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Ergodicity and invertibility of threshold moving-average models

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We investigate the first-order threshold moving-average model. We obtain a sufficient condition for a unique strictly stationary and ergodic solution of the model without the need to check irreducibility. We also establish necessary and sufficient conditions for its invertibility of first-order . Furthermore, we discuss the extension of the results to the first-order multiple threshold moving-average model and the higher-order threshold moving-average model.

Keywords: ergodicity; invertibility; strict stationarity; threshold moving-average model

1. Introduction

The threshold autoregressive (TAR) model introduced by Tong (1978) has been extensively investigated in the literature. Some basic results on the probabilistic structure of this class of models were given by Chan *et al.* (1985), Chan and Tong (1985) and Tong (1990). Further related results can be found in Chen and Tsay (1991), Brockwell *et al.* (1992), Liu and Susko (1992), An and Huang (1996), An and Chen (1997), Liu *et al.* (1997), Ling (1999), Cline and Pu (1999, 2004), Lanne and Saikkonen (2005), Liebscher (2005) and others.

This paper considers the first-order threshold moving-average (TMA(1)) model, which generates the time series $\{y_t : t = 0, \pm 1, \ldots\}$ by the following equation:

$$y_t = [\phi + \psi I(y_{t-1} \leqslant r)]\varepsilon_{t-1} + \varepsilon_t, \tag{1}$$

where ε_t is a sequence of independent and identically distributed (i.i.d.) random variables, with mean zero and a density function $f(x)$. Ling and Tong (2005) proposed a likelihood ratio test for a linear moving-average model against TMA models. However, the basic structure of the TMA model remains unclear.

Brockwell *et al.* (1992) studied a threshold autoregressive moving-average (ARMA) model and obtained a strictly stationary and ergodic solution for it. However, their result only covers the case when the moving-average part does not have any threshold component, and hence their threshold ARMA model does not include the TMA model as a special case. Liu and Susko (1992) proved that for the TMA model there always exists a strictly stationary solution without any restriction on the coefficients of the model. A similar result is obtained by Ling (1999) for the threshold ARMA model but under much stronger conditions. Thus, conditions under which the solution of the TMA model is unique and ergodic remain an open problem. Ling and Tong (2005) obtained a sufficient condition for the invertibility of a general TMA model. However, their condition is far from being necessary when compared with that for a linear moving-average model.

In this paper, we provide a sufficient condition for the ergodicity of the TMA(1) model in Section 2. In Section 3 we give the necessary and sufficient conditions for the invertibility of the TMA(1) model. Extension to the general threshold moving-average models is discussed in Section 4.

2. Ergodicity of TMA(1) models

A standard method in studying ergodicity of a nonlinear time series model is to put the model into the form of a Markov chain and then check the criterion given in Tweedie (1983); see also Tjøstheim (1990) and Tweedie (2001). This method comprises two parts. The first part is to check the ψ-irreducibility of the Markov chain. The second part is to check the so-called drift criterion. Once we have the irreducibility, the second part is more or less algebraic.

The challenge is irreducibility. Under certain conditions, Feigin and Tweedie (1985) showed that the Markov chain from the random coefficient autoregressive model is irreducible. Chan and Tong (1985) showed the irreduciblity of a class of Markov chains in terms of a suitable measure. When we put the TMA(1) model into a Markov chain (see Ling 1999), it is not hard to see that it neither fits into the Chan and Tong (1985) framework nor satisfies the condition given in Feigin and Tweedie (1985). Thus, it seems decidedly difficult to establish irreducibility. However, as we shall see, it is not necessary for us to check the irreducibility defined in Feigin and Tweedie (1985).

Let the time series $\{(y_t, \varepsilon_t)\}$ be defined on the probability space $(\Omega, \mathcal{F}, P)$. We now develop an alternative way to study ergodicity. First, define a random sequence

$$S_n(t) = \begin{cases} \varepsilon_t, & \text{if } n = 0, \\ \{\phi + \psi I(S_{n-1}(t-1) \leqslant r)\}\varepsilon_{t-1} + \varepsilon_t, & \text{if } n \geqslant 1. \end{cases}$$

Lemma 1. *If $|\psi|\sup_x|xf(x)| < 1$ and $\mathrm{E}|\varepsilon_t| < \infty$, then the limit of $S_n(t)$ exists almost surely (a.s.) as $n \to \infty$ for each t.*

Proof. Let $u_n(t) = [\phi + \psi I(S_n(t) \leqslant r)]\varepsilon_t$ and $b = |\psi|\sup_x|xf(x)|$. We have

$$\begin{aligned} A_n(t) &\equiv |\psi|\mathrm{E}[|I(S_n(t) \leqslant r) - I(S_{n-1}(t) \leqslant r)|\,|\varepsilon_t|] \\ &= |\psi|\mathrm{E}[|I(\varepsilon_t \leqslant r - u_{n-1}(t-1)) - I(\varepsilon_t \leqslant r - u_{n-2}(t-1))|\,|\varepsilon_t|] \\ &= |\psi|\mathrm{E}|\int_{r-u_{n-2}(t-1)}^{r-u_{n-1}(t-1)} |x|f(x)\,\mathrm{d}x| \\ &\leqslant |\psi|\sup_x|xf(x)|\mathrm{E}|u_{n-1}(t-1) - u_{n-2}(t-1)| = bA_{n-1}(t-1). \end{aligned} \tag{2}$$

Since $A_1(t-n+1) \leqslant |\psi|\mathrm{E}|\varepsilon_t| \equiv c$, it follows that

$$A_n(t) \leqslant bA_{n-1}(t-1) \leqslant \ldots \leqslant b^{n-1}A_1(t-n+1) \leqslant cb^{n-1}. \tag{3}$$

Thus, we have $\mathrm{E}|S_n(t+1) - S_{n-1}(t+1)| = A_n(t) \leqslant cb^{n-1}$. Using this and Cauchy's criterion, we can show that $\lim_{n\to\infty} S_n(t)$ exists a.s. for each t. □

We can state our first main result as follows.

Theorem 1. *Under the assumption of Lemma 1,* $\{X_t\}$ *defined by* $X_t = \lim_{n\to\infty} S_n(t)$ *is the unique strictly stationary and ergodic solution of model* (1).

Proof. Since $\{\varepsilon_t\}$ is strictly stationary and ergodic, without loss of generality we assume that T is a measure-preserving and ergodic mapping from Ω to Ω such that $\varepsilon_t(\omega) = \varepsilon_0(T^t\omega)$. Thus $X_t(\omega) = \lim_{n\to\infty} S_n(t, \omega) = \lim_{n\to\infty} S_n(0, T^t\omega) = X_0(T^t\omega)$, and hence $\{X_t\}$ is strictly stationary and ergodic. Note that $I(x_n \leqslant r) \to I(x \leqslant r)$ if $x \neq r$ for any sequence $\{x_n\}$ such that $x_n \to x$ as $n \to \infty$. Thus, $I(S_n(t) \leqslant r) \not\to I(X_t \leqslant r)$ as $n \to \infty$ only if $X_t = r$. Since ε_t has a density function, $P(X_t = r) = 0$. We can claim that $\{X_t\}$ is a solution of model (1). To prove uniqueness, we assume that there exists another solution $\{X_t\}$ of model (1). Similar to (2), we have

$$\begin{aligned}
\mathrm{E}|X_t - X'_t| &= |\psi|\mathrm{E}[|I(X_{t-1} \leqslant r) - I(X'_{t-1} \leqslant r)|\,|\varepsilon_{t-1}|] \\
&\leqslant b\mathrm{E}|[\phi + \psi I(X_{t-2} \leqslant r)]\varepsilon_{t-2} - [\phi + \psi I(X'_{t-2} \leqslant r)]\varepsilon_{t-2}| \\
&= b\mathrm{E}|X_{t-1} - X'_{t-1}| \leqslant \ldots \leqslant b^n\mathrm{E}|X_{t-n} - X'_{t-n}| \leqslant cb^n,
\end{aligned}$$

where $c = |\psi|\mathrm{E}|\varepsilon_t|$. Thus, $\mathrm{E}|X_t - X_t| = \lim_{n\to\infty} b^n c = 0$ and hence $X_t = X_t$ a.s. □

It is interesting to see that the ergodicity of model (1) depends on ψ, ϕ being irrelevant. When $\varepsilon_t \sim N(0, \sigma^2)$, we have $\sup_x|xf(x)| = (\sqrt{2\pi\mathrm{e}})^{-1}$. When $\varepsilon_t \sim t_n$, we have

$$\sup_x|xf(x)| = (1 + n^{-1})^{-(n+1)/2}\Gamma\left(\frac{n+1}{2}\right) \bigg/ \left[\sqrt{n\pi}\,\Gamma\left(\frac{n}{2}\right)\right].$$

When ε_t follows a double exponential distribution with parameter $\lambda > 0$, we have $\sup_x|xf(x)| = (2\mathrm{e})^{-1}$. In these cases, ψ admits quite a large range. When $\psi = 0$, model (1) reduces to the linear MA(1) model and the result reduces to the standard one.

3. Invertibility of TMA(1) models

It is well known that the stationarity condition of an AR(p) model and the invertibility condition of an MA(p) model have the same restriction on the corresponding parameters. Under the same kind of restriction on the parameters for the stationarity of a TAR(p) model as given in Chan and Tong (1985), Ling and Tong (2005) showed that the TMA(p) model is invertible. Now, for the ergodicity of a TAR(1), Petruccelli and Woolford (1984) and Chan *et al*. (1985) have established much sharper results, indeed the sharpest to date. It is therefore interesting to see if similar conditions suffice for the invertibility of a TMA(1) model.

Let $A_{t-1} = -[\phi + \psi I(y_{t-1} \leqslant r)]$. We have

$$\varepsilon_t = y_t + A_{t-1}\varepsilon_{t-1} = y_t + \sum_{j=1}^{J}\left(\prod_{i=1}^{j} A_{t-i}\right)y_{t-j} + \left(\prod_{i=1}^{J+1} A_{t-i}\right)\varepsilon_{t-J-1}. \tag{4}$$

When $\{y_t\}$ is strictly stationary and ergodic, we have by the ergodic theorem

$$\frac{1}{J}\left[\sum_{t=1}^{J} \ln|A_t| + \ln|y_{t-J}|\right] \to \mathrm{E}\ln|A_t| \text{ a.s.},$$

as $J \to \infty$ if $\mathrm{E}|\ln|y_t|| < \infty$. When $\mathrm{E}\ln|A_t| < 0$, we have

$$\lim_{J\to\infty}\left(\prod_{i=1}^{J} |A_{t-i}|\,|y_{t-J}|\right)^{1/J} = \mathrm{e}^{\mathrm{E}\ln|A_t|} < 1 \text{ a.s.}$$

For any $\omega \in \Omega$ such that the previous inequality holds, there exists a $N(\omega)$ such that $(\prod_{i=1}^{J} |A_{t-i}||y_{t-j}|)^{1/J}(\omega) \leqslant \rho(\omega) < 1$. Thus, $[\sum_{j=1}^{J}(\prod_{i=1}^{j} A_{t-i})y_{t-j}](\omega)$ converges, and hence

$$\sum_{j=1}^{J}\left(\prod_{i=1}^{j} A_{t-i}\right)y_{t-j} \to \sum_{j=1}^{\infty}\left(\prod_{i=1}^{j} A_{t-i}\right)y_{t-j} \ \textit{a.s.}, \tag{5}$$

when $\mathrm{E}\ln|A_t| < 0$. Similarly, we can show that the second term in (4) converges to zero a.s. when $\mathrm{E}\ln|A_t| < 0$. In this case, we have

$$\varepsilon_t = y_t + \sum_{j=1}^{\infty}\left(\prod_{i=1}^{j} A_{t-i}\right)y_{t-j} \ \textit{a.s.}, \tag{6}$$

and call model (1) invertible. If (6) does not hold, then we say that model (1) is not invertible.

When $\mathrm{E}\ln|A_t| > 0$, we can show that (5) converges to infinity a.s. by using a similar method and hence model (1) is not invertible. It is not hard to see that $\mathrm{E}\ln|A_t| = \ln[|\phi|^{1-F_y(r)}|\phi+\psi|^{F_y(r)}]$, where $F_y(x)$ is the distribution of y_t. Thus, we can summarize the result as follows.

Theorem 2. *Let $\{y_t\}$ be the unique strictly stationary and ergodic solution of model* (1) *with* $\mathrm{E}|\ln|y_t|| < \infty$. *Then model* (1) *is invertible if $|\phi|^{1-F_y(r)}|\phi+\psi|^{F_y(r)} < 1$ and is not invertible if $|\phi|^{1-F_y(r)}|\phi+\psi|^{F_y(r)} > 1$.*

It is not clear if the TMA(1) model is invertible when $|\phi|^{1-F_y(r)}|\phi+\psi|^{F_y(r)} = 1$. We conjecture that it is not. The above condition is much weaker than the invertibility condition in Ling and Tong (2005), which is $|\psi| < 1$ and $|\phi+\psi| < 1$. When $r = -\infty$ or $+\infty$, the sufficient condition is the same as that for the invertibility of a linear MA(1) model. It is not hard to show that $\mathrm{E}|\ln|y_t|| < \infty$ if $\mathrm{E}|y_t| < \infty$ and $\sup_x f(x) < \infty$.

4. Extension to general TMA models

We first consider the multiple threshold MA$(1, k)$ model:

$$y_t = \left\{ \phi_0 + \sum_{j=1}^{k} \psi_j I(r_{j-1} < y_{t-1} \leqslant r_j) \right\} \varepsilon_{t-1} + \varepsilon_t, \tag{7}$$

where $-\infty = r_0 < r_1 < \ldots < r_k = \infty$. This is a special case of the self-exciting threshold autoregressive moving-average model introduced by Tong (1983). We will extend Theorems 1 and 2 to model (7).

First, define $u_n(t) = \{\phi_0 + \sum_{j=1}^{k} \psi_j I(r_{j-1} < \mathrm{S}_n(t) \leqslant r_j)\}\varepsilon_t$ and

$$S_n(t) = \begin{cases} \varepsilon_t, & \text{if } n = 0, \\ u_{n-1}(t-1) + \varepsilon_t, & \text{if } n \geqslant 1. \end{cases}$$

When $u_{n-3}(t-2) \geqslant u_{n-2}(t-2)$, we have the following decomposition:

$$\begin{aligned} &\sum_{j=1}^{k} \psi_j \{ I(r_{j-1} < S_{n-1}(t-1) \leqslant r_j) - I(r_{j-1} < S_{n-2}(t-1) \leqslant r_j) \} \\ &\quad = \sum_{j=1}^{k} \psi_j \{ I(r_{j-1} - u_{n-2}(t-2) < \varepsilon_{t-1} \leqslant r_j - u_{n-2}(t-2)) \\ &\qquad - I(r_{j-1} - u_{n-3}(t-2) < \varepsilon_{t-1} \leqslant r_j - u_{n-3}(t-2)) \} \\ &\quad = \sum_{j=1}^{k} \psi_j \{ I(r_j - u_{n-3}(t-2) < \varepsilon_{t-1} \leqslant r_j - u_{n-2}(t-2)) \\ &\qquad - I(r_{j-1} - u_{n-3}(t-2) < \varepsilon_{t-1} \leqslant r_{j-1} - u_{n-2}(t-2)) \} \\ &\quad = \sum_{j=1}^{k-1} (\psi_j - \psi_{j+1}) I(r_j - u_{n-3}(t-2) < \varepsilon_{t-1} \leqslant r_j - u_{n-2}(t-2)). \end{aligned}$$

A similar identity holds when $u_{n-3}(t-2) \leqslant u_{n-2}(t-2)$. Thus, we can show that

$$\begin{aligned} \mathrm{E}|S_n(t) - S_{n-1}(t)| &\leqslant b\mathrm{E}|u_{n-2}(t-2) - u_{n-3}(t-2)| \\ &= b\mathrm{E}|S_{n-1}(t-1) - S_{n-2}(t-1)|, \end{aligned}$$

where $b = \sum_{j=1}^{k-1} |\psi_j - \psi_{j+1}| \sup_x |x| f(x)$. Similarly to the arguments in Section 2, we can prove that $S_n(t)$ converges a.s. for each t as $n \to \infty$. Let $X_t = \lim_{n\to\infty} S_n(t)$. Similarly to

Theorem 1, we can show that $\{X_t\}$ is the unique strictly stationary and ergodic solution of model (7). We have therefore proved the following theorem.

Theorem 3. *If* $\sum_{j=1}^{k-1}|\psi_j - \psi_{j+1}|\sup_x|x|f(x) < 1$, *then* $\{X_t\}$ *defined by* $X_t = \lim_{n\to\infty} S_n(t)$ *is the unique strictly stationary and ergodic solution of model* (7).

Let $A_{t-1} = -[\phi_0 + \sum_{j=1}^{k}\psi_j I(r_{j-1} < y_{t-1} \leqslant r_j)]$. We say that model (7) is invertible if $\varepsilon_t = y_t + \sum_{j=1}^{\infty}(\prod_{i=1}^{j} A_{t-i})y_{t-j}$ a.s., otherwise it is not invertible. Note that $\mathrm{E}\ln|A_t| = \ln[\prod_{j=1}^{k}\{|\phi_0 + \psi_j|^{F_y(r_j)-F_y(r_{j-1})}\}]$. Similar to Theorem 2, we have:

Theorem 4. *Let* $\{y_t\}$ *be the unique strictly stationary and ergodic solution of model* (7) *with* $\mathrm{E}|\ln|y_t|| < \infty$. *Then model* (7) *is invertible if* $\prod_{j=1}^{k}\{|\phi_0 + \psi_j|^{F_y(r_j)-F_y(r_{j-1})}\} < 1$ *and is not invertible if* $\prod_{j=1}^{k}\{|\phi_0 + \psi_j|^{F_y(r_j)-F_y(r_{j-1})}\} > 1$.

When $k = 2$, model (7) reduces to model (1) with $\phi = \phi_0 + \psi_2$ and $\psi = \psi_1 - \psi_2$ and the results in Theorems 3 and 4 are the same as those in Theorems 1 and 2. Chan *et al.* (1985) showed that stationarity of the multiple threshold AR(1) model only depends on the coefficients of the AR(1) submodels in the two extreme regimes, independent of all other AR(1) submodels. In contrast, Theorem 3 shows that the coefficients in AR(1) submodels other than those in the extreme regimes also play a role in the invertibility of the multiple threshold MA(1) model.

We next consider the two-regime TMA(p, q) model:

$$y_t = \sum_{i=1}^{p}[\phi_j + \psi_j I(y_{t-q} \leqslant r)]\varepsilon_{t-i} + \varepsilon_t, \tag{8}$$

where $q \geqslant p$. The invertibility of model (8) has been given in Ling and Tong (2005). Here, we only study its stationarity and ergodicity. We define

$$S_n(t) = \begin{cases} \varepsilon_t, & \text{if } n = 0, \\ \sum_{i=1}^{p}[\phi_i + \psi_i I(S_{n-1}(t-q) \leqslant r)]\varepsilon_{t-i} + \varepsilon_t, & \text{if } n \geqslant 1. \end{cases}$$

Let $u_n(t) = \sum_{i=1}^{p}[\phi_i + \psi_i I(S_{n-1}(t-q) \leqslant r)]\varepsilon_{t-i}$, $b = \sup_x|xf(x)|$ and $c = \sup_x|f(x)| \times \mathrm{E}|\sum_{i=1}^{p-1}\psi_i\varepsilon_{t-i}|$. When $p = q$, we have

$$\begin{aligned} A_n(t) &\equiv \mathrm{E}\left[\left|\sum_{i=1}^{p}\psi_i\varepsilon_{t-i}\right||I(S_n(t-q) \leqslant r) - I(S_{n-1}(t-q) \leqslant r)|\right] \\ &\leqslant \mathrm{E}\left|\left(\sum_{i=1}^{p-1}\psi_i\varepsilon_{t-i}\right)\int_{r-u_{n-2}(t-q)}^{r-u_{n-1}(t-q)} f(x)\,\mathrm{d}x\right| + |\psi_p|\mathrm{E}\left|\int_{r-u_{n-2}(t-q)}^{r-u_{n-1}(t-q)}|x|f(x)\,\mathrm{d}x\right| \\ &\leqslant (b|\psi_p| + c)\mathrm{E}|u_{n-1}(t-q) - u_{n-2}(t-q)| \leqslant (b|\psi_p| + c)A_{n-1}(t-q). \end{aligned}$$

When $q > p$, we can replace the constant factor $b|\psi_p| + c$ by c. Using a similar method as for Theorem 1, we have the following result.

Theorem 5. *If* $|\psi_p|\sup_x|xf(x)| + \sup_x|f(x)|\mathrm{E}|\sum_{i=1}^{p-1}\psi_i\varepsilon_{t-i}| < 1$ *when* $p = q$ *or* $\sup_x|f(x)| \times \mathrm{E}|\sum_{i=1}^{p}\psi_i\varepsilon_{t-i}| < 1$ *when* $p < q$, *then* $\{X_t\}$ *defined by* $X_t = \lim_{n\to\infty} S_n(t)$ *is the unique strictly stationary and ergodic solution of model* (8).

However, the ergodicity for model (8) with $1 \leqslant q < p$ remains open.

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