\right)\right) \mathrm{d}x\right).$$

We recognize the finite-dimensional distribution of the process Γ (see equation (12)). Hence, G and Γ have the same finite-dimensional distributions and G is an integral representation of the process Γ. □

4.2. Probability tail of the maximum

Proof of Theorem 1.4. The integral representation of Γ stated in Theorem 1.3 is a key tool in the study of the path properties. A general theory is indeed available for the path properties of stable integral process. The process Γ has an almost surely continuous path and hence is almost

surely bounded on $[0, T]$, for every $T > 0$. We can hence apply Theorem 10.5.1 in [11] giving the probability tail of suprema of bounded β-stable process with index $0 < \beta < 2$. This yields

$$\lim_{u \to \infty} u^\beta \mathbb{P}\left(\sup_{t \in [0,T]} \Gamma(t) \geq u \right)$$

$$= C_\beta \frac{1}{2} \left[\mathbb{E}\left(\int_\mathbb{R} L_T^+(x)^\beta (1 + v)\, \mathrm{d}x \right) + \mathbb{E}\left(\int_\mathbb{R} L_T^-(x)^\beta (1 - v)\, \mathrm{d}x \right) \right],$$

where L_T^+ and L_T^- are defined by

$$L_T^+(x) = \sup_{0 \leq t \leq T} \max(L_t(x), 0) \quad \text{and} \quad L_T^-(x) = \sup_{0 \leq t \leq T} \max(-L_t(x), 0).$$

We obtain the desired result since $L_T^+ \equiv L_T$ and $L_T^- \equiv 0$. The estimation of probability tail of $\sup_{t \in [0,T]} |\Gamma(t)|$ is proved in the same way. $\qquad\qquad\square$

4.3. Hölder continuity of sample paths

Proof of Theorem 1.5. First consider the case $0 < \beta < 1$ and the process indexed by (s, t) in $T = \{(s, t), 0 \leq s < t \leq 1/2\}$ defined by

$$\frac{G(t) - G(s)}{(t-s)^{1-1/\alpha} |\log(t-s)|^{1/\alpha}} = \sigma^\beta \int_{\Omega'} \int_\mathbb{R} \frac{L_t(x)(\omega') - L_s(x)(\omega')}{(t-s)^{1-1/\alpha} |\log(t-s)|^{/1/\alpha}} M(\mathrm{d}x, \mathrm{d}\omega').$$

We have to prove that this process is sample bounded on T. Applying Theorem 10.4.2 in [11], it is sufficient to check that

$$\sigma^\beta \int_{\Omega'} \int_\mathbb{R} \left(\sup_{(s,t) \in T} \frac{|L_t^\beta(x)(\omega') - L_s^\beta(x)(\omega')|}{(t-s)^{1-1/\alpha} |\log(t-s)|^{1/\alpha}} \right)^\beta \mathrm{d}x\, \mathbb{P}'(\mathrm{d}\omega') < \infty. \tag{44}$$

Denote by

$$K(x) = \sup_{(s,t) \in T} \frac{|L_t^\beta(x)(\omega') - L_s^\beta(x)(\omega')|}{(t-s)^{1-1/\alpha} |\log(t-s)|^{1/\alpha}}$$

and

$$K = \sup_{x \in \mathbb{R}} K(x).$$

It follows from the Hölder properties of local times of stable Lévy motions that K is almost surely finite and has moments of all orders (see [3]). Using the Hölder inequality with $p > \frac{\alpha}{\alpha - 1}$,

the left-hand side of equation (44) is bounded from above

$$\mathbb{E}' \int_{\mathbb{R}} K(x)^{\beta}\, \mathrm{d}x \leq \mathbb{E}' \int_{\mathbb{R}} K^{\beta} 1_{\{\sup_{0\leq t\leq 1/2} |Y(t)|\geq |x|\}}\, \mathrm{d}x$$

$$\leq \mathbb{E}'(K^{\beta p})^{1/p} \int_{\mathbb{R}} \mathbb{P}\left(\sup_{0\leq t\leq 1/2} |Y(t)| \geq |x|\right)^{1-1/p}\, \mathrm{d}x$$

and this quantity is finite as soon as $(1 - 1/p)\alpha > 1$ since $\mathbb{E}'(K^{\beta p}) < \infty$ and $\mathbb{P}(\sup_{0\leq t\leq 1/2} |Y(t)| \geq |x|) \sim C|x|^{-\alpha}$ as $|x| \to \infty$.

It is worth noting that this method yields an estimation on the tail probability:

$$\mathbb{P}\left(\sup_{0\leq s<t\leq 1/2} \frac{|\Gamma(t) - \Gamma(s)|}{(t-s)^{1-1/\alpha}|\log(t-s)|^{1/\alpha}} \geq \lambda\right) \sim C\lambda^{-\beta}$$

as $\lambda \to \infty$, with C equal to the left-hand side term of equation (44). In the case $1 \leq \beta < 2$, equation (44) still holds, but this is not a sufficient condition for sample boundedness. Following Cohen and Samorodnitsky [2], we can use a series representation available in the symmetric case $\nu = 0$, which shows that the process is conditionally Gaussian and then use results on the moduli of continuity of the Gaussian process. The proof is the same, with the local time of fractional Brownian motion replaced by the local time of α-stable Lévy motion. □

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