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On layered stable processes

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Layered stable (multivariate) distributions and processes are defined and studied. A layered stable process combines stable trends of two different indices, one of them possibly Gaussian. More precisely, over short intervals it is close to a stable process, while over long intervals it approximates another stable (possibly Gaussian) process. The absolute continuity of a layered stable process with respect to its short-range limiting stable process is also investigated. A series representation of layered stable processes is derived, giving insights into the structure both of the sample paths and of the short- and long-range behaviours of the process. This series representation is further used for simulation of sample paths.

Keywords: layered stable distributions and processes; Lévy processes; stable distributions and processes;

1. Introduction and preliminaries

Stable processes form one of the simplest class of Lévy processes without Gaussian component. They have been thoroughly studied, by many authors, in several fields, such as statistical physics, queuing theory and mathematical finance. A major reason for their attractiveness is the scaling property induced by the structure of the corresponding Lévy measure. Sato (1999) and Samorodnitsky and Taqqu (1994) give many basic facts on stable distributions and processes. Recent generalizations of stable processes can also be found, for example, in Barndorff-Nielsen and Shepard (2002) and Rosiński (2004). These new classes, which are of great interest in applications, have motivated our study.

In the present paper, we introduce and study further generalizations which we call *layered stable*. These are defined in terms of the structure of their Lévy measure whose radial component behaves asymptotically as an inverse polynomial, of different orders, near zero and at infinity. The *inner* and *outer (stability) indices* correspond respectively to these orders of polynomial decay. This simple layering leads to the following properties: the moment properties are determined by the outer index (Proposition 2.1), while the variational properties depend on the inner index (Proposition 2.2). On the other hand, the inner and outer indices also correspond to the short- and long-range behaviour of the sample paths. Over short intervals, a layered stable process behaves like a stable one with the corresponding inner index (Theorem 3.1). The long-range behaviour has two modes

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depending on the outer index. When the outer index is strictly less than 2, a layered stable process is close to a stable process with this index, while it behaves like Brownian motion if the outer index is strictly greater than 2 (Theorem 3.2). In relation to the short-time behaviour, we also investigate the mutual absolute continuity of a layered stable process and of its short-time limiting stable process (Theorem 4.1). A shot noise series representation reveals the nature of layering and also gives direct insights into the properties of layered stable processes. We present typical sample paths of a layered stable process, which are simulated via the series representation, for various combinations of stability indices in order to cover all the possible short- and long-time behaviours.

Let us begin with some general notation which will be used throughout the text. $\mathbb{R}^d$ is the d-dimensional Euclidean space with norm $\|\cdot\|$, $\mathbb{R}_0^d := \mathbb{R}^d \backslash \{0\}$, $\mathcal{B}(\mathbb{R}_0^d)$ is the Borel σ-field of $\mathbb{R}_0^d$, $S^{d-1} := \{z \in \mathbb{R}^d : \|z\| = 1\}$. The symbol $'$ is used to denote transpose, so $z' = (z_1, \ldots, z_d) \in \mathbb{R}^d$, while A' is the transpose of the matrix A. We denote by $\|\cdot\|_o$ the operator norm of a linear transformation, so if $A \in \mathbb{R}^{d\times d}$, then $\|A\|_o = \sup_{\|x\|\leqslant 1}\|Ax\|$. $f(x) \sim g(x)$ indicates that $f(x)/g(x) \to 1$, as $x \to x_0 \in [-\infty, \infty]$, while $f(x) \asymp g(x)$ is used to mean that there exist two positive constants c_1 and c_2 such that $c_1 g(x) \leqslant f(x) \leqslant c_2 g(x)$, for all x in an appropriate set. $\mathcal{L}(X)$ is the law of the random vector X, while $\stackrel{\mathcal{L}}{=}$ and $\stackrel{\mathcal{L}}{\to}$ denote respectively equality and convergence in distribution, or of the finite-dimensional distributions when random processes are considered. Moreover, $\stackrel{d}{\to}$ is used for the weak convergence of random processes in the space $\mathbb{D}([0, \infty), \mathbb{R}^d)$ of cadlag functions from $[0, \infty)$ into $\mathbb{R}^d$ equipped with the Skorokhod topology, while $\stackrel{v}{\to}$ denotes convergence in the vague topology. For any $r > 0$, T_r is a transformation of measures on $\mathbb{R}^d$ given, for any positive measure ρ, by $(T_r\rho)(B) = \rho(r^{-1}B)$, $B \in \mathcal{B}(\mathbb{R}^d)$. $\mathbb{P}|_{\mathcal{F}_t}$ is the restriction of a probability measure $\mathbb{P}$ to the σ-field $\mathcal{F}_t$, while ΔX_t denotes the jump of X at time t, that is, $\Delta X_t := X_t - X_{t-}$. Finally, all multivariate or matrix integrals are defined componentwise.

Recall that an infinitely divisible probability measure μ on $\mathbb{R}^d$, without Gaussian component, is called *stable* if its Lévy measure is given by

$$\nu(B) = \int_{S^{d-1}} \sigma(\mathrm{d}\xi) \int_0^\infty \mathbf{1}_B(r\xi)\frac{\mathrm{d}r}{r^{1+\alpha}}, \qquad B \in \mathcal{B}(\mathbb{R}_0^d),$$

where $\alpha \in (0, 2)$ is the stability index and σ is a finite positive measure on S^{d-1}. It is well known that the characteristic function of μ is given by

$$\widehat{\mu}(y) = \exp\left[\mathrm{i}\langle y, \eta\rangle + \int_{\mathbb{R}_0^d} (\mathrm{e}^{\mathrm{i}\langle y,z\rangle} - 1 - \mathrm{i}\langle y, z\rangle \mathbf{1}_{\{\|z\|\leqslant 1\}}(z))\nu_\alpha(\mathrm{d}z)\right] \tag{1.1}$$

$$= \begin{cases} \exp\left[\mathrm{i}\langle y, \tau_\alpha\rangle - c_\alpha \int_{S^{d-1}} |\langle y, \xi\rangle|^\alpha \left(1 - \mathrm{i}\tan\frac{\pi\alpha}{2}\operatorname{sgn}\langle y, \xi\rangle\right)\sigma(\mathrm{d}\xi)\right], & \text{if } \alpha \neq 1, \\ \exp\left[\mathrm{i}\langle y, \tau_1\rangle - c_1 \int_{S^{d-1}} \left(|\langle y, \xi\rangle| + \mathrm{i}\frac{2}{\pi}\langle y, \xi\rangle \ln|\langle y, \xi\rangle|\right)\sigma(\mathrm{d}\xi)\right], & \text{if } \alpha = 1, \end{cases}$$

for some $\eta \in \mathbb{R}^d$, where $c_\alpha = |\Gamma(-\alpha)\cos(\pi\alpha/2)|$ when $\alpha \neq 1$ and $c_1 = \pi/2$, while

$\tau_\alpha = \eta - (1-\alpha)^{-1}\int_{S^{d-1}} \xi\sigma(\mathrm{d}\xi)$ when $\alpha \neq 1$ and $\tau_1 = \eta - (1-\gamma)\int_{S^{d-1}} \xi\sigma(\mathrm{d}\xi)$, γ $(= 0.5772\ldots)$ being the Euler constant. A Lévy process $\{X_t : t \geqslant 0\}$ such that $\mathcal{L}(X_1) \sim \mu$ is called a *stable process*. Stable processes enjoy the *self-similarity* property, that is, for any $a > 0$,

$$\{X_{at} : t \geqslant 0\} \stackrel{\mathcal{L}}{=} \{a^{1/\alpha} X_t + bt : t \geqslant 0\},$$

for some $b \in \mathbb{R}^d$. Next, we recall a shot noise series representation of stable processes on a fixed finite horizon $[0, T]$, $T > 0$. Related results can be found, for example, in Theorem 1.4.5 of Samorodnitsky and Taqqu (1994). The centring constants given below are obtained in Proposition 5.5 of Rosiński (2004).

Lemma 1.1. *Let $T > 0$. Let $\{T_i\}_{i\geqslant 1}$ be a sequence of independent and identically distributed (i.i.d.) uniform random variables on $[0, T]$, let $\{\Gamma_i\}_{i\geqslant 1}$ be the arrival times of a standard Poisson process, and let $\{V_i\}_{i\geqslant 1}$ a sequence of i.i.d. random vectors in S^{d-1} with common distribution $\sigma(\mathrm{d}\xi)/\sigma(S^{d-1})$. Also let*

$$z_0 = \begin{cases} 0, & \text{if } \alpha \in (0, 1), \\ \displaystyle\int_{S^{d-1}} \frac{\xi\sigma(\mathrm{d}\xi)}{\sigma(S^{d-1})}, & \text{if } \alpha \in [1, 2), \end{cases}$$

and

$$b_T = \begin{cases} 0, & \text{if } \alpha \in (0, 1), \\ \sigma(S^{d-1})T(\gamma + \ln(\sigma(S^{d-1})T)), & \text{if } \alpha = 1, \\ \left(\dfrac{\alpha}{\sigma(S^{d-1})T}\right)^{-1/\alpha} \zeta\left(\frac{1}{\alpha}\right), & \text{if } \alpha \in (1, 2), \end{cases}$$

where ζ denotes the Riemann zeta function. Then the stochastic process

$$\left\{\sum_{i=1}^{\infty}\left[\left(\frac{\alpha\Gamma_i}{\sigma(S^{d-1})T}\right)^{-1/\alpha} V_i \mathbf{1}(T_i \leqslant t) - \left(\frac{\alpha i}{\sigma(S^{d-1})T}\right)^{-1/\alpha} z_0 \frac{t}{T}\right] + b_T z_0 \frac{t}{T} : t \in [0, T]\right\},$$

converges almost surely uniformly in t to an α-stable process $\{X_t : t \in [0, T]\}$ satisfying $\mathbb{E}[\mathrm{e}^{\mathrm{i}\langle y, X_T\rangle}] = \hat{\mu}(y)^T$, where $\hat{\mu}$ is given by (1.1) with

$$\eta = \begin{cases} \dfrac{1}{1-\alpha}\displaystyle\int_{S^{d-1}} \xi\sigma(\mathrm{d}\xi), & \text{if } \alpha \neq 1, \\ 0, & \text{if } \alpha = 1. \end{cases}$$

2. Definition and basic properties

We begin with the definition of a layered stable multivariate distribution by making precise the structure of its Lévy measure in polar coordinates.

Definition 2.1. *On $\mathbb{R}^d$, let μ be an infinitely divisible probability measure without Gaussian component. Then μ is called* layered stable *if its Lévy measure on $\mathbb{R}_0^d$ is given by*

$$\nu(B) = \int_{S^{d-1}} \sigma(\mathrm{d}\xi) \int_0^\infty 1_B(r\xi) q(r, \xi) \mathrm{d}r, \qquad B \in \mathcal{B}(\mathbb{R}_0^d), \tag{2.1}$$

where σ is a finite positive measure on S^{d-1}, and q is a locally integrable function from $(0, \infty) \times S^{d-1}$ to $(0, \infty)$ such that as $r \to 0$,

$$q(r, \xi) \sim c_1(\xi) r^{-\alpha-1}, \tag{2.2}$$

and as $r \to \infty$,

$$q(r, \xi) \sim c_2(\xi) r^{-\beta-1}, \tag{2.3}$$

for σ-almost every $\xi \in S^{d-1}$, where c_1 and c_2 are positive, integrable (with respect to σ) functions on S^{d-1}, and where $(\alpha, \beta) \in (0, 2) \times (0, \infty)$.

Let us call $q(\cdot, \cdot)$ the *q-function* of μ or of its Lévy measure ν. Clearly, ν is well defined as a Lévy measure since it behaves like an α-stable Lévy measure near the origin while decaying like a β-Pareto density when sufficiently far away from the origin. Moreover, α and β are respectively called the *inner* and *outer (stability) index* of μ or of ν.

For convenience, we henceforth denote by σ_1 and σ_2 the finite positive measures on S^{d-1} defined respectively by

$$\sigma_1(B) := \int_B c_1(\xi)\sigma(\mathrm{d}\xi), \qquad B \in \mathcal{B}(S^{d-1}), \tag{2.4}$$

and

$$\sigma_2(B) := \int_B c_2(\xi)\sigma(\mathrm{d}\xi), \qquad B \in \mathcal{B}(S^{d-1}), \tag{2.5}$$

and by ν_σ^α the positive measure on $\mathbb{R}_0^d$ given by

$$\nu_\sigma^\alpha(B) := \int_{S^{d-1}} \sigma(\mathrm{d}\xi) \int_0^\infty 1_B(r\xi) \frac{\mathrm{d}r}{r^{\alpha+1}}, \quad B \in \mathcal{B}(\mathbb{R}_0^d), \tag{2.6}$$

where $\alpha \in (0, \infty)$ and σ is a finite positive measure on S^{d-1}. Note that if $\alpha \in (0, 2)$, ν_σ^α is simply an α-stable Lévy measure, while it is not well defined as a Lévy measure when $\alpha \geqslant 2$.

Example 2.1. The following example of a layered stable Lévy measure is simple, yet interesting:

$$\nu(B) = \int_B \mathbf{1}_{\{\|z\|\leqslant 1\}}(z)\nu_\sigma^\alpha(\mathrm{d}z) + \int_{\mathrm{B}} 1_{\{\|z\|>1\}}(z)\nu_\sigma^\beta(\mathrm{d}z)$$
$$= \int_{S^{d-1}} \sigma(\mathrm{d}\xi)\int_0^\infty \mathbf{1}_B(r\xi)\frac{\mathrm{d}r}{r^{\alpha+1}\mathbf{1}_{(0,1]}(r) + r^{\beta+1}\mathbf{1}_{(1,\infty)}(r)}, \qquad B \in \mathcal{B}(\mathbb{R}_0^d). \tag{2.7}$$

Its q-function is given by

$$q(r, \xi) = r^{-\alpha-1}\mathbf{1}_{(0,1]}(r) + r^{-\beta-1}\mathbf{1}_{(1,\infty)}(r), \qquad \xi \in S^{d-1},$$

which is independent of ξ. The measure ν consists of two disjoint domains of stability, and this construction results in two layers for the radial component associated with each respective stability index. The name '*layered stable*' originates from this special structure.

Recall that an infinitely divisible probability measure μ on $\mathbb{R}^d$ is said to be of class L_0, or self-decomposable, if, for any $b > 1$, there exists a probability measure ϱ_b such that $\widehat{\mu}(z) = \widehat{\mu}(b^{-1}z)\widehat{\varrho_b}(z)$. Equivalently, the Lévy measure of μ has the form

$$\int_{S^{d-1}} \sigma(\mathrm{d}\xi)\int_0^\infty \mathbf{1}_B(r\xi)k_\xi(r)\frac{\mathrm{d}r}{r}, \qquad B \in \mathcal{B}(\mathbb{R}_0^d),$$

where σ is a finite positive measure on S^{d-1} and $k_\xi(r)$ is a non-negative function measurable in $\xi \in S^{d-1}$ and decreasing in $r > 0$. Clearly, the Lévy measure (2.7) induces a self-decomposable measure. Moreover, the classes L_m, $m = 1, 2, \ldots,$ are defined recursively as follows: $\mu \in L_m$ if, for every $b > 1$, there exists $\varrho_b \in L_{m-1}$ such that $\widehat{\mu}(z) = \widehat{\mu}(b^{-1}z)\widehat{\varrho_b}(z)$. It is then also clear that $L_0 \supset L_1 \supset L_2 \supset \ldots$. Let $h_\xi(u) := k_\xi(\mathrm{e}^{-u})$, be the so-called *h-function* of μ, or of its Lévy measure. Then, alternatively, $\mu \in L_0$ is shown to be in L_m if and only if $h_\xi(u) \in C^{m-1}$ and $h^{(j)} \geqslant 0$, for $j = 0, 1, \ldots, m-1$; see Sato (1980) for more details. The h-function of the Lévy measure (2.7) is given by

$$h_\xi(u) = \mathrm{e}^{\alpha u}\mathbf{1}_{(0,\infty)}(u) + \mathrm{e}^{\beta u}\mathbf{1}_{(-\infty,0]}(u),$$

which is in C^0 but not in C^1. Therefore, the infinitely divisible probability measure induced by (2.7) is in L_1 but not in L_2.

The following result asserts that layered stable distributions have the same probability tail behaviour as β-Pareto distributions, or β-stable distributions if $\beta \in (0, 2)$.

Proposition 2.1 (Moments). *Let μ be a layered stable distribution with Lévy measure ν given by* (2.1) *and let σ_2 be the measure given by* (2.5). *If $\sigma_2(S^{d-1}) \neq 0$, then*

$$\int_{\mathbb{R}^d} \|x\|^p \mu(\mathrm{d}x)\begin{cases} < +\infty, & p \in (0, \beta), \\ = +\infty, & p \in [\beta, \infty). \end{cases}$$

Moreover, $\int_{\mathbb{R}^\mathrm{d}} \|x\|^p\mu(\mathrm{d}x) < \infty$, $p \geqslant \beta$, and $\int_{\mathbb{R}^\mathrm{d}} \mathrm{e}^{\theta\|x\|}\mu(\mathrm{d}x) < \infty$, $\theta > 0$, if and only if $\sigma_2(S^{d-1}) = 0$.

Proof. By Theorem 25.3 of Sato (1999), it is enough to show that the restriction of ν to the set $\{z \in \mathbb{R}_0^d : \|z\| > 1\}$ has the corresponding moment properties.

First, assume that $\sigma_2(S^{d-1}) \neq 0$. Observe that $\int_{\|z\|>1} \|z\|^p \nu(\mathrm{d}z) = \int_{S^{d-1}} \sigma(\mathrm{d}\xi) \int_1^\infty r^p q(r, \xi)\mathrm{d}r$, and then by (2.3), the right-hand side is bounded from above and below by constant multiples of $\sigma_2(S^{d-1}) \int_1^\infty r^p \mathrm{d}r / r^{\beta+1}$, if $p \in (0, \beta)$, while it is otherwise clearly infinite.

Next, assume that $\sigma_2(S^{d-1}) = 0$ and let $p \in [\beta, \infty)$. Then there exists $M > 0$ such that $\int_{\|z\|>1} \|z\|^p \nu(\mathrm{d}z) \asymp \int_{S^{d-1}} \sigma(\mathrm{d}\xi) \int_1^M r^p q(r, \xi)\mathrm{d}r$ and $\int_{\|z\|>1} \mathrm{e}^{\theta\|z\|} \nu(\mathrm{d}z) \asymp \int_{S^{d-1}} \sigma(\mathrm{d}\xi) \int_1^M \mathrm{e}^{\theta r} \times q(r, \xi)\mathrm{d}r$. Conversely, if $\sigma_2(S^{d-1}) \neq 0$ and $p \in [\beta, \infty)$, then $\int_{\|z\|>1} \|z\|^p \nu(\mathrm{d}z) = +\infty$ as already shown and, once again, by (2.3), $\int_{\|z\|>1} \mathrm{e}^{\theta\|z\|} \nu(\mathrm{d}z) = \int_{S^{d-1}} \sigma(\mathrm{d}\xi) \int_1^\infty \mathrm{e}^{\theta r} q(r, \xi)\mathrm{d}r = +\infty$. □

Let us define the associated Lévy processes.

Definition 2.2. *A Lévy process, without Gaussian component, is called layered stable if its Lévy measure is given by* (2.1).

Henceforth, $\{X_t^{LS} : t \geqslant 0\}$ denotes a layered stable process in $\mathbb{R}^d$. Its characteristic function at time 1 is given by

$$\mathbb{E}[\mathrm{e}^{\mathrm{i}\langle y, X_1^{LS}\rangle}] = \exp\left[\mathrm{i}\langle y, \eta\rangle + \int_{\mathbb{R}_0^d} (\mathrm{e}^{\mathrm{i}\langle y,z\rangle} - 1 - \mathrm{i}\langle y, z\rangle \mathbf{1}_{\{\|z\|\leqslant 1\}}(z))\nu(\mathrm{d}z)\right], \tag{2.8}$$

where ν is the Lévy measure given by (2.1) and $\eta \in \mathbb{R}^d$. For convenience of notation, we write $\{X_t^{LS} : t \geqslant 0\} \sim LS_{\alpha,\beta}(\sigma, q; \eta)$ when (2.8) holds. Similarly, for $\alpha \in (0, 2)$, $\{X_t^{(\alpha)} : t \geqslant 0\}$ denotes an α-stable Lévy process. Its characteristic function at time 1 is given by

$$\mathbb{E}[\mathrm{e}^{\mathrm{i}\langle y, X_1^{(\alpha)}\rangle}] = \begin{cases} \exp\left[\mathrm{i}\langle y, \eta\rangle + \int_{\mathbb{R}_0^d} (\mathrm{e}^{\mathrm{i}\langle y,z\rangle} - 1)\nu_\sigma^\alpha(\mathrm{d}z)\right], & \text{if } \alpha \in (0, 1), \\ \exp\left[\mathrm{i}\langle y, \eta\rangle + \int_{\mathbb{R}_0^d} (\mathrm{e}^{\mathrm{i}\langle y,z\rangle} - 1 - \mathrm{i}\langle y, z\rangle \mathbf{1}_{\{\|z\|\leqslant 1\}}(z))\nu_\sigma^1(\mathrm{d}z)\right], & \text{if } \alpha = 1, \\ \exp\left[\mathrm{i}\langle y, \eta\rangle + \int_{\mathbb{R}_0^d} (\mathrm{e}^{\mathrm{i}\langle y,z\rangle} - 1 - \mathrm{i}\langle y, z\rangle)\nu_\sigma^\alpha(\mathrm{d}z)\right], & \text{if } \alpha \in (1, 2), \end{cases} \tag{2.9}$$

where ν_σ^α is given by (2.6), and we write $\{X_t^{(\alpha)} : t \geqslant 0\} \sim S_\alpha(\sigma; \eta)$ when (2.9) holds.

Recall that a stochastic process $\{X_t : t \in [0, T]\}$ is said to have almost surely finite pth variation if

$$\mathbb{P}\left(\sup_{\pi\in\mathcal{P}} \sum_{t_i\in\pi} \|X_{t_i} - X_{t_{i-1}}\|^p < \infty\right) = 1,$$

where $\mathcal{P}$ is the set of all the finite partitions π of $[0, T]$, of the form $0 = t_0$

$\leqslant t_1 \leqslant \ldots \leqslant t_{n-1} \leqslant t_n = T$, $n \geqslant 1$. A layered stable process shares the variational properties of a stable process with inner index α.

Proposition 2.2 (*p*th variation). *Let* $X := \{X_t^{LS} : t \geqslant 0\} \sim LS_{\alpha,\beta}(\sigma, q; \eta)$.

(i) *If* $\sigma_1(S^{d-1}) > 0$, *then* X *has almost surely finite first variation on every interval of positive length if and only if* $\alpha \in (0, 1)$.
(ii) *If* $\sigma_1(S^{d-1}) > 0$, $(\alpha, \beta) \in [1, 2) \times (1, \infty)$ *and* $\eta = -\int_{S^{d-1}} \xi\sigma(\mathrm{d}\xi) \int_1^\infty rq(r, \xi)\mathrm{d}r$, *then* X *has almost surely finite* p*th variation on every interval of positive length if and only if* $p > \alpha$.
(iii) *If* $\sigma_1(S^{d-1}) = 0$, *then* X *has almost surely finite first variation on every interval of positive length.*

Proof. (i) Recall that, near the origin, the radial component of the layered stable Lévy measure behaves like an α-stable Lévy measure. The first claim then follows immediately from Theorem 3 of Gikhman and Skorokhod (1969).

(ii) Since X is now centred, III b of Bretagnolle (1972) directly applies.

(iii) Letting ν be the Lévy measure of X, there exists $\varepsilon \in (0, 1)$ such that $\nu(\{z \in \mathbb{R}_0^d : \|z\| \leqslant \varepsilon\}) < +\infty$ and so $\int_{\|z\|\leqslant 1} \|z\|^p \nu(\mathrm{d}z) < +\infty$, $p \geqslant 1$. As in (i), the result follows from Theorem 3 of Gikhman and Skorokhod (1969). □

Let us now consider a series representation for a general layered stable process $\{X_t^{LS} : t \geqslant 0\} \sim LS_{\alpha,\beta}(\sigma, q; 0)$. Fix $T > 0$. Let $\{T_i\}_{i\geqslant 1}$ be a sequence of i.i.d. uniform random variables on $[0, T]$, let $\{\Gamma_i\}_{i\geqslant 1}$ be Poisson arrivals with rate 1, and let $\{V_i\}_{i\geqslant 1}$ be a sequence of i.i.d. random vectors in S^{d-1} with common distribution $\sigma(\mathrm{d}\xi)/\sigma(S^{d-1})$. Assume, moreover, that the random sequences $\{T_i\}_{i\geqslant 1}$, $\{\Gamma_i\}_{i\geqslant 1}$, and $\{V_i\}_{i\geqslant 1}$ are all mutually independent. Also, let

$$\overleftarrow{q}(u, \xi) := \inf\left\{ r > 0 : \sigma(S^{d-1}) \int_r^\infty q(s, \xi)\mathrm{d}s < u \right\},$$

and let $\{b_i\}_{i\geqslant 1}$ be a sequence of constants given by

$$b_i = \int_{i-1}^{i} \mathbb{E}\left[\overleftarrow{q}\left(\frac{s}{T}, V_1\right) V_1 \mathbf{1}\left(\overleftarrow{q}\left(\frac{s}{T}, V_1\right) \leqslant 1 \right) \right] \mathrm{d}s.$$

Then, by Theorem 5.1 of Rosiński (2001), the stochastic process

$$\left\{ \sum_{i=1}^{\infty} \left[\overleftarrow{q}\left(\frac{\Gamma_i}{T}, V_i\right) V_i \mathbf{1}(T_i \leqslant t) - b_i \frac{t}{T} \right] : t \in [0, T] \right\}, \tag{2.10}$$

converges almost surely uniformly in t to a Lévy process whose marginal law at time 1 is $LS_{\alpha,\beta}(\sigma, q; 0)$.

Example 2.2. The Lévy measure (2.7) leads to a highly illustrative series representation. Indeed,

$$\tilde{q}(r,\xi)=\left(\frac{\beta r}{\sigma(S^{d-1})}\right)^{-1/\beta}\mathbf{1}_{(0,\sigma(S^{d-1})/\beta]}(r)+\left(\frac{\alpha r}{\sigma(S^{d-1})}+1-\frac{\alpha}{\beta}\right)^{-1/\alpha}\mathbf{1}_{(\sigma(S^{d-1})/\beta,\infty)}(r),$$

and so the stochastic process

$$\left\{\sum_{i=1}^{\infty}\left[\left(\left(\frac{\beta\Gamma_i}{\sigma(S^{d-1})T}\right)^{-1/\beta}1_{(0,\sigma(S^{d-1})T/\beta]}(\Gamma_i)\right.\right.\right.$$
$$\left.\left.\left.+\left(\frac{\alpha\Gamma_i}{\sigma(S^{d-1})T}+1-\frac{\alpha}{\beta}\right)^{-1/\alpha}\mathbf{1}_{(\sigma(S^{d-1})T/\beta,\infty)}(\Gamma_i)\right)V_i\mathbf{1}(T_i\leqslant t)-b_iz_0\frac{t}{T}\right]\,:\,t\in[0,T]\right\},\quad(2.11)$$

where

$$b_i=\left(\frac{\beta}{\sigma(S^{d-1})T}\right)^{-1/\beta}\frac{(i\wedge\sigma(S^{d-1})T/\beta)^{1-1/\beta}-((i-1)\wedge\sigma(S^{d-1})T/\beta)^{1-1/\beta}}{1-1/\beta},$$

converges almost surely uniformly in t to a Lévy process whose marginal law at time 1 is $LS_{\alpha,\beta}(\sigma,q;0)$, with $z_0=\int_{S^{d-1}}\xi\sigma(\mathrm{d}\xi)/\sigma(S^{d-1})$. This series representation directly reveals the nature of the layering; all the jumps with absolute size greater than 1 are due to the β-stable shot noise $(\beta\Gamma_i/\sigma(S^{d-1}))^{-1/\beta}V_i$, while smaller jumps come from the terms $(\alpha\Gamma_i/\sigma(S^{d-1})+1-\alpha/\beta)^{-1/\alpha}V_i$, whose jump size is very close to the α-stable shot noise when Γ_i is sufficiently large.

3. Short- and long-range behaviour

We now present the first main result of this section by giving the short-range behaviour of a layered stable process. The results of this section were motivated by Section 3 of Rosiński (2004). Recall that σ_1 and σ_2 are the finite positive measures respectively given in (2.4) and (2.5), and that, for any $r>0$, T_r transforms the positive measure ρ via $(T_r\rho)(B)=\rho(r^{-1}B)$, $B\in\mathcal{B}(\mathbb{R}^d)$. For convenience, we will use, throughout this section, the notation $\nu_{\sigma,q}^{\alpha,\beta}$ for the Lévy measure of a layered stable process $LS_{\alpha,\beta}(\sigma,q;\eta)$.

Theorem 3.1 (Short-range behaviour). *Let* $\{X_t^{LS}:t\geqslant 0\}\sim LS_{\alpha,\beta}(\sigma,q;0)$*, let*

$$\eta_{\alpha,\beta}=\begin{cases}\int_{S^{d-1}}\xi\sigma(\mathrm{d}\xi)\int_0^1 rq(r,\xi)\mathrm{d}r, & \text{if } \alpha\in(0,1),\\ -\int_{S^{d-1}}\xi\sigma(\mathrm{d}\xi)\int_1^\infty rq(r,\xi)\mathrm{d}r, & \text{if } (\alpha,\beta)\in(1,2)\times(1,\infty),\\ 0, & \text{otherwise,}\end{cases}$$

and let

$$b_{\alpha,\beta} = \begin{cases} \dfrac{1}{\alpha - 1}\displaystyle\int_{S^{d-1}} \xi\sigma_1(\mathrm{d}\xi), & \text{if } (\alpha, \beta) \in (1, 2) \times (0, 1], \\ 0, & \text{otherwise.} \end{cases}$$

Then, as $h \to 0$, $h > 0$,

$$\{h^{-1/\alpha}(X_{ht}^{LS} + ht\eta_{\alpha,\beta}) - tb_{\alpha,\beta} : t \geqslant 0\} \xrightarrow{d} \{X_t^{(\alpha)} : t \geqslant 0\},$$

where $\{X_t^{(\alpha)} : t \geqslant 0\} \sim S_\alpha(\sigma_1; 0)$.

Proof. Since a layered stable process is a Lévy process, by a theorem of Skorokhod (see Theorem 15.17 of Kallenberg 2002), it suffices to show the weak convergence of its marginals at time 1. To this end, we will show the proper convergence of the generating triplet of the infinitely divisible law, following Theorem 15.14 of Kallenberg (2002).

For the convergence of the Lévy measure, we need to show that as $h \to 0$,

$$h(T_{h^{-1/\alpha}}\nu_{\sigma,q}^{\alpha,\beta}) \xrightarrow{v} \nu_{\sigma_1}^{\alpha},$$

or equivalently that

$$\lim_{h\to 0}\int_{\mathbb{R}_0^d} f(z)h(T_{h^{-1/\alpha}}\nu_{\sigma,q}^{\alpha,\beta})(\mathrm{d}z) = \int_{\mathbb{R}_0^d} f(z)\nu_{\sigma_1}^{\alpha}(\mathrm{d}z),$$

for all bounded continuous functions $f : \mathbb{R}_0^d \to \mathbb{R}$ vanishing in a neighbourhood of the origin. Let f be such a function with $|f| \leqslant C < \infty$ and $f(z) \equiv 0$ on $\{z \in \mathbb{R}_0^d : \|z\| \leqslant \varepsilon\}$, for some $\varepsilon > 0$; then by (2.2) we obtain

$$\int_{\mathbb{R}_0^d} f(z)h(T_{h^{-1/\alpha}}\nu_{\sigma,q}^{\alpha,\beta})(\mathrm{d}z) = h\int_{S^{d-1}} \sigma(\mathrm{d}\xi)\int_0^\infty f(h^{-1/\alpha}r\xi)q(r, \xi)\mathrm{d}r$$

$$\to \int_{S^{d-1}} c_1(\xi)\sigma(\mathrm{d}\xi)\int_0^\infty f(r\xi)\frac{\mathrm{d}r}{r^{\alpha+1}},$$

as $h \to 0$, where the last convergence is justified as follows. For $h \in (0, 1)$, we have

$$h\int_{S^{d-1}} \sigma(\mathrm{d}\xi)\int_0^\infty f(h^{-1/\alpha}r\xi)q(r, \xi)\mathrm{d}r$$

$$= h\int_{S^{d-1}} \sigma(\mathrm{d}\xi)\int_{h^{1/\alpha}\varepsilon}^{\varepsilon} f(h^{-1/\alpha}r\xi)q(r, \xi)\mathrm{d}r + h\int_{S^{d-1}} \sigma(\mathrm{d}\xi)\int_{\varepsilon}^{\infty} f(h^{-1/\alpha}r\xi)q(r, \xi)\mathrm{d}r.$$

Since $|f| \leqslant C$, and since the function q is locally integrable, that is, integrable over any compact subset of $(0, \infty) \times S^{d-1}$, condition (2.3) ensures that the second iterated integral above is bounded independently of h. Hence, as $h \to 0$, the second term above goes to zero. For the first term, note that the conditions on q and f ensure that

$$\lim_{h\to 0} h\int_{h^{1/\alpha}\varepsilon}^{\varepsilon} f(h^{-1/\alpha}r\xi)q(r, \xi)\mathrm{d}r = \int_{\varepsilon}^{\infty} f(r\xi)c_1(\xi)\frac{\mathrm{d}r}{r^{\alpha+1}},$$

for σ-almost every $\xi \in S^{d-1}$, and then dominated convergence allows us to conclude.

Next, the convergence of the Gaussian component holds since, for each $\kappa > 0$,

$$\begin{aligned}\int_{\|z\|\leqslant\kappa} zz'\, h(T_{h^{-1/\alpha}}\nu^{\alpha,\beta}_{\sigma,q})(\mathrm{d}z) &= \int_{S^{d-1}} \xi\xi'\sigma(\mathrm{d}\xi)\int_0^{h^{1/\alpha}\kappa} r^2 h^{1-2/\alpha} q(r,\xi)\mathrm{d}r\\ &= \int_{S^{d-1}} \xi\xi'\sigma(\mathrm{d}\xi)\int_0^{\kappa} r^2 h^{1+1/\alpha} q(h^{1/\alpha}r,\xi)\mathrm{d}r\\ &\to \int_{S^{d-1}} \xi\xi'\sigma_1(\mathrm{d}\xi)\int_0^{\kappa} r^2 \frac{\mathrm{d}r}{\mathrm{r}^{\alpha+1}} = \int_{\|z\|\leqslant\kappa} zz'\nu^{\alpha}_{\sigma_1}(\mathrm{d}z),\end{aligned}$$

as $h \to 0$. The passage to the limit is justified here by dominated convergence, since condition (2.2) ensures that

$$\lim_{h\to 0}\int_0^{\kappa} r^2 h^{1+1/\alpha} q(h^{1/\alpha}r,\xi)\mathrm{d}r = c_1(\xi)\int_0^{\kappa} r^2 \frac{\mathrm{d}r}{r^{\alpha+1}},$$

for σ-almost every $\xi \in S^{d-1}$, and since, moreover, for sufficiently small h,

$$\left\| \int_{S^{d-1}} \xi\xi'\sigma(\mathrm{d}\xi)\int_0^{\kappa} r^2 h^{1+1/\alpha} q(h^{1/\alpha}r,\xi)d\mathrm{r} \right\|_o \leqslant \frac{2\kappa^{2-\alpha}}{2-\alpha}\left\| \int_{S^{d-1}} \xi\xi'\sigma_1(\mathrm{d}\xi)\right\|_o < +\infty.$$

For the convergence of the drift part, assume first that $(\alpha,\beta) \notin (1,2)\times(0,1]$. For a σ-finite positive measure ν on $\mathbb{R}^d_0$, let

$$C_\alpha(\nu) := \begin{cases} \int_{\|z\|\leqslant 1} z\nu(\mathrm{d}z), & \text{if } \alpha \in (0,1),\\ 0, & \text{if } \alpha = 1,\\ -\int_{\|z\|>1} z\nu(\mathrm{d}z), & \text{if } \alpha \in (1,2).\end{cases} \tag{3.1}$$

Clearly $\eta_{\alpha,\beta} = C_\alpha(\nu^{\alpha,\beta}_{\sigma,q})$, and we show that for each $\kappa > 0$, as $h \to 0$,

$$C_\alpha(h(T_{h^{-1/\alpha}}\nu^{\alpha,\beta}_{\sigma,q})) - \int_{\kappa<\|z\|\leqslant 1} zh(T_{h^{-1/\alpha}}\nu^{\alpha,\beta}_{\sigma,q})(\mathrm{d}z) \to C_\alpha(\nu^{\alpha}_{\sigma_1}) - \int_{\kappa<\|z\|\leqslant 1} z\nu^{\alpha}_{\sigma_1}(\mathrm{d}z),$$

where the integral $\int_{\kappa<\|z\|\leqslant}$ is understood to be $-\int_{1<\|z\|\leqslant\kappa}$ when $\kappa > 1$. We have

$$C_\alpha(h(T_{h^{-1/\alpha}}\nu^{\alpha,\beta}_{\sigma,q})) - \int_{\kappa<\|z\|\leqslant 1} zh(T_{h^{-1/\alpha}}\nu^{\alpha,\beta}_{\sigma,q})(\mathrm{d}z)$$

$$= \begin{cases} \displaystyle\int_{S^{d-1}} \xi\sigma(\mathrm{d}\xi)\int_0^{h^{1/\alpha}\kappa} rh^{1-1/\alpha}q(r,\xi)\mathrm{d}r, & \text{if } \alpha\in(0,1), \\ \displaystyle\int_{S^{d-1}} \xi\sigma(\mathrm{d}\xi)\int_{h\kappa}^{h} rq(r,\xi)\mathrm{d}r, & \text{if } \alpha=1, \\ \displaystyle\int_{S^{d-1}} \xi\sigma(\mathrm{d}\xi)\int_{h^{1/\alpha}\kappa}^{\infty} rh^{1-1/\alpha}q(r,\xi)\mathrm{d}r, & \text{if } \alpha\in(1,2), \end{cases}$$

and it remains to show that, as $h\to 0$, each term above converges respectively to

$$\begin{cases} \displaystyle\int_{S^{d-1}} \xi\sigma_1(\mathrm{d}\xi)\int_0^{\kappa} r\frac{\mathrm{d}r}{r^{\alpha+1}}, & \text{if } \alpha\in(0,1), \\ \displaystyle\int_{S^{d-1}} \xi\sigma_1(\mathrm{d}\xi)\int_{\kappa}^{1} r\frac{\mathrm{d}r}{r^{2}}, & \text{if } \alpha=1, \\ \displaystyle\int_{S^{d-1}} \xi\sigma_1(\mathrm{d}\xi)\int_{\kappa}^{\infty} r\frac{\mathrm{d}r}{r^{\alpha+1}}, & \text{if } \alpha\in(1,2). \end{cases}$$

First, for $\alpha\in(0,1)$,

$$h^{1-1/\alpha}\int_0^{h^{1/\alpha}\kappa} rq(r,\xi)\mathrm{d}r = \int_0^{\kappa} rh^{1+1/\alpha}q(h^{1/\alpha}r,\xi)\mathrm{d}r \to c_1(\xi)\int_0^{\kappa} r\frac{\mathrm{d}r}{r^{\alpha+1}},$$

while, for $\alpha=1$,

$$\int_{h\kappa}^{h} rq(r,\xi)\mathrm{d}r = \int_{\kappa}^{1} rh^2q(hr,\xi)\mathrm{d}r \to c_1(\xi)\int_{\kappa}^{1} r\frac{\mathrm{d}r}{r^2},$$

making use of the conditions on q. Next, for $\alpha\in(1,2)$, and for $h\in(0,1)$,

$$h^{1-1/\alpha}\int_{h^{1/\alpha}\kappa}^{\infty} rq(r,\xi)\mathrm{d}r = h^{1-1/\alpha}\int_{\kappa}^{\infty} rq(r,\xi)\mathrm{d}r + h^{1-1/\alpha}\int_{h^{1/\alpha}\kappa}^{\kappa} rq(r,\xi)\mathrm{d}r.$$

The first integral on the right-hand side is bounded independently of h (by the conditions on q and since $\beta\in(1,\infty)$), and thus the corresponding first term converges to zero with h. For the second integral,

$$h^{1-1/\alpha}\int_{h^{1/\alpha}\kappa}^{\kappa} rq(r,\xi)\mathrm{d}r = \int_{\kappa}^{h^{-1/\alpha}\kappa} rh^{1+1/\alpha}q(h^{1/\alpha}r,\xi)\mathrm{d}r \to c_1(\xi)\int_{\kappa}^{\infty} r\frac{\mathrm{d}r}{r^{\alpha+1}} < +\infty,$$

where the conditions on q justify the convergence. Finally, assume $(\alpha,\beta)\in(1,2)\times(0,1]$. Then, for each $\kappa>0$, as $h\to 0$,

$$\int_{\kappa<\|z\|\leqslant 1} zh(T_{h^{-1/\alpha}}\nu^{\alpha,\beta}_{\sigma,q})(\mathrm{d}z) = \int_{S^{d-1}} \xi\sigma(\mathrm{d}\xi)\int_{\kappa}^{1} rh^{1+1/\alpha}q(h^{1/\alpha}r,\, \xi)\mathrm{d}r \to \int_{S^{d-1}} \xi\sigma_1(\mathrm{d}\xi)\int_{\kappa}^{1} r\frac{\mathrm{d}r}{r^{\alpha+1}},$$

where the convergence can be justified as before, and we thus obtain,

$$-b_{\alpha,\beta} - \int_{\kappa<\|z\|\leqslant 1} zh(T_{h^{-1/\alpha}}\nu^{\alpha,\beta}_{\sigma,q})(\mathrm{d}z) \to -\int_{\|z\|>\kappa} z\nu^{\alpha}_{\sigma_1}(\mathrm{d}z),$$

which completes the proof. □

Our next result is also of importance. Unlike the short-range behaviour, the long-range behaviour of a layered stable process depends on its outer stability index β. This behaviour is akin to a β-stable process if $\beta \in (0, 2)$, and to a Brownian motion if $\beta \in (2, \infty)$.

Theorem 3.2 (Long-range behaviour). *Let* $\{X^{LS}_t : t \geqslant 0\} \sim LS_{\alpha,\beta}(\sigma, q; 0)$.

(i) Let $\beta \in (0, 2)$, *let*

$$\eta_{\alpha,\beta} = \begin{cases} \int_{S^{d-1}} \xi\sigma(\mathrm{d}\xi)\int_0^1 rq(r,\, \xi)\mathrm{d}r, & \text{if } (\alpha, \beta) \in (0, 1)\times(0, 1), \\ -\int_{S^{d-1}} \xi\sigma(\mathrm{d}\xi)\int_1^{\infty} rq(r,\, \xi)\mathrm{d}r, & \text{if } \beta \in (1, 2), \\ 0, & \text{otherwise,} \end{cases}$$

and let

$$b_{\alpha,\beta} = \begin{cases} \dfrac{1}{1-\beta}\int_{S^{d-1}} \xi\sigma_2(\mathrm{d}\xi), & \text{if } (\alpha, \beta) \in [1, 2)\times(0, 1), \\ 0, & \text{otherwise.} \end{cases}$$

Then, as $h \to +\infty$,

$$\{h^{-1/\beta}(X^{LS}_{ht} + ht\eta_{\alpha,\beta}) + tb_{\alpha,\beta} : t \geqslant 0\}\{X^{(\beta)}_t : t \geqslant 0\},$$

where $\{X^{(\beta)}_t : t \geqslant 0\} \sim S_\beta(\sigma_2; 0)$.

(ii) Let $\beta \in (2, \infty)$ *and let*

$$\eta = -\int_{S^{d-1}} \xi\sigma(\mathrm{d}\xi)\int_1^{\infty} rq(r,\, \xi)\mathrm{d}r. \tag{3.2}$$

Then, as $h \to +\infty$,

$$\{h^{-1/2}(X^{LS}_{ht} + ht\eta) : t \geqslant 0\}\{W_t : t \geqslant 0\}, \tag{3.3}$$

where $\{W_t : t \geqslant 0\}$ *is a centred Brownian motion with covariance matrix* $\int_{\mathbb{R}^d_0} zz'\nu^{\alpha,\beta}_{\sigma,q}(\mathrm{d}z)$.

Proof. Claim (i) can be proved similarly to Theorem 3.1. For the convergence of the Lévy measure, we will show that

$$\lim_{h\to\infty}\int_{\mathbb{R}_0^d} f(z)h(T_{h^{-1/\beta}}\nu_{\sigma,q}^{\alpha,\beta})(\mathrm{d}z) = \int_{\mathbb{R}_0^d} f(z)\nu_{\sigma_2}^{\beta}(\mathrm{d}z),$$

for all bounded continuous functions $f : \mathbb{R}_0^d \to \mathbb{R}$ vanishing in a neighbourhood of the origin. Let f be such a function with $|f| \leqslant C < \infty$ and $f(z) \equiv 0$ on $\{z \in \mathbb{R}_0^d : \|z\| \leqslant \varepsilon\}$, for some $\varepsilon > 0$; then by (2.3) we obtain

$$\begin{aligned}\int_{\mathbb{R}_0^d} f(z)h(T_{h^{-1/\beta}}\nu_{\sigma,q}^{\alpha,\beta})(\mathrm{d}z) &= \int_{S^{d-1}}\sigma(\mathrm{d}\xi)\int_0^\infty f(r\xi)h^{1+1/\beta}q(h^{1/\beta}r,\xi)\mathrm{d}r\\ &\to \int_{S^{d-1}} c_2(\xi)\sigma(\mathrm{d}\xi)\int_0^\infty f(r\xi)\frac{\mathrm{d}r}{r^{\beta+1}},\end{aligned}$$

as $h \to \infty$, where the passage to the limit is, again, justified by dominated convergence with the conditions on q and f.

For the convergence of the Gaussian component, we have, as $h \to \infty$ and for each $\kappa > 0$,

$$\begin{aligned}\int_{\|z\|\leqslant\kappa} zz' h(T_{h^{-1/\beta}}\nu_{\sigma,q}^{\alpha,\beta})(\mathrm{d}z) &= \int_{S^{d-1}}\xi\xi'\sigma(\mathrm{d}\xi)\int_0^\kappa r^2 h^{1+1/\beta}q(h^{1/\beta}r,\xi)\mathrm{d}r\\ &\to \int_{S^{d-1}}\xi\xi'\sigma_2(\mathrm{d}\xi)\int_0^\kappa r^2\frac{\mathrm{d}r}{r^{\beta+1}} = \int_{\|z\|\leqslant\kappa} zz'\nu_{\sigma_2}^{\beta}(\mathrm{d}z),\end{aligned}$$

where the limit is obtained as in Theorem 3.1, making use of condition (2.3).

Finally, we prove the convergence of the drift part. Assume first that $(\alpha,\beta) \notin [1,2)\times(0,1)$. Let $C_\beta(\nu)$ be the defined as in (3.1), but with β replacing α. Clearly $\eta_{\alpha,\beta} = C_\beta(\nu_{\sigma,q}^{\alpha,\beta})$. We then show that for each $\kappa > 0$, as $h \to \infty$,

$$C_\beta(h(T_{h^{-1/\beta}}\nu_{\sigma,q}^{\alpha,\beta})) - \int_{\kappa<\|z\|\leqslant 1} zh(T_{h^{-1/\beta}}\nu_{\sigma,q}^{\alpha,\beta})(\mathrm{d}z) \to \mathrm{C}_\beta(\nu_{\sigma_2}^{\beta}) - \int_{\kappa<\|z\|\leqslant 1} z\nu_{\sigma_2}^{\beta}(\mathrm{d}z),$$

where the integral $\int_{\kappa<\|z\|\leqslant 1}$ is again understood to be $-\int_{1<\|z\|\leqslant\kappa}$ when $\kappa > 1$. As in Theorem 3.1, we obtain, as $h \to \infty$,

$$C_\beta(h(T_{h^{-1/\beta}}\nu_{\sigma,q}^{\alpha,\beta})) - \int_{\kappa<\|z\|\leqslant 1} zh(T_{h^{-1/\beta}}\nu_{\sigma,q}^{\alpha,\beta})(\mathrm{d}z) \to \begin{cases}\displaystyle\int_{S^{d-1}}\xi\sigma_2(\mathrm{d}\xi)\int_0^\kappa r\frac{\mathrm{d}r}{\mathrm{r}^{\beta+1}}, & \text{if } \beta\in(0,1),\\ \displaystyle\int_{S^{d-1}}\xi\sigma_2(\mathrm{d}\xi)\int_\kappa^1 r\frac{\mathrm{d}r}{\mathrm{r}^2}, & \text{if } \beta = 1,\\ \displaystyle\int_{S^{d-1}}\xi\sigma_2(\mathrm{d}\xi)\int_\kappa^\infty r\frac{\mathrm{d}r}{\mathrm{r}^{\beta+1}}, & \text{if } \beta\in(1,2).\end{cases}$$

Next, let $(\alpha,\beta) \in [1,2)\times(0,1)$. Then observe that, for each $\kappa > 0$ and as $h \to \infty$,

$$-b_{\alpha,\beta} - \int_{\kappa<\|z\|\leqslant 1} zh(T_{h^{-1/\alpha}}\nu_{\sigma,q}^{\alpha,\beta})(\mathrm{d}z) \to -\int_{\|z\|>\kappa} z\nu_{\sigma_1}^{\alpha}(\mathrm{d}z),$$

where the convergence holds true as before. This completes the proof of (i).

(ii) The random vector $h^{-1/2}X_h^{LS}$ is infinitely divisible with generating triplet

$$\left(-\int_{\|z\|\geqslant 1} zh(T_{h^{-1/2}}\nu^{\alpha,\beta}_{\sigma,q})(\mathrm{d}z),\ 0,\ (T_{h^{-1/2}}\nu^{\alpha,\beta}_{\sigma,q}) \right).$$

We first prove the vague convergence, to zero, of the Lévy measure $h(T_{h^{-1/2}}\nu^{\alpha,\beta}_{\sigma,q})$. Let f be a bounded continuous function from $\mathbb{R}^d_0$ to $\mathbb{R}$ such that $|f| \leqslant C < \infty$ and $f(z) \equiv 0$ on $\{z \in \mathbb{R}^d : \|z\| \leqslant \varepsilon\}$, for some $\varepsilon > 0$. Then, as $h \to \infty$,

$$h^{\beta/2-1}\int_{\mathbb{R}^d_0} f(z)h(T_{h^{-1/2}}\nu^{\alpha,\beta}_{\sigma,q})(\mathrm{d}z) = \int_{S^{d-1}} \sigma(\mathrm{d}\xi)\int_{\varepsilon}^{\infty} f(r\xi)h^{\beta/2+1/2}q(h^{1/2}r,\ \xi)\mathrm{d}r$$

$$\to \int_{S^{d-1}} \sigma(\mathrm{d}\xi)\int_{\varepsilon}^{\infty} f(r\xi)\frac{\mathrm{d}r}{r^{\beta+1}},$$

where the conditions on q and f ensure the passage to the limit. Since $\beta > 2$, we conclude that, as $h \to \infty$, $h(T_{h^{-1/2}}\nu^{\alpha,\beta}_{\sigma,q}) \xrightarrow{v} 0$.

For the convergence of the Gaussian component, we have, as $h \to +\infty$ and for each $\kappa > 0$,

$$\int_{\|z\|\leqslant\kappa} zz'h(T_{h^{-1/2}}\nu^{\alpha,\beta}_{\sigma,q})(\mathrm{d}z) = \int_{\|z\|\leqslant h^{1/2}\kappa} zz'\nu^{\alpha,\beta}_{\sigma,q}(\mathrm{d}z) \to \int_{\mathbb{R}^d_0} zz'\nu^{\alpha,\beta}_{\sigma,q}(\mathrm{d}z), \tag{3.4}$$

which is clearly well defined since $\int_{\mathbb{R}^d_0} \|z\|^2\nu^{\alpha,\beta}_{\sigma,q}(\mathrm{d}z) < \infty$. Finally, write

$$\int_{\|z\|>\kappa} zh(T_{h^{-1/2}}\nu^{\alpha,\beta}_{\sigma,q})(\mathrm{d}z) = \int_{S^{d-1}} \xi\sigma(\mathrm{d}\xi)\int_{\kappa}^{\infty} rh^{3/2}q(h^{1/2}r,\ \xi)\mathrm{d}r, \tag{3.5}$$

and with the conditions imposed on q, the dominated convergence theorem ensures that, as $h \to \infty$,

$$h^{\beta/2-1}\int_{\kappa}^{\infty} rh^{3/2}q(h^{1/2}r,\ \xi)\mathrm{d}r \to c_2(\xi)\int_{\kappa}^{\infty} r\frac{\mathrm{d}r}{r^{\beta+1}}. \tag{3.6}$$

Since $\beta > 2$, the convergence of the drift term is proved. □

For $\beta = 2$, layered stable processes do not seem to possess any nice long-range property, and this can be seen from the improper convergence of the Lévy measure, that is, as $h \to \infty$, $h(T_{h^{-1/2}}\nu^{\alpha,2}_{\sigma,q})$ converges vaguely to

$$\int_{S^{d-1}} \sigma_2(\mathrm{d}\xi)\int_0^{\infty} 1_B(r\xi)\frac{\mathrm{d}r}{r^{2+1}}, \qquad B \in \mathcal{B}(\mathbb{R}^d_0),$$

which is not well defined as a Lévy measure. However, additional assumptions on σ_2 lead to the weak convergence towards a Brownian motion as β approaches 2.

Proposition 3.3. *Let* $\{X^{LS}_t : t \geqslant 0\} \sim LS_{\alpha,\beta}(\sigma,\ q;\ 0)$ *in* $\mathbb{R}^d$.

(i) Let $\beta \in (1, 2)$ *and let* $\eta = -\int_{S^{d-1}} \xi\sigma(\mathrm{d}\xi)\int_1^{\infty} rq(r, \xi)\mathrm{d}r$. *If* σ_2 *is uniform on* S^{d-1} *such that* $\sigma_2(S^{d-1}) = d(2-\beta)$, *then*

$$\{h^{-1/\beta}(X_{ht}^{LS} + ht\eta) : t \geqslant 0\} \xrightarrow{d} \{W_t : t \geqslant 0\}, \qquad \text{as } h \to \infty, \beta \uparrow 2,$$

where $\{W_t : t \geqslant 0\}$ *is a d-dimensional (centred) standard Brownian motion. (Here, the limit is taken over* $h \to \infty$ *first.)*

(ii) Let $\beta \in (2, \infty)$ *and let* η *be the constant defined in* (3.2). *If* σ_2 *is symmetric such that* $\sigma_2(S^{d-1}) = \beta - 2$, *then*

$$\{h^{-1/2}(X_{ht}^{LS} + ht\eta) : t \geqslant 0\} \xrightarrow{d} \{W_t : t \geqslant 0\}, \qquad \text{as } h \to \infty, \beta \downarrow 2,$$

where $\{W_t : t \geqslant 0\}$ *is a centred Brownian motion with covariance matrix* $\int_{\mathbb{R}_0^d} zz'\nu_{\sigma,q}^{\alpha,2}(\mathrm{d}z)$. *(Here, the limit can be taken either over* $h \to \infty$ *or over* $\beta \downarrow 2$ *first.)*

Proof. (i) By Theorem 3.1, $h^{-1/\beta}(X_h^{LS} + h\eta) \xrightarrow{\mathcal{L}} X_1^{(\beta)}$, as $h \to \infty$, where $\{X_t^{(\beta)} : t \geqslant 0\} \sim S_\beta(\sigma_2; 0)$. Then, by E.18.8–18.9 of Sato (1999), we obtain $\mathbb{E}[\mathrm{e}^{\mathrm{i}\langle y, X_1^{(\beta)}\rangle}] = \exp[-c_{\beta,d}\|y\|^\beta]$, where

$$c_{\beta,d} = \frac{\Gamma(d/2)\Gamma((2-\beta)/2)}{2^\beta \beta \Gamma((\beta + d)/2)}\sigma_2(S^{d-1}).$$

Taking $\beta \uparrow 2$ and since $\Gamma(x+1) = x\Gamma(x)$, $x > 0$, we obtain the result.

(ii) The vague convergence, to zero, of the Lévy measure $h(T_{h^{-1/2}}\nu_{\sigma,q}^{\alpha,\beta})$ can be proved just as in Theorem 3.2(ii). Next, in view of (3.5)–(3.6), we have

$$\lim_{h\to\infty} h^{\beta/2-1}\left\|\int_{\|z\|>\kappa} zh(T_{h^{-1/2}}\nu_{\sigma,q}^{\alpha,\beta})(\mathrm{d}z)\right\| = \frac{\kappa^{1-\beta}}{\beta - 1}\left\|\int_{S^{d-1}} \xi\sigma_2(\mathrm{d}\xi)\right\| = 0,$$

where the last equality holds by the symmetry of σ_2. Since $\beta > 2$, we obtain

$$\lim_{h\to\infty}\left\|\int_{\|z\|>\kappa} zh(T_{h^{-1/2}}\nu_{\sigma,q}^{\alpha,\beta})(\mathrm{d}z)\right\| = 0,$$

which proves the convergence of the drift component. Finally, in view of (3.4), we have

$$\int_{\mathbb{R}_0^d} \|z\|^2 \nu_{\sigma,q}^{\alpha,\beta}(\mathrm{d}z) = \int_{S^{d-1}} \sigma(\mathrm{d}\xi)\int_0^\infty r^2 q(r, \xi)\mathrm{d}r < \infty,$$

using the conditions on q and since $\beta > 2$. The proof is complete. □

Remark 3.1. The short-range behaviour (Theorem 3.1) and the (non-Gaussian) long-range behaviour (Theorem 3.2(i)) can also be inferred from the series representation (2.10). For simplicity, consider the symmetric case and only the short-range behaviour. Letting $X_t := \sum_{i=1}^\infty \overleftarrow{q}(\Gamma_i/T, V_i)V_i\mathbf{1}(T_i \leqslant t)$, we have

$$h^{-1/\alpha}X_{ht} = \sum_{i=1}^\infty h^{-1/\alpha}\overleftarrow{q}\left(\frac{\Gamma_i}{hT}, V_i\right)V_i\mathbf{1}(hT_i \leqslant ht),$$

and so, for each $u > 0$ and each $\xi \in S^{d-1}$ such that $c_1(\xi) \in [0, \infty)$, bounded convergence gives

$$h^{-1/\alpha}\tilde{q}(h^{-1}u, \xi) = h^{-1/\alpha}\inf\left\{ r > 0 : \sigma(S^{d-1})\int_r^\infty q(s, \xi)\mathrm{d}s < h^{-1}u \right\}$$

$$= \inf\left\{ r > 0 : \sigma(S^{d-1})\int_r^\infty h^{1+1/\alpha} q(h^{1/\alpha}s, \xi)\mathrm{d}s < u \right\}$$

$$\to \inf\left\{ r > 0 : c_1(\xi)\sigma(S^{d-1})\int_r^\infty s^{-\alpha-1}\mathrm{d}s < u \right\} = \left(\frac{\alpha u}{c_1(\xi)\sigma(S^{d-1})} \right)^{-1/\alpha},$$

as $h \to 0$, which is indeed an α-stable shot noise.

4. Absolute continuity with respect to short-range limiting stable process

Two Lévy processes, which are mutually absolutely continuous, share any almost sure local behaviour. The next theorem confirms this fact in relation to the short-range behaviour result obtained in Theorem 3.1. Indeed, given any layered stable process with respect to some probability measure, one can find a probability measure under which the layered stable process is identical in law to its short-range limiting stable process. This result should be compared with Section 4 of Rosiński (2004).

Recall that c_1 and c_2 are the integrable (with respect to σ) functions on S^{d-1} appearing in (2.2) and (2.3), while σ_1 and σ_2 are the finite positive measures (2.4) and (2.5), respectively. As before, we use the notation $\nu_{\sigma,q}^{\alpha,\beta}$ for the Lévy measure of a layered stable process $X := \{X_t : t \geqslant 0\} \sim LS_{\alpha,\beta}(\sigma, q; \eta)$, while ν_σ^α is the measure (2.6).

Theorem 4.1. *Let* $\mathbb{P}$, $\mathbb{Q}$ *and* $\mathbb{T}$ *be probability measures on* $(\Omega, \mathcal{F})$ *such that under* $\mathbb{P}$ *the canonical process* $\{X_t : t \geqslant 0\}$ *is a Lévy process in* $\mathbb{R}^d$ *with* $\mathcal{L}(X_1) \sim LS_{\alpha,\beta}(\sigma, q; k_0)$, *while under* $\mathbb{Q}$ *it is a Lévy process with* $\mathcal{L}(X_1) \sim S_\alpha(\sigma_1; k_1)$, *and let* $(\mathcal{F}_t)_{t\geqslant 0}$ *be the natural filtration of* $\{X_t : t \geqslant 0\}$. *Moreover, when* $\beta \in (0, 2)$ *and under* $\mathbb{T}$, $\{X_t : t \geqslant 0\}$ *is a Lévy process with* $\mathcal{L}(X_1) \sim S_\beta(\sigma_2; \eta)$, *for some* $\eta \in \mathbb{R}^d$. *Then the following results hold:*

(i) $\mathbb{P}|_{\mathcal{F}_t}$ *and* $\mathbb{Q}|_{\mathcal{F}_t}$ *are mutually absolutely continuous, for every* $t > 0$, *if and only if*

$$k_0 - k_1 = \begin{cases} \displaystyle\int_{S^{d-1}} \xi\sigma(\mathrm{d}\xi)\int_0^1 rq(r, \xi)\mathrm{d}r, & \text{if } \alpha \in (0, 1), \\ \displaystyle\int_{S^{d-1}} \xi\sigma(\mathrm{d}\xi)\int_0^1 r(q(r, \xi) - c_1(\xi)r^{-\alpha-1})\mathrm{d}r, & \text{if } \alpha = 1, \\ \displaystyle\frac{1}{\alpha-1}\int_{S^{d-1}} \xi\sigma_1(\mathrm{d}\xi) + \int_{S^{d-1}} \xi\sigma(\mathrm{d}\xi)\int_0^1 r(q(r, \xi) - c_1(\xi)r^{-\alpha-1})\mathrm{d}r, & \text{if } \alpha \in (1, 2). \end{cases}$$

(ii) *If* $\alpha \neq \beta$, *then for any choice of* $\eta \in \mathbb{R}^d$, $\mathbb{P}|_{\mathcal{F}_t}$ *and* $\mathbb{T}|_{\mathcal{F}_t}$ *are singular, for every* $t > 0$.

(iii) *For every* $t > 0$,

$$\frac{d\mathbb{Q}}{d\mathbb{P}}|_{\mathcal{F}_t} = e^{U_t},$$

where $\{U_t : t \geqslant 0\}$ *is a Lévy process defined on* $(\Omega, \mathcal{F}, \mathbb{P})$ *by*

$$U_t := \lim_{\varepsilon \downarrow 0} \sum_{\{s \in (0,t] : \|\Delta X_s\| > \varepsilon\}} \times \left[\ln\left(\frac{q(\|\Delta X_s\|, \Delta X_s/\|\Delta X_s\|)}{c_1(\Delta X_s/\|\Delta X_s\|)\|\Delta X_s\|^{-\alpha-1}} \right) - t(\nu^{\alpha,\beta}_{\sigma,q} - \nu^{\alpha}_{\sigma_1})(\{z \in \mathbb{R}^d_0 : \|z\| > \varepsilon\}) \right], \quad (4.1)$$

and where the convergence holds $\mathbb{P}$*-almost surely, uniformly in* t *on every interval of positive length.*

Proof. (i) By Theorem 33.1 and Remark 33.3 of Sato (1999), to prove the result it is necessary and sufficient to show that the following three conditions hold:

$$\int_{\{z:|\varphi(z)|\leqslant 1\}} \varphi(z)^2 \nu^{\alpha}_{\sigma_1}(dz) < \infty, \quad (4.2)$$

$$\int_{\{z:\varphi(z)>1\}} e^{\varphi(z)} \nu^{\alpha}_{\sigma_1}(dz) < \infty, \quad (4.3)$$

$$\int_{\{z:\varphi(z)<-1\}} \nu^{\alpha}_{\sigma_1}(dz) < \infty, \quad (4.4)$$

where the function $\varphi : \mathbb{R}^d_0 \to \mathbb{R}$ is defined by $(d\nu^{\alpha,\beta}_{\sigma,q}/d\nu^{\alpha}_{\sigma_1})(z) = e^{\varphi(z)}$, that is,

$$\varphi(z) = \ln\left(\frac{q(\|z\|, z/\|z\|)}{c_1(z/\|z\|)\|z\|^{-\alpha-1}} \right), \qquad z \in \mathbb{R}^d_0.$$

Now observe that

$$\lim_{\|z\|\to 0} \varphi(z) = \lim_{\|z\|\to 0} \ln\left(\frac{c_1(z/\|z\|)\|z\|^{-\alpha-1}}{c_1(z/\|z\|)\|z\|^{-\alpha-1}} \right) = 0, \quad (4.5)$$

and that, as $\|z\| \to \infty$,

$$\varphi(z) \sim \ln\left(\frac{c_2(z/\|z\|)\|z\|^{-\beta-1}}{c_1(z/\|z\|)\|z\|^{-\alpha-1}} \right) = \ln\left(\frac{c_2(z/\|z\|)}{c_1(z/\|z\|)} \right) + (\alpha - \beta)\ln\|z\|$$
$$\to \begin{cases} -\infty, & \text{if } \alpha < \beta, \\ +\infty, & \text{if } \alpha > \beta. \end{cases} \quad (4.6)$$

Conditions (4.2) and (4.4) are thus immediately satisfied, via (4.5) and (4.6) with $\alpha < \beta$. In view of (4.6) with $\alpha > \beta$, condition (4.3) is also satisfied since $\int_{\{z:\varphi(z)>1\}} \mathrm{e}^{\varphi(z)}\nu^{\alpha}_{\sigma_1}(\mathrm{d}z)$ is bounded from above and below by constant multiples of

$$\int_{\|z\|>1} \frac{q(\|z\|, z/\|z\|)}{c_1(z/\|z\|)\|z\|^{-\alpha-1}}\nu^{\alpha}_{\sigma_1}(\mathrm{d}z) = \nu^{\alpha,\beta}_{\sigma,q}(\{z \in \mathbb{R}^d_0 : \|z\| > 1\}).$$

When $\alpha = \beta \in (0, 2)$, we have, by (4.5) and (4.6),

$$\lim_{\|z\|\to 0} \varphi(z) = 0, \qquad \lim_{\|z\|\to\infty} |\varphi(z)| = \lim_{\|z\|\to\infty} \left| \ln\left(\frac{c_2(z/\|z\|)}{c_1(z/\|z\|)}\right)\right| < +\infty.$$

Condition (4.2) is then satisfied since $\int_{\{z:|\varphi(z)|\leqslant 1\}} \varphi(z)^2\nu^{\alpha}_{\sigma_1}(\mathrm{d}z)$ is bounded from above and below by constant multiples of $\int_{\|z\|>1} \varphi(z)^2\nu^{\alpha}_{\sigma_1}(\mathrm{d}z)$, which is further bounded by a constant multiple of $\nu^{\alpha}_{\sigma_1}(\{z \in \mathbb{R}^d_0 : \|z\| > 1\})$. Conditions (4.3) and (4.4) are also satisfied since the domains $\{z \in \mathbb{R}^d_0 : \varphi(z) > 1\}$ and $\{z \in \mathbb{R}^d_0 : \varphi(z) < -1\}$ are contained in compact subsets of $\mathbb{R}^d_0$.

(ii) It suffices to show that either one of the following two conditions always fails:

$$\int_{\{z\,:\psi(z)>1\}} \mathrm{e}^{\psi(z)}\nu^{\beta}_{\sigma_2}(\mathrm{d}z) < +\infty, \tag{4.7}$$

$$\int_{\{z\,:\psi(z)<-1\}} \nu^{\beta}_{\sigma_2}(\mathrm{d}z) < +\infty, \tag{4.8}$$

where the function ψ: $S^{d-1} \to \mathbb{R}$ is defined via $(\mathrm{d}\nu^{\alpha,\beta}_{\sigma,q}/\mathrm{d}\nu^{\beta}_{\sigma_2})(z) = \mathrm{e}^{\psi(z)}$, that is,

$$\psi(z) = \ln\left(\frac{q(\|z\|, z/\|z\|)}{c_2(z/\|z\|)\|z\|^{-\beta-1}}\right), \qquad z \in \mathbb{R}^d_0.$$

As in the proof of (i), observe that

$$\lim_{\|z\|\to\infty} \psi(z) = \lim_{\|z\|\to\infty} \ln\left(\frac{c_2(z/\|z\|)\|z\|^{-\alpha-1}}{c_2(z/\|z\|)\|z\|^{-\alpha-1}}\right) = 0,$$

and that, as $\|z\| \to 0$,

$$\psi(z) \sim \ln\left(\frac{c_1(z/\|z\|)\|z\|^{-\alpha-1}}{c_2(z/\|z\|)\|z\|^{-\beta-1}}\right) = \ln\left(\frac{c_1(z/\|z\|)}{c_2(z/\|z\|)}\right) + (\beta-\alpha)\ln\|z\| \to \begin{cases} +\infty, & \text{if } \alpha > \beta, \\ -\infty, & \text{if } \alpha < \beta. \end{cases}$$

Therefore, condition (4.7) fails when $\alpha > \beta$ since

$$\int_{\{z:\psi(z)>1\}} \mathrm{e}^{\psi(z)}\nu^{\beta}_{\sigma_2}(\mathrm{d}z) = \nu^{\alpha,\beta}_{\alpha,q}(\{z \in \mathbb{R}^d_0 : \varphi(z) > 1\}) = +\infty,$$

while (4.8) fails when $\alpha < \beta$ since $\nu^{\beta}_{\sigma_2}(\{z \in \mathbb{R}^d_0 : \psi(z) < -1\}) = +\infty$.

(iii) This is a direct consequence of (i) with the help of Theorem 33.2 of Sato (1999). □

Remark 4.1. As in Example 2.1, let

$$q(r, \xi) = r^{-\alpha-1}\mathbf{1}_{(0,1]}(r) + r^{-\beta-1}\mathbf{1}_{(1,\infty)}(r), \qquad \xi \in S^{d-1}.$$

Then the Lévy process $\{U_t : t \geqslant 0\}$ given in (4.1) becomes

$$U_t = (\alpha - \beta) \sum_{\{s \in (0,t]: \|\Delta X_s\| > 1\}} \ln(\|\Delta X_s\|) - t\left(\frac{1}{\beta} - \frac{1}{\alpha}\right)\sigma(S^{d-1}).$$

Intuitively speaking, the density transformation $(\mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P})|_{\mathcal{F}_t}$ replaces all the β-stable jumps of a layered stable process up to time t (i.e., the jumps with absolute size greater than 1) by the corresponding α-stable jumps with the same jump direction. Moreover, when $\alpha < \beta$, the Lévy measure ν of $\mathcal{L}(U_1)$ is concentrated on $(-\infty, 0)$ and is given by

$$\nu(-\infty, y) = \alpha^{-1}\sigma(S^{d-1})\exp\left(\frac{\alpha}{\beta - \alpha}\right)y, \qquad y < 0,$$

while when $\alpha > \beta$, it is concentrated on $(0, \infty)$ and is given by

$$\nu(y, \infty) = \alpha^{-1}\sigma(S^{d-1})\exp\left(\frac{\alpha}{\beta - \alpha}\right)y, \qquad y > 0.$$

Let us next restate the absolute continuity result (Theorem 4.1) based on the fact that a series representation generates sample paths of a Lévy process directly by generating every single jump. For simplicity, we consider the symmetric case. Let $\{Y_t : t \geqslant 0\}$ be an α-stable process with $\mathcal{L}(Y_1) \sim S_\alpha(\sigma; k_1)$. By Lemma 1.1, there exists a version of $\{Y_t : t \in [0, T]\}$ given by

$$Y'_t = \sum_{i=1}^{\infty}\left(\frac{\alpha\Gamma_i}{\sigma(S^{d-1})T}\right)^{-1/\alpha} V_i \mathbf{1}(T_i \leqslant t) + k_1 t.$$

Also, let $\{X_t : t \geqslant 0\}$ be a layered stable process with $\mathcal{L}(X_1) \sim LS_{\alpha,\beta}(\sigma, q; k_0)$. In view of the series representation (2.11), there exists a version of $\{X_t : t \in [0, T]\}$ given by

$$X'_t = \sum_{i=1}^{\infty}\left[\left(\frac{\beta\Gamma_i}{\sigma(S^{d-1})T}\right)^{-1/\beta}\mathbf{1}_{(0,\sigma(S^{d-1})T/\beta)}(\Gamma_i) + \left(\frac{\alpha\Gamma_i}{\sigma(S^{d-1})T} + 1 - \frac{\alpha}{\beta}\right)^{-1/\alpha}\mathbf{1}_{(\sigma(S^{d-1})T/\beta,\infty)}(\Gamma_i)\right] V_i \mathbf{1}(T_i \leqslant t) + k_0 t,$$

where all the random sequences are the same as those appearing in $\{Y'_t : t \in [0, T]\}$ above. By Theorem 4.1, they are mutually absolutely continuous if and only if

$$k_0 - k_1 = \begin{cases} \dfrac{1}{\alpha - 1}\displaystyle\int_{S^{d-1}} \xi\sigma_1(\mathrm{d}\xi), & \text{if } \alpha \in (0, 1) \cup (1, 2), \\ 0, & \text{if } \alpha = 1. \end{cases}$$

We infer that the Lévy process $\{U_t : t \in [0, T]\}$ in the Radon–Nikodym derivative of Theorem 4.1(iii), that is,

$$\left.\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_t} = \mathrm{e}^{U_t},$$

has a version given by

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}|_{\mathcal{F}_t} = \mathrm{e}^{U_t},$$

has a version given by

$$U'_t = -\frac{\alpha - \beta}{\alpha} \sum_{i=1}^{\infty} \ln\left(\frac{\alpha \Gamma_i}{\sigma(S^{d-1})T}\right) \mathbf{1}_{(0, \sigma(S^{d-1})T/\alpha]}(\Gamma_i) \mathbf{1}(T_i \leqslant t) - t\left(\frac{1}{\beta} - \frac{1}{\alpha}\right)\sigma(S^{d-1}).$$

As a direct consequence, we have

$$\mathbb{P}(X \in B) = \mathbb{E}_{\mathbb{P}}[\mathrm{e}^{U'_T} 1_B(Y')], \qquad B \in \mathcal{B}(\mathbb{D}([0, T], \mathbb{R}^d)).$$

Moreover, in view of Theorem 33.2 of Sato (1999),

$$\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{Q}}|_{\mathcal{F}_t} = \mathrm{e}^{-U_t},$$

and so we can derive a version of $\{U_t : t \in [0, T]\}$ in terms of the jumps of the layered stable process as follows:

$$U''_t = -\frac{\alpha - \beta}{\beta} \sum_{i=1}^{\infty} \ln\left(\frac{\beta \Gamma_i}{\sigma(S^{d-1})T}\right) \mathbf{1}_{(0, \sigma(S^{d-1})T/\beta]}(\Gamma_i \mathbf{1}(T_i \leqslant t) - t\left(\frac{1}{\beta} - \frac{1}{\alpha}\right)\sigma(S^{d-1}).$$

Similarly, we have

$$\mathbb{Q}(Y \in B) = \mathbb{E}_{\mathbb{Q}}[\mathrm{e}^{-U''_T} 1_B(X')], \qquad B \in \mathcal{B}(\mathbb{D}([0, T], \mathbb{R}^d)).$$

5. Concluding remarks

We conclude this paper with a number of observations.

First, the weak convergence towards a Brownian motion, proved in Proposition 3.3(i), is interesting in the sense that a stable process with uniformly dependent components converges in law to a standard Brownian motion. It is also interesting to see how a stable process with independent components can converge towards a Brownian motion. To this end, for $i = 1, \ldots, d$, let $a_i \in [0, \infty)$, let

$$b'_{i+} := (0, \ldots, 0, +1, 0, \ldots, 0), \qquad b'_{i-} := (0, \ldots, 0, -1, 0, \ldots, 0),$$

where $+1$ and -1 are located at the ith component, and set

$$\sigma(\mathrm{d}\xi) := \sum_{i=1}^{d} \frac{2-\alpha}{2} a_i (\delta_{b_{i+}}(\mathrm{d}\xi) + \delta_{b_{i-}}(\mathrm{d}\xi)), \qquad \xi \in S^{d-1},$$

where δ is the Dirac measure. Clearly, σ is a symmetric finite positive measure on S^{d-1}. Also, let $\{X_t^{(\alpha)} : t \geqslant 0\} \sim S_\alpha(\sigma; 0)$. Then, if y_i is the ith component of y, we have, by E.18.8–18.9 of Sato (1999) and using $\Gamma(x+1) = x\Gamma(x)$, $x > 0$,

$$\mathbb{E}[\mathrm{e}^{\mathrm{i}\langle y, X_1^{(\alpha)}\rangle}] = \exp\left[-\frac{\Gamma(1/2)\Gamma((2-\alpha)/2)}{2^\alpha \alpha \Gamma((1+\alpha)/2)}\int_{S^{d-1}} |\langle y, \xi\rangle|^\alpha \sigma(\mathrm{d}\xi)\right]$$

$$= \exp\left[-\frac{1}{2}\sum_{i=1}^{d} \frac{\Gamma(1/2)\Gamma(1+(2-\alpha)/2)}{2^{\alpha-2}\alpha\Gamma((1+\alpha)/2)} a_i |y_i|^\alpha\right]$$

$$\to \exp\left[-\frac{1}{2}\sum_{i=1}^{d} a_i |y_i|^2\right], \qquad \text{as } \alpha \uparrow 2.$$

Therefore, as $\alpha\uparrow 2$, we obtain $\{X_t^{(\alpha)} : t \geqslant 0\}\{W_t : t \geqslant 0\}$, where $\{W_t : t \geqslant 0\}$ is a Brownian motion with covariance matrix

$$\begin{pmatrix} a_1 & 0 & \dots & 0 \\ 0 & a_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_d \end{pmatrix}.$$

Second, by making use of the absolute continuity of Lévy measures, we can derive two more forms of the series representation of a layered stable process corresponding to the Lévy measure (2.7), with $\alpha < \beta$. With the notation of Theorem 4.1, we obtain, for $z \in \mathbb{R}_0^d$,

$$\frac{\mathrm{d}\nu_{\sigma,q}^{\alpha,\beta}}{\mathrm{d}\nu_\sigma^\alpha}(z) = \mathbf{1}_{(0,1]}(\|z\|) + \|z\|^{\alpha-\beta}\mathbf{1}_{(1,\infty)}(\|z\|) \leqslant 1$$

and

$$\frac{\mathrm{d}\nu_{\sigma,q}^{\alpha,\beta}}{\mathrm{d}\nu_\sigma^\beta}(z) = \|z\|^{\beta-\alpha}\mathbf{1}_{(0,1]}(\|z\|) + \mathbf{1}_{(1,\infty)}(\|z\|) \leqslant 1.$$

Then, by the rejection method of Rosiński (2001), the summands $\{\overleftarrow{q}(\Gamma_i/T, V_i)V_i\}_{i\geqslant 1}$ in (2.11) can be respectively replaced by

$$\left\{\left(\frac{\alpha\Gamma_i}{\sigma(S^{d-1})T}\right)^{-1/\alpha}\mathbf{1}\left(\frac{\mathrm{d}\nu_{\sigma,q}^{\alpha,\beta}}{\mathrm{d}\nu_\sigma^\alpha}\left(\left(\frac{\alpha\Gamma_i}{\sigma(S^{d-1})T}\right)^{-1/\alpha}V_i\right) \geqslant U_i\right)V_i\right\}_{i\geqslant 1}$$

and

$$\left\{\left(\frac{\beta\Gamma_i}{\sigma(S^{d-1})T}\right)^{-1/\beta}\mathbf{1}\left(\frac{\mathrm{d}\nu_{\sigma,q}^{\alpha,\beta}}{\mathrm{d}\nu_\sigma^\beta}\left(\left(\frac{\beta\Gamma_i}{\sigma(S^{d-1})T}\right)^{-1/\beta}V_i\right) \geqslant U_i\right)V_i\right\}_{i\geqslant 1},$$

where $\{U_i\}_{i\geqslant 1}$ is a sequence of i.i.d. uniform random variables on [0, 1], independent of all the other random sequences.

Third, in complete similarity to the work presented in Houdré and Kawai (2006), it is possible to define a notion of fractional layered stable motion. Then, as in that work, over short intervals, fractional layered stable motion will be close to fractional stable motion (with inner index α) while over long intervals it will be close to either fractional Brownian motion (if $\beta > 2$) or to fractional stable motion (with index $\beta < 2$).

Fourth, let us observe some sample paths of a layered stable process, generated via the series representation (2.11). By Theorems 3.1 and 3.2, the entire situation is exhausted by the following three cases:

(i) $\alpha < \beta < 2$,
(ii) $\beta \in (2, \infty)$,
(iii) $\alpha > \beta$ with $\beta \in (0, 2)$.

Figure 1 corresponds to case (i) and typical sample paths of a symmetric layered stable process with $(\alpha, \beta) = (1.3, 1.9)$ are drawn in short-range, regular and long-range settings. For better comparison, we also drew its corresponding 1.3-stable and 1.9-stable processes. All these sample paths are generated via the series representation (2.11) for a layered stable process, or the one given in Lemma 1 for stable processes. Three sample paths within each figure are generated on a common probability space in the sense that a common set of random sequences $\{\Gamma_i\}_{i\geqslant 1}$, $\{V_i\}_{i\geqslant 1}$ and $\{T_i\}_{i\geqslant 1}$ is used. The desired short- and long-range behaviours are apparent.

For case (ii), we draw in Figure 2 typical sample paths of a symmetric layered stable process with $(\alpha, \beta) = (1.1, 2.5)$, along with its corresponding 1.1-stable process and a Brownian motion with a suitable variance. The layered stable process and the 1.1-stable process are generated as before, while the Brownian motion is independent of the others. As expected, the long-range Gaussian-type behaviour (Theorem 3.2(ii)) is clearly apparent. These stable-type short-range and Gaussian-type long-range behaviours have long been considered to be very appealing in applications. At the top in Figure 2, the layered stable process and its short-range limiting stable process are almost indistinguishable in a graphical sense (of course, not probabilistically).

Finally, for case (iii), we give in Figure 3 typical sample paths of a symmetric layered stable process with $(\alpha, \beta) = (1.9, 1.3)$, along with its corresponding 1.9-stable and 1.3-stable processes. Unlike the sample path behaviours observed in Figure 1, the path of the layered stable processes behaves more continuously (like a 1.9-stable one) over short intervals, but more discontinuously over long intervals (like a 1.3-stable one).

Fifth, we briefly introduce another generalization of stable processes. Again, on $\mathbb{R}^d$, let μ be an infinitely divisible probability measure without Gaussian component. Then μ is *mixed stable* if its Lévy measure is given by

$$\nu(B) = \int_{(0,2)} \int_{S^{d-1}} \sigma(\mathrm{d}\xi) \int_0^\infty 1_B(r\xi) \frac{\mathrm{d}r}{r^{\alpha+1}} \varphi(\mathrm{d}\alpha), \qquad B \in \mathcal{B}(\mathbb{R}_0^d), \tag{5.1}$$

where φ is a probability measure on (0, 2) such that

$$\int_{(0,2)} \frac{1}{\alpha(2-\alpha)} \varphi(\mathrm{d}\alpha) < \infty.$$

The simplest example of a mixed stable distribution is formed by convolution of stable distributions of different orders. This can be constructed by setting $\varphi(\mathrm{d}\alpha) = \sum_k c_k \delta_{\alpha_k}(\mathrm{d}\alpha)$, where for all k, $\alpha_k \in (0, 2)$, $c_k \geqslant 0$, and $\sum_k c_k = 1$.

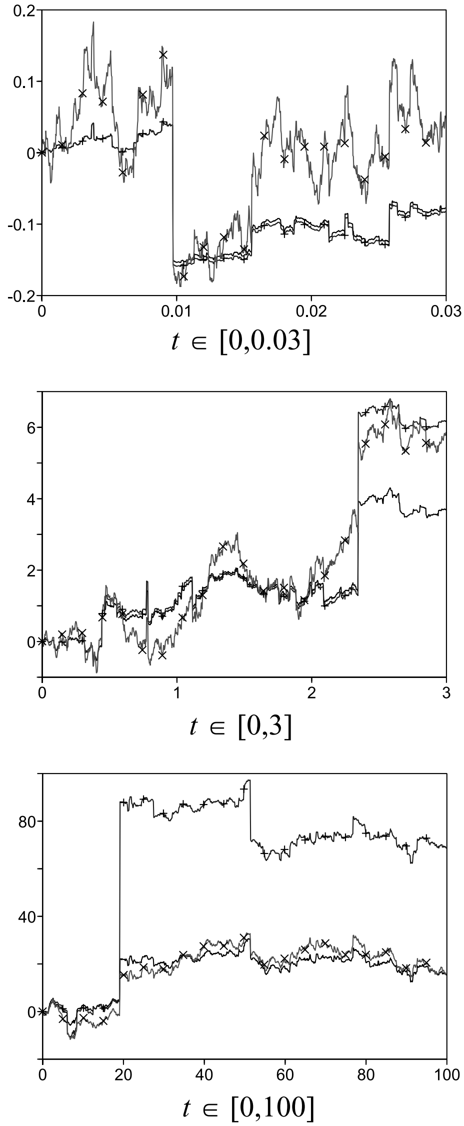

Figure 1. Typical sample paths of layered stable process with $(\alpha, \beta) = (1.3, 1.9)$, 1.3-stable process (+), and 1.9-stable process (×)

Recall that in Example 2.1 we defined the classes L_m, $m = 0, 1, \ldots$. Let $L_\infty := \cap_{m=0}^{\infty} L_m$. It is proved in Theorem 3.4 of Sato (1980) that an infinitely divisible probability measure without Gaussian component is in L_∞ if and only if its Lévy measure has the form (5.1), and that its characteristic function is given by

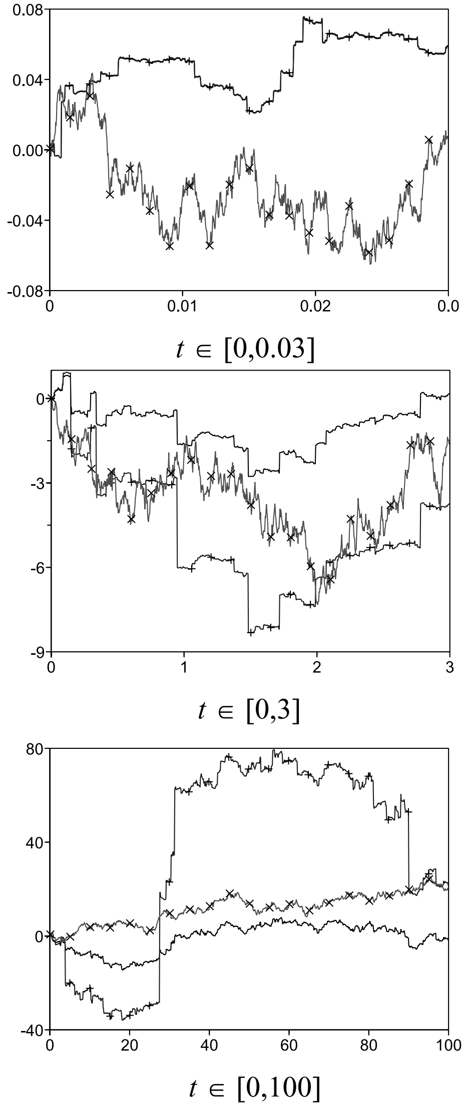

Figure 2. Typical sample paths of layered stable process with $(\alpha, \beta) = (1.1, 2.5)$, 1.1-stable process (+), and a Brownian motion ($\times$)

$$\widehat{\mu}(y) = \exp\biggl[\mathrm{i}\langle y, \eta\rangle - \int_{(0,2)} c_\alpha \int_{S^{d-1}} |\langle y, \xi\rangle|^\alpha \Bigl(1 - \mathrm{i}\tan\frac{\pi\alpha}{2}\operatorname{sgn}\langle y, \xi\rangle\Bigr)\sigma(\mathrm{d}\xi)\varphi(\mathrm{d}\alpha)$$

$$- \varphi(\{1\})c_1 \int_{S^{d-1}} \biggl(|\langle y, \xi\rangle| + i\frac{2}{\pi}\langle y, \xi\rangle \ln|\langle y, \xi\rangle|\biggr)\sigma(\mathrm{d}\xi)\biggr],$$

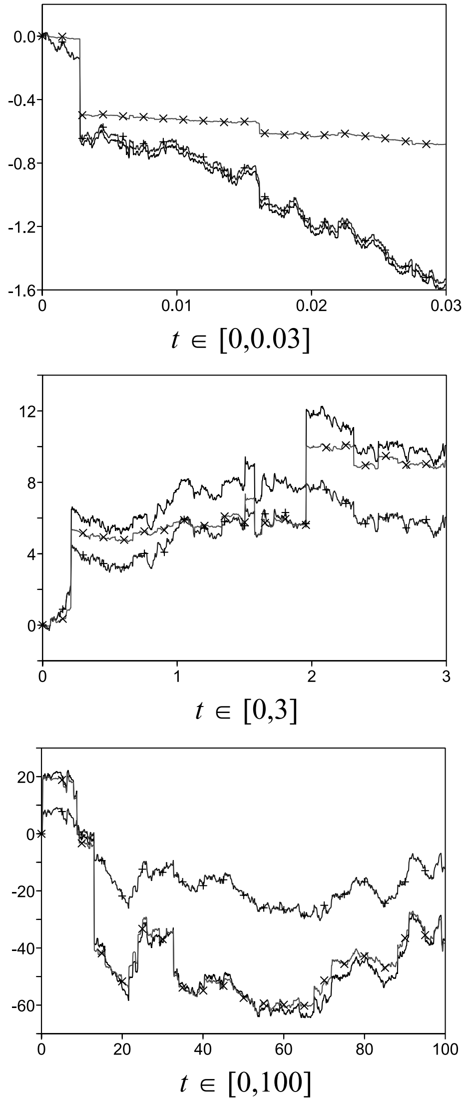

Figure 3. Typical sample paths of layered stable process with $(\alpha, \beta) = (1.9, 1.3)$, 1.9-stable process (+), and 1.3-stable process ($\times$)

for some $\eta \in \mathbb{R}^d$, and where $c_\alpha = |\Gamma(-\alpha)\cos(\pi\alpha/2)|$ when $\alpha \neq 1$ while $c_1 = \pi/2$. We have seen in Example 2.1 that an infinitely divisible probability measure is in L_0 if and only if the corresponding Lévy measure has the form

$$\int_{S^{d-1}} \sigma(\mathrm{d}\xi) \int_0^\infty \mathbf{1}_B(r\xi) k_\xi(r) \frac{\mathrm{d}r}{r}, \qquad B \in \mathcal{B}(\mathbb{R}_0^d),$$

where σ is a finite positive measure on S^{d-1} and where $k_\xi(r)$ is a non-negative function measurable in $\xi \in S^{d-1}$ and decreasing in $r > 0$. Recently, Barndorff-Nielsen *et al.* (2006) defined a new class of infinitely divisible distributions by further requiring that the function $k_\xi(r)$ be completely monotone in r for σ-almost every ξ. Mixed stable distributions are indeed in this class since $\int_{(0,2)} r^{-\alpha}\varphi(\mathrm{d}\alpha)$ is completely monotone.

Finally, note that the associated Lévy process that we call a *mixed stable process* possesses an interesting series representation. For simplicity, assume that σ in (5.1) is symmetric. Let $\{\Gamma_i\}_{i\geq 1}$, $\{T_i\}_{i\geq 1}$ and $\{V_i\}_{i\geq 1}$ be random sequences defined as before. In addition, let $\{\alpha_i\}_{i\geq 1}$ be a sequence of i.i.d. random variables with common distribution φ. Assume, moreover, that all these random sequences are mutually independent. Then, with the help of the generalized shot noise method of Rosiński (2001), it can be shown that the stochastic process

$$\left\{\sum_{i=1}^{\infty}\left(\frac{\alpha_i\Gamma_i}{\sigma(S^{d-1})T}\right)^{-1/\alpha_i} V_i\mathbf{1}(T_i \leq t) : t \in [0, T]\right\}$$

converges almost surely uniformly in t to a mixed stable process whose marginal law at time 1 is mixed stable with the Lévy measure (5.1). Comparing this result with the series representation of a stable process given in Lemma 1.1, a mixed stable process can be thought of as a stable process with each of its jumps obeying a randomly chosen stability index. This jump structure also implies that, unlike the layered stable process, the mixed stable process does not alter its behaviour in terms of the time range.

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